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The evolution of H-surfaces with a Plateau boundary condition

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Abstract

In this paper we consider the heat flow associated to the classical Plateau problem for surfaces of prescribed mean curvature. To be precise, for a given Jordan curve $\Gamma \subset \mathbb{R}^3$, a given prescribed mean curvature function $H: \mathbb{R}^3 \to \mathbb{R}$ and an initial datum $u_o: B \to \mathbb{R}^3$ satisfying the Plateau boundary condition, i.e. that $u_o|_{\partial B}: \partial B \to \Gamma$ is a homeomorphism, we consider the geometric flow

$$\partial_t u - \Delta u = -2(H \circ u) D_1 u \times D_2 u \quad \text{in } B \times (0, \infty),$$
$$u(\cdot, 0) = u_o \quad \text{on } B, \qquad u(\cdot, t)|_{\partial B}: \partial B \to \Gamma \text{ is weakly monotone for all } t > 0.$$

We show that an isoperimetric condition on H ensures the existence of a global weak solution. Moreover, we establish that these global solutions sub-converge as $t \to \infty$ to a conformal solution of the classical Plateau problem for surfaces of prescribed mean curvature.

1. Introduction

1.1. The history of the problem

The classical Plateau problem for H-surfaces consists in the construction of parametric surfaces $u: B \to \mathbb{R}^3$ with prescribed mean curvature H and with boundary Γ; here Γ is a given closed, rectifiable Jordan curve in $\mathbb{R}^3$. For parametric surfaces $u \in C^2(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ defined on the unit disk B in $\mathbb{R}^2$ it has the following formulation:

$$\begin{cases} \Delta u = 2(H \circ u) D_1 u \times D_2 u \quad \text{on } B, \\ u|_{\partial B}: \partial B \to \Gamma \text{ is a homeomorphism}, \\ |D_1 u|^2 - |D_2 u|^2 = 0 = D_1 u \cdot D_2 u \quad \text{on } B. \end{cases} \tag{1.1}$$

Here $(1.1)_1$ is called the H-surface equation and $(1.1)_3$ are the conformality relations. Non-constant C^2-solutions u to $(1.1)_1$ and $(1.1)_3$ are usually called H-surfaces in $\mathbb{R}^3$. The geometric significance of $(1.1)_1$ and $(1.1)_3$ is that

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its solutions are 2-dimensional immersed surfaces in $\mathbb{R}^3$ with mean curvature given by H. The Plateau boundary condition $(1.1)_2$ is a free boundary condition with one degree of freedom. Problem (1.1) has been treated by many authors, e.g. by Heinz [19], Hildebrandt [21,22], Gulliver and Spruck [16,17], Steffen [38,39] and Wente [45]. Several optimal results have been obtained in the seventies and these results essentially settle the existence problem (1.1) for disk type surfaces in $\mathbb{R}^3$. One prominent example is the result of Hildebrandt [21,22] which ensures the existence of an H-surface contained in a ball B_R of radius R in $\mathbb{R}^3$ whenever Γ is a closed, rectifiable Jordan curve contained in B_R and the prescribed mean curvature function satisfies $|H| \leqslant \frac{1}{R}$ on B_R.

In contrast to the Plateau problem for H-surfaces, much less is known for the associated flow to (1.1). This geometric flow can be formulated as follows:

$$\begin{cases} \partial_t u - \Delta u = -2(H \circ u) D_1 u \times D_2 u & \text{in } B \times (0,\infty), \\ u(\cdot,0) = u_o & \text{on } B, \\ u(\cdot,t)|_{\partial B} : \partial B \to \Gamma \text{ is weakly monotone for all } t > 0. \end{cases} \tag{1.2}$$

For the precise definition we refer to (1.8). In the special case $H \equiv 0$, i.e. the evolutionary Plateau problem for minimal surfaces, this flow was considered by Chang and Liu in [6–8]. Their main result ensures the existence of a global weak solution which sub-converges asymptotically as $t \to \infty$ to a conformal solution of the Plateau problem for minimal surfaces, i.e. a solution of (1.1) with $H \equiv 0$. Moreover, the same authors treated the case $H \equiv \text{const}$, see [7]. In this case, existence of a global weak solution with image contained in a ball of radius R, was shown under the Hildebrandt type condition $|H| < \frac{1}{R}$. Finally, in [42] Struwe considered the H-surface flow subject to a free boundary condition of the type $u(\cdot,t) \in S$ on ∂B and the orthogonality condition $\partial_r u(\cdot,t) \perp T_{u(\cdot,t)} S$ on ∂B for all $t > 0$. In this context S is assumed to be a sufficiently regular surface in $\mathbb{R}^3$ which is diffeomorphic to the standard sphere S^2.

With respect to the associated flow for a Dirichlet boundary condition on the lateral boundary, several results ensure the existence of global weak, respectively smooth classical solutions. In this case the problem can be formulated as follows:

$$\begin{cases} \partial_t u - \Delta u = -2(H \circ u) D_1 u \times D_2 u & \text{in } B \times (0,\infty), \\ u(\cdot) = u_o & \text{on } B \times \{0\} \cup \partial B \times (0,\infty), \end{cases} \tag{1.3}$$

for given initial and boundary values $u_o \in W^{1,2}(B,\mathbb{R}^3)$. In [33], Rey showed that the Hildebrandt type condition $|u_o| < R$ on B and $|H| < \frac{1}{R}$ for a Lipschitz continuous prescribed mean curvature function $H: B_R \to \mathbb{R}$ is sufficient to guarantee the existence of a smooth global solution of (1.3). For an existence result for short time existence of classical solutions without any assumption on H and u_o we refer to Chen and Levine [9]. In this paper also the bubbling phenomenon at a first singular time is analyzed. Such a bubbling was ruled out by Rey in [33] for the proof of the long time existence using the Hildebrandt condition. The previous papers rely on methods developed by Struwe [41] for the harmonic map heat flow. In recent papers Hong and Hsu [24] respectively Leone, Misawa and Verde [27] established the existence of a global weak solution for the evolutionary flow to higher dimensional H-surfaces by different methods; in the first paper the authors were also able to show that the solutions are of class $C^{1,\alpha}$, which is the best regularity one can expect for systems including the parabolic n-Laplacian as leading term. Again a Hildebrandt type condition serves to exclude the occurrence of H-bubbles during the flow. We note that all mentioned papers rely on the strong assumption of Lipschitz continuity for H and the Hildebrandt type condition for the existence proof of global solutions. These strong assumptions were considerably weakened in a previous paper [3], in the sense that an *isoperimetric condition* for bounded and continuous prescribed mean curvature functions $H: \mathbb{R}^3 \to \mathbb{R}$ is sufficient for the existence of global solutions to (1.3). Such an isoperimetric condition relates the weighted H-volume of a set $E \subset \mathbb{R}^3$ to its perimeter via

$$2\left|\int_E H\,d\xi\right| \leqslant c\mathbf{P}(E) \tag{1.4}$$

for any set $E \subset \mathbb{R}^3$ with finite perimeter $\mathbf{P}(E) \leqslant s$. The condition (1.4) is termed isoperimetric condition of type (c,s). In [38,39], Steffen showed that such a condition with $c < 1$ is sufficient for the existence of solutions to (1.1), and moreover that all known classical existence results can be deduced from such a condition. The paper [3] gives the full parabolic analogue of this result for the flow (1.3), which yields global solutions under a large variety of conditions. Moreover, the same isoperimetric condition allows to analyze the asymptotic behavior as $t \to \infty$, to be precise, global solutions sub-converge as $t \to \infty$ to solutions of the stationary Dirichlet problem for the H-surface equation. Under the Dirichlet boundary condition, these solutions of course cannot be expected to be conformal and therefore they admit no differential geometric meaning. For this reason we are here interested in the flow (1.2) under the geometrically more natural Plateau boundary condition. We prove that the free boundary condition $(1.2)_3$ allows the surfaces $u(\cdot,t)$ to adjust themselves conformally as $t \to \infty$, so that global solutions to (1.2) sub-converge to classical *conformal* solutions of the Plateau problem, which actually parametrize immersed surfaces with prescribed mean curvature.

1.2. Formulation of the problem and results

The aim of the present paper is to give a suitable meaning to the heat flow associated to the classical Plateau problem (1.1). In order to formulate this evolution problem, we need to explain to a certain extent some notations from the classical theory. Let $\Gamma \subset \mathbb{R}^3$ be a Jordan curve such that a C^3-parametrization $\gamma: S^1 \to \Gamma$ exists. By $\widehat{\gamma}: \mathbb{R} \to \Gamma$, we denote the corresponding map on the universal cover $\mathbb{R}$ of S^1, defined by $\widehat{\gamma}(\varphi) = \gamma(e^{i\varphi})$. Associated with the Jordan curve Γ we consider the following class of mappings from the unit disk $B \subset \mathbb{R}^2$ into $\mathbb{R}^3$ defined by

$$\mathcal{S}(\Gamma) := \left\{ u \in W^{1,2}\big(B, \mathbb{R}^3\big) \;\middle|\; \begin{array}{c} u|_{\partial B}: \partial B \to \Gamma \text{ is a continuous,} \\ \text{weakly monotone parametrization of } \Gamma \end{array} \right\}.$$

The monotonicity condition on $u|_{\partial B}$ means precisely that $u|_{\partial B}$ is the uniform limit of orientation preserving homeomorphisms from ∂B onto Γ. This class allows the action of the non-compact Möbius group of conformal diffeomorphisms of the disk into itself, i.e. with $u \in \mathcal{S}(\Gamma)$ we have $u \circ g \in \mathcal{S}(\Gamma)$ whenever $g \in \mathcal{G}$, where $\mathcal{G}$ denotes the Möbius group defined by

$$\mathcal{G} = \left\{ g: w \mapsto e^{i\varphi} \frac{a+w}{1+\bar{a}w}: \; a \in \mathbb{C},\ |a| < 1,\ \varphi \in \mathbb{R} \right\}.$$

In order to factor out the action of the Möbius group it is standard to impose a *three-point condition*. More precisely, we fix three arbitrary distinct points $P_1, P_2, P_3 \in \partial B$ – for convenience we may choose $P_k = e^{i\Theta_k}$ with $\Theta_k := \frac{2\pi k}{3}$ for $k = 1, 2, 3$ – and three distinct points $Q_1, Q_2, Q_3 \in \Gamma$ and impose the condition $u(P_k) = Q_k$ for $k = 1, 2, 3$. The corresponding function space we denote by

$$\mathcal{S}^*(\Gamma) := \big\{ u \in \mathcal{S}(\Gamma): \; u(P_k) = Q_k \text{ for } k = 1, 2, 3 \big\}. \tag{1.5}$$

We note that $u \in W^{1,2}(B, \mathbb{R}^3)$ is contained in $\mathcal{S}^*(\Gamma)$ if and only if $u(e^{i\vartheta}) = \widehat{\gamma}(\varphi(\vartheta))$ for all $\vartheta \in \mathbb{R}$ and some function $\varphi: \mathbb{R} \to \mathbb{R}$ that is contained in the space

$$\mathcal{T}^*(\Gamma) := \left\{ \varphi \in C^0 \cap W^{\frac{1}{2},2}(\mathbb{R}) \;\middle|\; \begin{array}{c} \varphi \text{ is non-decreasing, } \varphi(\cdot + 2\pi) = \varphi + 2\pi \\ \text{and } \widehat{\gamma}(\varphi(\Theta_k)) = Q_k \text{ for } k = 1, 2, 3 \end{array} \right\},$$

where here, $\Theta_k \in [0, 2\pi)$ is characterized by $e^{i\Theta_k} = P_k$ for $k = 1, 2, 3$. We can always achieve $\mathcal{T}^*(\Gamma) \neq \varnothing$ by changing the orientation of the parametrization $\gamma: S^1 \to \Gamma$ if necessary. The space of admissible testing functions for a given surface $u \in \mathcal{S}^*(\Gamma)$ with $u(e^{i\vartheta}) = \widehat{\gamma}(\varphi(\vartheta))$, is then given by

$$T_u\mathcal{S}^* := \big\{ w \in L^\infty \cap W^{1,2}\big(B, \mathbb{R}^3\big): \; w\big(e^{i\vartheta}\big) = \widehat{\gamma}'(\varphi)(\psi - \varphi) \text{ for some } \psi \in \mathcal{T}^*(\Gamma) \big\}.$$

We note that $T_u\mathcal{S}^*$ is a convex cone. The significance of this set becomes clear from Lemma 2.1 which ensures that a given $w \in T_u\mathcal{S}^*$ is the variation vector field of an admissible variation of u; here admissible has to be understood in the sense that the variation is contained in $\mathcal{S}^*(\Gamma)$ along the variation. The class $\mathcal{S}^*(\Gamma)$ also allows so-called *inner variations*. These variations are generated by vector fields η belonging to the class $\mathcal{C}^*(B)$ (cf. (2.5)), the class of all C^1-vector fields η on $\overline{B}$ which are tangential along ∂B and vanish at the three points P_1, P_2 and P_3.

Finally, for a given closed, convex obstacle $A \subset \mathbb{R}^3$ with $\Gamma \subset A^\circ$, we define

$$\mathcal{S}^*(\Gamma, A) := \big\{ u \in \mathcal{S}^*(\Gamma): \; u(x) \in A \text{ for a.e. } x \in B \big\}. \tag{1.6}$$

As already mentioned before our goal is to define a geometric flow associated with the classical Plateau problem (1.1) for surfaces with prescribed mean curvature function $H: A \to \mathbb{R}$ that is continuous and bounded in A. This geometric flow should allow the existence of global (weak) solutions which at least sub-converge asymptotically as $t \to \infty$ to solutions of the stationary Plateau problem (1.1). Our definition of this flow is as follows: For a given obstacle A, a given Jordan curve Γ contained in A° and an initial datum $u_o \in \mathcal{S}^*(\Gamma, A)$ we are looking for a global weak solution

$$u \in L^\infty\big(0,\infty; W^{1,2}\big(B,\mathbb{R}^3\big)\big) \quad \text{with } \partial_t u \in L^2\big(0,\infty; L^2\big(B,\mathbb{R}^3\big)\big) \tag{1.7}$$

to the following *evolutionary Plateau problem for H-surfaces*:

$$\begin{cases} \partial_t u - \Delta u = -2(H\circ u)D_1 u \times D_2 u & \text{weakly in } B\times(0,\infty),\\ u(\cdot,0) = u_o & \text{in } B,\\ u(\cdot,t)\in\mathcal{S}^*(\Gamma,A) & \text{for a.e. } t\in(0,\infty),\\ \displaystyle\int_B \big[Du(\cdot,t)\cdot Dw + \Delta u(\cdot,t)\cdot w\big]\,dx \geqslant 0 & \text{for a.e. } t\in(0,\infty) \text{ and all } w\in T_{u(\cdot,t)}\mathcal{S}^*,\\ \displaystyle\int_B \operatorname{Re}\big(\mathfrak{h}\big[u(\cdot,t)\big]\bar\partial\eta\big) + (\partial_t u\cdot Du)(\cdot,t)\eta\,dx = 0 & \text{for a.e. } t\in(0,\infty) \text{ and all } \eta\in\mathcal{C}^*(B). \end{cases} \tag{1.8}$$

In $(1.8)_5$ we have identified $\mathbb{R}^2$ with $\mathbb{C}$ and abbreviated $\bar\partial\eta := \frac{1}{2}(D_1\eta + \mathbf{i}D_2\eta)$. Further, for a map $w \in W^{1,2}(B,\mathbb{R}^3)$ we use the abbreviation

$$\mathfrak{h}[w] := |D_1 w|^2 - |D_2 w|^2 - 2\mathbf{i}D_1 w\cdot D_2 w. \tag{1.9}$$

We point out that for sufficiently regular u, by the Gauss–Green formula the inequality $(1.8)_4$ is equivalent to

$$\int_{\partial B} \frac{\partial u}{\partial r}(x,t)w(x,t)\,d\mathcal{H}^1 x \geqslant 0 \quad \text{for all } w\in T_{u(\cdot,t)}\mathcal{S}^*. \tag{1.10}$$

We therefore interpret $(1.8)_4$ as a weak formulation of (1.10). It is well defined in our situation because $\Delta u(\cdot,t)\in L^1(B)$ for a.e. t as a consequence of (1.7) and $(1.8)_1$, while (1.10) cannot be used in the general case since the radial derivative $\frac{\partial u}{\partial r}$ might not be well defined on ∂B. With this respect $(1.8)_4$ can be interpreted as a weak form of the Neumann boundary condition (1.10) and henceforth we shall denote $(1.8)_4$ *weak Neumann boundary condition*. The last property $(1.8)_5$ can be viewed as a type of conformality condition. For a stationary solution, i.e. a time independent solution, $(1.8)_5$ yields the conformality in B, that is we have $\mathfrak{h}[u]\equiv 0$ in B which is equivalent to $(1.1)_3$. For a weak solution of the evolutionary Plateau problem, starting with an initial datum u_o, we cannot expect the solution to be conformal for every time slice $t>0$. However, the asymptotic behavior as $t\to\infty$ should enforce the solution to become conformal. This can actually be shown for a sequence of time slices $t_j\to\infty$, since the constructed weak solutions obey the property $\partial_t u\in L^2(B\times(0,\infty))$. Therefore, weak solutions of (1.8) sub-converge as $t\to\infty$ asymptotically to a solution of the classical Plateau problem (1.1). In this sense (apart from the three-point condition which is inherited in $(1.8)_3$) the flow from (1.8) is a natural geometric flow associated to the classical Plateau problem for surfaces of prescribed mean curvature.

We also note that $(1.8)_1$ and $(1.8)_4$ can be combined to

$$\int_{B\times(0,\infty)} Du\cdot Dw + \partial_t u\cdot w + 2(H\circ u)D_1 u\times D_2 u\cdot w\,dz \geqslant 0 \tag{1.11}$$

for all $w\in L^\infty(B\times(0,\infty),\mathbb{R}^3)\cap L^2(0,\infty;W^{1,2}(B,\mathbb{R}^3))$ with $w(\cdot,t)\in T_{u(\cdot,t)}\mathcal{S}^*$ for a.e. $t\in(0,\infty)$. In order to keep the presentation more intuitive we prefer to use the H-surface system and weak Neumann type boundary condition separately, instead of the unified variational inequality (1.11).

To explain the main results of the present paper, we start by specifying the hypotheses. For the obstacle $A\subseteq\mathbb{R}^3$ we suppose that

$$A\subseteq\mathbb{R}^3 \text{ is closed, convex, with } C^2\text{-boundary and bounded principal curvatures.} \tag{1.12}$$

By $\mathcal{H}_{\partial A}(a)$ we denote the minimum of the principal curvatures of ∂A in the point $a\in\partial A$, taken with respect to the inward pointing unit normal vector. Moreover, we assume that

$$H: A \to \mathbb{R} \text{ is a bounded, continuous function} \tag{1.13}$$

and satisfies

$$|H| \leqslant \mathcal{H}_{\partial A} \quad \text{on } \partial A. \tag{1.14}$$

As before, we assume that

$$\Gamma \subset A^\circ \text{ is a Jordan curve parametrized by } \gamma \in C^3\big(S^1, \Gamma\big). \tag{1.15}$$

Furthermore, we suppose that H satisfies a *spherical isoperimetric condition of type* (c, s) *on* A, for parameters $0 < s \leqslant \infty$ and $0 < c < 1$. This means that for every spherical 2-current T (cf. Definition 3.2) with $\operatorname{spt} T \subseteq A$ and $\mathbf{M}(T) \leqslant s$ there holds

$$2\big|\langle Q, H\Omega\rangle\big| = 2\left|\int_A i_Q H\Omega\right| \leqslant c\mathbf{M}(T), \tag{1.16}$$

where Q denotes the unique integer multiplicity rectifiable 3-current with $\partial Q = T$, $\mathbf{M}(Q) < \infty$ and $\operatorname{spt} Q \subseteq A$. Moreover, i_Q denotes the integer valued multiplicity function of Q and Ω the volume form on $\mathbb{R}^3$. Finally, for the initial values $u_o \in \mathcal{S}^*(\Gamma, A)$, we assume that they satisfy

$$\int_B |Du_o|^2\,dx \leqslant s(1-c). \tag{1.17}$$

Note that this is automatically satisfied in the case $s = \infty$. Under this set of assumptions, we have the following general existence result.

Theorem 1.1. *Assume that* $A \subseteq \mathbb{R}^3$ *and* $H: A \to \mathbb{R}$ *satisfy the assumptions* (1.12)–(1.16) *and let* $u_o \in \mathcal{S}^*(\Gamma, A)$ *be given with* (1.17). *Then there exists a global weak solution*

$$u \in C^0\big([0,\infty); L^2(B, A)\big) \cap L^\infty\big((0,\infty); W^{1,2}(B, A)\big)$$

with $\partial_t u \in L^2(B \times (0,\infty), \mathbb{R}^3)$ *to* (1.8). *Moreover, the initial datum is achieved as usual in the* L^2*-sense, that is* $\lim_{t\downarrow 0} \|u(\cdot,t) - u_o\|_{L^2(B,\mathbb{R}^3)} = 0$.

With respect to the asymptotic behavior as $t \to \infty$ we have the following

Theorem 1.2. *Under the assumptions of Theorem 1.1 there exist a map* $u_* \in \mathcal{S}^*(\Gamma, A)$ *and a sequence* $t_j \to \infty$ *such that* $u(\cdot, t_j) \rightharpoonup u_*$ *weakly in* $W^{1,2}(B, \mathbb{R}^3)$ *and such that* u_* *is a solution of the Plateau problem for surfaces of prescribed mean curvature*

$$\begin{cases} \Delta u_* = 2(H \circ u_*) D_1 u_* \times D_2 u_* & \text{weakly in } B, \\ u_* \in \mathcal{S}^*(\Gamma, A), \\ |D_1 u_*|^2 - |D_2 u_*|^2 = 0 = D_1 u_* \cdot D_2 u_* & \text{in } B. \end{cases} \tag{1.18}$$

The solution satisfies $u_* \in C^0(\overline{B}, \mathbb{R}^3) \cap C^{1,\alpha}_{\mathrm{loc}}(B, \mathbb{R}^3)$ *for every* $\alpha \in (0,1)$, *and if* H *is Hölder continuous, then* $u_* \in C^{2,\beta}(\overline{B}, \mathbb{R}^3)$ *for some* $\beta \in (0,1)$ *and* u_* *is a classical solution of* (1.1).

1.3. Technical aspects of the proofs

In the present section, we briefly comment on the several different aspects that are joined to the existence proof.

Variational formulation via Geometric Measure Theory The starting point of our considerations is the observation that the geometric flow (1.8) admits a variational structure. This means that $u \mapsto -\Delta u + 2(H \circ u) D_1 u \times D_2 u$ can be interpreted as the Euler–Lagrange operator of the energy functional $\mathbf{E}_H(v) := \mathbf{D}(u) + 2\mathbf{V}_H(u, u_o)$ defined on the class $\mathcal{S}^*(\Gamma, A)$. Here, $\mathbf{V}_H(u, u_o)$ measures the oriented volume (taken with multiplicities as in (1.16)) enclosed by the surfaces u and u_o and weighted with the prescribed mean curvature function $H: A \to \mathbb{R}$. The definition of the volume term can be made rigorous by methods from Geometric Measure Theory, and at this stage we follow ideas introduced by Steffen [38,39]. Minimizers of such energy functionals are in particular stationary with respect to inner variations, i.e. $\frac{\partial}{\partial s}|_{s=0}\mathbf{E}_H(u \circ \phi_s) = 0$ whenever ϕ_s is the flow generated by a vector field $\eta \in \mathcal{C}^*(B)$. Since the volume term is invariant under inner transformations, minimizers of $\mathbf{E}_H$ satisfy $\partial\mathbf{D}(u;\eta) = \int_B \operatorname{Re}(\mathfrak{h}[u]\bar{\partial}\eta)\,dx = 0$, which leads to conformal solutions. The conformality is geometrically significant since it implies that the minimizers parametrize an immersed surface with mean curvature given by the prescribed function H. Finally, variations which take into account the possibility to vary minimizers along ∂B tangential to Γ give rise to a weak Neumann type boundary condition as $(1.8)_4$. Therefore, (1.8) can be interpreted as the gradient flow associated with the classical Plateau problem (1.1). For the construction of solutions to this gradient flow, we use the following time discretization approach.

Time discretization – Rothe's method This approach has been successfully carried out for the construction of weak solutions for the harmonic map heat flow by Haga, Hoshino and Kikuchi [18] and Kikuchi [26] (see also Moser [30] for an application of the technique to the biharmonic heat flow). For a fixed step size $h > 0$ we sub-divide $(0,\infty)$ into $((j-1)h, jh]$ for $j \in \mathbb{N}$. We fix a closed, convex subset $A \subseteq \mathbb{R}^3$ and a datum $u_o \in \mathcal{S}^*(\Gamma, A)$. For $j = 0$ we let $u_{o,h} := u_o$. Then, for $j \in \mathbb{N}$ we recursively define time-discretized energy functionals according to

$$\mathbf{F}_{j,h}(w) := \mathbf{D}(w) + \mathbf{V}_H(w, u_o) + \frac{1}{2h}\int_B |w - u_{j-1,h}|^2\,dx.$$

We construct $u_{j,h}$ as a minimizer of the functional $\mathbf{F}_{j,h}$ in a fixed sub-class of $\mathcal{S}^*(\Gamma, A)$, which may be defined for example by a further energy restriction such as $\mathbf{D}(u) \leqslant \sigma\mathbf{D}(u_o)$. At this stage, we impose a spherical isoperimetric condition on the prescribed mean curvature function $H: A \to \mathbb{R}$ to ensure the existence of an $\mathbf{F}_{j,h}$-minimizer. Moreover, since the leading terms $\mathbf{D}(w)$ and $\mathbf{V}_H(w, u_o)$ of the energy functional are conformally invariant, we impose the classical three-point condition of the type $u(P_k) = Q_k$, $k = 1, 2, 3$ for three points $P_k \in \partial B$, to factor out the action of the Möbius group in the leading terms of the functional. In this setting, we can ensure the existence of minimizers in $\mathcal{S}^*(\Gamma, A)$ to $\mathbf{F}_{j,h}$ by modifying the methods developed in [38] (see also [13,3]). Having the sequence of $\mathbf{F}_{j,h}$-minimizers $u_{j,h}$ at hand one defines an approximative solution to the Plateau H-flow from (1.8) by letting

$$u_h(x,t) := u_{j,h}(x) \quad \text{for all } x \in B \text{ and } j \in \mathbb{N} \text{ with } t \in \big((j-1)h, jh\big].$$

The constructed minimizers $u_{j,h}$ are actually Hölder continuous in the interior of B and continuous up to the boundary ∂B. This follows by using the $\mathbf{F}_{j,h}$-minimality along the lines of an old device of Morrey based on the harmonic replacement and comparison of energies. The lower order L^2-term, i.e. the term playing the role of the discrete time derivative, is at this stage harmless. This term has however a certain draw back. It is responsible for the fact that the Hölder estimates cannot be achieved uniformly in h when $h \downarrow 0$.

The obstacle condition $u_{j,h}(B) \subseteq A$ and the possible energy restriction of the form $\mathbf{D}(u_{j,h}) \leqslant \sigma\mathbf{D}(u_o)$ in principle only allow to derive certain variational inequalities for minimizers. However, if one imposes a condition relating the absolute value of the prescribed mean curvature function H along the boundary ∂A of the obstacle to the principle curvatures $\mathcal{H}_{\partial A}$ of ∂A, then by some sort of maximum principle the minimizers $u_{j,h}$ fulfill the Euler–Lagrange system associated with the functional $\mathbf{F}_{j,h}$. Formulated in terms of the function u_h, this system reads as

$$\Delta_t^h u_h - \Delta u_h + 2(H \circ u_h)D_1u_h \times D_2u_h = 0 \quad \text{weakly on } B \times (0,\infty) \tag{1.19}$$

if we abbreviate

$$\Delta_t^h w(x,t) := \frac{w(x,t) - w(x,t-h)}{h}$$

for the finite difference quotient in time. We mention that $u_h(\cdot,t) \in \mathcal{S}^*(\Gamma, A)$ for any $t \geqslant 0$, by construction. Moreover, varying the minimizers $u_{j,h}$ tangentially to Γ along ∂B yields the weak Neumann type boundary condition for the map u_h:

$$0 \leqslant \int_{B\times\{t\}} [Du_h \cdot Dw + \Delta u_h \cdot w]\,dx \tag{1.20}$$

for any $w \in T_{u_h(\cdot,t)}\mathcal{S}^*$ and $t > 0$. Finally, inner variations lead to some kind of perturbed conformality condition, more precisely

$$\int_{B\times\{t\}} \left[\mathrm{Re}\big(\mathfrak{h}[u_h]\bar{\partial}\eta\big) + \Delta_t^h u_h \cdot Du_h \eta\right] dx = 0 \tag{1.21}$$

whenever $\eta \in \mathcal{C}^*(B)$ and $t > 0$. The combination of (1.19), (1.20) and (1.21) means that u_h solves the time-discretized Plateau flow for surfaces of prescribed mean curvature, and the main effort of the paper is to show that the constructed solutions u_h actually converge to a solution of (1.8) as $h \downarrow 0$.

An ε-regularity result Due to the non-linear character of the time-discrete H-flow system (1.19), the (non-linear) Plateau type boundary condition appearing in (1.20) and the perturbed conformality condition (1.21), the analysis of the convergence is a non-trivial task and needs several technically involved tools. The major obstructions stem from three facts. Firstly, the non-linear H-term, i.e. $2(H \circ w)D_1 w \times D_2 w$, is not continuous with respect to weak convergence in $W^{1,2}$. Secondly, the weak boundary condition (1.20) associated with the Plateau problem contains a hidden non-linearity in the constraint $w \in T_{u_h(\cdot,t)}\mathcal{S}^*$ and therefore is also not compatible with weak convergence. Finally, the non-linear term $\mathfrak{h}[u_h]\bar{\partial}\eta$ also causes problems in the limit $h \downarrow 0$. For these reasons, one would need at least uniform local $W^{2,2}$-estimates up to the boundary in order to achieve local strong convergence in $W^{1,2}$.

However, the approximation scheme only yields uniform L^∞–$W^{1,2}$-bounds for u_h and L^2-bounds for the discrete time derivative $\Delta_t^h u_h$. Therefore, one can only conclude that a subsequence u_{h_i} converges in C^0–L^2 and weakly* in L^∞–$W^{1,2}$ to a limit map $u \in L^\infty$–$W^{1,2} \cap C^{0,\frac{1}{2}}$–$L^2$, and furthermore that the weak limit admits a time derivative $\partial_t u \in L^2$ and that $\Delta_t^{h_i} u_{h_i}$ converges weakly to $\partial_t u$ in L^2. These convergence properties are not sufficient, though, to pass to the limit neither in the non-linear H-term $H(u_h)D_1 u_h \times D_2 u_h$, nor in the boundary condition (1.20), nor in the non-linear term $\mathfrak{h}[u_h]$. For the treatment of these terms, we employ ideas used by Moser for the construction of a biharmonic map heat flow [30]. These methods have been successfully adapted in [3], where a related H-surface flow with a Dirichlet boundary condition on the lateral boundary has been studied (see also [4] for an application to the heat flow for n-harmonic maps).

First of all one argues slice-wise, that is for a fixed time t. Then the sequence $u_{h_i}(\cdot,t)$ is composed by different minimizers, all of them in $\mathcal{S}^*(\Gamma, A)$, and each of them satisfies (1.19), (1.20) and (1.21) on the fixed time slice. In particular, the maps u_{h_i} satisfy the three-point condition and have uniformly bounded Dirichlet energy and therefore are equicontinuous on ∂B. The idea now is to establish some sort of ε-regularity result. By this we mean an assertion of the form

$$\sup_{i\in\mathbb{N}} \int_{B_\varrho^+(x_o)} |Du_{h_i}|^2\, dx < \varepsilon \quad \Longrightarrow \quad \sup_{i\in\mathbb{N}} \|u_{h_i}\|_{W^{2,2}(B^+_{\varrho/2}(x_o))} < \infty, \tag{1.22}$$

where $\varepsilon > 0$ is a universal constant which can be determined in dependence on the data. Here $B_\varrho^+(x_o)$ denotes either an interior disk $B_\varrho(x_o) \subset B$ or a half-disk centered at a boundary point $x_o \in \partial B$. In any case we only consider disks such that $B_\varrho^+(x_o) \cap \{P_1, P_2, P_3\} = \varnothing$. The proof of statement (1.22) is the core of our construction of weak solutions and consists of two steps, which we summarize next.

A priori $W^{2,2}$-estimates up to the Plateau boundary The first step of the proof of (1.22) consists of proving a priori estimates under additional regularity assumptions. We establish them for general solutions which satisfy $\Delta u = F$ in B, together with a Plateau type boundary condition and the weak Neumann type condition (1.20). Here, we need to consider right-hand sides of critical growth $|F| \leqslant C(|Du|^2 + f)$ for some $f \in L^2(B)$. This is the reason why we can establish $W^{2,2}$-estimates in a first step only under the additional assumption $|Du| \in L^4_{\mathrm{loc}}$, which implies $F \in L^2_{\mathrm{loc}}$. In the interior, the local $W^{2,2}$-estimate (1.22) then follows via the difference quotient technique and an application of the Gagliardo–Nirenberg interpolation inequality in a standard way. However, the boundary version of this result is much more involved. Here we need local versions of global $W^{2,2}$-estimates which have been derived for minimal surfaces with a Plateau type boundary condition by Struwe in [43,25] (see also [6–8]). The local $W^{2,2}$-estimate follows by a technically involved angular difference quotient argument. For its implementation, additionally to $Du \in L^4(B_\varrho^+(x_o))$ we also need to assume that the oscillation of u on $B_\varrho^+(x_o)$ is small enough. This is needed in order to ensure that the image of u is contained in a tubular neighborhood of Γ, so that the nearest-point retraction onto Γ is well defined. In this situation, it is possible to adapt the standard variations that are used in the difference quotient argument in such a way that they are admissible under the Plateau boundary constraint. The additional assumption of small oscillation can be established by a Courant–Lebesgue type argument, once the local interior $W^{2,2}$-estimate is known. This is a consequence of an argument by Hildebrandt and Kaul [23] and has been exploited before in the situation of a free boundary condition in [36]. Therefore it only remains to establish the local $W^{1,4}$-estimate at the boundary in order to justify the application of the above $W^{2,2}$-estimates to the time-discretized H-surface flow.

Calderón–Zygmund estimates up to the boundary for systems with critical growth Here we use a Calderón–Zygmund type argument for solutions of systems of the type $\Delta u = F$ which satisfy a Plateau type boundary condition, where the right-hand side has critical growth as above. Our arguments are inspired by methods which have been developed for elliptic and parabolic p-Laplacian type systems by Acerbi and Mingione [1] (see also the paper by Caffarelli and Peral [5]). In order to deal with the critical growth of the inhomogeneity, we again need a small oscillation assumption for the derivation of suitable comparison estimates. The small oscillation is guaranteed by the continuity of the minimizers $u_{j,h}$. As local comparison problems, we consider the system $\Delta w = 0$ on $B^+_\varrho(x_o)$, together with the boundary condition $w = u$ on $B \cap \partial B_\varrho(x_o)$ and a Plateau type condition on $\partial B \cap B_\varrho(x_o)$. For such solutions local $W^{2,2}$-estimates hold, which allow an improvement of integrability of the gradient of u on its level sets. This improvement yields a quantitative Calderón–Zygmund estimate of the form

$$\int_{B^+_{\varrho/4}(x_o)} |Du|^4\,dx \leqslant \frac{C}{\varrho^2}\biggl(\int_{B^+_{\varrho/2}(x_o)} |Du|^2\,dx\biggr)^2 + C\int_{B^+_{\varrho/2}(x_o)} |f|^2\,dx,$$

for some universal constant C, provided $\operatorname{osc}_{B^+_\varrho(x_o)} u$ is small enough and $\|Du\|_{L^2}$ is bounded from above. For our applications however, we are only interested in the qualitative regularity $u \in W^{1,4}(B^+_{\varrho/2}(x_o), \mathbb{R}^3)$, which enables us to apply the a priori $W^{2,2}$-estimates from above and thereby to establish the ε-regularity result (1.22).

Concentration compactness arguments Next, we apply (1.22) to the sequence (u_{h_i}) on a fixed time slice $t > 0$. Since the smallness assumption on the left-hand side of (1.22) is satisfied for all but finitely many points $x_o \in \overline{B} \setminus \{P_1, P_2, P_3\}$ for a sufficiently small radius $\varrho(x_o) > 0$, we infer uniform $W^{2,2}$-estimates and therefore strong $W^{1,q}$-convergence for any $q \geqslant 1$ away from finitely many concentration points. Since anyway we have to deal with finitely many exceptional points, we can also exclude the points P_1, P_2, P_3 from the three-point condition from our considerations. The local strong convergence suffices to conclude that the non-linear terms in (1.19), (1.20) and (1.21) locally converge to the corresponding terms for the limit map u. Assuming that $\Delta^h_t u_h \to -f$ weakly in L^2, we infer that $u(\cdot,t)$ solves (1.19), (1.20) and (1.21) away from finitely many singular points if we replace u_h by u and $\Delta^h_t u_h$ by $-f$ in all three formulae. The finite singular set obviously is a set of vanishing $W^{1,2}$-capacity, and this enables us to deduce that $u(\cdot,t)$ is a weak solution to (1.19), (1.20) and (1.21) on all of B. It is worth to note, that in the capacity argument for the perturbed conformality relation (1.21) we have to utilize the regularity result by Rivière [34] for the H-surface equation and then the Calderón–Zygmund estimate mentioned above in order to have $\mathfrak{h}[u_h] \in L^2_{\mathrm{loc}}$. To conclude that u actually is a weak solution of (1.8) we need to have the identification $f = -\partial_t u$. This assertion can be achieved along the replacement argument by Moser [30].

Asymptotics as $t \to \infty$: Convergence to a conformal solution The strategy for the proof of the asymptotic behavior is similar, i.e. a concentration compactness argument combined with a capacity argument. The only major difference occurs since we can choose the time slices $t_i \to \infty$ in such a way that $\int_B |\partial_t u|^2(\cdot,t_i)\,dx \to 0$ as $i \to \infty$. Therefore, for the weak limit map $u_* := \lim_{i\to\infty} u(\cdot,t_i)$ the weak conformality condition (1.21) becomes

$$\partial\mathbf{D}(u_*;\eta) = \int_B \operatorname{Re}\bigl(\mathfrak{h}[u_*]\bar\partial\eta\bigr)\,dx = 0$$

for all $\eta \in \mathcal{C}^*(B)$. It is well known from the theory of H-surfaces that this identity implies the conformality of the limit map u_*. Moreover, the regularity result by Rivière [34] combined with classical arguments yield that u_* is regular up to the boundary. As a result, the flow sub-converges as $t \to \infty$ to a classical solution of the Plateau problem for H-surfaces, i.e. to a map that parametrizes an immersed surface with prescribed mean curvature and boundary contour given by Γ.

1.4. Applications

In this section we give some sufficient conditions ensuring the existence of a weak solution to the heat flow for surfaces with prescribed mean curvature satisfying a Plateau boundary condition (1.8). They follow from Theorem 1.1 and known criteria guaranteeing the validity of an isoperimetric condition, cf. [38,39,12,13].

Theorem 1.3. *Let A be convex and the closure of a C^2-domain in $\mathbb{R}^3$ and let the principal curvatures of ∂A be bounded. By $\mathcal{H}_{\partial A}$ we denote the minimum of the principal curvatures of ∂A. Further, we consider initial data $u_o \in \mathcal{S}^*(\Gamma, A)$ and $H \in L^\infty(A) \cap C^0(A)$. Then each of the following conditions*

$$\sup_A |H| \leqslant \sqrt{\frac{2\pi}{3\mathbf{D}(u_o)}}, \tag{1.23}$$

$$A \subseteq B_R \quad \text{and} \quad \int_{\{\xi \in A:\, |H(\xi)| \geqslant \frac{3}{2R}\}} |H|^3\, dx < \frac{9\pi}{2}, \tag{1.24}$$

$$\sup_A |H| < \frac{3}{2} \sqrt[3]{\frac{4\pi}{3\mathcal{L}^3(A)}}, \tag{1.25}$$

$$\mathcal{L}^3\big\{a \in A:\, \big|H(a)\big| \geqslant \tau\big\} \leqslant c\frac{4\pi}{3}\tau^{-3} \quad \text{for some } c < 1 \text{ and any } \tau > 0 \tag{1.26}$$

together with the curvature assumption

$$\big|H(a)\big| \leqslant \mathcal{H}_{\partial A}(a) \quad \text{for } a \in \partial A, \tag{1.27}$$

ensure the existence of a weak solution of (1.8) with the properties described in Theorem 1.1. The same conditions guarantee the sub-convergence of $u(\cdot, t)$ to a solution of the Plateau problem (1.1).

In the case $A \equiv \overline{B}_R(0) \subseteq \mathbb{R}^3$ the conditions (1.25) and (1.27) simplify to

$$\sup_{B_R(0)} |H| < \frac{3}{2}\frac{1}{R}, \qquad \big|H(a)\big| \leqslant \frac{1}{R} \quad \text{for } a \in \partial B_R(0).$$

Moreover, in this case we have that (1.24) is fulfilled. Consequently, both of the assumptions (1.24) and (1.25) contain the preceding Hildebrandt type assumptions as special cases and ensure the existence of a weak solution in the sense of Theorem 1.1 to the parabolic H-flow system (1.8). Finally, we note that (1.23) can be improved by choosing u_o to be an area minimizing disk type surface spanned by the Jordan curve Γ. Then, in (1.23) the Dirichlet energy of u_o equals the minimal area A_Γ spanned by Γ and the condition (1.23) turns into

$$\sup_A |H| \leqslant \sqrt{\frac{2\pi}{3A_\Gamma}}, \tag{1.28}$$

allowing large values of H for Jordan curves with small minimal area.

2. Notation and preliminaries

In this section we collect the main notation and some results needed in the proofs later.

2.1. Notation

Throughout this article, we write B for the open unit disk in $\mathbb{R}^2$. More generally, by $B_r(x_o) \subset \mathbb{R}^2$ we denote the open disk with center $x_o \in \mathbb{R}^2$ and radius $r > 0$. Moreover, we use the notation $B_r^+(x_o) := B \cap B_r(x_o)$ for the interior part of the disk $B_r(x_o)$, which will frequently be used in particular in the case for a center $x_o \in \partial B$. Furthermore, we use the abbreviations $S_r^+(x_o) := \partial B_r^+(x_o) \cap \overline{B}$ and $I_r(x_o) := B_r^+(x_o) \cap \partial B$, so that

$$\partial B_r^+(x_o) = S_r^+(x_o) \cup I_r(x_o).$$

For the *Dirichlet energy* of a map $u \in W^{1,2}(B, \mathbb{R}^3)$, we write

$$\mathbf{D}(u) := \frac{1}{2}\int_B |Du|^2\,dx \quad \text{and} \quad \mathbf{D}_G(u) := \frac{1}{2}\int_G |Du|^2\,dx$$

for any measurable subset $G \subset B$.

2.2. The chord-arc condition

Any Jordan curve Γ of class C^1 satisfies a (δ, M)-*chord-arc condition*, i.e. there are constants $\delta > 0$ and $M \geqslant 1$ such that for each pair of distinct points $p, q \in \Gamma$ we have

$$\min\{L(\Gamma_{p,q}), L(\Gamma^*_{p,q})\} \leqslant M|p-q| \quad \text{provided } |p-q| \leqslant \delta, \tag{2.1}$$

where $\Gamma_{p,q}$, $\Gamma^*_{p,q}$ denote the two sub-arcs of Γ that connect p with q, and $L(\cdot)$ is their length.

2.3. Admissible variations and variation vector fields

There are two possible types of variations for a given surface $u \in \mathcal{S}^*(\Gamma)$. The first type – called *outer variations* or *variations of the dependent variables* – are those performing a deformation of the surface in the ambient space $\mathbb{R}^3$. The initial vector field of the variation should be a map $w \in T_u\mathcal{S}^*$. However, it is not clear at this stage that such a vector field yields a one-sided variation $u_s \in \mathcal{S}^*(\Gamma)$ for values $0 \leqslant s \ll 1$ with $u_o = u$. Since we are dealing with surfaces contained in a closed, convex subset $A \subset \mathbb{R}^3$ we also need a version respecting the obstacle condition $u_s(B) \subset A$ along the variation. The existence of these kind of variations is granted by the following

Lemma 2.1. *Let $u \in \mathcal{S}^*(\Gamma)$ and $w \in T_u\mathcal{S}^*$ be given. Then there hold*:

(i) *There exists a one-sided variation $[0, \varepsilon) \ni s \mapsto u_s \in \mathcal{S}^*(\Gamma)$ with $u_o = u$ and $\frac{\partial}{\partial s}u_s|_{s=0} = w$.*
(ii) *If $\Gamma \subset A^\circ$, there exists a one-sided variation $[0, \varepsilon) \ni s \mapsto u_s \in \mathcal{S}^*(\Gamma, A)$ with $u_o = u$ and $\frac{\partial}{\partial s}u_s|_{s=0} \in (w + W_0^{1,2}(B, \mathbb{R}^3)) \cap C^0(\overline{B}, \mathbb{R}^3)$.*

In both cases, the variations u_s satisfy $\frac{\partial}{\partial s}u_s \in L^\infty(B, \mathbb{R}^3) \cap C^0(\partial B, \mathbb{R}^3)$ for all $s \in [0, \varepsilon)$ and moreover we have the following bounds:

$$\sup_{0\leqslant s<\varepsilon}\left(\|u_s\|_{W^{1,2}(B)} + \left\|\frac{\partial}{\partial s}u_s\right\|_{L^\infty(B)} + \left\|\frac{\partial}{\partial s}u_s\right\|_{C^0(\partial B)}\right) < \infty. \tag{2.2}$$

Proof. By $\varphi, \psi \in \mathcal{T}^*(\Gamma)$ we denote functions that are determined by the properties

$$u(e^{i\vartheta}) = \widehat{\gamma}(\varphi), \quad \text{respectively by} \quad w(e^{i\vartheta}) = \widehat{\gamma}'(\varphi)(\psi - \varphi).$$

For $s \in [0, \varepsilon)$, we define $h_s \in W^{1,2}(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ as the harmonic extension of the boundary data on ∂B given by $\widehat{\gamma}(\varphi + s(\psi - \varphi))$. These boundary data are bounded in $W^{\frac{1}{2},2}_{\mathrm{loc}}(\mathbb{R})$, uniformly in $s \in [0, \varepsilon)$, and therefore, its harmonic extensions satisfy

$$\sup_{0\leqslant s<\varepsilon}\|h_s\|_{W^{1,2}(B)} < \infty. \tag{2.3}$$

The derivative $\frac{\partial}{\partial s}h_s$ is the harmonic extension of the boundary values $\widehat{\gamma}'(\varphi + s(\psi - \varphi))(\psi - \varphi)$, which are uniformly bounded with respect to s in $C^0 \cap W^{\frac{1}{2},2}$. From the maximum principle we thereby infer

$$\sup_{0\leqslant s<\varepsilon}\left\|\frac{\partial}{\partial s}h_s\right\|_{C^0(\overline{B})} < \infty. \tag{2.4}$$

In particular, the function $\widetilde{w} := \frac{\partial}{\partial s} h_s|_{s=0}$ is the harmonic extension of the boundary values given by $\widehat{\gamma}'(\varphi)(\psi - \varphi)$ and therefore $\widetilde{w} \in (w + W_0^{1,2}(B, \mathbb{R}^3)) \cap C^0(\overline{B}, \mathbb{R}^3)$. Next, since $\varphi, \psi \in \mathcal{T}^*(\Gamma)$, which is a convex set, and $s \in [0, \varepsilon)$, we also have $\varphi + s(\psi - \varphi) \in \mathcal{T}^*(\Gamma)$, which means $h_s \in \mathcal{S}^*(\Gamma)$. Now we distinguish between the two cases stated in the lemma.

For the proof of (i), we define the variation u_s by

$$u_s := h_s + s(w - \widetilde{w}) - (h_0 - u).$$

Since $h_0 - u \in W_0^{1,2}(B, \mathbb{R}^3)$ and $w - \widetilde{w} \in W_0^{1,2}(B, \mathbb{R}^3)$, we conclude $u_s \in \mathcal{S}^*(\Gamma)$ for all $s \in [0, \varepsilon)$, and a straightforward calculation gives $\frac{\partial}{\partial s} u_s|_{s=0} = w$. The claimed bounds (2.2) follow from (2.3), (2.4) and $w, \widetilde{w} \in L^\infty \cap W^{1,2}(B, \mathbb{R}^3)$ with $w|_{\partial B} = \widetilde{w}|_{\partial B} \in C^0(\partial B, \mathbb{R}^3)$.

In the case of (ii), we choose a cut-off function $\zeta \in C^\infty(A, [0,1])$ with $\zeta \equiv 1$ on a neighborhood of Γ and $\operatorname{spt}\zeta \subset A^\circ$, which is possible by our assumption $\Gamma \subset A^\circ$. Then we define u_s by

$$u_s := u + \zeta(u)(h_s - h_0).$$

Because of (2.4), we can choose $\varepsilon > 0$ so small that $\|h_s - h_0\|_{L^\infty} < \operatorname{dist}(\operatorname{spt}\zeta, \partial A)$ for all $s \in [0, \varepsilon)$. Distinguishing between the cases $u(x) \in \operatorname{spt}\zeta$ and $u(x) \in A \setminus \operatorname{spt}\zeta$, we deduce $u_s(B) \subset A$ for any $s \in [0, \varepsilon)$. In order to compute the boundary values of $\frac{\partial}{\partial s}|_{s=0} u_s$, we note that $u(\partial B) \subset \Gamma$ and therefore $\zeta(u) \equiv 1$ on ∂B. We conclude $\frac{\partial}{\partial s}|_{s=0} u_s = \frac{\partial}{\partial s}|_{s=0} h_s$ on ∂B in the sense of traces and consequently, $\frac{\partial}{\partial s}|_{s=0} u_s = \widetilde{w} \in w + W_0^{1,2}(B, \mathbb{R}^3)$, as claimed. Again, the assertion (2.2) follows from (2.3) and (2.4). □

The second class of variations are the so-called *inner variations* or *variations of the independent variables*, which are re-parametrizations of the surfaces $u: B \to \mathbb{R}^3$ in the domain of definition. For the variation vector fields for this kind of variations we define the classes

$$\begin{cases} \mathcal{C}(B) := \{\eta \in C^1(\overline{B}, \mathbb{R}^3): \ \eta \text{ is tangential to } \partial B \text{ along } \partial B\}, \\ \mathcal{C}^*(B) := \{\eta \in \mathcal{C}(B): \ \eta(P_k) = 0 \text{ for } k = 1,2,3\}. \end{cases} \tag{2.5}$$

For $\eta \in \mathcal{C}^*(B)$ we consider the associated flow ϕ_s with $\phi_0 = \mathrm{id}$. Our assumptions on η ensure that $\phi_s(\overline{B}) \subset \overline{B}$ and $\phi_s(P_k) = P_k$ for all $s \in (-\varepsilon, \varepsilon)$ and $k \in \{1,2,3\}$. Moreover, since ϕ_s is an orientation preserving diffeomorphism for sufficiently small $|s|$, we know for $u \in \mathcal{S}^*(\Gamma, A)$ that $u \circ \phi_s|_{\partial B}$ is a weakly monotone parametrization of Γ and therefore $u_s := u \circ \phi_s \in \mathcal{S}^*(\Gamma, A)$. The first variation of the Dirichlet integral with respect to such inner variations is given by

$$\partial \mathbf{D}(u; \eta) := \frac{d}{ds}\bigg|_{s=0} \mathbf{D}(u \circ \varphi_s) = \int_B \operatorname{Re}\big(\mathfrak{h}[u] \bar{\partial}\eta\big)\, dx. \tag{2.6}$$

The following well-known compactness result is crucial for the existence of solutions to the Plateau problem. Its proof, which is based on the Courant–Lebesgue Lemma, can be found e.g. in [43, Lemma I.4.3].

Lemma 2.2. *The injection $\mathcal{S}^*(\Gamma) \hookrightarrow C^0(\partial B, \mathbb{R}^3)$ is compact, that is bounded subsets of $\mathcal{S}^*(\Gamma)$ (with respect to the $W^{1,2}$-norm) have equicontinuous traces on ∂B.*

2.4. An elementary iteration lemma

The following standard iteration result will be used in order to re-absorb certain terms.

Lemma 2.3. *For $0 < r < R$, let $f: [r, R] \to [0, \infty)$ be a bounded function with*

$$f(s) \leqslant \vartheta f(t) + \frac{A}{(t-s)^\alpha} + B \quad \text{for all } r \leqslant s < t \leqslant R,$$

for constants $A, B \geqslant 0$, $\alpha > 0$ and $\vartheta \in (0,1)$. Then we have

$$f(r) \leqslant c(\alpha, \vartheta)\left[\frac{A}{(R-r)^\alpha} + B\right].$$

2.5. An interpolation inequality

The following Gagliardo–Nirenberg interpolation inequality plays a central role in the proof of our regularity results and thereby for the construction of global weak solutions to our parabolic free boundary problem of Plateau type.

Lemma 2.4. *(See [32].) Let $B_\varrho(x_o) \subset \mathbb{R}^n$ with $0 < \varrho \leqslant 1$ and $B_\varrho^+(x_o) := B_\varrho(x_o) \cap B$. For any parameters $1 \leqslant \sigma, q, r < \infty$ and $\vartheta \in (0,1)$ such that $-\frac{n}{\sigma} \leqslant \vartheta(1-\frac{n}{q}) - (1-\vartheta)\frac{n}{r}$, there is a constant $C = C(n,q,r)$ such that for any $v \in W^{1,q}(B_\varrho^+(x_o))$ there holds*

$$\fint_{B_\varrho^+(x_o)} \left|\frac{v}{\varrho}\right|^\sigma dx \leqslant C\left(\fint_{B_\varrho^+(x_o)} \left|\frac{v}{\varrho}\right|^q + |Dv|^q\,dx\right)^{\frac{\vartheta\sigma}{q}} \left(\fint_{B_\varrho^+(x_o)} \left|\frac{v}{\varrho}\right|^r dx\right)^{\frac{(1-\vartheta)\sigma}{r}}.$$

For a map $u \in W^{2,2}(B_\varrho^+(x_o), \mathbb{R}^N)$, we may apply this to $v = |Du| \in W^{1,2}(B_\varrho^+(x_o))$, with the parameters $\sigma = 4$, $n = 2$, $q = r = 2$, $\vartheta = \frac{1}{2}$. This yields, with a universal constant C, the following interpolation estimate

$$\int_{B_\varrho^+(x_o)} |Du|^4\,dx \leqslant C \int_{B_\varrho^+(x_o)} \left|D^2u\right|^2 + \left|\frac{Du}{\varrho}\right|^2 dx \int_{B_\varrho^+(x_o)} |Du|^2\,dx. \tag{2.7}$$

The following lemma is due to Morrey [29, Lemma 5.4.1].

Lemma 2.5. *Assume that $v \in W_0^{1,2}(\Omega)$ for a domain $\Omega \subset \mathbb{R}^2$ and that $w \in L^1(\Omega)$ satisfies the Morrey growth condition*

$$\int_{B_r(y)\cap\Omega} |w|\,dx \leqslant C_o r^{2\alpha}$$

for all radii $r > 0$ and center $y \in \Omega$, with constants $C_o > 0$ and $\alpha > 0$. Then there holds $v^2 w \in L^1(\Omega)$ with

$$\int_{B_r(y)\cap\Omega} \left|v^2 w\right| dx \leqslant C_1 C_o |\Omega|^{\alpha/2} r^\alpha \int_\Omega |Dv|^2\,dx$$

for all $r > 0$, $y \in \Omega$ and a universal constant $C_1 = C_1(\alpha) > 0$.

2.6. A generalization of Rivière's result

The following result, which is a slight improvement of Rivière's fundamental paper [34], can be retrieved from [31].

Theorem 2.6. *Let $\Omega \in L^2(B, \mathfrak{so}(m) \otimes \mathbb{R}^2)$ and $f \in L^s(B, \mathbb{R}^m)$ with $s > 1$ be given. Then, any weak solution $u \in W^{1,2}(B, \mathbb{R}^m)$ of*

$$-\Delta u = \Omega \cdot Du + f \quad \text{on } B$$

is Hölder continuous in B for some Hölder exponent $\alpha \in (0,1)$. Moreover, if u admits a continuous boundary trace $u|_{\partial B}$, then u is also continuous up to the boundary, that is $u \in C^{0,\alpha}(B, \mathbb{R}^m) \cap C^0(\overline{B}, \mathbb{R}^m)$.

This result is important for our purposes since as noted by Rivière [34], the right-hand side of the H-surface equation $(1.1)_1$ can be written in the form $\Omega \cdot Du$. The difference of the above statement to the one in [34] stems from the fact that an L^s-perturbation with $s > 1$ of the critical right-hand side $\Omega \cdot Du \in L^1$ is considered. This generalization is necessary for our purposes. In our setting f plays the role of the time derivative $\partial_t u$ which by our construction will be an L^2-map on almost every time slice $B \times \{t\}$. The statement concerning the boundary regularity goes indeed back to [23, Lemma 3]. Once the interior regularity is established the assumption of a continuous boundary trace can be used to conclude the regularity up to the boundary by a simple lemma concerning Sobolev maps.

3. The H-volume functional

Here, we briefly recall the definition of the H-volume functional and some of its properties. For a more detailed treatment of the topic, we refer to [38] or [13]. The definition of the H-volume functional that we present here relies on the theory of currents. The standard references are [15] and [37].

3.1. Definitions

We write $\mathcal{D}^k(\mathbb{R}^3)$, $k \in \{0, 1, 2, 3\}$, for the space of smooth k-forms with compact support in $\mathbb{R}^3$. A distribution $T: \mathcal{D}^k(\mathbb{R}^3) \to \mathbb{R}$ is called *k-current* on $\mathbb{R}^3$. The *mass* of T is defined by

$$\mathbf{M}(T) := \sup\{T(\omega):\ \omega \in \mathcal{D}^k(\mathbb{R}^3),\ \|\omega\|_\infty \leqslant 1\}.$$

The boundary of a k-current T is the $(k-1)$-current ∂T given by $\partial T(\alpha) := T(d\alpha)$ for $\alpha \in \mathcal{D}^{k-1}(\mathbb{R}^3)$. A current T is called *closed* if $\partial T \equiv 0$. For the definition of the H-volume functional, the following sub-class of currents will be crucial.

Definition 3.1. A k-current T on $\mathbb{R}^3$ is called an *integer multiplicity rectifiable k-current* if it can be represented as

$$T(\omega) := \int_M \langle \omega(x), \xi(x)\rangle \theta(x)\, d\mathcal{H}^k(x) \quad \text{for all } \omega \in \mathcal{D}^k(\mathbb{R}^3),$$

where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure, $M \subset \mathbb{R}^3$ is an $\mathcal{H}^k$-measurable, countably k-rectifiable subset, $\theta: M \to \mathbb{N}$ is a locally $\mathcal{H}^k$-integrable function and $\xi: M \to \bigwedge_k \mathbb{R}^3$ is an $\mathcal{H}^k$-measurable function of the form $\xi(x) = \tau_1(x) \wedge \cdots \wedge \tau_k(x)$, where $\tau_1(x), \ldots, \tau_k(x)$ form an orthonormal basis of the approximate tangent space $T_x M$ for $\mathcal{H}^k$-a.e. $x \in M$.

The preceding definition follows the terminology of Simon [37]. In the language of Federer [15], the currents defined above are called locally rectifiable k-currents. Examples of integer multiplicity rectifiable 2-currents are induced by any map $u \in W^{1,2}(B, \mathbb{R}^3)$ via integration of 2-forms over the surface u as follows:

$$J_u(\omega) := \int_B u^\# \omega = \int_B \langle \omega \circ u, D_1 u \wedge D_2 u\rangle\, dx \quad \forall \omega \in \mathcal{D}^2(\mathbb{R}^3).$$

The fact that J_u is an integer multiplicity rectifiable 2-current in $\mathbb{R}^3$ can be checked by a Lusin type approximation argument as in [14, Section 6.6.3]. Moreover, the current J_u has finite mass since

$$\mathbf{M}(J_u) := \sup\{J_u(\omega):\ \omega \in \mathcal{D}^2(\mathbb{R}^3),\ \|\omega\|_\infty \leqslant 1\} \leqslant \int_B |D_1 u \wedge D_2 u|\, dx \leqslant \mathbf{D}(u).$$

If $v \in W^{1,2}(B, \mathbb{R}^3)$ is a parametric surface with associated 2-current J_v then $(J_u - J_v)(\omega)$ is determined by integration of $u^\# \omega - v^\# \omega$ over the set $G := \{x \in B:\ u(x) \neq v(x)\}$, and therefore we have

$$\mathbf{M}(J_u - J_v) \leqslant \mathbf{D}_G(u) + \mathbf{D}_G(v). \tag{3.1}$$

The main idea for the definition of the oriented H-volume $\mathbf{V}_H(u, v)$ enclosed by two surfaces $u, v \in \mathcal{S}^*(\Gamma, A)$ is to interpret the 2-current $J_u - J_v$ as the boundary of an integer multiplicity rectifiable 3-current Q of finite mass in $\mathbb{R}^3$, i.e. to write $J_u - J_v = \partial Q$. Such 3-currents can be interpreted as a set with integer multiplicities and finite (absolute) volume, more precisely, they can be written as

$$Q(\gamma) = \int_{\mathbb{R}^3} i_Q \gamma \quad \text{for all } \gamma \in \mathcal{D}^3(\mathbb{R}^3)$$

with an integer valued multiplicity function $i_Q \in L^1(\mathbb{R}^3, \mathbb{Z})$. Since in the present situation, the boundary ∂Q has finite mass, the multiplicity function i_Q turns out to be a BV-function on $\mathbb{R}^3$. The oriented H-volume enclosed by u and v can then be defined by

$$\mathbf{V}_H(u,v) := \int_{\mathbb{R}^3} i_Q H\Omega,$$

where Ω denotes the standard volume form on $\mathbb{R}^3$. We interpret this term as the volume of the set $\operatorname{spt} i_Q$, whose boundary is parametrized by the mappings u and v, where the multiplicities and the orientation are taken into account. In order to make this idea precise, we need to ensure the existence and the uniqueness of the 3-current Q with $\partial Q = J_u - J_v$ from above. We first note that the 2-currents $J_u - J_v$ considered here are spherical in the sense of

Definition 3.2. A 2-current T with support in A is called *spherical* iff it can be represented by a map $f \in W^{1,2}(S^2, A)$ in the form $T = f_\#[\![S^2]\!]$, i.e.

$$T(\omega) = \int_{S^2} f^\# \omega \quad \text{for all } \omega \in \mathcal{D}^2(\mathbb{R}^3). \tag{3.2}$$

From [13, Lemma 3.3] we recall the following fact.

Lemma 3.3. *For any $u, v \in \mathcal{S}^*(\Gamma, A)$ the current $J_u - J_v$ is a spherical 2-current in A.*

Since $T := J_u - J_v$ can be written in the form (3.2), it is in particular closed because

$$\partial T(\omega) = \int_{S^2} f^\# d\omega = \int_{S^2} d(f^\# \omega) = 0 \quad \text{for all } \omega \in \mathcal{D}^1(\mathbb{R}^3).$$

Therefore, for all $u, v \in \mathcal{S}^*(\Gamma, A)$, the current $T = J_u - J_v$ is a closed, integer multiplicity rectifiable 2-current of finite mass with $\operatorname{spt} T \subseteq A$. By the deformation theorem, we conclude the existence of an integer multiplicity rectifiable 3-current Q of finite mass with $\partial Q = T$ (see [37, Theorem 29.1] or [15, 4.2.9]). Furthermore, the constancy theorem implies that Q is unique up to integer multiples of $[\![\mathbb{R}^3]\!]$, which makes Q the unique current of finite mass with $\partial Q = T$. In order to prove $\operatorname{spt} Q \subseteq A$, we consider the nearest-point retraction $\pi : \mathbb{R}^3 \to A$ onto the convex set A. From $\partial \pi_\# Q = \pi_\# \partial Q = T$ and $\mathbf{M}(\pi_\# Q) \leqslant (\operatorname{Lip} \pi)^3 \mathbf{M}(Q) \leqslant \mathbf{M}(Q)$, we infer in view of the uniqueness established above that $\pi_\# Q = Q$. This means that $\operatorname{spt} Q \subseteq A$, as claimed. The above reasoning leads us to

Lemma 3.4. *Let $A \subseteq \mathbb{R}^3$ be a closed convex set. Then for every spherical 2-current T on $\mathbb{R}^3$ with $\operatorname{spt} T \subseteq A$, there exists a* unique *integer multiplicity rectifiable 3-current Q with the properties $\mathbf{M}(Q) < \infty$, $\partial Q = T$ and $\operatorname{spt} Q \subseteq A$.*

This result allows us to define the oriented H-volume enclosed by two maps $u, v \in \mathcal{S}^*(\Gamma, A)$.

Definition 3.5. For $u, v \in \mathcal{S}^*(\Gamma, A)$, we write $J_u - J_v$ for the associated spherical 2-current and $I_{u,v}$ for the unique integer multiplicity rectifiable 3-current with boundary $\partial I_{u,v} = J_u - J_v$, finite mass $\mathbf{M}(I_{u,v}) < \infty$ and $\operatorname{spt} I_{u,v} \subseteq A$. Then the *$H$-volume* enclosed by u and v is defined by

$$\mathbf{V}_H(u,v) := I_{u,v}(H\Omega) = \int_A i_{u,v} H\Omega.$$

Here, $i_{u,v}$ denotes the multiplicity function of $I_{u,v}$, and Ω the standard volume form of $\mathbb{R}^3$.

3.2. Some important properties of the H-volume

Throughout this work, we assume that H satisfies a spherical isoperimetric condition of type (c, s) on A as defined in (1.16). This condition can be re-written in terms of the H-volume as follows: Consider any $u, v \in \mathcal{S}^*(\Gamma, A)$ with $\mathbf{D}(u) + \mathbf{D}(v) \leqslant s$, so that in particular $\mathbf{M}(J_u - J_v) \leqslant s$. Then the H-volume enclosed by u and v is bounded by

$$2\big|\mathbf{V}_H(u,v)\big| \leqslant c\mathbf{M}(J_u - J_v) \leqslant c\big(\mathbf{D}_G(u) + \mathbf{D}_G(v)\big), \tag{3.3}$$

where $G = \{x \in B:\ u(x) \neq v(x)\}$. For the second inequality we refer to (3.1). Next, we state the following well-known invariance of the volume functional, cf. [13, (2.12)].

Lemma 3.6. *The H-volume is invariant under orientation preserving C^1-diffeomorphisms $\varphi, \psi: \overline{B} \to \overline{B}$ in the sense that for all $u, v \in \mathcal{S}^*(\Gamma, A)$, there holds*

$$\mathbf{V}_H(u \circ \varphi, v \circ \psi) = \mathbf{V}_H(u, v).$$

The next lemma states that the H-volume functional admits all the properties to derive the variational (in-)equality (first variation formula) later on. We have

Lemma 3.7. *Let $u, v \in \mathcal{S}^*(\Gamma, A)$ so that the H-volume $\mathbf{V}_H(u, v)$ is defined (cf. Definition 3.5). Then there hold:*

(i) *Assume that $\tilde{u} \in \mathcal{S}^*(\Gamma, A)$ is given. Then $\mathbf{V}_H(\tilde{u}, v)$ and $\mathbf{V}_H(\tilde{u}, u)$ (that are also well defined by Lemma 3.4) satisfy*

$$\mathbf{V}_H(\tilde{u}, u) + \mathbf{V}_H(u, v) = \mathbf{V}_H(\tilde{u}, v),$$

and

$$\big|\mathbf{V}_H(\tilde{u}, u)\big| \leqslant \|H\|_{L^\infty}\|u - \tilde{u}\|_{L^\infty}\big[\mathbf{D}_G(u) + \mathbf{D}_G(\tilde{u})\big],$$

where $G = \{x \in B:\ u(x) \neq \tilde{u}(x)\}$.

(ii) *Consider a one-sided variation $u_\tau \in \mathcal{S}^*(\Gamma, A)$, $\tau \in [0, \varepsilon)$, for which the bound*

$$\sup_{0\leqslant\tau<\varepsilon}\left(\|u_\tau\|_{W^{1,2}(B)} + \left\|\frac{\partial}{\partial\tau}u_\tau\right\|_{L^\infty(B)} + \left\|\frac{\partial}{\partial\tau}u_\tau\right\|_{C^0(\partial B)}\right) < \infty \tag{3.4}$$

holds true. Then $\mathbf{V}_H(u_\tau, u)$ and $\mathbf{V}_H(u_\tau, v)$ are defined for $\tau \in [0, \varepsilon)$ and with the abbreviation $U(\tau, x) := u_\tau(x)$, the following homotopy formula holds:

$$\mathbf{V}_H(u_\tau, v) - \mathbf{V}_H(u, v) = \mathbf{V}_H(u_\tau, u) = \int_B \int_0^\tau (H \circ U)\langle \Omega \circ U, U_s \wedge U_{x_1} \wedge U_{x_2}\rangle\, ds\, dx. \tag{3.5}$$

Proof. For the proof of (i), we refer to [13, Lemma 3.6 (i)]. We turn our attention to the proof of (ii). We define a 3-current by

$$Q_U(\phi) := \int_B \int_0^\tau \langle \phi \circ U, U_s \wedge U_{x_1} \wedge U_{x_2}\rangle\, ds\, dx = \int_{[0,\tau]\times B} U^\# \phi$$

for every $\phi \in \mathcal{D}^3(\mathbb{R}^3)$. The idea of the proof is to apply a construction similar to the one from [13, Lemma 3.3 (i)] to each of the functions $u_s := U(s, \cdot)$ for any $s \in [0, \tau]$. To this end, we note that since $u_s \in \mathcal{S}^*(\Gamma, A) \subseteq \mathcal{S}^*(\Gamma)$, we can find $\varphi_s \in \mathcal{T}^*(\Gamma)$ with

$$u_s\big(e^{i\vartheta}\big) = \widehat{\gamma}\big(\varphi_s(\vartheta)\big) \quad \text{for each } \vartheta \in [0, 2\pi].$$

Because $\widehat{\gamma}: \mathbb{R} \to \Gamma$ is a local C^1-diffeomorphism, the assumption (3.4) implies

$$\sup_{0\leqslant s\leqslant\tau}\left(\|\varphi_s\|_{W^{1/2,2}(0,2\pi)} + \left\|\frac{\partial}{\partial s}\varphi_s\right\|_{C^0([0,2\pi])}\right) < \infty. \tag{3.6}$$

Now we choose an arbitrary $\delta > 0$ and define $h_s: [1-\delta, 1] \times S^1 \to \mathbb{R}$ as the unique harmonic function with boundary values given by

$$h_s\big(1-\delta, e^{i\vartheta}\big) = \varphi_s(\vartheta) - \vartheta \quad \text{and} \quad h_s\big(1, e^{i\vartheta}\big) = 0.$$

We note that these boundary traces are well defined since $\varphi_s(\cdot + 2\pi) = \varphi_s + 2\pi$ for every $\varphi_s \in \mathcal{T}^*(\Gamma)$. As a consequence of (3.6), this function satisfies

$$\sup_{0\leqslant s\leqslant\tau}\|h_s\|_{W^{1,2}} \leqslant c \sup_{0\leqslant s\leqslant\tau}\big(\|\varphi_s\|_{W^{1/2,2}(0,2\pi)} + \|\mathrm{id}\|_{W^{1/2,2}(0,2\pi)}\big) < \infty. \tag{3.7}$$

Moreover, the derivative $\frac{\partial}{\partial s}h_s$ is again a harmonic function, with the boundary values given by $\frac{\partial}{\partial s}\varphi_s$ on $\{1-\delta\}\times\partial B$ and by zero on $\{1\}\times\partial B$. The maximum principle and (3.6) therefore imply

$$\sup_{0\leqslant s\leqslant\tau}\left\|\frac{\partial}{\partial s}h_s\right\|_{L^\infty}\leqslant\sup_{0\leqslant s\leqslant\tau}\left\|\frac{\partial}{\partial s}\varphi_s\right\|_{C^0([0,2\pi])}<\infty. \tag{3.8}$$

Now we are in a position to define the functions $\tilde{u}_s: B\to\mathbb{R}^3$ by

$$\tilde{u}_s\big(\varrho e^{i\vartheta}\big):=\begin{cases}u_s\big(\frac{\varrho}{1-\delta}e^{i\vartheta}\big) & \text{for } 0\leqslant\varrho<1-\delta,\\ \widehat{\gamma}(h_s(\varrho,e^{i\vartheta})+\vartheta) & \text{for } 1-\delta\leqslant\varrho\leqslant 1.\end{cases}$$

We note that the definition of h_s ensures that $\tilde{u}_s\in W^{1,2}(B,\mathbb{R}^3)$ for each $s\in[0,\tau]$. Since $\widehat{\gamma}$ is a local C^1-diffeomorphism, the bounds (3.4), (3.7) and (3.8) imply

$$\sup_{0\leqslant s<\tau}\left(\|\tilde{u}_s\|_{W^{1,2}(B)}+\left\|\frac{\partial}{\partial s}\tilde{u}_s\right\|_{L^\infty(B)}\right)<\infty. \tag{3.9}$$

Moreover, $\tilde{u}_s(e^{i\vartheta})=\widehat{\gamma}(\vartheta)$ for each $s\in[0,\tau]$ and $\vartheta\in\mathbb{R}$, so that $\tilde{u}_s|_{\partial B}$ is of class C^1. Finally, since $\tilde{u}_s$ is constructed as a re-parametrization of the original variation u_s and the C^1-diffeomorphism $\widehat{\gamma}$ we also have $\tilde{u}_s(B)\subseteq A$ for $s\in[0,\tau]$. We abbreviate $\widetilde{U}(s,x):=\tilde{u}_s(x)$ and observe that

$$\widetilde{U}\big([0,\tau]\times(B\setminus B_{1-\delta})\big)\subset\Gamma.$$

Since $\widetilde{U}|_{[0,\tau]\times B_{1-\delta}}$ is defined as a rescaled version of U and Γ is a 1-dimensional curve, the above construction does not change the corresponding currents, more precisely we have

$$Q_{\widetilde{U}}=Q_U,\qquad J_{\tilde{u}_o}=J_u\quad\text{and}\quad J_{\tilde{u}_\tau}=J_{u_\tau}.$$

We claim that $\partial Q_U=J_{u_\tau}-J_u$. To this end, we choose $\omega\in\mathcal{D}^2(\mathbb{R}^3)$ and calculate, using Stokes' theorem:

$$\begin{aligned}\partial Q_U(\omega)=Q_U(d\omega)=Q_{\widetilde{U}}(d\omega)&=\int_{[0,\tau]\times B}d\big(\widetilde{U}^{\#}\omega\big)\\ &=\int_B\tilde{u}_\tau^{\#}\omega-\int_B\tilde{u}_o^{\#}\omega+\int_{[0,\tau]\times\partial B}\widetilde{U}^{\#}\omega.\end{aligned}$$

Here, the application of Stokes' theorem can be justified by an approximation argument since we have (3.9) and $\widetilde{U}$ is of class C^1 on $[0,\tau]\times\partial B$. Next, we observe that the last integral vanishes because $\frac{\partial}{\partial s}\widetilde{U}$ vanishes on $[0,\tau]\times\partial B$. We thereby deduce

$$\partial Q_U(\omega)=J_{u_\tau}(\omega)-J_u(\omega)\quad\text{for all }\omega\in\mathcal{D}^2\big(\mathbb{R}^3\big).$$

The definition of the H-volume now yields the claim (3.5). $\square$

4. The time-discrete variational formulation

To set up the approximation scheme by time discretization we shall use H-energy functionals with a suitable lower order perturbation term of the form

$$\mathbf{F}(u):=\mathbf{D}(u)+2\mathbf{V}_H(u,u_o)+\frac{1}{2h}\int_B|u-z|^2\,dx\equiv\mathbf{F}^{(h)}_{u_o,z}(u) \tag{4.1}$$

defined for $u\in\mathcal{S}^*(\Gamma,A)$, where $u_o\in\mathcal{S}^*(\Gamma,A)$ is a given fixed reference surface; see Definition 3.5 for the notion of the volume functional. Here, $h>0$ and $z\in\mathcal{S}^*(\Gamma,A)$ are given. The H-volume term measures the oriented volume enclosed by u and the given fixed reference surface u_o weighted with H. In order not to overburden the presentation of the results and proofs we prefer not to indicate the dependence of the functional on the data u_o, z and $h>0$. We start with the following assertion concerning the first variation formulae.

Lemma 4.1.

(i) *Let $u_\tau \in \mathcal{S}^*(\Gamma, A)$, $\tau \in [0, \varepsilon)$ by a one-sided variation of $u \in \mathcal{S}^*(\Gamma, A)$ with initial vector field $\varphi \in L^\infty(B, \mathbb{R}^3) \cap W^{1,2}(B, \mathbb{R}^3)$ and assume that it satisfies the bounds (3.4). Then we have*

$$\lim_{\tau\downarrow 0} \frac{\mathbf{F}(u_\tau) - \mathbf{F}(u)}{\tau} = \int_B \left[\frac{u-z}{h} \cdot \varphi + Du \cdot D\varphi + 2(H \circ u) D_1 u \times D_2 u \cdot \varphi \right] dx. \tag{4.2}$$

(ii) *If $u \in \mathcal{S}^*(\Gamma, A)$ and φ_τ is the flow generated by a vector field $\eta \in \mathcal{C}^*(B)$, then*

$$\frac{d}{d\tau}\bigg|_{\tau=0} \mathbf{F}(u \circ \varphi_\tau) = \int_B \operatorname{Re}\big(\mathfrak{h}[u]\bar\partial\eta\big)\, dx + \frac{1}{h}\int_B (u-z) \cdot Du\eta\, dx. \tag{4.3}$$

Proof. The assertion (i) follows from a straightforward calculation, using the homotopy formula (3.5). For the claim (ii), in view of Lemma 3.6 and (2.6) we only have to compute

$$\frac{d}{d\tau}\bigg|_{\tau=0} \frac{1}{2h}\int_B |u \circ \varphi_\tau - z|^2\, dx = \frac{1}{h}\int_B (u-z) \cdot Du\eta\, dx. \qquad \square$$

The integral $\delta\mathbf{F}(u; \varphi)$ in (4.2) is called the *first variation* of the functional $\mathbf{F}$ in direction φ and the integral $\partial\mathbf{F}(u; \eta)$ from (ii) the *first variation of independent variables* (inner first variation) of $\mathbf{F}$ at u in the direction η. The preceding lemma leads to the following

Lemma 4.2. *Let $A \subseteq \mathbb{R}^3$ be the closure of a convex C^2-domain in $\mathbb{R}^3$ and assume that the principal curvatures of ∂A are bounded with*

$$\big|H(a)\big| \leqslant \mathcal{H}_{\partial A}(a) \quad \text{for } a \in \partial A, \tag{4.4}$$

where $\mathcal{H}_{\partial A}(a)$ denotes the minimum of the principle curvatures of ∂A at the point a with respect to the inner unit normal $\nu(a)$. Assume that $u \in \mathcal{S}^*(\Gamma, A)$ minimizes $\mathbf{F}$ in the class $\mathcal{S}^*(\Gamma, A)$. Then it satisfies the variational inequality

$$\int_B \left[\frac{u-z}{h} \cdot \varphi + Du \cdot D\varphi + 2(H \circ u) D_1 u \times D_2 u \cdot \varphi \right] dx \geqslant 0 \tag{4.5}$$

for all $\varphi \in T_u\mathcal{S}^*$, and moreover, the stationarity condition

$$\int_B \operatorname{Re}\big(\mathfrak{h}[u]\bar\partial\eta\big)\, dx + \frac{1}{h}\int_B (u-z) \cdot Du\eta\, dx = 0 \tag{4.6}$$

holds true for every $\eta \in \mathcal{C}^*(B)$.

Proof. The case of variation vector fields with zero boundary values is covered in our earlier work [3], cf. Lemma 4.3, Lemma 4.4 (iii). More precisely, under the condition (4.4), we derived the Euler–Lagrange equation

$$\int_B \left[\frac{u-z}{h} \cdot \varphi + Du \cdot D\varphi + 2(H \circ u) D_1 u \times D_2 u \cdot \varphi \right] dx = 0 \tag{4.7}$$

for every $\mathbf{F}$-minimizer $u \in \mathcal{S}^*(\Gamma, A)$ of $\mathbf{F}$ and any $\varphi \in L^\infty(B, \mathbb{R}^3) \cap W_0^{1,2}(B, \mathbb{R}^3)$. Therefore, it remains to prove the corresponding inequality for vector fields $w \in T_u\mathcal{S}^*$. To this end, we employ Lemma 2.1 (ii) to construct a one-sided variation $u_\tau \in \mathcal{S}^*(\Gamma, A)$, $\tau \in [0, \varepsilon)$ with the properties (3.4), $u_o = u$ and $\widetilde{w} := \frac{\partial}{\partial\tau}|_{\tau=0} u_\tau \in w + W_0^{1,2}(B, \mathbb{R}^3)$. Since the maps u_τ are admissible as comparison maps for u, we infer from (4.2) that

$$0 \leqslant \lim_{\tau\downarrow 0} \frac{\mathbf{F}(u_\tau) - \mathbf{F}(u)}{\tau} = \int_B \left[\frac{u-z}{h} \cdot \widetilde{w} + Du \cdot D\widetilde{w} + 2(H \circ u) D_1 u \times D_2 u \cdot \widetilde{w} \right] dx.$$

Moreover, $\varphi := w - \widetilde{w} \in L^\infty \cap W_0^{1,2}(B, \mathbb{R}^3)$ is admissible in (4.7). Joining this with the above inequality, we infer (4.5) for $\varphi = w$, which completes the proof of (4.5).

For the second assertion (4.6), we consider the flow ϕ_τ of the vector field $\eta \in \mathcal{C}^*(B)$ with $\phi_0 = \mathrm{id}$. Then, the maps $u \circ \phi_\tau \in \mathcal{S}^*(\Gamma, A)$ are admissible competitors for u (see the derivation of (2.6)), and (4.6) follows from the inner variation formula (4.3). □

5. Existence of minimizers to the time-discrete problem

The next lemma will be crucial for the construction of minimizers to the time-discrete volume functional by the direct method of the calculus of variations. It was proven in [13, Lemma 4.1], see also [11, Lemma 4.1] for a version in higher dimensions. The main idea is to control the volume of possible bubbles occurring in a minimizing sequence u_i by replacing it by a new sequence $\tilde{u}_i$. This new sequence agrees with the limit map u outside of a small set G on which bubbles may evolve, while on this set, the energy of the $\tilde{u}_i$ is controlled by the bubble energy (cf. (vi) below). The term $|V_H(\tilde{u}_i, u)|$ – which can be interpreted as the volume of the bubbles – can be bounded in terms of the Dirichlet energy by use of the isoperimetric condition. This enables us to establish a lower semicontinuity property of the time-discrete volume functional and thereby to prove the existence of **F**-minimizers.

Lemma 5.1. *Assume that $u_i \rightharpoonup u$ weakly in $W^{1,2}(B, \mathbb{R}^3)$ and $u_i|_{\partial B} \to u|_{\partial B}$ uniformly on ∂B. Then for every $\varepsilon > 0$ there exist $R > 0$, a measurable set $G \subseteq B$, and maps $\tilde{u}_i \in W^{1,2}(B, \mathbb{R}^3)$, such that after extraction of a subsequence there hold*:

(i) $\tilde{u}_i = u$ *on* $B \setminus G$ *with* $\mathcal{L}^2(G) < \varepsilon$;
(ii) $\tilde{u}_i|_{\partial B} = u|_{\partial B}$ *on* ∂B;
(iii) $\tilde{u}_i(x) = u_i(x)$ *if* $|u_i(x)| \geqslant R$ *or* $|u_i(x) - u(x)| \geqslant 1$;
(iv) $\lim_{i\to\infty} \|\tilde{u}_i - u_i\|_{L^\infty(B,\mathbb{R}^3)} = 0$;
(v) $\tilde{u}_i \rightharpoonup u$ *weakly in* $W^{1,2}(B, \mathbb{R}^3)$ *in the limit* $i \to \infty$;
(vi) $\limsup_{i\to\infty}[\mathbf{D}_G(\tilde{u}_i) + \mathbf{D}_G(u)] \leqslant \varepsilon + \liminf_{i\to\infty}[\mathbf{D}(u_i) - \mathbf{D}(u)]$;
(vii) *If the u_i take values in a closed convex subset $A \subseteq \mathbb{R}^3$, then the $\tilde{u}_i$ can also be chosen to have values in A.*

The proof of (i) to (vi) was carried out in [13, Lemma 4.1]. The assertion (vii) follows immediately from the construction in [13], since the maps $\tilde{u}_i$ are defined as convex combinations of u_i and u, whose images are contained in the convex set A.

Lemma 5.1 enables us to prove the existence of **F**-minimizers in the class

$$\mathcal{S}^*(\Gamma, A, \sigma) := \{w \in \mathcal{S}^*(\Gamma, A): \mathbf{D}(w) \leqslant \sigma \mathbf{D}(u_o)\},$$

where we choose $\sigma := \frac{1+c}{1-c}$ if $s < \infty$ and $\sigma = \infty$ otherwise. This choice of σ is made in such a way that (1.17) implies $(1+\sigma)\mathbf{D}(u_o) \leqslant s$.

Lemma 5.2. *Suppose $A \subseteq \mathbb{R}^3$ is a closed convex set and the function $H: A \to \mathbb{R}$ is bounded and continuous and satisfies a spherical isoperimetric condition of type (c, s) on A with $0 < s \leqslant \infty$ and $0 < c < 1$. Moreover let $u_o \in \mathcal{S}^*(\Gamma, A)$ be a fixed reference surface with (1.17) and z a fixed surface in $\mathcal{S}^*(\Gamma, A, \sigma)$ for σ defined as above. Then for every $h > 0$, the variational problem*

$$\mathbf{F}(w) := \mathbf{D}(w) + 2\mathbf{V}_H(u, u_o) + \frac{1}{2h}\int_B |u - z|^2\,dx \to \min \quad \text{in } \mathcal{S}^*(\Gamma, A, \sigma)$$

has a solution.

Proof. We first observe that for any $w \in \mathcal{S}^*(\Gamma, A, \sigma)$ by the choice of σ there holds

$$\mathbf{M}(J_w - J_{u_o}) \leqslant \mathbf{D}(w) + \mathbf{D}(u_o) \leqslant (\sigma + 1)\mathbf{D}(u_o) \leqslant s.$$

Hence, the spherical isoperimetric condition of type (c, s) gives

$$2\left|\mathbf{V}_H(w, u_o)\right| \leqslant c\mathbf{M}(J_w - J_{u_o}) \leqslant c\big(\mathbf{D}(w) + \mathbf{D}(u_o)\big).$$

This implies in particular that the functional is bounded from below on $\mathcal{S}^*(\Gamma, A, \sigma)$ by

$$\mathbf{F}(w) \geqslant (1-c)\mathbf{D}(w) - c\mathbf{D}(u_o) \geqslant -c\mathbf{D}(u_o). \tag{5.1}$$

We now consider an $\mathbf{F}$-minimizing sequence (u_i) of maps in $\mathcal{S}^*(\Gamma, A, \sigma)$, that is

$$\lim_{i\to\infty} \mathbf{F}(u_i) = \inf\big\{\mathbf{F}(w)\colon\ w \in \mathcal{S}^*(\Gamma, A, \sigma)\big\}.$$

Applying (5.1) to $w = u_i$, we infer

$$\sup_{i\in\mathbb{N}} \mathbf{D}(u_i) \leqslant \frac{1}{1-c}\Big[\sup_{i\in\mathbb{N}} \mathbf{F}(u_i) + c\mathbf{D}(u_o)\Big] < \infty.$$

Lemma 2.2 thus implies that the boundary traces $u_i|_{\partial B}$ are equicontinuous. Passing to a subsequence and taking Rellich's theorem into account we may therefore assume that the maps u_i converge weakly in $W^{1,2}(B, \mathbb{R}^3)$, strongly in $L^2(B, \mathbb{R}^3)$, and almost everywhere on B to a surface $u \in W^{1,2}(B, A)$ with $\mathbf{D}(u) \leqslant \sigma \mathbf{D}(u_o)$. Moreover, we have that $u_i|_{\partial B} \to u|_{\partial B}$ holds uniformly on ∂B. Due to the uniform convergence on ∂B and the fact that the sequences u_i satisfy the three-point condition $u_i(P_k) = Q_k$, also the limit surface fulfills the three-point condition $u(P_k) = Q_k$ for $k = 1, 2, 3$, and therefore we have $u \in \mathcal{S}^*(\Gamma, A, \sigma)$. We now apply Lemma 5.1 with a given $0 < \varepsilon < \frac{1}{2}\mathbf{D}(u)$ to obtain, after passage to another subsequence, surfaces $\tilde{u}_i \in \mathcal{S}^*(\Gamma, A)$. Since u, u_i and u_o are in the class $\mathcal{S}^*(\Gamma, A)$, Lemma 3.7 (i) and Lemma 5.1 (iv) and (vi) yield that

$$\big|\mathbf{V}_H(\tilde{u}_i, u_i)\big| \leqslant \|H\|_{L^\infty}\|\tilde{u}_i - u_i\|_{L^\infty}\big[\mathbf{D}_G(\tilde{u}_i) + \mathbf{D}_G(u_i)\big] \to 0 \quad \text{as } i \to \infty. \tag{5.2}$$

Now, we infer from Lemma 5.1 (vi) and (3.1) that for i large enough there holds

$$\mathbf{M}(J_{\tilde{u}_i} - J_u) \leqslant \mathbf{D}_G(\tilde{u}_i) + \mathbf{D}_G(u) \leqslant 2\varepsilon + \mathbf{D}(u_i) - \mathbf{D}(u) < \sigma\mathbf{D}(u_o) \leqslant s.$$

Therefore, by the spherical isoperimetric condition with $c < 1$ we have

$$2\big|\mathbf{V}_H(\tilde{u}_i, u)\big| \leqslant 2\varepsilon + \mathbf{D}(u_i) - \mathbf{D}(u).$$

Moreover we have

$$\mathbf{V}_H(u_i, u_o) - \mathbf{V}_H(u, u_o) = \mathbf{V}_H(\tilde{u}_i, u) - \mathbf{V}_H(\tilde{u}_i, u_i).$$

This allows us to conclude – with the help of the strong convergence $u_i \to u$ in $L^2(B, \mathbb{R}^3)$ and (5.2) – that there holds

$$\begin{aligned}
\mathbf{F}(u_i) &= \mathbf{D}(u_i) + 2\mathbf{V}_H(u_i, u_o) + \frac{1}{2h}\int_B |u_i - z|^2\,dx \\
&= \mathbf{F}(u) - \mathbf{D}(u) + \mathbf{D}(u_i) + 2\mathbf{V}_H(\tilde{u}_i, u) - 2\mathbf{V}_H(\tilde{u}_i, u_i) + \frac{1}{2h}\int_B |u_i - z|^2\,dx - \frac{1}{2h}\int_B |u - z|^2\,dx \\
&\geqslant \mathbf{F}(u) - 3\varepsilon
\end{aligned}$$

for sufficiently large $i \in \mathbb{N}$. Since $\varepsilon > 0$ was arbitrary we conclude that $u \in \mathcal{S}^*(\Gamma, A, \sigma)$ minimizes the variational functional $\mathbf{F}$. □

The following regularity result for minimizers was established in [3, Theorem 6.1]. We note that it is also a special case of the much more involved result by Rivière (see Theorem 2.6) for solutions to the Euler–Lagrange equation (4.7).

Lemma 5.3. *Suppose A, H, u_o and z satisfy the hypotheses of Lemma 5.2 with parameters σ, s and c. Then, every minimizer of the functional $\mathbf{F}$ in the class $\mathcal{S}^*(\Gamma, A, \sigma)$ is Hölder continuous in B and continuous up to the boundary, that is $u \in C^0(\overline{B}, \mathbb{R}^3)$.*

6. A priori estimates

In this section, we derive local a priori estimates for solutions of problems satisfying a Plateau boundary condition – i.e. for solutions to inequalities of the type (4.5) – under the additional assumption $u \in W^{1,4}(B_R^+(x_o), \mathbb{R}^3)$.

The validity of the $W^{1,4}$ assumption will be justified later. Moreover, we restrict ourselves to the case $B_R^+(x_o) \cap \{P_1, P_2, P_3\} = \varnothing$. This will be sufficient for our purposes since the set $\{P_1, P_2, P_3\}$ has vanishing capacity. More precisely, we consider maps $u \in \mathcal{S}^*(\Gamma) \cap W^{1,4}(B_R^+(x_o), \mathbb{R}^3)$ with $\Delta u = F$ on $B_R^+(x_o) \subset B$ for some $x_o \in \partial B$ and $F \in L^1(B_R^+(x_o), \mathbb{R}^3)$. Additionally, we assume that u satisfies the natural boundary condition associated with the Plateau problem on $I_R(x_o) \subset \partial B_R^+(x_o)$, i.e. we assume that

$$\int_{B_R^+(x_o)} Du \cdot Dw \, dx + \int_{B_R^+(x_o)} F \cdot w \, dx \geqslant 0 \tag{6.1}$$

holds true for all $w \in T_u\mathcal{S}^*$ with $w = 0$ on $S_R^+(x_o)$ in the sense of traces. We recall that the condition $w \in T_u\mathcal{S}^*$ is equivalent to the boundary representation $w(e^{i\vartheta}) = \widehat{\gamma}'(\varphi)(\psi - \varphi)$ for some $\psi \in \mathcal{T}^*(\Gamma)$. Here, $\varphi \in \mathcal{T}^*(\Gamma)$ is defined by $u(e^{i\vartheta}) = \widehat{\gamma}(\varphi(\vartheta))$ for all $\vartheta \in \mathbb{R}$. On the inhomogeneity F, we impose the growth condition

$$|F| \leqslant C_1\big(|Du|^2 + |f|\big) \quad \text{in } B_R^+(x_o), \tag{6.2}$$

where $C_1 > 0$ and $f \in L^2(B_R^+(x_o), \mathbb{R}^3)$ are given. Moreover, we assume

$$\int_{B_R^+(x_o)} |f|^2 \, dx \leqslant K^2 \tag{6.3}$$

for some constant $K > 0$. We start with the interior a priori estimate.

Theorem 6.1. *On $B_R(x_o) \Subset B$ consider a solution $u \in W^{1,4}(B_R(x_o), \mathbb{R}^3)$ of $\Delta u = F$, where F satisfies (6.2) for $C_1 > 0$ and $f \in L^2(B_R(x_o), \mathbb{R}^3)$. There exist $\varepsilon_o = \varepsilon_o(C_1) > 0$ and $C = C(C_1)$ such that the smallness condition*

$$\int_{B_R(x_o)} |Du|^2 \, dx \leqslant \varepsilon_o^2 \tag{6.4}$$

implies $u \in W^{2,2}(B_{R/2}(x_o), \mathbb{R}^3)$ with

$$\int_{B_{R/2}(x_o)} \big|D^2u\big|^2 \, dx \leqslant \frac{C}{R^2} \int_{B_R(x_o)} |Du|^2 \, dx + C \int_{B_R(x_o)} |f|^2 \, dx.$$

We note that in [3, Lemma 7.3], this result was established for more regular right-hand sides with $f \in W^{1,2}(B_R(x_o), \mathbb{R}^3)$. Here, we shall weaken this property to the natural assumption $f \in L^2(B_R(x_o), \mathbb{R}^3)$.

Proof. A standard application of the difference quotient technique yields the following estimate for any radii s, t with $\frac{R}{2} \leqslant s < t \leqslant R$:

$$\int_{B_s(x_o)} \big|D^2u\big|^2 \, dx \leqslant \frac{C}{(t-s)^2} \int_{B_t(x_o)} |Du|^2 \, dx + C \int_{B_t(x_o)} |F|^2 \, dx.$$

Using the growth condition (6.2), the regularity assumption $u \in W^{1,4}(B_R(x_o), \mathbb{R}^3)$ and the Gagliardo–Nirenberg interpolation inequality (2.7), we can further estimate

$$\begin{aligned} \int_{B_t(x_o)} |F|^2 \, dx &\leqslant C \int_{B_t(x_o)} |Du|^4 + |f|^2 \, dx \\ &\leqslant C \int_{B_t(x_o)} \big|D^2u\big|^2 + \left|\frac{Du}{R/2}\right|^2 dx \int_{B_R(x_o)} |Du|^2 \, dx + C \int_{B_R(x_o)} |f|^2 \, dx, \end{aligned}$$

for a constant $C = C(C_1)$. Combining the preceding two estimates, using the smallness assumption (6.4) and also $t - s \leqslant R/2$, we arrive at

$$\int_{B_s(x_o)} |D^2u|^2\,dx \leqslant C\varepsilon_o^2 \int_{B_t(x_o)} |D^2u|^2\,dx + \frac{C}{(t-s)^2}\int_{B_R(x_o)} |Du|^2\,dx + C\int_{B_R(x_o)} |f|^2\,dx.$$

Choosing $\varepsilon_o \in (0,1)$ small enough in dependence on $C = C(C_1)$, we can therefore derive the claim by an application of the Iteration Lemma 2.3. □

For the boundary analogue of Theorem 6.1, we additionally have to assume small oscillation of u. This is needed for the following extension result which we shall employ for the construction of admissible testing functions.

Lemma 6.2. *There is a radius $\varrho_o = \varrho_o(\Gamma) \in (0,1)$ such that the following holds. Consider maps $u_k \in W^{1,2}(B_r^+(x_o), \mathbb{R}^3)$ for $k \in \mathcal{I}$ with an index set $\mathcal{I}$ and $x_o \in \partial B$, such that for some $p_o \in \Gamma$ we have*

$$u_k\big(B_r^+(x_o)\big) \subset B_{\varrho_o}(p_o) \subset \mathbb{R}^3 \quad \text{for all } k \in \mathcal{I}. \tag{6.5}$$

Then there are maps $\Phi_k \in W^{1,2}(B_r^+(x_o), \mathbb{R})$ with

$$\widehat{\gamma} \circ \Phi_k = u_k \quad \text{on } I_r(x_o) \text{ for } k \in \mathcal{I},$$

and which satisfy a.e. on $B_r^+(x_o)$ the estimates

$$\begin{aligned} &|\Phi_k - \Phi_\ell| \leqslant C|u_k - u_\ell|,\\ &|D\Phi_k| \leqslant C|Du_k|,\\ &|D\Phi_k - D\Phi_\ell| \leqslant C|Du_k - Du_\ell| + |Du_k||u_k - u_\ell| \end{aligned}$$

for all $k, \ell \in \mathcal{I}$, where C denotes a universal constant that depends only on Γ.

Proof. Since $\Gamma \subset \mathbb{R}^3$ is a Jordan curve of class C^3, there is a tubular neighborhood $U \subset \mathbb{R}^3$ of Γ such that the nearest-point retraction $\pi : U \to \Gamma$ is well defined and of class C^2 with $\|\pi\|_{C^2} \leqslant C(\Gamma)$. For a $\varrho_o > 0$ sufficiently small, the assumption (6.5) implies in particular $u_k(B_r^+(x_o)) \subset U$ for $k \in \mathcal{I}$. Consequently, we may define

$$\widetilde{\Phi}_k := \gamma^{-1} \circ \pi \circ u_k : B_r^+(x_o) \to S^1 \quad \text{for } k \in \mathcal{I}.$$

Since $\gamma^{-1} \circ \pi$ is Lipschitz continuous with Lipschitz constant L depending only on Γ, we may once more diminish $\varrho_o > 0$ in dependence on Γ such that $\gamma^{-1} \circ \pi(B_{\varrho_o}(p_o))$ is contained in a half sphere $S^+ := \{e^{i\vartheta} \in S^1 : \vartheta_o < \vartheta < \vartheta_o + \pi\}$ for some $\vartheta_o \in \mathbb{R}$. On this half sphere, the function $\arg : S^+ \ni e^{i\vartheta} \mapsto \vartheta \in (\vartheta_o, \vartheta_o + \pi)$ is a C^2-diffeomorphism, which implies that $\widehat{\gamma}^{-1} = \arg \circ \gamma^{-1}$ is of class C^2 with $\|\widehat{\gamma}^{-1}\|_{C^2} \leqslant C(\Gamma)$ on $\pi(B_{\varrho_o}(p_o))$. Now, for $k \in \mathcal{I}$ we define

$$\Phi_k := \arg \circ \widetilde{\Phi}_k = \widehat{\gamma}^{-1} \circ \pi \circ u_k : B_r^+(x_o) \to \mathbb{R}.$$

Since $\widehat{\gamma}^{-1} \circ \pi$ is Lipschitz continuous on $B_{\varrho_o}(p_o)$, we deduce the first two of the asserted estimates. For the last one, we calculate $D\Phi_k = (T \circ u_k)Du_k$, where $T = D(\widehat{\gamma}^{-1} \circ \pi)$ is of class C^1 with $\|T\|_{C^1} \leqslant C(\Gamma)$. This implies the remaining claim by straightforward calculations. □

Now we are in a position to prove the a priori estimates up to the boundary.

Theorem 6.3. *There is a constant $\varepsilon_1 = \varepsilon_1(C_1, \Gamma) \in (0,1)$ for which the following two criteria for $W^{2,2}$-regularity hold true: Whenever $u \in \mathcal{S}^*(\Gamma)$ satisfies (6.1) on a half-disk around $x_o \in \partial B$ with $B_R^+(x_o) \cap \{P_1, P_2, P_3\} = \varnothing$ and $R < \frac{1}{2}$, then we have:*

(i) *If $F = 0$ and the solution satisfies $\operatorname{osc}_{B_R^+(x_o)} u \leqslant \varepsilon_1$ as well as the Morrey space estimate*

$$\int_{B_r^+(y)} |Du|^2\,dx \leqslant C_M\left(\frac{r}{R}\right)^{2\alpha} \quad \text{for all } y \in B_{R/2}^+(x_o) \text{ and } 0 < r < \frac{R}{2} \tag{6.6}$$

for some $\alpha \in (0,1)$ and $C_M \geqslant 1$, then we have $u \in W^{2,2}(B^+_{R/4}(x_o), \mathbb{R}^3)$ and with a constant $C = C(\Gamma, \alpha)$ furthermore the quantitative estimate

$$\int_{B^+_{R/4}(x_o)} \left|D^2 u\right|^2 dx \leqslant C C_M^{1/\alpha} \frac{1}{R^2} \int_{B^+_{R/2}(x_o)} |Du|^2 \, dx.$$

(ii) *If F satisfies (6.2) and u satisfies $u \in W^{1,4}(B^+_{R/2}(x_o), \mathbb{R}^3)$ and the smallness conditions*

$$\int_{B^+_R(x_o)} |Du|^2 \, dx < \varepsilon_1^2 \quad \textit{and} \quad \operatorname*{osc}_{B^+_R(x_o)} u \leqslant \varepsilon_1 \tag{6.7}$$

we have $u \in W^{2,2}(B^+_{R/4}(x_o), \mathbb{R}^3)$ together with the quantitative estimate

$$\int_{B^+_{R/4}(x_o)} \left|D^2 u\right|^2 dx \leqslant \frac{C}{R^2} \int_{B^+_{R/2}(x_o)} |Du|^2 \, dx + C \int_{B^+_{R/2}(x_o)} |f|^2 \, dx$$

with a universal constant $C = C(C_1, \Gamma)$.

Proof. The first part of the proof is identical for both cases. We will later choose $\varepsilon_1 \in (0, \varrho_o)$ with the radius $\varrho_o(\Gamma) > 0$ from Lemma 6.2. This implies in view of our assumption $\operatorname{osc}_{B^+_R(x_o)} u \leqslant \varepsilon_1$, that we have

$$u\big(B^+_R(x_o)\big) \subset B_{\varrho_o}\big(u(x_o)\big). \tag{6.8}$$

For $h \in (-\frac{R}{8}, \frac{R}{8})$, we introduce the notation

$$u_h\big(re^{i\vartheta}\big) := u\big(re^{i(\vartheta+h)}\big) \quad \text{whenever } re^{i\vartheta} \in B^+_{3R/8}(x_o).$$

We note that $re^{i\vartheta} \in B^+_{3R/8}(x_o)$ implies $re^{i(\vartheta+h)} \in B^+_R(x_o)$, so that the preceding definition makes sense. Further, the inclusion (6.8) yields

$$u\big(B^+_{3R/8}(x_o)\big) \subset B_{\varrho_o}\big(u(x_o)\big) \quad \text{and} \quad u_{\pm h}\big(B^+_{3R/8}(x_o)\big) \subset B_{\varrho_o}\big(u(x_o)\big).$$

Therefore, with Lemma 6.2 we find maps $\Phi, \Phi_h, \Phi_{-h} \in W^{1,2}(B^+_{3R/8}(x_o))$ such that

$$\begin{cases} |\Phi - \Phi_{\pm h}| \leqslant C|u - u_{\pm h}|, \\ |D\Phi| \leqslant C|Du|, \\ |D\Phi - D\Phi_{\pm h}| \leqslant C|Du - Du_{\pm h}| + C|Du||u - u_{\pm h}| \end{cases} \tag{6.9}$$

hold true a.e. on $B^+_{3R/8}(x_o)$, where $C = C(\Gamma)$, and moreover

$$\widehat{\gamma} \circ \Phi = u, \qquad \widehat{\gamma} \circ \Phi_{\pm h} = u_{\pm h} \quad \text{on } I_{3R/8}(x_o).$$

Since $u \in \mathcal{S}^*(\Gamma)$, we can thus find a map $\varphi \in \mathcal{T}^*(\Gamma)$ with

$$\varphi(\vartheta) = \Phi\big(e^{i\vartheta}\big) \quad \text{whenever } e^{i\vartheta} \in I_{3R/8}(x_o).$$

Similarly, we define

$$\varphi_{\pm h}(\vartheta) := \Phi_{\pm h}\big(e^{i\vartheta}\big) \quad \text{whenever } e^{i\vartheta} \in I_{3R/8}(x_o).$$

Next, for $v \colon B^+_{3R/8}(x_o) \to \mathbb{R}^k$ we define the *angular difference quotient* by

$$\partial_h v := \frac{1}{h}(v_h - v).$$

Now we let s, t be arbitrary radii with $\frac{R}{4} \leqslant s < t \leqslant \frac{3R}{8}$ and choose a cut-off function $\eta \in C_0^\infty(B_t(x_o), [0,1])$ with $\eta \equiv 1$ on $B_s(x_o)$ and $|D\eta| \leqslant \frac{2}{t-s}$. Then we would like to test the inequality (6.1) with the testing function

$$\partial_{-h}\big(\eta^2 \partial_h u\big) = \frac{1}{h^2}\big[\eta^2(u_h - u) + \eta_{-h}^2(u_{-h} - u)\big].$$

With the abbreviation $\widetilde{w} := \eta^2(u_h - u)$, this identity becomes

$$\partial_{-h}\big(\eta^2\partial_h u\big) = \frac{1}{h^2}[\widetilde{w} - \widetilde{w}_{-h}] = \frac{1}{h}\partial_{-h}\widetilde{w}.$$

Unfortunately, $\widetilde{w}$ and $-\widetilde{w}_{-h}$ are not admissible in (6.1) since they might not attain the right boundary values. Therefore, following Struwe [43] we modify the function $\widetilde{w}$ to

$$w := \widetilde{w} - \eta^2 \int_{\Phi}^{\Phi_h}\int_{\Phi}^{t} \widehat{\gamma}''(s)\,ds\,dt =: \widetilde{w} - \eta^2 g_+ \quad \text{on } B^+_{3R/8}(x_o)$$

and extend w by zero outside of $B^+_{3R/8}(x_o)$. From $(6.9)_1$ we infer

$$|g_+| \leqslant C|\Phi - \Phi_h|^2 \leqslant C|u - u_h|^2, \tag{6.10}$$

for a constant $C = C(\Gamma)$. In order to estimate Dg_+, we calculate

$$\begin{aligned} Dg_+ &= D\big[\widehat{\gamma}(\Phi_h) - \widehat{\gamma}(\Phi) - \widehat{\gamma}'(\Phi)(\Phi_h - \Phi)\big] \\ &= \big(\widehat{\gamma}'(\Phi_h) - \widehat{\gamma}'(\Phi) - \widehat{\gamma}''(\Phi)(\Phi_h - \Phi)\big)D\Phi + \big(\widehat{\gamma}'(\Phi_h) - \widehat{\gamma}'(\Phi)\big)(D\Phi_h - D\Phi). \end{aligned}$$

Since $\widehat{\gamma}$ is of class C^3 we use the bounds (6.9) to obtain

$$\begin{aligned} |Dg_+| &\leqslant C|\Phi - \Phi_h|^2|D\Phi| + C|\Phi - \Phi_h||D\Phi_h - D\Phi| \\ &\leqslant C|u - u_h|^2|Du| + C|u - u_h||Du_h - Du| \end{aligned} \tag{6.11}$$

a.e. on $B^+_{3R/8}(x_o)$, where $C = C(\Gamma)$. Next, we calculate the boundary values of w. By the choice of φ and φ_h, we have for any $e^{i\vartheta} \in I_{3R/8}(x_o)$

$$\begin{aligned} w\big(e^{i\vartheta}\big) &= \widetilde{w}\big(e^{i\vartheta}\big) - \eta^2\big(e^{i\vartheta}\big) \int_{\varphi(\vartheta)}^{\varphi_h(\vartheta)}\int_{\varphi(\vartheta)}^{t} \widehat{\gamma}''(s)\,ds\,dt \\ &= \eta^2\big(e^{i\vartheta}\big)\left[\widehat{\gamma}\big(\varphi_h(\vartheta)\big) - \widehat{\gamma}\big(\varphi(\vartheta)\big) - \int_{\varphi(\vartheta)}^{\varphi_h(\vartheta)}\int_{\varphi(\vartheta)}^{t} \widehat{\gamma}''(s)\,ds\,dt\right] \\ &= \eta^2\big(e^{i\vartheta}\big)\widehat{\gamma}'\big(\varphi(\vartheta)\big)\big(\varphi_h(\vartheta) - \varphi(\vartheta)\big). \end{aligned}$$

On the other hand, for $e^{i\vartheta} \notin I_{3R/8}(x_o)$, the choice of η implies $w(e^{i\vartheta}) = 0$. Defining

$$\psi(\vartheta) := \eta^2\big(e^{i\vartheta}\big)\varphi_h(\vartheta) + \big(1 - \eta^2\big(e^{i\vartheta}\big)\big)\varphi(\vartheta),$$

we thereby deduce

$$w\big(e^{i\vartheta}\big) = \widehat{\gamma}'\big(\varphi(\vartheta)\big)\big(\psi(\vartheta) - \varphi(\vartheta)\big) \quad \text{for all } \vartheta \in \mathbb{R}.$$

We observe that $\psi \in C^0 \cap W^{1/2,2}(\mathbb{R})$ is weakly monotone and satisfies the periodicity condition $\psi(\cdot + 2\pi) = \psi + 2\pi$ because it is a convex combination of two functions with these properties. Moreover, it satisfies the three-point condition $\widehat{\gamma}(\psi(\Theta_k)) = Q_k$ since η vanishes in $P_k = e^{i\Theta_k}$ because of the assumption $B^+_R(x_o) \cap \{P_1, P_2, P_3\} = \varnothing$. We conclude $\psi \in \mathcal{T}^*(\Gamma)$, which in turn implies that $w = \widetilde{w} - \eta^2 g_+ \in T_u\mathcal{S}^*$ is an admissible testing function in (6.1). From this we infer

$$-\int_{B^+_R(x_o)} Du \cdot D\widetilde{w}\,dx \leqslant \int_{B^+_R(x_o)} F \cdot \widetilde{w}\,dx + \int_{B^+_R(x_o)} |Du|\big|D\big(\eta^2 g_+\big)\big| + \eta^2|F||g_+|\,dx. \tag{6.12}$$

With $-\widetilde{w}_{-h} = \eta^2_{-h}(u_{-h} - u)$ we can repeat the preceding arguments with h, η^2 replaced by $-h$, η^2_{-h}. We obtain a function $g_-: B^+_{3R/8}(x_o) \to \mathbb{R}^3$ that satisfies the estimates

$$|g_-| \leqslant C|\Phi - \Phi_{-h}|^2 \leqslant C|u - u_{-h}|^2, \tag{6.13}$$

$$|Dg_-| \leqslant C|u - u_{-h}|^2|Du| + C|u - u_{-h}||Du_{-h} - Du|, \tag{6.14}$$

with a constant $C = C(\Gamma)$, and for which $-\widetilde{w}_{-h} - \eta_{-h}^2 g_-$ is admissible in the variational inequality (6.1). This leads to

$$\int_{B_R^+(x_o)} Du \cdot D\widetilde{w}_{-h}\,dx \leqslant -\int_{B_R^+(x_o)} F \cdot \widetilde{w}_{-h}\,dx + \int_{B_R^+(x_o)} |Du|\big|D\big(\eta_{-h}^2 g_-\big)\big| + \eta_{-h}^2|F||g_-|\,dx.$$

Adding the preceding inequality to (6.12) and dividing by h^2, we deduce

$$-\frac{1}{h}\int_{B_R^+(x_o)} Du \cdot D\partial_{-h}\widetilde{w}\,dx \leqslant \frac{1}{h}\int_{B_R^+(x_o)} F \cdot \partial_{-h}\widetilde{w}\,dx + \frac{1}{h^2}\int_{B_R^+(x_o)} |F|\big(\eta^2|g_+| + \eta_{-h}^2|g_-|\big)\,dx$$
$$+ \frac{1}{h^2}\int_{B_R^+(x_o)} |Du|\big(\big|D\big(\eta^2 g_+\big)\big| + \big|D\big(\eta_{-h}^2 g_-\big)\big|\big)\,dx.$$

Taking into account the definition $\frac{1}{h}\widetilde{w} = \eta^2\partial_h u$, we can re-write the preceding inequality in the form

$$\begin{aligned}
-\int_{B_R^+(x_o)} Du \cdot D\partial_{-h}\big(\eta^2\partial_h u\big)\,dx &\leqslant \int_{B_R^+(x_o)} |F|\big|\partial_{-h}\big(\eta^2\partial_h u\big)\big|\,dx + \frac{1}{h^2}\int_{B_R^+(x_o)} |F|\big(\eta^2|g_+| + \eta_{-h}^2|g_-|\big)\,dx \\
&\quad + \frac{1}{h^2}\int_{B_R^+(x_o)} |Du|\big(\eta^2|Dg_+| + \eta_{-h}^2|Dg_-|\big)\,dx \\
&\quad + \frac{2}{h^2}\int_{B_R^+(x_o)} |Du|\big(\eta|D\eta||g_+| + \eta_{-h}|D\eta_{-h}||g_-|\big)\,dx \\
&= I + II + III + IV,
\end{aligned} \tag{6.15}$$

with the obvious meaning of I–IV. In the sequel we estimate these terms separately. We start with the *estimate of II*. Here we use (6.10) and (6.13) and the fact that $0 \leqslant \eta \leqslant 1$ to obtain

$$II \leqslant C\int_{B_R^+(x_o)} |F|\big(\eta|\partial_h u|^2 + \eta_{-h}|\partial_{-h}u|^2\big)\,dx.$$

Next we deduce the *estimate for IV*. Here we use again (6.10) and (6.13) to obtain

$$\begin{aligned}
IV &\leqslant C\int_{B_R^+(x_o)} |Du|\big(\eta|D\eta||\partial_h u|^2 + \eta_{-h}|D\eta_{-h}||\partial_{-h}u|^2\big)\,dx \\
&\leqslant C\int_{B_R^+(x_o)} \frac{|Du|}{t-s}\big(\eta|\partial_h u|^2 + \eta_{-h}|\partial_{-h}u|^2\big)\,dx.
\end{aligned}$$

The *estimate of III* is achieved as follows. Using (6.11), (6.14) and Young's inequality we find for any $\mu > 0$

$$\begin{aligned}
III &\leqslant C\int_{B_R^+(x_o)} |Du|\big(\eta^2|\partial_h u||D\partial_h u| + \eta_{-h}^2|\partial_{-h}u||D\partial_{-h}u|\big)\,dx + C\int_{B_R^+(x_o)} |Du|^2\big(\eta|\partial_h u|^2 + \eta_{-h}|\partial_{-h}u|^2\big)\,dx \\
&\leqslant \mu\int_{B_R^+(x_o)} \eta^2|D\partial_h u|^2 + \eta_{-h}^2|D\partial_{-h}u|^2\,dx + C_\mu\int_{B_R^+(x_o)} |Du|^2\big(\eta|\partial_h u|^2 + \eta_{-h}|\partial_{-h}u|^2\big)\,dx.
\end{aligned}$$

Adding the estimates for II, III and IV and using $2\frac{|Du|}{t-s} \leqslant |Du|^2 + \frac{1}{(t-s)^2}$ we deduce

$$II + III + IV \leqslant \mu \int_{B_R^+(x_o)} \eta^2 |D\partial_h u|^2 + \eta_{-h}^2 |D\partial_{-h} u|^2 \, dx$$
$$+ C_\mu \int_{B_R^+(x_o)} \Big(|Du|^2 + \frac{1}{(t-s)^2} + |F| \Big) \big(\eta |\partial_h u|^2 + \eta_{-h} |\partial_{-h} u|^2 \big) \, dx.$$

Next, we consider the term I. Here, by Young's inequality and a standard estimate for difference quotients we find

$$I = \int_{B_R^+(x_o)} |F| \big| \partial_{-h} \big(\eta^2 \partial_h u \big) \big| \, dx$$
$$\leqslant \mu \int_{B_R^+(x_o)} \big| D\big(\eta^2 \partial_h u \big) \big|^2 \, dx + C_\mu \int_{B_{t+h}^+(x_o)} |F|^2 \, dx.$$

In the last line we used the fact that $\operatorname{spt}(\eta^2 \partial_h u) \subset B_t^+(x_o)$ and $e^{-ih} B_t^+(x_o) \subset B_{t+h}^+(x_o)$. For the estimate of the left-hand side of (6.15) from below we compute

$$- \int_{B_R^+(x_o)} Du \cdot D\partial_{-h} \big(\eta^2 \partial_h u \big) \, dx = \int_{B_R^+(x_o)} D\partial_h u \cdot D\big(\eta^2 \partial_h u \big) \, dx$$
$$= \int_{B_R^+(x_o)} \eta^2 |D\partial_h u|^2 + 2\eta D\partial_h u \cdot \partial_h u \otimes D\eta \, dx$$
$$\geqslant \frac{1}{2} \int_{B_R^+(x_o)} \eta^2 |D\partial_h u|^2 \, dx - C \int_{B_R^+(x_o)} |\partial_h u|^2 |D\eta|^2 \, dx$$
$$\geqslant \frac{1}{2} \int_{B_R^+(x_o)} \eta^2 |D\partial_h u|^2 \, dx - C \int_{B_t^+(x_o)} \frac{|\partial_h u|^2}{(t-s)^2} \, dx.$$

Joining the preceding estimates we arrive at

$$\frac{1}{2} \int_{B_R^+(x_o)} \eta^2 |D\partial_h u|^2 \, dx \leqslant 2\mu \int_{B_R^+(x_o)} \eta^2 |D\partial_h u|^2 + \eta_{-h}^2 |D\partial_{-h} u|^2 \, dx$$
$$+ C_\mu \int_{B_R^+(x_o)} \Big(|Du|^2 + \frac{1}{(t-s)^2} + |F| \Big) \big(\eta |\partial_h u|^2 + \eta_{-h} |\partial_{-h} u|^2 \big) \, dx$$
$$+ C_\mu \int_{B_{t+h}^+(x_o)} |F|^2 \, dx + C \int_{B_t^+(x_o)} \frac{|\partial_h u|^2}{(t-s)^2} \, dx. \tag{6.16}$$

From the transformation $x = ye^{-ih}$ we infer the identity

$$\int_{B_R^+(x_o)} \eta^2 |D\partial_h u|^2 \, dx = \int_{B_R^+(x_o)} \eta_{-h}^2 |D\partial_{-h} u|^2 \, dy.$$

Therefore, we can re-absorb the first integral from the right-hand side of (6.16) into the left after choosing $\mu \in (0, 1)$ sufficiently small. Keeping in mind the properties of the cut-off function η, we deduce

$$\int_{B_s^+(x_o)} |D\partial_h u|^2\,dx \leqslant C \int_{B_R^+(x_o)} \big(|Du|^2+|F|\big)\big(\eta|\partial_h u|^2+\eta_{-h}|\partial_{-h}u|^2\big)\,dx + C\int_{B_{t+h}^+(x_o)} |F|^2\,dx + C\int_{B_t^+(x_o)} \frac{|\partial_h u|^2}{(t-s)^2}\,dx \tag{6.17}$$

with a constant $C=C(\Gamma)$. For the bound of the right-hand side we distinguish between the two cases (i) and (ii). We begin with

Proof of (i). In this case, we follow the strategy from [43, p. 73] and first extend the map u by reflection $u(x):=u(x/|x|^2)$ onto the full disk $B_{R/2}(x_o)$ and then cover $B_t(x_o)$ with disks $B_\varrho(x_i)$ of radius $\varrho\in(0,\frac{R}{16})$. This can be done in such a way that every point $x\in B_t(x_o)$ is contained in at most N of the disks $B_{2\varrho}(x_i)$ with N independent from ϱ. We choose a standard cut-off function $\zeta\in C_0^\infty(B_2(0))$ with $\zeta\equiv 1$ on $B_1(0)$ and let $\zeta_i(x):=\zeta(\frac{x-x_i}{\varrho})$. Then we estimate

$$\begin{aligned}\int_{B_R^+(x_o)} |Du|^2\big(\eta|\partial_h u|^2+\eta_{-h}|\partial_{-h}u|^2\big)\,dx &= \int_{B_t^+(x_o)} \eta\big(|Du|^2+|Du_h|^2\big)|\partial_h u|^2\,dx \\ &\leqslant \sum_i \int_{B_{2\varrho}(x_i)\cap B_t(x_o)} \big(|Du|^2+|Du_h|^2\big)|\partial_h u|^2\zeta_i^2\,dx.\end{aligned}$$

Because of the Morrey type assumption (6.6), each of the latter integrals can be estimated by Lemma 2.5 with $\Omega=B_{2\varrho}(x_i)\cap B_t(x_o)$, $w=|Du|^2+|Du_h|^2$, $v_i=|\partial_h u|\zeta_i$ and $C_o=C_M R^{-2\alpha}$. This leads us to

$$\begin{aligned}\int_{B_R^+(x_o)} |Du|^2\big(\eta|\partial_h u|^2+\eta_{-h}|\partial_{-h}u|^2\big)\,dx &\leqslant CC_M\Big(\frac{\varrho}{R}\Big)^{2\alpha}\sum_i \int_{B_{2\varrho}(x_i)\cap B_t(x_o)} |Dv_i|^2 \\ &\leqslant CC_M N\Big(\frac{\varrho}{R}\Big)^{2\alpha}\int_{B_t^+(x_o)} |D\partial_h u|^2+\frac{1}{\varrho^2}|\partial_h u|^2\,dx.\end{aligned}$$

At this stage we fix $\varrho\in(0,\frac{R}{16})$ in the form $\varrho^{2\alpha}:=R^{2\alpha}/(2CC_M N)$. Plugging the resulting estimate into (6.17) and keeping in mind $F=0$ and $C_M\geqslant 1$, we arrive at

$$\int_{B_s^+(x_o)} |D\partial_h u|^2\,dx \leqslant \frac12\int_{B_t^+(x_o)} |D\partial_h u|^2\,dx + C\frac{C_M^{1/\alpha}}{(t-s)^2}\int_{B_{R/2}^+(x_o)} |\partial_h u|^2\,dx.$$

Here, we can re-absorb the first integral on the right-hand side by means of Lemma 2.3. Letting $h\downarrow 0$, we thereby deduce

$$\int_{B_{R/4}^+(x_o)} |D_\vartheta Du|^2\,dx \leqslant CC_M^{1/\alpha}\frac{1}{R^2}\int_{B_{R/2}^+(x_o)} |Du|^2\,dx, \tag{6.18}$$

where D_ϑ denotes the angular derivative. It remains to estimate the second radial derivative D_r^2u. For this we re-write the Laplacian in polar coordinates as $\Delta=D_r^2u+\frac1r D_r u+\frac{1}{r^2}D_\vartheta^2 u$. Since u is harmonic on $B_R^+(x_o)$, this implies $|D_r^2u|^2\leqslant C|D_\vartheta Du|^2+C|Du|^2$ and the claim (i) follows from (6.18) with a constant $C=C(\Gamma,\alpha)$. Now we proceed to

Proof of (ii). Under the assumption $u\in W^{1,4}(B_R^+(x_o),\mathbb{R}^3)$, the estimate (6.17) readily implies

$$\int_{B_s^+(x_o)} |\partial_h Du|^2\,dx \leqslant C\int_{B_{t+h}^+(x_o)} |f|^2+|Du|^4+\frac{1}{(t-s)^2}|Du|^2\,dx$$

by the bound (6.2) on F, where $C = C(C_1, \Gamma)$. Since the right-hand side is bounded independently from h, we infer after letting $h \to 0$

$$\int_{B_s^+(x_o)} |D_\vartheta Du|^2\,dx \leqslant C \int_{B_t^+(x_o)} |f|^2 + |Du|^4 + \frac{1}{(t-s)^2}|Du|^2\,dx. \tag{6.19}$$

Similarly as in the proof of (i), we use $\Delta = D_r^2 u + \frac{1}{r} D_r u + \frac{1}{r^2} D_\vartheta^2 u$, the equation $\Delta u = F$ on $B_R^+(x_o)$ in the sense of distributions and also the assumption $R < \frac{1}{2}$ which implies $\frac{1}{r} < 2$ on $B_R^+(x_o)$. Combining this with (6.19), we deduce

$$\begin{aligned}\int_{B_s^+(x_o)} \left|D_r^2 u\right|^2 dx &\leqslant C \int_{B_s^+(x_o)} |F|^2 + |D_r u|^2 + \left|D_\vartheta^2 u\right|^2 dx \\ &\leqslant C \int_{B_t^+(x_o)} |f|^2 + |Du|^4 + \frac{1}{(t-s)^2}|Du|^2\,dx,\end{aligned}$$

with $C = C(C_1, \Gamma)$, where here, the last estimate follows from assumption (6.2) and (6.19). Combining this with (6.19) and applying the interpolation inequality (2.7), we arrive at

$$\begin{aligned}\int_{B_s^+(x_o)} \left|D^2 u\right|^2 dx &\leqslant C \int_{B_t^+(x_o)} |f|^2 + |Du|^4 + \frac{1}{(t-s)^2}|Du|^2\,dx \\ &\leqslant C\varepsilon_1^2 \int_{B_t^+(x_o)} \left|D^2 u\right|^2 + \frac{1}{R^2}|Du|^2\,dx + C \int_{B_t^+(x_o)} |f|^2 + \frac{1}{(t-s)^2}|Du|^2\,dx.\end{aligned}$$

In the last step, we also used the assumption (6.7). If we decrease once more the value of ε_1 in such a way that $C\varepsilon_1^2 \leqslant \frac{1}{2}$, we arrive at

$$\int_{B_s^+(x_o)} \left|D^2 u\right|^2 dx \leqslant \frac{1}{2} \int_{B_t^+(x_o)} \left|D^2 u\right|^2 dx + \frac{C}{(t-s)^2} \int_{B_{R/2}^+(x_o)} |Du|^2\,dx + C \int_{B_{R/2}^+(x_o)} |f|^2\,dx.$$

Since this inequality holds for all s, t with $\frac{R}{4} \leqslant s < t \leqslant \frac{3R}{8} < \frac{R}{2}$, the Iteration Lemma 2.3 now implies the claim (ii). □

7. Calderón–Zygmund estimates for solutions

In this section we remove the $W^{1,4}$-hypothesis from the last section which was needed to establish the local $W^{2,2}$-estimates in Theorems 6.1 and 6.3 (ii).

7.1. *Results for comparison problems*

In this section, we provide some results for harmonic maps with a partial Plateau boundary condition. These maps will serve as comparison maps later. More precisely, we consider minimizers of the Dirichlet energy in the class

$$\mathcal{S}_u^*(\Gamma) := \left\{ w \in \mathcal{S}^*(\Gamma)\colon\ w = u \text{ on } B \setminus B_R^+(x_o) \right\}$$

for some $x_o \in \partial B$. Minimizers $v \in \mathcal{S}^*(\Gamma)$ of this problem are harmonic on $B_R^+(x_o)$ and satisfy a weak Neumann type boundary condition on $I_R(x_o)$. More precisely, we have

Lemma 7.1. *Every minimizer v of* $\mathbf{D}$ *in the class $\mathcal{S}_u^*(\Gamma)$ satisfies*

$$\int_{B_R^+(x_o)} Dv \cdot Dw\,dx \geqslant 0 \tag{7.1}$$

for all $w \in T_v\mathcal{S}^*$ with $w = 0$ on $S_R^+(x_o)$ in the sense of traces.

Proof. The proof is a slight modification of Lemma 2.1 (i). We assume that the boundary values of v are given by $\varphi \in \mathcal{T}^*(\Gamma)$ in the sense $v(e^{i\vartheta}) = \widehat{\gamma}(\varphi(\vartheta))$. For a given $w \in T_v\mathcal{S}^*$ with $w = 0$ on $S_R^+(x_o)$ in the sense of traces, we can find a $\psi \in \mathcal{T}^*(\Gamma)$ with $w(e^{i\vartheta}) = \widehat{\gamma}'(\varphi)(\psi - \phi)$ for all $\vartheta \in \mathbb{R}$ and $\psi(\vartheta) = \varphi(\vartheta)$ whenever $e^{i\vartheta} \notin I_R(x_o)$. We define $h_s : B \to \mathbb{R}^3$, $0 \leqslant s \ll 1$ as the unique minimizer of the Dirichlet energy with $h_s = u$ on $B \setminus B_R^+(x_o)$ and $h_s(e^{i\vartheta}) = \widehat{\gamma}(\varphi + s(\psi - \varphi))$ for all $e^{i\vartheta} \in I_R(x_o)$. As in the proof of Lemma 2.1 (i), we then define

$$v_s := h_s + s\left(w - \frac{\partial h_s}{\partial s}\bigg|_{s=0}\right) - (h_0 - u)$$

and check that the maps $v_s \in \mathcal{S}_u^*(\Gamma)$ are admissible as competitors for v with $\frac{\partial}{\partial s}|_{s=0} v_s = w$. We conclude the claim (7.1) by $0 \leqslant \frac{d}{ds}|_{s=0}\mathbf{D}(v_s) = \int_{B_R^+(x_o)} Dv \cdot Dw\, dx$. □

Next, we give an existence result concerning $\mathbf{D}$-minimizers in the class $\mathcal{S}_u^*(\Gamma)$.

Lemma 7.2. *For every map $u \in \mathcal{S}^*(\Gamma)$ and every disk $B_R^+(x_o)$ with center $x_o \in \partial B$, there is a minimizer $v \in \mathcal{S}^*(\Gamma)$ of the Dirichlet energy $\mathbf{D}$ in the class $\mathcal{S}_u^*(\Gamma)$.*

Proof. We choose a minimizing sequence $v_k \in \mathcal{S}_u^*(\Gamma)$ for $\mathbf{D}$. Since the boundary traces of the v_k are contained in the compact set Γ, the $W^{1,2}$-norms of v_k are uniformly bounded. Therefore, we can assume $v_k \rightharpoonup v$ in $W^{1,2}(B, \mathbb{R}^3)$ and almost everywhere, as $k \to \infty$. Furthermore, Lemma 2.2 implies uniform convergence $v_k|_{\partial B} \to v|_{\partial B}$ of the boundary traces. From this we deduce that the limit map again satisfies $v \in \mathcal{S}_u^*(\Gamma)$. The lower semicontinuity of the Dirichlet energy $\mathbf{D}$ with respect to weak $W^{1,2}$-convergence then yields the claim. □

The main result of this section are the following $W^{2,2}$-estimates for solutions of the comparison problem.

Lemma 7.3. *For a map $u \in \mathcal{S}^*(\Gamma)$, a center $x_o \in \partial B$ and a radius $R \in (0, \frac{1}{2})$ with $B_R^+(x_o) \cap \{P_1, P_2, P_3\} = \varnothing$, we consider a minimizer v of the Dirichlet energy in the class $\mathcal{S}_u^*(\Gamma)$ with $\int_{B_R^+(x_o)} |Du|^2\, dx \leqslant E_o$ for some constant $E_o \geqslant 1$. There is a constant $\varepsilon_2 = \varepsilon_2(\delta, M, \{Q_i\}) > 0$ such that the smallness condition*

$$\operatorname*{osc}_{B_R^+(x_o)} v \leqslant \varepsilon_2$$

implies $v \in W^{2,2}(B_{R/4}^+(x_o), \mathbb{R}^3)$, and for some constant $C = C(\Gamma, E_o)$, we have the quantitative estimate

$$\int_{B_{R/4}^+(x_o)} \left|D^2 v\right|^2 dx \leqslant \frac{C}{R^2} \int_{B_{R/2}^+(x_o)} |Dv|^2\, dx.$$

Proof. Since the minimizer v satisfies the Euler–Lagrange equations (7.1), the claim follows from Theorem 6.3 (i) as soon as we have established the Morrey type bound (6.6). To this end, we choose a radius $\varrho_1 = \varrho_1(\Gamma) > 0$ so small that every ball $B_{\varrho_1}^3(p) \subset \mathbb{R}^3$ contains at most one of the points Q_1, Q_2, Q_3. Then we define

$$\varepsilon_2 := \min\left\{\varepsilon_1, \delta, \frac{\varrho_1}{M}\right\} \tag{7.2}$$

with the constant ε_1 from Theorem 6.3 (i) and the parameters $\delta > 0$ and $M \geqslant 1$ from the chord-arc condition (2.1) of Γ. For a fixed $y \in I_{R/2}(x_o) \subset \partial B$ we consider the function

$$\left[0, \frac{R}{2}\right] \ni r \mapsto \Phi(r) := \mathbf{D}_{B_r^+(y)}(v) = \frac{1}{2}\int_0^r \int_{S_\varrho^+(y)} \left[\left|\frac{\partial v}{\partial \varrho}\right|^2 + \left|\frac{\partial v}{\partial \omega}\right|^2\right] d\mathcal{H}^1 \omega\, d\varrho,$$

where $S_\varrho^+(y) = \partial B_\varrho(y) \cap \overline{B}$ and $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure on $\mathbb{R}^2$. Since the function Φ is absolutely continuous, we know that for almost every $r \in [0, \frac{R}{2}]$ there holds

$$\Psi(r) := \frac{r}{2} \int_{S_r^+(y)} \left|\frac{\partial v}{\partial \omega}\right|^2 d\mathcal{H}^1\omega \leqslant r\Phi'(r). \tag{7.3}$$

From now on we consider only such r for which (7.3) holds, so that the minimizer v is continuous on $S_r^+(y)$ by the Sobolev embedding theorem. Writing $\{x_r, y_r\} := S_r^+(y) \cap \partial B$, we can thereby estimate

$$\left|v(x_r) - v(y_r)\right|^2 \leqslant \left(\int_{S_r^+(y)} \left|\frac{\partial v}{\partial \omega}\right| d\mathcal{H}^1\omega\right)^2 \leqslant 2\pi\Psi(r) \leqslant 2\pi r\Phi'(r). \tag{7.4}$$

Since $\operatorname{osc}_{B_R^+(x_o)} v \leqslant \varepsilon_2 \leqslant \delta$, the chord-arc condition (2.1) implies the existence of a sub-arc $\Gamma_r \subset \Gamma$ connecting the points $v(x_r)$ and $v(y_r)$ with

$$L(\Gamma_r) \leqslant M\left|v(x_r) - v(y_r)\right| \leqslant M\varepsilon_2 \leqslant \varrho_1, \tag{7.5}$$

where we used the choice of ε_2 in the last step. From the choice of ϱ_1 we thereby infer that this sub-arc Γ_r contains at most one of the points Q_1, Q_2, Q_3, which implies $\Gamma_r = v(I_r(y))$. Indeed, if this was not the case, the sub-arc $\Gamma_r = v(\partial B \setminus I_r(y))$ would contain all three of the points Q_1, Q_2, Q_3, which is a contradiction. Combining (7.5) and (7.4), we thereby deduce

$$L^2\big(v\big(I_r(y)\big)\big) \leqslant M^2\left|v(x_r) - v(y_r)\right|^2 \leqslant 2\pi M^2 r\Phi'(r). \tag{7.6}$$

Our next goal is to estimate $\mathbf{D}_{B_r^+(y)}(v)$ by constructing a suitable comparison map. To this end, we define $c_r : I_r(y) \to \Gamma_r$ as the orientation preserving parametrization of the sub-arc $\Gamma_r = v(I_r(y))$ proportionally to arc length. From this choice of the parametrization, we infer

$$r \int_{I_r(y)} \left|c_r'\right|^2 d\mathcal{H}^1 = \frac{rL^2(c_r)}{L(I_r(y))} \leqslant L^2\big(v\big(I_r(y)\big)\big) \leqslant 2\pi M^2 r\Phi'(r), \tag{7.7}$$

where we employed (7.6) in the last step. Next we define Dirichlet boundary values on $\partial B_r^+(y) = S_r^+(y) \cup I_r(y)$ by

$$g(x) := \begin{cases} v(x) & \text{for } x \in S_r^+(y), \\ c_r(x) & \text{for } x \in I_r(y), \end{cases}$$

and define $w \in W^{1,2}(B_r^+(y), \mathbb{R}^3)$ as the minimizer of the Dirichlet energy with boundary data g. By comparing w with a suitable cone with the same boundary values g, we infer

$$\begin{aligned} \mathbf{D}_{B_r^+(y)}(w) &\leqslant Cr \int_{\partial B_r^+(y)} \left|\frac{\partial g}{\partial \omega}\right|^2 d\mathcal{H}^1\omega \\ &= Cr \int_{S_r^+(y)} \left|\frac{\partial v}{\partial \omega}\right|^2 d\mathcal{H}^1\omega + Cr \int_{I_r(y)} \left|c_r'\right|^2 d\mathcal{H}^1\omega \\ &\leqslant C\Psi(r) + CM^2 r\Phi'(r) \leqslant CM^2 r\Phi'(r), \end{aligned} \tag{7.8}$$

where we used the definition of Ψ and the estimates (7.7) and (7.3) in the last two steps. On the other hand, extending w by v outside of $B_r^+(y)$, we get an admissible comparison map for v, so that the minimizing property of v implies

$$\Phi(r) = \mathbf{D}_{B_r^+(y)}(v) \leqslant \mathbf{D}_{B_r^+(y)}(w) \leqslant mr\Phi'(r)$$

with a constant $m = m(\Gamma)$, or equivalently

$$\frac{d}{d\tau}\big(\tau^{-\frac{1}{m}}\Phi(\tau)\big) = \frac{1}{m}\tau^{-\frac{1}{m}-1}\big[m\tau\Phi'(\tau) - \Phi(\tau)\big] \geqslant 0.$$

Integrating over $[\varrho, \frac{R}{2}]$ and abbreviating $\alpha := \frac{1}{2m}$, we arrive at

$$\varrho^{-2\alpha} \int_{B_\varrho^+(y)} |Dv|^2\,dx \leqslant \left(\frac{R}{2}\right)^{-2\alpha} \Phi\left(\frac{R}{2}\right) \leqslant C R^{-2\alpha} \int_{B_R^+(x_o)} |Dv|^2\,dx. \tag{7.9}$$

This is the desired Morrey estimate for points $y \in I_{R/2}(x_o) \subset \partial B$ and radii $\varrho \in (0, \frac{R}{2}]$. Now we consider an arbitrary point $y \in B_{R/2}^+(x_o)$ and arbitrary radii $\varrho \in (0, \frac{R}{4}]$. We abbreviate $R_y := 1 - |y| \in (0, \frac{R}{2}]$ and $y' := \frac{y}{|y|}$ and distinguish between the cases $0 < \varrho < R_y$ and $\varrho \geqslant R_y$. In the latter case, there holds $B_\varrho^+(y) \subset B_{2\varrho}^+(y')$ and therefore, we deduce from (7.9)

$$\varrho^{-2\alpha} \int_{B_\varrho^+(y)} |Dv|^2\,dx \leqslant \varrho^{-2\alpha} \int_{B_{2\varrho}^+(y')} |Dv|^2\,dx \leqslant C R^{-2\alpha} \int_{B_R^+(x_o)} |Dv|^2\,dx. \tag{7.10}$$

We turn our attention to the remaining case $\varrho < R_y$. First, for $\varrho \leqslant \frac{1}{2} R_y$ we use the mean value property of harmonic maps with the result

$$\int_{B_\varrho(y)} |Dv|^2\,dx \leqslant C\varrho^2 \|Dv\|^2_{L^\infty(B_{R_y/2}(y))} \leqslant C\left(\frac{\varrho}{R_y}\right)^2 \int_{B_{R_y}(y)} |Dv|^2\,dx.$$

For $\varrho \in (\frac{1}{2}R_y, R_y)$, the same estimate holds trivially. Combining this with (7.10) for $\varrho = R_y$, we deduce

$$\int_{B_\varrho(y)} |Dv|^2\,dx \leqslant C\left(\frac{\varrho}{R_y}\right)^2 \left(\frac{R_y}{R}\right)^{2\alpha} \int_{B_R^+(x_o)} |Dv|^2\,dx \leqslant C\left(\frac{\varrho}{R}\right)^{2\alpha} \int_{B_R^+(x_o)} |Dv|^2\,dx,$$

where we used $\varrho < R_y$ in the last step. Summarizing, for every $y \in B_{R/2}^+(x_o)$ and every $\varrho \in (0, \frac{R}{4}]$ we infer the bound

$$\varrho^{-2\alpha} \int_{B_\varrho^+(y)} |Dv|^2\,dx \leqslant C R^{-2\alpha} \int_{B_R^+(x_o)} |Dv|^2\,dx,$$

and for $\varrho \in (\frac{R}{4}, \frac{R}{2}]$, it holds trivially. This completes the proof of the desired Morrey estimate (6.6) with $C_M = C \int_{B_R^+(x_o)} |Dv|^2\,dx \leqslant C E_o$ and therefore, Theorem 6.3 (i) yields the claimed $W^{2,2}$-estimate. □

7.2. $W^{1,4}$-regularity for solutions

We start with a comparison estimate for two solutions of (6.1).

Lemma 7.4. *We consider a disk $B_r^+(x_o)$ with $x_o \in \partial B$ and $F_1, F_2 \in L^1(B_r^+(x_o), \mathbb{R}^3)$. Assume that $u_k \in \mathcal{S}^*(\Gamma)$, $k = 1, 2$, are solutions to*

$$\int_{B_r^+(x_o)} Du_k \cdot Dw\,dx + \int_{B_r^+(x_o)} F_k \cdot w\,dx \geqslant 0 \tag{7.11}$$

for all $w \in T_{u_k}\mathcal{S}^*$ with $w = 0$ on $S_r^+(x_o)$ in the sense of traces. Moreover, we assume $u_1 = u_2$ on $S_r^+(x_o)$ in the trace sense and

$$u_k\big(B_r^+(x_o)\big) \subset B_{\varrho_o}(p_o), \quad k = 1, 2, \text{ for some } p_o \in \Gamma,$$

where $\varrho_o = \varrho_o(\Gamma)$ is the radius determined in Lemma 6.2. Then, with a universal constant $C = C(\Gamma)$ the following comparison estimate holds true:

$$\int_{B_r^+(x_o)} |Du_1 - Du_2|^2\,dx \leqslant C\|u_1 - u_2\|_{L^\infty(B_r^+(x_o))} \int_{B_r^+(x_o)} |F_1| + |F_2| + |Du_1|^2 + |Du_2|^2\,dx.$$

Proof. Lemma 6.2 guarantees the existence of maps $\Phi_1, \Phi_2 \in W^{1,2}(B_r^+(x_o), \mathbb{R})$ with

$$\begin{cases} |\Phi_1 - \Phi_2| \leqslant C|u_1 - u_2|, \\ |D\Phi_1| + |D\Phi_2| \leqslant C\big(|Du_1| + |Du_2|\big) \end{cases} \tag{7.12}$$

a.e. on $B_r^+(x_o)$ with a constant $C = C(\Gamma)$, and

$$\widehat{\gamma} \circ \Phi_k = u_k, \quad \text{on } I_r(x_o) \text{ for } k = 1, 2.$$

Since $u_k \in \mathcal{S}^*(\Gamma)$, we can find maps $\varphi_k \in \mathcal{T}^*(\Gamma)$ with

$$\varphi_k(\vartheta) = \Phi_k\big(e^{i\vartheta}\big) \quad \text{provided } e^{i\vartheta} \in I_r(x_o).$$

We define a testing function by

$$w_1 := u_2 - u_1 - \int_{\Phi_1}^{\Phi_2}\int_{\Phi_1}^{t} \widehat{\gamma}''(s)\,ds\,dt =: u_2 - u_1 - g_1.$$

In order to check that this function is admissible in the inequality for u_1, we calculate the boundary values of w in points $e^{i\vartheta} \in I_r(x_o)$ by

$$\begin{aligned} w_1\big(e^{i\vartheta}\big) &= \widehat{\gamma}\big(\varphi_2(\vartheta)\big) - \widehat{\gamma}\big(\varphi_1(\vartheta)\big) - \int_{\varphi_1(\vartheta)}^{\varphi_2(\vartheta)}\int_{\varphi_1(\vartheta)}^{t} \widehat{\gamma}''(s)\,ds\,dt \\ &= \widehat{\gamma}'\big(\varphi_1(\vartheta)\big)\big(\varphi_2(\vartheta) - \varphi_1(\vartheta)\big). \end{aligned}$$

Since $\varphi_2 \in \mathcal{T}^*(\Gamma)$, this implies that $w_1 \in T_{u_1}\mathcal{S}^*$. Moreover, from (7.12) and $u_1 = u_2$ on $S_r^+(x_o)$ we infer that also $\Phi_1 = \Phi_2$ on $S_r^+(x_o)$, in the sense of traces. By definition, we thus have $w_1 = 0$ on $S_r^+(x_o)$ in the trace sense. Therefore, $w_1 = u_2 - u_1 - g_1$ is an admissible testing function in the inequality (7.11) for u_1. This provides us with the estimate

$$\int_{B_r^+(x_o)} Du_1 \cdot D(u_1 - u_2)\,dx \leqslant \int_{B_r^+(x_o)} F_1 \cdot (u_2 - u_1 - g_1) - Du_1 \cdot Dg_1\,dx. \tag{7.13}$$

Similarly as above, one checks that

$$w_2 := u_1 - u_2 - \int_{\Phi_2}^{\Phi_1}\int_{\Phi_2}^{t} \widehat{\gamma}''(s)\,ds\,dt =: u_1 - u_2 - g_2$$

is an admissible testing function in the inequality (7.11) for u_2. This implies

$$-\int_{B_r^+(x_o)} Du_2 \cdot D(u_1 - u_2)\,dx \leqslant \int_{B_r^+(x_o)} F_2 \cdot (u_1 - u_2 - g_2) - Du_2 \cdot Dg_2\,dx. \tag{7.14}$$

Adding the inequalities (7.13) and (7.14), we arrive at

$$\int_{B_r^+(x_o)} |Du_1 - Du_2|^2\,dx \leqslant \int_{B_r^+(x_o)} (F_1 - F_2) \cdot (u_2 - u_1) - \sum_{k=1}^{2}(F_k \cdot g_k + Du_k \cdot Dg_k)\,dx. \tag{7.15}$$

Next, we observe that the definition of g_k and the bounds (7.12) imply for $k = 1, 2$ almost everywhere on $B_r^+(x_o)$ that

$$\begin{aligned} &|g_k| \leqslant C|\Phi_1 - \Phi_2|^2 \leqslant C|u_1 - u_2|^2, \\ &|Dg_k| \leqslant C\big(|D\Phi_1| + |D\Phi_2|\big)|\Phi_1 - \Phi_2| \leqslant C\big(|Du_1| + |Du_2|\big)|u_1 - u_2|, \end{aligned}$$

holds true. Here $C = C(\Gamma)$. In particular we have $|u_1 - u_2| \leqslant 2\varrho_o = C(\Gamma)$. Using the preceding bounds in (7.15), we obtain

$$\int_{B_r^+(x_o)} |Du_1 - Du_2|^2\,dx \leqslant C\|u_1 - u_2\|_{L^\infty(B_r^+(x_o))} \int_{B_r^+(x_o)} |F_1| + |F_2| + |Du_1|^2 + |Du_2|^2\,dx,$$

and this establishes the claimed comparison estimate. □

We use the preceding comparison estimate in the following theorem for the derivation of Calderón–Zygmund type estimates for the gradient. For the proof, we use techniques going back to Caffarelli and Peral [5]. Actually, our proof is inspired by arguments of Acerbi and Mingione [1,28]. In the following theorem, we are dealing both with the boundary case $x_o \in \partial B$ and the interior case $x_o \in B$.

Theorem 7.5. *Assume that $u \in \mathcal{S}^*(\Gamma)$ satisfies (6.1) on a half-disk $B_R^+(x_o)$ with $x_o \in \overline{B}$, $R \in (0,1)$ and $B_R^+(x_o) \cap \{P_1, P_2, P_3\} = \emptyset$, under the assumption (6.2). We assume that $\mathbf{D}(u) \leqslant E_o$ for some constant $E_o \geqslant 1$. Then there is a constant $\varepsilon_3 = \varepsilon_3(C_1, \Gamma, E_o) > 0$ such that the smallness condition*

$$\operatorname*{osc}_{B_R^+(x_o)} u \leqslant \varepsilon_3$$

implies $u \in W^{1,4}(B_{R/4}^+(x_o), \mathbb{R}^3)$, with the corresponding quantitative estimate

$$\int_{B_{R/4}^+(x_o)} |Du|^4\,dx \leqslant \frac{C}{R^2}\Bigg(\int_{B_{R/2}^+(x_o)} |Du|^2\,dx\Bigg)^2 + C\int_{B_{R/2}^+(x_o)} |f|^2\,dx$$

for a universal constant $C = C(C_1, \Gamma, E_o)$.

Proof. We shall later fix the constant $\varepsilon_3 > 0$ such that $\varepsilon_3 \leqslant \min\{\varepsilon_2, \varrho_o\}$ with the constant ε_2 from Lemma 7.3 and the radius ϱ_o from Lemma 6.2. In particular, this implies

$$u\big(B_R^+(x_o)\big) \subset B^3_{\varepsilon_3}(p_o) \subset B^3_{\varrho_o}(p_o) \tag{7.16}$$

for $p_o = u(x_o) \in \mathbb{R}^3$.

Step 1: *Covering of super-level sets*. For every $r \in (0, \frac{R}{2})$ and $\lambda > 0$ we define super-level sets

$$E(r,\lambda) := \big\{x \in B_r^+(x_o):\ x \text{ is a Lebesgue point of } Du \text{ with } \big|Du(x)\big| > \lambda\big\}.$$

We let

$$\lambda_o^2 := \fint_{B_{R/2}^+(x_o)} |Du|^2 + |f|\,dx \tag{7.17}$$

and fix two radii $s, t \in [\frac{R}{4}, \frac{R}{2}]$ with $s < t$. Then we define

$$\lambda_1 := \frac{35R}{t-s}\lambda_o. \tag{7.18}$$

For a fixed $\lambda \geqslant \lambda_1$, we consider a point $x_1 \in E(s,\lambda)$. For any radius r with $\frac{1}{70}(t-s) < r < t-s$, we can estimate by the definition of λ_o

$$\fint_{B_r^+(x_1)} |Du|^2 + |f|\,dx \leqslant \left(\frac{R}{2r}\right)^2 \lambda_o^2 < \left(\frac{35R}{t-s}\right)^2 \lambda_o^2 = \lambda_1^2 \leqslant \lambda^2.$$

On the other hand, since $x_1 \in E(s,\lambda)$, the definition of $E(s,\lambda)$ implies

$$\liminf_{r\downarrow 0} \fint_{B_r^+(x_1)} |Du|^2 + |f|\,dx \geqslant \big|Du(x_1)\big|^2 > \lambda^2.$$

The preceding two estimates and the absolute continuity of the integral enable us to define $r_1 \in (0, \frac{1}{70}(t-s))$ (depending on x_1) as the maximal radius with the property

$$\fint_{B^+_{r_1}(x_1)} |Du|^2 + |f|\,dx = \lambda^2. \tag{7.19}$$

The maximality of the radius implies in particular

$$\fint_{B^+_{r}(x_1)} |Du|^2 + |f|\,dx < \lambda^2 \quad \text{for all } r_1 < r \leqslant 70 r_1. \tag{7.20}$$

Proceeding in this way with every $x \in E(s,\lambda)$, we obtain a family of disks covering $E(s,\lambda)$, each of which satisfies (7.19) and (7.20). By Vitali's covering theorem, we may extract countably many, pairwise disjoint disks $B^+_k := B^+_{r_k}(x_k)$ with centers $x_k \in E(s,\lambda)$ and $0 < r_k < \frac{1}{70}(t-s)$ for $k \in \mathbb{N}$, and with

$$E(s,\lambda) \subset \bigcup_{k\in\mathbb{N}} 5B^+_k \subset B^+_t(x_o).$$

Here and in what follows, we use the notation $\sigma B^+_r(x) = B^+_{\sigma r}(x)$ for any $\sigma > 0$. By the choice of B^+_k, the formulae (7.19) and (7.20) are valid for each of the B^+_k, which means in particular that for each $k \in \mathbb{N}$ there holds

$$\fint_{B^+_k} |Du|^2 + |f|\,dx = \lambda^2 \tag{7.21}$$

and at the same time,

$$\fint_{\sigma B^+_k} |Du|^2 + |f|\,dx < \lambda^2 \quad \text{for } \sigma \in \{5, 10, 70\}. \tag{7.22}$$

Step 2: *Comparison estimates*. For each $k \in \mathbb{N}$, we distinguish whether we are in the *interior situation* $10B_k \Subset B$ or in the *boundary situation* $10\overline{B}_k \cap \partial B \neq \varnothing$. We first consider the interior situation, in which $10B^+_k = 10B_k$. In this case, we choose the comparison map as the harmonic function with $w_k \in u + W^{1,2}_0(10B_k, \mathbb{R}^3)$. Since w_k is harmonic and its boundary values are contained in $B_{\varepsilon_3}(p_o)$ by (7.16), the maximum principle implies $w_k(10B_k) \subset B_{\varepsilon_3}(p_o)$. Testing the equations $\Delta u = F$ and $\Delta w_k = 0$ on $10B_k$ with $w_k - u \in W^{1,2}_0(10B_k, \mathbb{R}^3)$, we therefore infer the *comparison estimate*

$$\fint_{10B_k} |Dw_k - Du|^2\,dx \leqslant 2\varepsilon_3 \fint_{10B_k} |F|\,dx \leqslant 2\varepsilon_3 C_1 \fint_{10B_k} |Du|^2 + |f|\,dx \leqslant C(C_1)\varepsilon_3\lambda^2, \tag{7.23}$$

where we used assumption (6.2) and (7.22) for the two last estimates. Furthermore, since w_k is harmonic and therefore energy minimizing, we have for every $q \in [1, \infty)$

$$\left(\fint_{5B_k} |Dw_k|^q\,dx\right)^{\frac{2}{q}} \leqslant \sup_{5B_k} |Dw_k|^2 \leqslant C \fint_{10B_k} |Dw_k|^2\,dx \leqslant C \fint_{10B_k} |Du|^2\,dx \leqslant C\lambda^2, \tag{7.24}$$

where we used (7.22) in the last step. Next, we turn our attention to the boundary case, in which there exists a point $y_k \in 10\overline{B}_k \cap \partial B$. Writing $\widetilde{B}^+_k := B^+_{r_k}(y_k)$, we have

$$5B^+_k \subset 15\widetilde{B}^+_k \subset 60\widetilde{B}^+_k \subset 70B^+_k.$$

As comparison map on $60\widetilde{B}^+_k$ we choose a minimizer $w_k \in W^{1,2}(B, \mathbb{R}^3)$ of the Dirichlet energy in the class

$$\{w \in \mathcal{S}^*(\Gamma):\ w = u \text{ on } B \setminus 60\widetilde{B}^+_k\}.$$

This minimizer w_k exists by Lemma 7.2 and by Lemma 7.1, it satisfies the differential inequality (7.11) on $60\widetilde{B}_k^+$ with $F = 0$. Moreover, its image is contained in the ball $B_{\varepsilon_3}(p_o)$ by the convex hull property of the Dirichlet energy. We thus infer from the Comparison Lemma 7.4 that

$$\begin{aligned}
\fint_{5B_k^+} |Dw_k - Du|^2\,dx &\leqslant C \fint_{60\widetilde{B}_k^+} |Dw_k - Du|^2\,dx \\
&\leqslant C(\Gamma)\varepsilon_3 \fint_{60\widetilde{B}_k^+} |F| + |Du|^2 + |Dw_k|^2\,dx \\
&\leqslant C(C_1,\Gamma)\varepsilon_3 \fint_{60\widetilde{B}_k^+} |f| + |Du|^2\,dx \leqslant C\varepsilon_3\lambda^2,
\end{aligned} \tag{7.25}$$

where in the last line, we used first the minimizing property of w_k and then the bound (7.22) with $\sigma = 70$ together with the inclusion $60\widetilde{B}_k^+ \subset 70B_k^+$. Moreover, from Lemma 7.3, applied on $60\widetilde{B}_k^+$, we infer the following bound for every $q \in [1,\infty)$.

$$\begin{aligned}
\biggl(\fint_{5B_k^+} |Dw_k|^q\,dx\biggr)^{\frac{2}{q}} &\leqslant C_q\biggl(\fint_{15\widetilde{B}_k^+} |Dw_k|^q\,dx\biggr)^{\frac{2}{q}} \\
&\leqslant C_q \fint_{15\widetilde{B}_k^+} r_k^2\bigl|D^2 w_k\bigr|^2 + |Dw_k|^2\,dx \\
&\leqslant C_q(E_o,\Gamma) \fint_{30\widetilde{B}_k^+} |Dw_k|^2\,dx + C_q \fint_{15\widetilde{B}_k^+} |Dw_k|^2\,dx \\
&\leqslant C_q(E_o,\Gamma) \fint_{60\widetilde{B}_k^+} |Du|^2\,dx \leqslant C_q(E_o,\Gamma)\lambda^2,
\end{aligned} \tag{7.26}$$

where the last bound is a consequence of (7.22) with $\sigma = 70$, since $60\widetilde{B}_k^+ \subset 70B_k^+$.

Step 3: *Energy estimates on super-level sets*. The property (7.21) of the sets B_k^+ implies

$$\bigl|B_k^+\bigr| = \frac{1}{\lambda^2}\int_{B_k^+} |Du|^2\,dx + \frac{1}{\lambda^2}\int_{B_k^+} |f|\,dx. \tag{7.27}$$

In the first integral on the right-hand side, we decompose the domain of integration into $B_k^+ \cap \{|Du| > \lambda/2\}$ and $B_k^+ \cap \{|Du| \leqslant \lambda/2\}$, with the result

$$\frac{1}{\lambda^2}\int_{B_k^+} |Du|^2\,dx \leqslant \frac{1}{\lambda^2}\int_{B_k^+\cap\{|Du|>\lambda/2\}} |Du|^2\,dx + \frac{1}{4}\bigl|B_k^+\bigr|.$$

Similarly, by distinguishing the cases $|f| > \lambda^2/4$ and $|f| \leqslant \lambda^2/4$, we deduce

$$\frac{1}{\lambda^2}\int_{B_k^+} |f|\,dx \leqslant \frac{1}{\lambda^2}\int_{B_k^+\cap\{|f|>\lambda^2/4\}} |f|\,dx + \frac{1}{4}\bigl|B_k^+\bigr|.$$

Plugging the preceding two estimates into (7.27) and re-absorbing the resulting term $\frac{1}{2}|B_k^+|$ into the left-hand side, we arrive at

$$\bigl|B_k^+\bigr| \leqslant \frac{2}{\lambda^2}\int_{B_k^+\cap\{|Du|>\lambda/2\}} |Du|^2\,dx + \frac{2}{\lambda^2}\int_{B_k^+\cap\{|f|>\lambda^2/4\}} |f|\,dx \tag{7.28}$$

for every $k \in \mathbb{N}$. Since the sets $5B_k^+$ cover the super-level set $E(s,\lambda) \supset E(s,L\lambda)$ for every parameter $L \geqslant 1$, there holds

$$\int_{E(s,L\lambda)} |Du|^2\,dx \leqslant \sum_{k\in\mathbb{N}} \int_{5B_k^+\cap E(s,L\lambda)} |Du|^2\,dx.$$

Each of the terms in the above sum can be estimated as follows:

$$\begin{aligned}\int_{5B_k^+\cap E(s,L\lambda)} |Du|^2\,dx &\leqslant 2\int_{5B_k^+} |Du-Dw_k|^2\,dx + \frac{2}{(L\lambda)^{4/3}} \int_{5B_k^+\cap E(s,L\lambda)} |Dw_k|^2|Du|^{4/3}\,dx \\ &\leqslant 2\int_{5B_k^+} |Du-Dw_k|^2\,dx + \frac{C}{(L\lambda)^4}\int_{5B_k^+} |Dw_k|^6\,dx + \frac{1}{2}\int_{5B_k^+\cap E(s,L\lambda)} |Du|^2\,dx,\end{aligned}$$

where we used Young's inequality in the last step. Here, we re-absorb the last integral into the left-hand side and estimate the other two integrals in the preceding line by (7.23) and (7.24) if we are in the interior situation, respectively by (7.25) and (7.26) in the boundary situation. This leads us to

$$\int_{5B_k^+\cap E(s,L\lambda)} |Du|^2\,dx \leqslant C\big(\varepsilon_3 + L^{-4}\big)\lambda^2\big|B_k^+\big|,$$

with $C = C(C_1, \Gamma, E_o)$. Summing over $k \in \mathbb{N}$ and then applying (7.28), we arrive at

$$\begin{aligned}\int_{E(s,L\lambda)} |Du|^2\,dx &\leqslant C\big(\varepsilon_3+L^{-4}\big)\lambda^2 \sum_{k\in\mathbb{N}} \big|B_k^+\big| \\ &\leqslant C\big(\varepsilon_3+L^{-4}\big)\sum_{k\in\mathbb{N}}\Bigg[\int_{B_k^+\cap\{|Du|>\lambda/2\}} |Du|^2\,dx + \int_{B_k^+\cap\{|f|>\lambda^2/4\}} |f|\,dx\Bigg] \\ &\leqslant C\big(\varepsilon_3+L^{-4}\big)\Bigg[\int_{B_t^+(x_o)\cap\{|Du|>\lambda/2\}} |Du|^2\,dx + \int_{B_t^+(x_o)\cap\{|f|>\lambda^2/4\}} |f|\,dx\Bigg].\end{aligned} \tag{7.29}$$

In the last step we used the fact that the sets B_k^+ are pairwise disjoint and contained in $B_t^+(x_o)$. We recall that this estimate holds true for all $\lambda \geqslant \lambda_1$.

Step 4: *The final estimate*. We define truncations

$$|Du|_\ell := \min\big\{|Du|, \ell\big\} \quad \text{for every } \ell \in \mathbb{N}.$$

Fubini's theorem yields for every $\ell \in \mathbb{N}$ that there holds

$$\begin{aligned}\int_{B_s^+(x_o)} |Du|_\ell^2 |Du|^2\,dx &= 2\int_{B_s^+(x_o)} \int_0^{|Du|_\ell} \lambda\,d\lambda\,|Du|^2\,dx \\ &= 2\int_0^\ell \lambda \int_{B_s^+(x_o)\cap\{|Du|_\ell>\lambda\}} |Du|^2\,dx\,d\lambda.\end{aligned}$$

Clearly, for $\lambda \leqslant \ell$, the condition $|Du|_\ell > \lambda$ is equivalent to $|Du| > \lambda$. We use this to calculate by a change of variables

$$\int_{B_s^+(x_o)} |Du|_\ell^2 |Du|^2\,dx = 2L^2\int_0^{\ell/L} \lambda \int_{E(s,L\lambda)} |Du|^2\,dx\,d\lambda$$

$$\leqslant 2L^2 \int_{\lambda_1}^{\ell/L} \lambda \int_{E(s,L\lambda)} |Du|^2\,dx\,d\lambda + L^2\lambda_1^2 \int_{B^+_{R/2}(x_o)} |Du|^2\,dx$$
$$=: II + III.$$

It remains to estimate the term II. For this aim we recall the estimate (7.29), which holds for any $\lambda \geqslant \lambda_1$. This leads us to

$$II \leqslant C\big(\varepsilon_3 L^2 + L^{-2}\big) \int_{\lambda_1}^{2\ell} \lambda \int_{B^+_t(x_o)\cap\{|Du|>\lambda/2\}} |Du|^2\,dx\,d\lambda + C\big(\varepsilon_3 L^2 + L^{-2}\big) \int_{\lambda_1}^{\infty} \lambda \int_{B^+_t(x_o)\cap\{|f|>\lambda^2/4\}} |f|\,dx\,d\lambda$$
$$=: C\big(\varepsilon_3 L^2 + L^{-2}\big)(II_1 + II_2),$$

with the obvious labeling of II_1 and II_2. For the estimation of the first term, we calculate by a change of variables and Fubini's theorem

$$II_1 \leqslant C \int_0^{\ell} \lambda \int_{B^+_t(x_o)\cap\{|Du|_\ell>\lambda\}} |Du|^2\,dx\,d\lambda$$
$$= C \int_{B^+_t(x_o)} \int_0^{|Du|_\ell} \lambda\,d\lambda\,|Du|^2\,dx = C \int_{B^+_t(x_o)} |Du|_\ell^2 |Du|^2\,dx.$$

Similarly, now by the change of variables $\mu = \lambda^2/4$ we estimate

$$II_2 \leqslant C \int_0^{\infty} \int_{B^+_t(x_o)\cap\{|f|>\mu\}} |f|\,dx\,d\mu$$
$$= C \int_{B^+_t(x_o)} \int_0^{|f|} d\mu\,|f|\,dx = C \int_{B^+_t(x_o)} |f|^2\,dx.$$

Collecting the estimates, we arrive at

$$\int_{B^+_s(x_o)} |Du|_\ell^2 |Du|^2\,dx \leqslant C\big(\varepsilon_3 L^2 + L^{-2}\big)\Big(\int_{B^+_t(x_o)} |Du|_\ell^2 |Du|^2\,dx + \int_{B^+_t(x_o)} |f|^2\,dx\Big)$$
$$+ L^2\lambda_1^2 \int_{B^+_{R/2}(x_o)} |Du|^2\,dx,$$

where here, $C = C(C_1, \Gamma, E_o)$. Now we choose first the parameter $L \geqslant 1$ so large that $CL^{-2} \leqslant \frac{1}{4}$ and then $\varepsilon_3 \in (0, 1)$ so small that $C\varepsilon_3 L^2 \leqslant \frac{1}{4}$. This fixes the parameters L and ε_3 in dependence on C_1, Γ and E_o. Using the above choice of parameters and the choice of λ_1 in (7.18), the preceding inequality becomes

$$\int_{B^+_s(x_o)} |Du|_\ell^2 |Du|^2\,dx \leqslant \frac{1}{2} \int_{B^+_t(x_o)} |Du|_\ell^2 |Du|^2\,dx + \frac{1}{2} \int_{B^+_{R/2}(x_o)} |f|^2\,dx + C\frac{R^2}{(t-s)^2}\lambda_o^2 \int_{B^+_{R/2}(x_o)} |Du|^2\,dx,$$

whenever $\frac{R}{4} \leqslant s < t \leqslant \frac{R}{2}$. Therefore, the Iteration Lemma 2.3 is applicable and yields

$$\int_{B^+_{R/4}(x_o)} |Du|_\ell^2 |Du|^2\,dx \leqslant C \int_{B^+_{R/2}(x_o)} |f|^2\,dx + C\lambda_o^2 \int_{B^+_{R/2}(x_o)} |Du|^2\,dx$$

for each $\ell \in \mathbb{N}$. Letting $\ell \to \infty$, we deduce by Fatou's Lemma, keeping in mind the definition of λ_o in (7.17),

$$\int_{B^+_{R/4}(x_o)} |Du|^4\,dx \leqslant C \int_{B^+_{R/2}(x_o)} |f|^2\,dx + C \fint_{B^+_{R/2}(x_o)} |Du|^2 + |f|\,dx \int_{B^+_{R/2}(x_o)} |Du|^2\,dx.$$

This implies the claim by Young's and Jensen's inequalities with a constant C having the dependencies indicated in the formulation of the lemma. □

8. Uniform $W^{2,2}$-estimates

We begin with the interior $W^{2,2}$-estimates, which will be crucial for the boundary estimates since they will imply continuity of the solutions up to the boundary. A similar result was proven in [3, Lemma 7.3] for right-hand sides with $f \in W^{1,2}(B_R(x_o), \mathbb{R}^3)$. Here, we weaken this assumption to $f \in L^2(B_R(x_o), \mathbb{R}^3)$.

Lemma 8.1. *On $B_R(x_o) \Subset B$, consider a weak solution $u \in W^{1,2} \cap C^0(B_R(x_o), \mathbb{R}^3)$ of $\Delta u = F$, where F satisfies (6.2) for some constant $C_1 > 0$ and $f \in L^2(B_R(x_o), \mathbb{R}^3)$. There exists $\varepsilon_o = \varepsilon_o(C_1) > 0$ and $C = C(C_1)$ such that the smallness condition*

$$\int_{B_R(x_o)} |Du|^2\,dx \leqslant \varepsilon_o^2 \tag{8.1}$$

implies $u \in W^{2,2}(B_{R/2}(x_o), \mathbb{R}^3)$ with the quantitative estimate

$$\int_{B_{R/2}(x_o)} \left|D^2u\right|^2 dx \leqslant \frac{C}{R^2} \int_{B_R(x_o)} |Du|^2\,dx + C \int_{B_R(x_o)} |f|^2\,dx. \tag{8.2}$$

Proof. We choose the constant $\varepsilon_o > 0$ as in Theorem 6.1. In view of this theorem, it only remains to establish $u \in W^{1,4}_{\mathrm{loc}}(B_R(x_o), \mathbb{R}^3)$. To this end, for any $y \in B_R(x_o)$, we first exploit the continuity of u in order to choose a radius $\varrho > 0$ small enough to have that $\operatorname{osc}_{B_\varrho(y)} u \leqslant \varepsilon_3$ for the constant ε_3 determined in Theorem 7.5. From this lemma, we then infer $u \in W^{1,4}(B_{\varrho/4}(y), \mathbb{R}^3)$. Since the point $y \in B_R(x_o)$ was arbitrary, this implies $u \in W^{1,4}_{\mathrm{loc}}(B_R(x_o), \mathbb{R}^3)$. Therefore, we may apply the a priori estimates from Theorem 6.1 for any radius $\widetilde{R} < R$ and let $\widetilde{R} \uparrow R$ in order to arrive at the claimed estimate. □

The first important implication of the preceding lemma is the following result that will guarantee small oscillation of the solutions, which we assumed in the preceding sections. A similar result has been used in [36, Lemma 3.1] in the context of a free boundary condition. We point out that similar arguments yield continuity of u up to the boundary if the boundary values are continuous, cf. Hildebrandt and Kaul [23], but the modulus of continuity would depend on the absolute continuity of $\mathcal{L}^2 \llcorner |Du|^2$ and would therefore not be suitable for our purposes.

Lemma 8.2. *Assume that $u \in W^{1,2}(B^+_R(x_o), \mathbb{R}^3) \cap C^0(B^+_R(x_o), \mathbb{R}^3)$ weakly solves $\Delta u = F$ on $B^+_R(x_o)$, where $x_o \in \partial B$ and $R \in (0,1)$, and suppose that F satisfies (6.2) and (6.3) for some constants $C_1, K > 0$. Moreover, we assume that u maps $I_{R/2}(x_o)$ into a subset $G \subset \mathbb{R}^3$. Then there exists $\varepsilon_o = \varepsilon_o(C_1) > 0$ and $C = C(C_1)$ such that the smallness condition*

$$\varepsilon^2 := \int_{B^+_R(x_o)} |Du|^2\,dx \leqslant \varepsilon_o^2$$

implies

$$\operatorname{dist}\bigl(u(y), G\bigr) \leqslant C(\varepsilon + RK) \quad \text{for all } y \in B^+_{R/2}(x_o). \tag{8.3}$$

If additionally, the boundary trace $u|_{I_R(x_o)}$ is continuous with modulus of continuity $\omega: [0,\infty) \to [0,\infty)$, then there holds

$$\operatorname*{osc}_{B^+_{R/2}(x_o)} u \leqslant C(\varepsilon + RK) + 2\omega(R).$$

Proof. We fix an arbitrary $y \in B^+_{R/2}(x_o)$ and let $r := \frac{1}{4}\operatorname{dist}(y, \partial B) < \frac{R}{4}$. We choose the constant $\varepsilon_o = \varepsilon_o(C_1) > 0$ as in Lemma 8.1, which enables us to apply this lemma on $B_{2r}(y) \Subset B^+_R(x_o)$. We infer $u \in W^{2,2}(B_r(y), \mathbb{R}^3) \hookrightarrow C^{0,\alpha}(B_r(y), \mathbb{R}^3)$ for an arbitrary $\alpha \in (0, 1)$, with the corresponding estimate

$$
\begin{aligned}
r^{2\alpha}[u]^2_{C^{0,\alpha}(B_r(y))} &\leqslant C(\alpha) \int_{B_r(y)} r^2 \big|D^2u\big|^2 + |Du|^2\,dx \\
&\leqslant C(\alpha, C_1)\Bigg[\int_{B_{2r}(y)} |Du|^2\,dx + r^2 \int_{B_{2r}(y)} |f|^2\,dx\Bigg] \\
&\leqslant C(\alpha, C_1)\big(\varepsilon^2 + R^2K^2\big).
\end{aligned}
$$

In particular, we know that for every $x \in B_r(y)$, there holds

$$
\big|u(x) - u(y)\big| \leqslant Cr^{-\alpha}(\varepsilon + RK)|x - y|^\alpha \leqslant C(\varepsilon + RK). \tag{8.4}
$$

Here, we may eliminate the dependence of the constant on α by fixing $\alpha = \frac{1}{2}$. Moreover, since $s \mapsto u(sx)$ is absolutely continuous for a.e. $x \in B^+_R(x_o)$ and $u(\frac{x}{|x|}) \in G$, we conclude

$$
\begin{aligned}
\fint_{B_r(y)\cap\partial B_{|y|}} \operatorname{dist}\big(u(x), G\big)\,d\mathcal{H}^1 &\leqslant \fint_{B_r(y)\cap\partial B_{|y|}} \left|u(x) - u\left(\frac{x}{|x|}\right)\right| d\mathcal{H}^1 x \\
&\leqslant \frac{C}{r} \int_{B_r(y)\cap\partial B_{|y|}} \int_{|y|}^1 \left|\frac{\partial u}{\partial r}\left(\varrho\frac{x}{|x|}\right)\right| d\varrho\,d\mathcal{H}^1 x \\
&\leqslant C\Bigg(\int_{B^+_R(x_o)} |Du|^2\,dx\Bigg)^{1/2} \leqslant C\varepsilon,
\end{aligned}
$$

where we used $1 - |y| = \operatorname{dist}(y, \partial B) = 4r$. Combining this with (8.4), we arrive at

$$
\operatorname{dist}\big(u(y), G\big) \leqslant \fint_{B_r(y)\cap\partial B_{|y|}(0)} \big[\operatorname{dist}\big(u(x), G\big) + \big|u(x) - u(y)\big|\big]\,dx \leqslant C(\varepsilon + RK),
$$

which is the first assertion (8.3). If we assume moreover that $u|_{I_R(x_o)}$ is continuous with modulus of continuity ω, then $u(I_{R/2}(x_o))$ is contained in $G \cap B_{\omega(R)}(p)$ for $p = u(x_o)$. Consequently, we infer the estimate (8.3) with $G \cap B_{\omega(R)}(p)$ instead of G, which implies that $u(B^+_{R/2}(x_o))$ is contained in a ball of radius $C(\varepsilon + RK) + \omega(R)$. This yields the second assertion of the lemma. □

Now we are in a position to extend the $W^{2,2}$-estimates of Lemma 8.1 up to the boundary.

Theorem 8.3. *Consider a solution $u \in \mathcal{S}^*(\Gamma) \cap C^0(B, \mathbb{R}^3)$ of (6.1) on a half-disk centered in $x_o \in \partial B$ with $R \in (0, \frac{1}{2})$ and $B^+_R(x_o) \cap \{P_1, P_2, P_3\} = \varnothing$. We suppose that the assumptions (6.2) and (6.3) are in force and that $u|_{I_R(x_o)}$ is continuous with modulus of continuity $\omega: [0, \infty) \to [0, \infty)$. Then there is a constant $\varepsilon_4 = \varepsilon_4(C_1, \Gamma, \omega(\cdot)) \in (0, 1)$ and a radius $R_o = R_o(C_1, \Gamma, K, \omega(\cdot)) \in (0, 1)$ such that the smallness conditions*

$$
\int_{B^+_R(x_o)} |Du|^2\,dx \leqslant \varepsilon_4^2 \quad \text{and} \quad R \leqslant R_o \tag{8.5}
$$

imply $u \in W^{2,2}(B^+_{R/4}(x_o), \mathbb{R}^3)$. Moreover, with a universal constant $C = C(C_1, \Gamma)$ we have the quantitative estimate

$$
\int_{B^+_{R/4}(x_o)} \big|D^2u\big|^2\,dx \leqslant \frac{C}{R^2} \int_{B^+_R(x_o)} |Du|^2\,dx + C \int_{B^+_R(x_o)} |f|^2\,dx. \tag{8.6}
$$

Proof. We will later fix $\varepsilon_4 \in (0,1)$ so small that $\varepsilon_4 \leqslant \min\{\varepsilon_o, \varepsilon_1\}$ with the constants ε_o and ε_1 determined in Lemma 8.2, respectively in Theorem 6.3. Lemma 8.2 then implies

$$\operatorname*{osc}_{B^+_{R/2}(x_o)} u \leqslant C(\varepsilon_4 + R_o K) + 2\omega(R_o).$$

Therefore we can achieve – by choosing $0 < \varepsilon_4 < \min\{\varepsilon_o, \varepsilon_1\}$ and $R_o \in (0,1)$ sufficiently small – that

$$\operatorname*{osc}_{B^+_{R/2}(x_o)} u \leqslant \min\{\varepsilon_1, \varepsilon_3\}, \tag{8.7}$$

where ε_3 denotes the constant from Theorem 7.5 for the choice $E_o = 1$. We note that the choice of ε_4 can be performed in dependence on C_1, Γ and $\omega(\cdot)$, while R_o may depend additionally on K. The small oscillation property (8.7) together with the fact $\int_{B^+_R(x_o)} |Du|^2\,dx \leqslant 1$ enables us to apply the Calderón–Zygmund Theorem 7.5, from which we infer $u \in W^{1,4}_{\mathrm{loc}}(B^+_{R/2}(x_o), \mathbb{R}^3)$. Therefore and because of the smallness properties (8.5) and (8.7), we may apply Theorem 6.3 (ii), which yields the desired estimate (8.6). This concludes the proof of the theorem. □

9. Concentration compactness principle

In this section we consider sequences of maps $u_k \in \mathcal{S}^*(\Gamma, A)$ satisfying the Euler–Lagrange system, the weak Neumann type boundary condition and the stationarity condition. To be precise, for $f \in L^2(B, \mathbb{R}^3)$ we consider solutions $u \in \mathcal{S}^*(\Gamma, A)$ of the system

$$\int_B Du \cdot D\varphi + 2(H \circ u) D_1 u \times D_2 u \cdot \varphi - f \cdot \varphi\,dx = 0 \tag{9.1}$$

for any $\varphi \in C_0^\infty(B, \mathbb{R}^3)$. We note that by Rivière's result in the form of Theorem 2.6, such maps are of class $C^{0,\alpha}(B, \mathbb{R}^3)$ for some $\alpha \in (0,1)$, and by the result of Hildebrandt and Kaul from [23, Lemma 3] also continuous up to the boundary of B, i.e. $u \in C^0(\overline{B}, \mathbb{R}^3)$. Further, we say that $u \in \mathcal{S}^*(\Gamma, A)$ satisfies the Neumann type boundary condition associated to the Plateau boundary condition in the weak sense, if for any $w \in T_u\mathcal{S}^*$ there holds

$$0 \leqslant \int_B [Du \cdot Dw + \Delta u \cdot w]\,dx. \tag{9.2}$$

Finally, we call $u \in \mathcal{S}^*(\Gamma, A)$ stationary (with respect to inner variations), if for any vector field $\eta \in \mathcal{C}^*(B)$ there holds

$$\int_B \operatorname{Re}\big(\mathfrak{h}[u]\bar\partial\eta\big)\,dx - \int_B f \cdot Du\eta\,dx = 0. \tag{9.3}$$

We note that (9.1), (9.2) and (9.3) are satisfied for minimizers of the functionals $\mathbf{F}$ defined in (4.1) with $f = \frac{1}{h}(u - z) \in L^2(B, \mathbb{R}^3)$, see Lemma 4.2.

Lemma 9.1. *Assume that $u_k \in \mathcal{S}^*(\Gamma, A) \cap C^0(\overline{B}, \mathbb{R}^3)$ and $f_k \in L^2(B, \mathbb{R}^3)$ for $k \in \mathbb{N}$. Moreover, suppose that u_k fulfills the Euler–Lagrange system (9.1), the weak Neumann condition (9.2) and the stationarity condition (9.3) with (u_k, f_k) instead of (u, f). Finally, suppose that $u_k \to u$ strongly in $L^2(B, \mathbb{R}^3)$ and*

$$\sup_{k \in \mathbb{N}} \int_B |Du_k|^2 + |f_k|^2\,dx < \infty. \tag{9.4}$$

Then the following hold:

(i) *If furthermore*

$$f_k \rightharpoonup f \quad \text{weakly in } L^2\big(B, \mathbb{R}^3\big), \tag{9.5}$$

then the limit map $u \in \mathcal{S}^*(\Gamma, A)$ solves the Euler–Lagrange system (9.1), fulfills the boundary condition (9.2) and the stationarity condition (9.3).

(ii) *The non-linear H-term converges in the sense of distributions* (*even without the assumption* (9.5)), *that is for every* $\varphi \in C_0^\infty(B, \mathbb{R}^3)$ *we have*

$$\int_B (H \circ u) D_1 u \times D_2 u \cdot \varphi \, dx = \lim_{k \to \infty} \int_B (H \circ u_k) D_1 u_k \times D_2 u_k \cdot \varphi \, dx. \tag{9.6}$$

Proof. We first prove the claim (i) and therefore assume that (9.5) is valid. We start with the observation that by (9.4), the maps $u_k \in \mathcal{S}^*(\Gamma, A)$ satisfy $\sup_{k\in\mathbb{N}} \mathbf{D}(u_k) < \infty$. Moreover, by the definition of the class $\mathcal{S}^*(\Gamma, A)$ they also satisfy the three-point condition, that is $u_k(P_j) = Q_j$ for $j = 1, 2, 3$. As is well known from the theory of parametric minimal surfaces, it then follows from the Courant–Lebesgue Lemma and the Jordan curve property of Γ that the sequence of boundary traces $u_k|_{\partial B}$ is equicontinuous (cf. Lemma 2.2) and therefore all the maps u_k have the same modulus of continuity ω on ∂B. Therefore, we may assume that $u \in \mathcal{S}^*(\Gamma, A)$ and $u_k \to u$ uniformly on ∂B.

We define a sequence of Radon measures μ_k on $\mathbb{R}^2$ by

$$\mu_k := \mathcal{L}^2 \llcorner |Du_k|^2.$$

Since $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,2}(B, \mathbb{R}^3)$ by (9.4), we have

$$\sup_{k\in\mathbb{N}} \mu_k(\mathbb{R}^2) = 2 \sup_{k\in\mathbb{N}} \mathbf{D}(u_k) < \infty.$$

Therefore, passing to a not relabeled subsequence we can assume that $\mu_k \rightharpoonup \mu$ in the sense of Radon measures, for a Radon measure μ on $\mathbb{R}^2$ with $\mu(\mathbb{R}^2) < \infty$. We note that $\mu \llcorner (\mathbb{R}^2 \setminus \overline{B}) = 0$ by construction. Next we define the *singular set* Σ of μ by

$$\Sigma := \{x_o \in \overline{B}: \mu(\{x_o\}) \geqslant \varepsilon\} \cup \{P_1, P_2, P_3\},$$

where $\varepsilon := \min\{\varepsilon_o, \varepsilon_4\} > 0$ for the constants ε_o and ε_4 from Lemma 8.1, respectively Theorem 8.3. We mention that $\operatorname{card}(\Sigma) < \infty$, since $\mu(\overline{B}) < \infty$. Now, for any $x_o \in \overline{B} \setminus \Sigma$ there exists a radius $\varrho_{x_o} > 0$ such that $B_{\varrho_{x_o}}(x_o) \cap \Sigma = \varnothing$ and $\mu(\overline{B_{\varrho_{x_o}}(x_o)}) < \varepsilon$. Since $P_1, P_2, P_3 \in \Sigma$, we have in particular that $B_{\varrho_{x_o}}(x_o) \cap \{P_1, P_2, P_3\} = \varnothing$. In the case of a center $x_o \in B$ we choose the disk in such a way that $B_{\varrho_{x_o}}(x_o) \subseteq B$, while in the boundary case $x_o \in \partial B$ we choose $\varrho_{x_o} \leqslant R_o$, where R_o is the radius from Theorem 8.3. We note that the radius R_o can be chosen only in dependence on $\|H\|_{L^\infty}$, Γ, ω and $K := \sup_k \int_B |f_k|^2 \, dx$, and in particular independent from $k \in \mathbb{N}$. The dependence on $\|H\|_{L^\infty}$ results from the non-linear term $2(H \circ u) D_1 u \times D_2 u$ which can be estimated by $\|H\|_{L^\infty} |Du|^2$. Since

$$\limsup_{k\to\infty} \mu_k\big(B_{\varrho_{x_o}}(x_o)\big) \leqslant \mu\big(\overline{B_{\varrho_{x_o}}(x_o)}\big) < \varepsilon,$$

we can find $k_o \in \mathbb{N}$ such that

$$\int_{B^+_{\varrho_{x_o}}(x_o)} |Du_k|^2 \, dx = \mu_k\big(B_{\varrho_{x_o}}(x_o)\big) < \varepsilon \quad \text{for any } k \geqslant k_o.$$

Therefore, the smallness hypotheses (8.1) of Lemma 8.1 respectively (8.5) of Theorem 8.3 are fulfilled for u_k with $k \geqslant k_o$. Finally, by assumption we have $u_k \in C^0(\overline{B}, \mathbb{R}^3)$ and as stated above, the boundary traces $u_k|_{\partial B}$ are equicontinuous. Therefore, the application of Lemma 8.1, respectively of Theorem 8.3, yields the estimate

$$\begin{aligned} \int_{B^+_{\varrho_{x_o}/4}(x_o)} |D^2 u_k|^2 \, dx &\leqslant C\bigg[\varrho_{x_o}^{-2} \int_{B^+_{\varrho_{x_o}}(x_o)} |Du_k|^2 \, dx + \int_B |f_k|^2 \, dx\bigg] \\ &\leqslant C\bigg[\varrho_{x_o}^{-2} \varepsilon + \sup_{k\in\mathbb{N}} \int_B |f_k|^2 \, dx\bigg] =: C \end{aligned}$$

for any $k \geqslant k_o$, with a constant C independent from k. This implies the uniform bound

$$\sup_{k\geqslant k_o} \|u_k\|_{W^{2,2}(B^+_{\varrho_{x_o}/4}(x_o), \mathbb{R}^3)} < \infty. \tag{9.7}$$

Here we set $B^+_{\varrho_{x_o}/4}(x_o) = B_{\varrho_{x_o}/4}(x_o)$ for an interior point $x_o \in B$. Hence, passing again to a not relabeled subsequence we have $u_k \rightharpoonup u$ weakly in $W^{2,2}(B^+_{\varrho_{x_o}/4}(x_o), \mathbb{R}^3)$ and strongly in $W^{1,q}(B^+_{\varrho_{x_o}/4}(x_o), \mathbb{R}^3)$ for any $q \geqslant 1$. Because of the Sobolev embedding $W^{1,q} \hookrightarrow L^\infty$ that holds for $q > 2$, we moreover have $u_k \to u$ uniformly on $B^+_{\varrho_{x_o}/4}(x_o)$. Therefore, for any $\varphi \in C_0^\infty(B^+_{\varrho_{x_o}/4}(x_o), \mathbb{R}^3)$ we have

$$\int\limits_{B^+_{\varrho_{x_o}/4}(x_o)} Du \cdot D\varphi + 2(H \circ u) D_1 u \times D_2 u \cdot \varphi - f \cdot \varphi \, dx$$
$$= \lim_{k\to\infty} \int\limits_{B^+_{\varrho_{x_o}/4}(x_o)} Du_k \cdot D\varphi + 2(H \circ u_k) D_1 u_k \times D_2 u_k \cdot \varphi - f_k \cdot \varphi \, dx = 0. \tag{9.8}$$

Now we consider the case of a boundary point $x_o \in \partial B \setminus \Sigma$. We choose $w \in T_u^* \mathcal{S}$ and $\zeta \in C_0^\infty(B_{\varrho_{x_o}/4}(x_o), [0,1])$. By definition we have $w(e^{i\vartheta}) = \widehat{\gamma}'(\varphi)(\psi - \varphi)$, where φ is defined by $u(e^{i\vartheta}) = \widehat{\gamma}(\varphi(\vartheta))$ and $\psi \in \mathcal{T}^*(\Gamma)$. For the maps u_k we have the corresponding representations $u_k(e^{i\vartheta}) = \widehat{\gamma}(\varphi_k(\vartheta))$ for some $\varphi_k \in \mathcal{T}^*(\Gamma)$. Due to the uniform convergence $u_k \to u$ on ∂B we know $\varphi_k \to \varphi$ uniformly. We then define w_k on ∂B by $w_k(e^{i\vartheta}) := \widehat{\gamma}'(\varphi_k)(\psi - \varphi_k)$. Its harmonic extension, which we also denote by w_k, clearly is in $W^{1,2}(B, \mathbb{R}^3) \cap L^\infty(B, \mathbb{R}^3)$, because its boundary trace is contained in $W^{\frac{1}{2},2} \cap C^0$. Since

$$\zeta\big(e^{i\vartheta}\big) w_k\big(e^{i\vartheta}\big) = \widehat{\gamma}'(\varphi_k)\big(\zeta\big(e^{i\vartheta}\big)\psi + \big(1 - \zeta\big(e^{i\vartheta}\big)\big)\varphi_k - \varphi_k\big),$$

we see that $\zeta w_k \in T_{u_k}\mathcal{S}^*$ and $\operatorname{spt}(\zeta w_k) \subseteq B^+_{\varrho_{x_o}/4}(x_o)$. Testing the weak Neumann boundary condition (9.2) for u_k with ζw_k we deduce

$$0 \leqslant \int\limits_{B^+_{\varrho_{x_o}/4}(x_o)} \big[Du_k \cdot D(\zeta w_k) + \Delta u_k \cdot (\zeta w_k)\big] \, dx.$$

Since $u_k \in W^{2,2}(B^+_{\varrho_{x_o}/4}(x_o), \mathbb{R}^3)$ the Gauss–Green theorem leads us to

$$0 \leqslant \int\limits_{I_{\varrho_{x_o}/4}(x_o)} \frac{\partial u_k}{\partial r} \cdot (\zeta w_k) \, d\mathcal{H}^1.$$

In the boundary integral we can pass to the limit $k \to \infty$, since we have $\frac{\partial u_k}{\partial r} \to \frac{\partial u}{\partial r}$ in $L^2(I_{\varrho_{x_o}/4}(x_o), \mathbb{R}^3)$ and $w_k \to w$ uniformly on $I_{\varrho_{x_o}/4}(x_o)$. This yields

$$0 \leqslant \int\limits_{I_{\varrho_{x_o}/4}(x_o)} \frac{\partial u}{\partial r} \cdot (\zeta w) \, d\mathcal{H}^1.$$

Using again the Gauss–Green theorem we finally arrive at

$$0 \leqslant \int\limits_{B^+_{\varrho_{x_o}/4}(x_o)} \big[Du \cdot D(\zeta w) + \Delta u \cdot (\zeta w)\big] \, dx, \tag{9.9}$$

whenever $w \in T_u\mathcal{S}^*$ and $\zeta \in C_0^\infty(B_{\varrho_{x_o}/4}(x_o), [0,1])$. By a partition of unity argument we conclude from (9.8) and (9.9) that u solves

$$-\Delta u + 2(H \circ u) D_1 u \times D_2 u = f \quad \text{weakly in } B \setminus \Sigma \tag{9.10}$$

and for any $w \in T_u\mathcal{S}^*$ with $\operatorname{spt} w \subseteq \overline{B} \setminus \Sigma$ the Neumann type boundary condition

$$0 \leqslant \int\limits_B [Du \cdot Dw + \Delta u \cdot w] \, dx \tag{9.11}$$

holds true.

Next, we wish to establish that (9.10) holds on the whole of B. This can be shown by a capacity argument along the lines of the proof of [3, Lemma 7.5]. Once we know that (9.10) holds on B we can apply the modification of Rivière's result from Theorem 2.6 to conclude that $u \in C^{0,\alpha}_{\mathrm{loc}}(B, \mathbb{R}^3)$ for some $\alpha \in (0,1)$. Since $u|_{\partial B}$ is continuous, the same result yields $u \in C^0(\overline{B}, \mathbb{R}^3)$. From (9.10) we conclude that $\Delta u \in L^1(B, \mathbb{R}^3)$. This allows us to apply again a capacity argument to conclude that (9.11) holds for any $w \in T_u\mathcal{S}^*$, without any restriction on the support of w. To summarize, we have shown that (9.10) holds on B and (9.11) holds for any $w \in T_u\mathcal{S}^*$.

To conclude the proof of (i) we finally show that the limit u also fulfills the stationarity condition (9.3). This can be achieved as follows: By assumption we have the stationarity of the maps u_k in the sense that for any $k \in \mathbb{N}$ and every $\eta \in \mathcal{C}^*(B)$ there holds

$$\int_B \mathrm{Re}\big(\mathfrak{h}[u_k]\bar{\partial}\eta\big)\,dx - \int_B f_k \cdot Du_k \eta\,dx = 0. \tag{9.12}$$

Since $f_k \rightharpoonup f$ weakly in $L^2(B, \mathbb{R}^3)$ and $u_k \to u$ strongly in $W^{1,q}(B^+_{\varrho_{x_o}/4}(x_o), \mathbb{R}^3)$, we easily see that for any $\eta \in \mathcal{C}^*(B)$ with support in $\overline{B^+_{\varrho_{x_o}/4}(x_o)}$, the above identity is preserved in the limit, that is

$$\int_B \mathrm{Re}\big(\mathfrak{h}[u]\bar{\partial}\eta\big)\,dx - \int_B f \cdot Du \eta\,dx = 0, \tag{9.13}$$

provided $\operatorname{spt}\eta \subset \overline{B^+_{\varrho_{x_o}/4}(x_o)}$. A partition of unity argument then yields (9.13) for all vector fields $\eta \in \mathcal{C}^*(B)$ with support contained in $\overline{B} \setminus \Sigma$. Since $u \in C^0(\overline{B}, \mathbb{R}^3)$ as noted above, Theorem 7.5 yields $u \in W^{1,4}(\Omega, \mathbb{R}^3)$ for any $\Omega \Subset B \setminus \{P_1, P_2, P_3\}$, which implies in particular $\mathfrak{h}[u] \in L^2(\Omega)$ for any such Ω. In this situation again a capacity argument implies that u is stationary in the sense of (9.3) for any vector field η with support compactly contained in $\overline{B} \setminus \{P_1, P_2, P_3\}$. The case of a general vector field $\eta \in \mathcal{C}^*(B)$ is treated by the following approximation argument. We choose a cut-off function $0 \leqslant \xi \in C^1_0([0,1])$ with $\xi \equiv 1$ on $[0, \frac{1}{2}]$ and $|\xi'| \leqslant 3$. For $0 < \delta \ll 1$ we consider

$$\eta_\delta := \eta(1 - \xi_\delta) \quad \text{where } \xi_\delta(x) := \sum_{j=1}^{3} \xi\left(\frac{|x - P_j|}{\delta}\right).$$

Then, (9.3) holds true with η_δ. Since $\eta(P_j) = 0$ for $j = 1, 2, 3$ and $\operatorname{spt}(\eta \otimes D\xi_\delta) \subset \bigcup_{j=1}^3 B_\delta(P_j)$, we calculate $|\eta \otimes D\xi_\delta| \leqslant C\|D\eta\|_{L^\infty}$ and consequently, $\|D\eta_\delta\|_{L^\infty} \leqslant C\|D\eta\|_{L^\infty}$ for any $0 < \delta \ll 1$. Combining this with $D\eta_\delta \to D\eta$ on $\overline{B} \setminus \{P_1, P_2, P_3\}$ as $\delta \downarrow 0$, the dominated convergence theorem implies that (9.3) holds for $\eta \in \mathcal{C}^*(B)$. This proves (i).

Finally, the claim (ii) can be obtained as follows: Due to the bound (9.4), by passing to a non-relabeled subsequence, we may assume that $u_k \rightharpoonup u$ weakly in $W^{1,2}(B, \mathbb{R}^3)$ and $f_k \rightharpoonup f$ weakly in $L^2(B, \mathbb{R}^3)$. Therefore, we can apply the claim (i), which implies together with the Euler–Lagrange system (9.1) for the maps u_k that

$$\begin{aligned}
\int_B 2(H \circ u) D_1 u \times D_2 u \cdot \varphi\,dx &= \int_B -Du \cdot D\varphi + f \cdot \varphi\,dx \\
&= \lim_{k\to\infty} \int_B -Du_k \cdot D\varphi + f_k \cdot \varphi\,dx \\
&= \lim_{k\to\infty} \int_B 2(H \circ u_k) D_1 u_k \times D_2 u_k \cdot \varphi\,dx
\end{aligned}$$

holds true for any $\varphi \in C^\infty_0(B, \mathbb{R}^3)$. Since the left-hand side is independent from the subsequence, the last equality must hold for the whole sequence. This proves (ii). □

10. The approximation scheme

In this section we follow a method due to Moser [30], who adapted the time discretization approach known as Rothe's method to a biharmonic map heat flow. We use similar techniques for the construction of solutions to the evolutionary Plateau problem for H-surfaces by a time discretization approach. Rothe's method has been applied before in [3] for the construction of global weak solutions to the heat flow for surfaces with prescribed mean curvature with a Dirichlet boundary condition on the lateral boundary. Since the arguments in this section are somewhat similar to those in [3] we only sketch the proofs and avoid reproductions. Throughout this section, we suppose that the general assumptions listed in Section 1 are in force. In particular, we assume that the prescribed mean curvature function H satisfies an isoperimetric condition of type (c, s). By $u_o \in \mathcal{S}^*(\Gamma, A)$ we denoted a fixed reference surface for which the inequality $\mathbf{D}(u_o) \leqslant \frac{1}{2}s(1-c)$ holds true. We recall that by $\mathcal{S}^*(\Gamma, A, \sigma)$ we denoted the class of all surfaces $w \in \mathcal{S}^*(\Gamma, A)$ with $\mathbf{D}(w) \leqslant \sigma\mathbf{D}(u_o)$, for $\sigma = \frac{1+c}{1-c}$. Now, consider $j \in \mathbb{N}_0$ and $h > 0$. We define sequences of energy functionals $\mathbf{F}_{j,h}$ and maps $u_{j,h} \in \mathcal{S}^*(\Gamma, A, \sigma)$ according to the following recursive iteration scheme: We set $u_{o,h} = u_o$. Once $u_{j-1,h}$ is constructed, the map $u_{j,h} \in \mathcal{S}^*(\Gamma, A, \sigma)$ is chosen as a minimizer of the variational problem

$$\mathbf{F}_{j,h}(\tilde{u}) \to \min \quad \text{in } \mathcal{S}^*(\Gamma, A, \sigma), \tag{10.1}$$

for the energy functional

$$\mathbf{F}_{j,h}(\tilde{u}) := \mathbf{D}(\tilde{u}) + 2\mathbf{V}_H(\tilde{u}, u_o) + \frac{1}{2h}\int_B |\tilde{u} - u_{j-1,h}|^2\,dx.$$

Lemma 5.2 guarantees the existence of such a minimizer $u_{j,h} \in \mathcal{S}^*(\Gamma, A, \sigma)$. We have $\mathbf{D}(u_{j,h}) \leqslant \sigma\mathbf{D}(u_o)$. Actually, we have the strict inequality $\mathbf{D}(u_{j,h}) < \sigma\mathbf{D}(u_o)$ for any $j \in \mathbb{N}$, and the same proof also yields an estimate for the discrete time derivative. This follows exactly as in [3, Lemma 8.1] and therefore we state only the result.

Lemma 10.1. *Assume that the assumptions of Lemma 5.2 are in force and $\sigma = \frac{1+c}{1-c}$. Then the minimizers $u_{j,h} \in \mathcal{S}^*(\Gamma, A, \sigma)$ of $\mathbf{F}_{j,h}$ satisfy the strict inequality*

$$\mathbf{D}(u_{j,h}) < \frac{1+c}{1-c}\mathbf{D}(u_o) = \sigma\mathbf{D}(u_o) \tag{10.2}$$

and

$$\sum_{\ell=1}^{j} \frac{1}{2h}\int_B |u_{\ell,h} - u_{\ell-1,h}|^2\,dx \leqslant 2\mathbf{D}(u_o) \tag{10.3}$$

for any $j \in \mathbb{N}$.

Since $\mathbf{D}(u_{j,h}) < \sigma\mathbf{D}(u_o)$, all variations that were used in the proof of Lemma 4.2 remain admissible also under the additional constraint $\mathbf{D}(v) \leqslant \sigma\mathbf{D}(u_o)$. It follows that the minimizers $u_{j,h}$ are actually solutions of the Euler–Lagrange system as stated below.

Theorem 10.2. *Suppose that the assumptions of Lemma 5.2 are in force and that ∂A is of class C^2 with bounded principal curvatures. Further, assume that*

$$\big|H(a)\big| \leqslant \mathcal{H}_{\partial A}(a) \quad \text{for } a \in \partial A. \tag{10.4}$$

Then any minimizer $u_{j,h} \in \mathcal{S}^*(\Gamma, A, \sigma)$ with $j \in \mathbb{N}$ satisfies the time-discrete Euler–Lagrange system weakly on B, that is

$$\int_B \left[\frac{u_{j,h} - u_{j-1,h}}{h} \cdot \varphi + Du_{j,h} \cdot D\varphi + 2(H \circ u_{j,h})D_1 u_{j,h} \times D_2 u_{j,h} \cdot \varphi\right] dx = 0$$

whenever $\varphi \in L^\infty(B, \mathbb{R}^3) \cap W_0^{1,2}(B, \mathbb{R}^3)$. Moreover, $u_{j,h}$ fulfills the weak form of the Neumann type boundary condition, i.e. we have

$$0 \leqslant \int_B [Du_{j,h} \cdot Dw + \Delta u_{j,h} \cdot w]\,dx$$

for any $w \in T_{u_{j,h}}\mathcal{S}^*$. Finally, the maps $u_{j,h}$ are stationary in the sense that there holds

$$\int_B \operatorname{Re}\big(\mathfrak{h}[u_{j,h}]\bar\partial\eta\big)\,dx + \frac{1}{h}\int_B (u_{j,h} - u_{j-1,h})\cdot Du_{j,h}\eta\,dx = 0$$

whenever $\eta \in \mathcal{C}^*(B)$.

We now define the approximating sequence, which will lead to the desired global weak solution in the limit $h \downarrow 0$. We let

$$u_h(x,t) := u_{j,h}(x) \quad \text{for } (j-1)h < t \leqslant jh,\ j \in \mathbb{N} \text{ and } x \in B$$

and $u_h(\cdot,t) = u_o$ for $t \leqslant 0$. Using the finite difference quotient operator in time, that is

$$\Delta_t^h v(x,t) := \frac{v(x,t) - v(x,t-h)}{h},$$

we can re-write the Euler–Lagrange system from above in the form

$$\Delta_t^h u_h - \Delta u_h + 2(H\circ u_h)D_1u_h \times D_2u_h = 0 \quad \text{in } B\times(0,\infty). \tag{10.5}$$

Moreover, we have the stationarity of u_h in the form

$$\int_{B\times\{t\}} \operatorname{Re}\big(\mathfrak{h}[u_h]\bar\partial\eta\big)\,dx + \int_{B\times\{t\}} \Delta_t^h u_h \cdot Du_h\eta\,dx = 0$$

whenever $t > 0$ and $\eta \in \mathcal{C}^*(B)$. Here, $\mathfrak{h}_h := |D_1u_h|^2 - |D_2u_h|^2 - 2\mathbf{i}D_1u_h\cdot D_2u_h$. Finally, we have the weak Neumann type boundary condition for the map u_h for any $t > 0$, that is

$$0 \leqslant \int_{B\times\{t\}} [Du_h \cdot Dw + \Delta u_h \cdot w]\,dx$$

for any $w \in T_{u_h(\cdot,t)}\mathcal{S}^*$. We mention that $u_h(\cdot,t) \in \mathcal{S}^*(\Gamma, A)$ for any $t \geqslant 0$. The bounds (10.2) and (10.3) imply the energy estimate

$$\sup_{h>0}\sup_{T>0}\left[\mathbf{D}\big(u_h(\cdot,T)\big) + \frac{1}{2}\int_0^T\int_B \big|\Delta_t^h u_h\big|^2\,dx\,dt\right] \leqslant C\mathbf{D}(u_o). \tag{10.6}$$

A version of Poincaré's inequality moreover implies

$$\big\|u_h(\cdot,T)\big\|_{L^2(B)} \leqslant C\big\|Du_h(\cdot,T)\big\|_{L^2(B)} + C\big\|u_h(\cdot,T)\big\|_{L^2(\partial B)} \leqslant C\sqrt{\mathbf{D}\big(u_h(\cdot,T)\big)} + C(\Gamma)$$

for any $h, T > 0$, which combined with the uniform energy bound (10.6) yields

$$\sup_{h>0}\|u_h\|_{L^\infty((0,\infty),W^{1,2}(B,\mathbb{R}^3))} \leqslant C\|Du_o\|_{L^2(B,\mathbb{R}^3)} + C(\Gamma). \tag{10.7}$$

Next, arguing exactly as in [3, Chapter 8] we deduce the following continuity property of u_h with respect to the time direction:

$$\big\|u_h(\cdot,t) - u_h(\cdot,s)\big\|_{L^2(B)} \leqslant 4\sqrt{\mathbf{D}(u_o)}[\sqrt{t-s} + \sqrt{h}\,] \quad \forall h > 0,\ t > s \geqslant 0.$$

As in [2, Lemma 4.1] we can conclude from [35, Theorem 3] that there exists a sequence $h_i \downarrow 0$ and a map $u \in C^{0,\frac{1}{2}}([0,\infty); L^2(B,\mathbb{R}^3)) \cap L^\infty([0,\infty); W^{1,2}(B,\mathbb{R}^3))$ such that

$$u_{h_i} \to u \quad \text{in } C^0\big([0,T]; L^2\big(B,\mathbb{R}^3\big)\big) \text{ as } i \to \infty, \text{ for all } T > 0.$$

Further, we can also achieve $Du_{h_i} \rightharpoonup Du$ in $L^2(B\times(0,T),\mathbb{R}^{3\cdot 2})$ for every $T > 0$, as $i \to \infty$. Moreover, Lemma 2.2 implies $u_{h_i}(\cdot,t) \to u(\cdot,t)$ uniformly on ∂B, from which we infer $u(\cdot,t) \in \mathcal{S}^*(\Gamma, A)$ for a.e. $t > 0$.

Now, for $\varphi \in C_0^\infty(B \times (0,\infty), \mathbb{R}^3)$ we have

$$\int_0^\infty \int_B u \cdot \partial_t \varphi \, dx \, dt = - \lim_{i\to\infty} \int_0^\infty \int_B \Delta_t^{h_i} u_{h_i} \cdot \varphi \, dx \, dt \leqslant \sqrt{C\mathbf{D}(u_o)} \|\varphi\|_{L^2(B\times(0,\infty))}.$$

Here we performed a partial integration with respect to difference quotients in time, applied the Cauchy–Schwarz inequality and finally used the uniform bound (10.6). This implies the existence of the weak time derivative $\partial_t u \in L^2(B \times (0,\infty), \mathbb{R}^3)$ with

$$\int_0^\infty \int_B |\partial_t u|^2 \, dx \, dt \leqslant C\mathbf{D}(u_o). \tag{10.8}$$

Moreover, we have

$$\Delta_t^{h_i} u_{h_i} \rightharpoonup \partial_t u \quad \text{weakly in } L^2\big(B \times (0,\infty), \mathbb{R}^3\big). \tag{10.9}$$

Next, from (10.6) we conclude that for $0 \leqslant t_1 < t_2$ there holds

$$\sup_{i\in\mathbb{N}} \int_{t_1}^{t_2} \int_B |Du_{h_i}|^2 \, dx \, dt \leqslant C\mathbf{D}(u_o)(t_2 - t_1) \tag{10.10}$$

and

$$\sup_{i\in\mathbb{N}} \int_{t_1}^{t_2} \int_B \big|\Delta_t^{h_i} u_{h_i}\big|^2 \, dx \, dt \leqslant 2C\mathbf{D}(u_o). \tag{10.11}$$

By Fatou's Lemma we therefore have

$$\int_{t_1}^{t_2} \liminf_{i\to\infty} \int_B |Du_{h_i}|^2 + \big|\Delta_t^{h_i} u_{h_i}\big|^2 \, dx \, dt \leqslant C\mathbf{D}(u_o)(2 + t_2 - t_1) < \infty$$

and for almost every $t \in (t_1, t_2)$ we conclude

$$\liminf_{i\to\infty} \int_B \big|Du_{h_i}(\cdot,t)\big|^2 + \big|\Delta_t^{h_i} u_{h_i}(\cdot,t)\big|^2 \, dx < \infty. \tag{10.12}$$

Hence, for fixed $t \in (t_1, t_2)$ satisfying (10.12) and a non-relabeled subsequence (possibly depending on t) we have

$$\sup_{i\in\mathbb{N}} \int_B \big|Du_{h_i}(\cdot,t)\big|^2 + \big|\Delta_t^{h_i} u_{h_i}(\cdot,t)\big|^2 \, dx < \infty.$$

Next, we consider $k_i \in \mathbb{N}$ such that $(k_i - 1)h_i < t \leqslant k_i h_i$. Then $u_{h_i}(x,t) = u_{k_i,h_i}(x)$ is a minimizer of the functional

$$\mathbf{F}_{k_i,h_i}(\tilde{u}) = \mathbf{D}(\tilde{u}) + 2\mathbf{V}_H(\tilde{u}, u_o) + \frac{1}{2h_i} \int_B |\tilde{u} - u_{k_i-1,h_i}|^2 \, dx$$

in the class $\mathcal{S}^*(\Gamma, A, \sigma)$. From Theorem 10.2 we thereby infer that u_{k_i,h_i} solves the Euler–Lagrange system (9.1) with

$$f_k := -\frac{u_{k_i,h_i} - u_{k_i-1,h_i}}{h_i} = -\Delta_t^{h_i} u_{h_i}(\cdot,t),$$

and moreover, the weak form of the Neumann boundary condition (9.2) and the stationarity condition (9.3), again with f_k defined as above. Finally, for a fixed time $t \in (t_1, t_2)$ we can pass once more to a subsequence – which may depend on t – such that $f_{k_i} \rightharpoonup f(\cdot,t)$ weakly in $L^2(B, \mathbb{R}^3)$ as $i \to \infty$. Therefore, all assumptions of Lemma 9.1 (i) are fulfilled and we conclude that the limit $u(\cdot,t)$ satisfies the limit system

$$-\Delta u(\cdot,t)+2\big(H\circ u(\cdot,t)\big)D_1u(\cdot,t)\times D_2u(\cdot,t)=f(\cdot,t) \tag{10.13}$$

weakly on B. Moreover, $u(\cdot,t)$ fulfills the weak Neumann type boundary condition

$$0\leqslant\int_B\big[Du(\cdot,t)\cdot Dw+\Delta u(\cdot,t)\cdot w\big]\,dx \tag{10.14}$$

for any $w\in T_{u(\cdot,t)}\mathcal{S}^*$, and $u(\cdot,t)$ is stationary in the sense that

$$\int_B\operatorname{Re}\big(\mathfrak{h}\big[u(\cdot,t)\big]\bar\partial\eta\big)\,dx-\int_B f(\cdot,t)\cdot Du(\cdot,t)\eta\,dx=0 \tag{10.15}$$

holds true whenever $\eta\in\mathcal{C}^*(B)$. We note that this holds whenever $t>0$ is chosen such that (10.12) holds. However, since the subsequence chosen above may depend on t this is not enough to identify $-f(\cdot,t)$ as $\partial_t u(\cdot,t)$ and to guarantee that u is the desired global weak solution. Therefore, for given $a>0$ and $i\in\mathbb{N}$ we define the *set of bad time slices* by

$$\Lambda_{i,a}:=\left\{t\in(t_1,t_2):\int_B\big|Du_{h_i}(\cdot,t)\big|^2+\big|\Delta_t^{h_i}u_{h_i}(\cdot,t)\big|^2\,dx>a\right\}.$$

By (10.10) and (10.11), the measure $|\Lambda_{i,a}|$ is bounded by

$$|\Lambda_{i,a}|\leqslant\frac{C\mathbf{D}(u_o)(2+t_2-t_1)}{a}. \tag{10.16}$$

We now define modified sequences $(\tilde u_{h_i})_{i\in\mathbb{N}}$ and $(\tilde f_{h_i})_{i\in\mathbb{N}}$ according to

$$\tilde u_{h_i}(x,t):=\begin{cases}u(x,t) & \text{if } t\in\Lambda_{i,a},\\ u_{h_i}(x,t) & \text{if } t\notin\Lambda_{i,a},\end{cases}$$

and

$$\tilde f_{h_i}(x,t):=\begin{cases}f(x,t) & \text{if } t\in\Lambda_{i,a},\\ -\Delta_t^{h_i}u_{h_i}(x,t) & \text{if } t\notin\Lambda_{i,a}.\end{cases}$$

We observe that for each fixed $a>0$ we still have $\tilde u_{h_i}\to u$ in $L^\infty([t_1,t_2];L^2(B,\mathbb{R}^3))$. Furthermore, for a.e. $t\in(t_1,t_2)$ we have that $\tilde u_{h_i}(\cdot,t)$ solves

$$-\Delta\tilde u_{h_i}(\cdot,t)+2\big(H\circ\tilde u_{h_i}(\cdot,t)\big)D_1\tilde u_{h_i}(\cdot,t)\times D_2\tilde u_{h_i}(\cdot,t)=\tilde f_{h_i}(\cdot,t),$$

weakly on B. From Theorem 2.6, we therefore infer $\tilde u_{h_i}(\cdot,t)\in C^0(\overline{B},\mathbb{R}^3)$ for a.e. $t\in(t_1,t_2)$. Moreover, the maps $\tilde u_{h_i}(\cdot,t)$ fulfill the Neumann type boundary condition and are stationary in the sense

$$\int_B\operatorname{Re}\big[\mathfrak{h}\big[\tilde u_{h_i}(\cdot,t)\big]\bar\partial\eta\big]\,dx-\int_B\tilde f_{h_i}(\cdot,t)\cdot D\tilde u_{h_i}(\cdot,t)\eta\,dx=0$$

for any $\eta\in\mathcal{C}^*(B)$. From the definition of $\tilde u_{h_i}$ and $\tilde f_{h_i}$ it is clear that we have

$$\sup_{i\in\mathbb{N}}\int_B\big|D\tilde u_{h_i}(\cdot,t)\big|^2+\big|\tilde f_{h_i}(\cdot,t)\big|^2\,dx\leqslant\max\left\{a,\int_B\big|Du(\cdot,t)\big|^2+\big|f(\cdot,t)\big|^2\,dx\right\}<\infty$$

for a.e. $t\in(t_1,t_2)$. Therefore, we may apply Lemma 9.1 (ii) to the sequences $\tilde u_{h_i}\in\mathcal{S}^*(\Gamma,A)\cap C^0(\overline{B},\mathbb{R}^3)$ and $\tilde f_{h_i}\in L^2(B,\mathbb{R}^3)$ for $i\in\mathbb{N}$, with the result

$$\int_{B\times\{t\}}2(H\circ u)D_1u\times D_2u\cdot\varphi\,dx=\lim_{i\to\infty}\int_{B\times\{t\}}2(H\circ\tilde u_{h_i})D_1\tilde u_{h_i}\times D_2\tilde u_{h_i}\cdot\varphi\,dx$$

for all $\varphi\in C_0^\infty(B,\mathbb{R}^3)$. Furthermore, we have

$$\int_{B\times\{t\}} 2(H\circ\tilde u_{h_i})D_1\tilde u_{h_i}\times D_2\tilde u_{h_i}\cdot\varphi\,dx \leqslant \|H\|_{L^\infty}\|\varphi\|_{L^\infty}\int_B \left|D\tilde u_{h_i}(\cdot,t)\right|^2 dx$$
$$\leqslant \|H\|_{L^\infty}\|\varphi\|_{L^\infty}\max\left\{a,\int_B \left|Du(\cdot,t)\right|^2 dx\right\}$$

for a.e. $t\in(t_1,t_2)$. Since the right-hand side is in $L^1([t_1,t_2],\mathbb{R})$, the last two formulae imply by the dominated convergence theorem that we have the convergence

$$\int_{t_1}^{t_2}\int_B 2(H\circ u)D_1u\times D_2u\cdot\varphi\,dx\,dt = \lim_{i\to\infty}\int_{t_1}^{t_2}\int_B 2(H\circ\tilde u_{h_i})D_1\tilde u_{h_i}\times D_2\tilde u_{h_i}\cdot\varphi\,dx\,dt \tag{10.17}$$

whenever $\varphi\in C_0^\infty(B\times(t_1,t_2),\mathbb{R}^3)$. It remains to replace the functions $\tilde u_{h_i}$ on the right-hand side by the original sequence u_{h_i}. To this end, we recall the uniform bound (10.6) in order to estimate

$$\int_{B\times\{t\}} \left|2(H\circ u_{h_i})D_1u_{h_i}\times D_2u_{h_i}\cdot\varphi\right| dx \leqslant 2\|H\|_{L^\infty}\|\varphi\|_{L^\infty}\int_B \left|Du_{h_i}(\cdot,t)\right|^2 dx \leqslant C\|H\|_{L^\infty}\|\varphi\|_{L^\infty}\mathbf{D}(u_o).$$

We integrate this with respect to t over $\Lambda_{i,a}$ and use the measure estimate (10.16) to get

$$\int_{\Lambda_{i,a}}\int_B \left|2(H\circ u_{h_i})D_1u_{h_i}\times D_2u_{h_i}\cdot\varphi\right| dx\,dt \leqslant C\|H\|_{L^\infty}\|\varphi\|_{L^\infty}\mathbf{D}(u_o)|\Lambda_{i,a}| \leqslant \widetilde C\,\frac{2+t_2-t_1}{a}$$

with a constant $\widetilde C$ independent from i and a. Similarly, since $\tilde u_{h_i}(\cdot,t)\equiv u(\cdot,t)$ for $t\in\Lambda_{i,a}$, we have

$$\int_{\Lambda_{i,a}}\int_B \left|2(H\circ\tilde u_{h_i})D_1\tilde u_{h_i}\times D_2\tilde u_{h_i}\cdot\varphi\right| dx\,dt \leqslant \widetilde C\int_{\Lambda_{i,a}}\int_B \left|Du(\cdot,t)\right|^2 dx\,dt.$$

Joining the last two estimates we obtain

$$\left|\int_{t_1}^{t_2}\int_B \left[2(H\circ\tilde u_{h_i})D_1\tilde u_{h_i}\times D_2\tilde u_{h_i} - 2(H\circ u_{h_i})D_1u_{h_i}\times D_2u_{h_i}\right]\cdot\varphi\,dx\,dt\right|$$
$$= \left|\int_{\Lambda_{i,a}}\int_B [\dots]\,dx\,dt\right| \leqslant \widetilde C\left[\frac{1}{a}+\int_{\Lambda_{i,a}}\int_B \left|Du(\cdot,t)\right|^2 dx\,dt\right]$$

for a constant $\widetilde C$ independent of i and a. At this stage we let $a\to\infty$. In view of (10.17), we infer

$$\int_{t_1}^{t_2}\int_B 2(H\circ u)D_1u\times D_2u\cdot\varphi\,dx\,dt = \lim_{i\to\infty}\int_{t_1}^{t_2}\int_B 2(H\circ u_{h_i})D_1u_{h_i}\times D_2u_{h_i}\cdot\varphi\,dx\,dt$$

whenever $\varphi\in C_0^\infty(B\times(t_1,t_2),\mathbb{R}^3)$. The last identity, the weak convergence $Du_{h_i}\rightharpoonup Du$ in $L^2(B\times(t_1,t_2),\mathbb{R}^{3\cdot 2})$ and the weak convergence $\Delta_t^{h_i}u_{h_i}\rightharpoonup\partial_t u$ in $L^2(B\times(0,\infty),\mathbb{R}^3)$ from (10.9) allow us to pass to the limit $i\to\infty$ in (10.5), and this proves that u solves the limit system

$$-\Delta u + 2(H\circ u)D_1u\times D_2u = -\partial_t u \quad \text{weakly on } B\times(0,\infty). \tag{10.18}$$

The above construction yields $u\in C^0([0,\infty);L^2(B,\mathbb{R}^3))\cap L^\infty([0,\infty);W^{1,2}(B,\mathbb{R}^3))$ and $\partial_t u\in L^2(B\times(0,\infty),\mathbb{R}^3)$. Since the weak limit u satisfies the weak Neumann type boundary condition (10.14), we only have to show the stationarity condition

$$\int_B \mathrm{Re}\big(\mathfrak{h}\big[u(\cdot,t)\big]\bar\partial\eta\big)\,dx + \int_B \partial_t u(\cdot,t)\cdot Du(\cdot,t)\eta\,dx = 0 \tag{10.19}$$

for any $\eta \in \mathcal{C}^*(B)$ and a.e. $t > 0$. In view of (10.15), it suffices to show $f(\cdot,t) = -\partial_t u(\cdot,t)$ for a.e. $t > 0$. But this easily follows by joining Eqs. (10.13) and (10.18). The proof of Theorem 1.1 is thus completed.

11. Convergence to a stationary solution; proof of Theorem 1.2

Here we study the asymptotics of the flow as $t \to \infty$, more precisely, for a suitable sequence of times $t_k \to \infty$ we wish to show convergence of the maps $(u(\cdot,t_k))_{k\in\mathbb{N}}$ to a conformal H-surface u_* satisfying the Plateau boundary condition, i.e. a solution to (1.18). Since $\partial_t u \in L^2(B \times (0,\infty), \mathbb{R}^3)$ we can find a sequence of times $t_k \to \infty$ with

$$\int_B \left|\partial_t u(\cdot,t_k)\right|^2 dx \to 0 \quad \text{as } k \to \infty. \tag{11.1}$$

Further, we can choose the times t_k in such a way that the partial maps $u(\cdot,t_k)$ satisfy the Euler–Lagrange system (9.1), the weak Neumann type boundary condition (9.2) and the stationarity condition (9.3) with u replaced by $u(\cdot,t_k)$ and f replaced by $f_k := -\partial_t u(\cdot,t_k)$. Since $u \in L^\infty([0,\infty), W^{1,2}(B,\mathbb{R}^3))$ we have

$$\sup_{k\in\mathbb{N}} \int_B \left|Du(\cdot,t_k)\right|^2 + \left|u(\cdot,t_k)\right|^2 dx < \infty, \tag{11.2}$$

so that we can achieve strong convergence $u(\cdot,t_k) \to u_*$ with respect to the L^2-norm and almost everywhere on B for some limit map $u_* \in W^{1,2}(B,A)$. Lemma 2.2 implies the uniform convergence $u(\cdot,t_k)|_{\partial B} \to u_*|_{\partial B}$ of the boundary traces, from which we conclude $u_* \in \mathcal{S}^*(\Gamma, A)$. The property $(1.18)_1$ now follows from an application of Lemma 9.1 (i) with $f_k = -\partial_t u(\cdot,t_k) \to 0$ strongly in $L^2(B,\mathbb{R}^3)$ by (11.1). Furthermore, the same lemma yields the stationarity condition

$$0 = \int_B \operatorname{Re}\big(\mathfrak{h}[u_*]\bar\partial\eta\big)\, dx = \partial\mathbf{D}(u_*;\eta) \tag{11.3}$$

for any $\eta \in \mathcal{C}^*(B)$. Next, we claim that the conformal invariance of $\mathbf{D}$ yields this equation in fact for every $\eta \in \mathcal{C}(B)$. To this end, we define φ_τ as the flow generated by a general vector field $\eta \in \mathcal{C}(B)$ with $\varphi_0 = \mathrm{id}$. For every $\tau \in (-\varepsilon,\varepsilon)$ we choose the conformal diffeomorphism $g_\tau : \overline{B} \to \overline{B}$ defined by $g_\tau(P_j) = \varphi_\tau^{-1}(P_j)$ for $j = 1,2,3$ and define a new variational vector field $\tilde\eta := \frac{\partial}{\partial\tau}|_{\tau=0}(\varphi_\tau \circ g_\tau)$. We note that this definition implies $g_0 = \mathrm{id}$ and $\tilde\eta(P_j) = 0$ for $j = 1,2,3$, so that $\tilde\eta \in \mathcal{C}^*(B)$ is admissible in (11.3). Combining this fact with the conformal invariance of $\mathbf{D}$, we calculate

$$\partial\mathbf{D}(u_*;\eta) = \frac{d}{d\tau}\bigg|_{\tau=0} \mathbf{D}(u_* \circ \varphi_\tau) = \frac{d}{d\tau}\bigg|_{\tau=0} \mathbf{D}(u_* \circ \varphi_\tau \circ g_\tau) = \partial\mathbf{D}(u_*;\tilde\eta) = 0.$$

It is well known that the validity of this equation for every $\eta \in \mathcal{C}(B)$ implies the claimed conformality $(1.18)_3$ of the limit map u_*, cf. [10, Section 4.5] or [13, Corollary 2.2]. This completes the proof of (1.18).

Concerning the regularity of u_*, we first infer from the result of Rivière [34, Theorem I.2] that the limit map u_* is continuous in B. From classical elliptic bootstrap arguments one then concludes $u \in C^{1,\alpha}_{\mathrm{loc}}(B,\mathbb{R}^3)$ for every $\alpha \in (0,1)$, cf. e.g. [3, Lemma 7.2], and an argument by Hildebrandt and Kaul [23] implies even continuity up to the boundary, i.e. $u_* \in C^0(\overline{B},\mathbb{R}^3)$.

Assuming the prescribed mean curvature function H to be Hölder, the classical Schauder theory yields $u_* \in C^{2,\beta}_{\mathrm{loc}}(B,A)$ for some $\beta \in (0,1)$, and u_* is a surface with mean curvature given by H. The boundary regularity can then be retrieved from [10, Section 7.3, Theorem 2], with the result $u_* \in C^{2,\beta}(\overline{B},A)$. For classical solutions u_* to the H-surface equation it is moreover well known that $u_* \in \mathcal{S}^*(\Gamma)$ implies that $u|_{\partial B} : \partial B \to \Gamma$ is a homeomorphism, i.e. $(1.1)_2$, cf. [21, Proof of Satz 3] or [40, Proposition 2.7]. We refer to [38, Theorem 5.3] for a brief summary of regularity results on H-surfaces, as well as to [29,20,44].

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