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Periodic approximations of the ergodic constants in the stochastic homogenization of nonlinear second-order (degenerate) equations ☆

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Abstract

We prove that the effective nonlinearities (ergodic constants) obtained in the stochastic homogenization of Hamilton–Jacobi, "viscous" Hamilton–Jacobi and nonlinear uniformly elliptic pde are approximated by the analogous quantities of appropriate "periodizations" of the equations. We also obtain an error estimate, when there is a rate of convergence for the stochastic homogenization.
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1. Introduction

In this note we prove that the effective nonlinearities arising in the stochastic homogenization of Hamilton–Jacobi, "viscous" Hamilton–Jacobi and nonlinear uniformly elliptic pde can be approximated almost surely by the effective nonlinearities of appropriately chosen "periodizations" of the equations. We also establish an error estimate in settings for which a rate of convergence is known for the stochastic homogenization.

To facilitate the exposition, state (informally) our results in the introduction and put everything in context, we begin by recalling the basic stochastic homogenization results for the class of equations we are considering here. The linear uniformly elliptic problem was settled long ago by Papanicolaou and Varadhan [30,31] and Kozlov [19], while general uniformly variational problems were studied by Dal Maso and Modica [16,17] (see also Zhikov, Kozlov, and Oleĭnik [35]). Nonlinear, nonvariational problems were considered only relatively recently. Souganidis [33] and Rezakhanlou and Tarver [32] considered the stochastic homogenization of convex and coercive Hamilton–Jacobi equations. The homogenization of viscous Hamilton–Jacobi equations with convex and coercive nonlinearities was established by Lions and Souganidis [24,25] and Kosygina, Rezakhanlou, and Varadhan [20]. These equations in spatio-temporal media were studied by Kosygina and Varadhan [21] and Schwab [34]. A new proof for the homogenization yielding

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convergence in probability was found by Lions and Souganidis [26], and the argument was extended to almost sure by Armstrong and Souganidis [5] who also considered unbounded environments satisfying general mixing assumptions. Later Armstrong and Souganidis [6] put forward a new argument based on the so-called metric problem. The convergence rate for these problems was obtained, first in the framework of Hamilton–Jacobi equations by Armstrong, Cardaliaguet and Souganidis [1] and later extended to the viscous Hamilton–Jacobi case by Armstrong and Cardaliaguet [2], while Matic and Nolen [27] obtained variance estimates for a special class of first-order problems. All the above results assume that the Hamiltonians are convex and coercive. The only known result for stochastic homogenization of noncoercive Hamilton–Jacobi equations was obtained by Cardaliaguet and Souganidis [14] for the so-called G-equation (also see Nolen and Novikov [28] who considered the same in dimension $d = 2$ and under additional structure conditions). The stochastic homogenization of fully nonlinear uniformly elliptic second-order equations was established by Caffarelli, Souganidis, and Wang [12] and Caffarelli and Souganidis [10] obtained a rate of convergence in strongly mixing environments. Armstrong and Smart extended in [3] the homogenization result of [12] to some nonuniformly elliptic setting and, recently, improved the convergence rate in [4]. The results of [12,10] were extended to spatio-temporal setting by Lin [23].

The problem considered in this paper—approximation of the homogenized effective quantities by the effective quantities for periodic problems—is a classical one. Approximation by periodic problems as a way to prove random homogenization was used first in [31] for linear uniformly elliptic problems and later, in the context of random walks in random environments, by, for example, Lawler [22] and Guo and Zeitouni [18]. The approach of [31] can be seen as a particular case of the "principle of periodic localization" of Zhikov, Kozlov, and Oleĭnik [35] for linear, elliptic problems. Bourgeat and Piatnitski [8] gave the first convergence estimates for this approximation, while Owhadi [29] proved the convergence for the effective conductivity. As far as we know the results in this paper are the first for nonlinear problems and provide a complete answer to this very natural question.

We continue with an informal discussion of the results. Since the statements and arguments for the Hamilton–Jacobi and viscous Hamilton–Jacobi equations are similar here as well as in the rest of the paper we combine them under the heading viscous Hamilton–Jacobi–Bellman equations, for short viscous HJB. Hence our presentation consists of two parts, one for viscous HJB and one for fully nonlinear second-order uniformly elliptic problems. We begin with the former and continue with the latter.

Viscous Hamilton–Jacobi–Bellman equations We consider viscous HJB equations of the form

$$-\varepsilon \operatorname{tr}\left(A\left(\frac{x}{\varepsilon},\omega\right)D^2u^\varepsilon\right) + H\left(Du^\varepsilon, \frac{x}{\varepsilon},\omega\right) = 0, \tag{1.1}$$

with the possibly degenerate elliptic matrix $A = A(y,\omega)$ and the Hamiltonian $H = H(p,y,\omega)$ stationary ergodic with respect to ω and, moreover, H convex and coercive in p and A the "square" of a Lipschitz matrix. Precise assumptions are given in Section 2. Under these conditions, Lions and Souganidis [25] proved that almost sure homogenization holds. This means that there exists a convex and coercive Hamiltonian $\overline{H}$, which we call the ergodic constant, such that the solution $u^\varepsilon = u^\varepsilon(x,\omega)$ of (1.1), subject to appropriate boundary conditions, converges, as $\varepsilon \to 0$, locally uniformly and almost surely to the solution $\bar{u}$ of the deterministic equation, with the same boundary conditions, $\overline{H}(D\bar{u}) = 0$.

A very useful way to identify the effective Hamiltonian $\overline{H}(p)$ is to consider the approximate auxiliary problem

$$\delta v^\delta - \operatorname{tr}\big(A(x,\omega)D^2v^\delta\big) + H\big(Dv^\delta + p, x, \omega\big) = 0 \quad \text{in } \mathbb{R}^d, \tag{1.2}$$

which admits a unique stationary solution $v^\delta(\cdot,\omega)$, often refer to as an "approximate corrector". It was shown in [24] that, for each $p \in \mathbb{R}^d$, $c > 0$ and almost surely in ω, as $\delta \to 0$,

$$\sup_{y\in B_{c/\delta}} \left|\delta v^\delta(y,\omega) + \overline{H}(p)\right| \to 0. \tag{1.3}$$

If A and H in (1.1) are replaced by L-periodic in y maps $A_L(\cdot,\omega)$ and $H_L(\cdot,\cdot,\omega)$, the effective Hamiltonian $\overline{H}_L(\cdot,\omega)$ is, for any $(p,\omega) \in \mathbb{R}^d \times \Omega$, the unique constant $\overline{H}_L(p,\omega)$ for which the problem

$$-\operatorname{tr}\big(A_L(x,\omega)D^2\chi\big) + H_L(D\chi + p, x, \omega) = \overline{H}_L(p,\omega) \quad \text{in } \mathbb{R}^d, \tag{1.4}$$

has a continuous, L-periodic solution χ. In the context of periodic homogenization, (1.4) and χ are called respectively the corrector equation and corrector. Without any periodicity, for the constant in (1.4) to be unique, it is necessary for χ

to be strictly sublinear at infinity. As it was shown by Lions and Souganidis [24], in general it is not possible to find such solutions. The nonexistence of correctors is the main difference between the periodic and the stationary ergodic settings, a fact which leads to several technical difficulties as well as new qualitative behaviors.

A natural question is whether it is possible to come up with A_L and H_L such that, as $L \to \infty$, $\overline{H}_L(\cdot, \omega)$ converges locally uniformly in p and almost surely in ω to $\overline{H}$. For this it is necessary to choose A_L and H_L carefully. The intuitive idea, and this was done in the linear uniformly elliptic setting [8,29,35], is to take $A_L = A$ and $H_L = H$ in $[-L/2, L/2)^d$ and then to extend them periodically in $\mathbb{R}^d$. Unfortunately, such obvious as well as simple choice cannot work for viscous HJB equations for two reasons. The first one is that (1.4), with the appropriate boundary/initial conditions, does not have a solution unless A_L and H_L are at least continuous, a property that is not satisfied by the simple choice described above. The second one, which is more subtle, is intrinsically related to the convexity and the coercivity of the Hamiltonian. Indeed it turns out that viscous HJB equations are very sensitive to large values of the Hamiltonian. As a consequence, the H_L's must be substantially smaller than H at places where H and H_L differ.

To illustrate the need to come up with suitable periodizations, we discuss the elementary case when $A \equiv 0$ and $H(p, x, \omega) = |p|^2 - V(x, \omega)$ with V stationary, bounded and uniformly continuous. This is one of the very few examples for which the homogenized Hamiltonian is explicitly known for some values of p. Indeed $\overline{H}(0) = \inf_{x \in \mathbb{R}^d} V(x, \omega)$, a quantity which is independent on ω in view of the stationarity of V and the assumed ergodicity. If $H_L(p, x, \omega) = |p|^2 - V_L(x, \omega)$, with V_L is L-periodic, then $\overline{H}_L(0, \omega) = \inf_{x \in \mathbb{R}^d} V_L(x, \omega)$. It follows that V_L cannot just be any regularized truncation of $V(\cdot, \omega)$, since it must satisfy, in addition, the condition $\inf_{x \in \mathbb{R}^d} V_L(x, \omega) \to \inf_{x \in \mathbb{R}^d} V(x, \omega)$ as $L \to \infty$. To illustrate further this restriction, let us naively define V_L in $[-(L-1)/2, (L-1)/2)^d$ as a smooth interpolation between $V_L = V$ in $[-(L-1)/2, (L-1)/2)^d$ and $V_L = 0$ on $\partial[-L/2, L/2)^d$, and extend V_L periodically. Then this approximation would not always be suitable for the approximation because it implies $\overline{H}_L(0, \omega) \leqslant 0$ whatever the map V is.

Here we show that it is possible to choose periodic A_L and H_L so that the ergodic constant $\overline{H}_L(p, \omega)$ converges, as $L \to +\infty$ to $\overline{H}(p)$ locally uniformly in p and almost surely in ω. One direction of the convergence is based on the knowledge of homogenization, while the other relies on the construction of subcorrectors (i.e., subsolutions) to (1.4) using approximate correctors for the original system. We also provide an error estimate for the convergence provided a rate is known for the homogenization, which is the case for "i.i.d. environments" [1,2].

Fully nonlinear, uniformly elliptic equations We consider fully nonlinear uniformly elliptic equations of the form

$$F\left(D^2 u^\varepsilon, \frac{x}{\varepsilon}, \omega\right) = 0. \tag{1.5}$$

Following Caffarelli, Souganidis and Wang [12], it turns out that there exists a uniformly elliptic $\overline{F}$, which we call again the ergodic constant of the homogenization, such that the solution of (1.5)—complemented with suitable nonoscillatory boundary conditions—converges, as $\varepsilon \to 0$ and almost surely, to the solution of $\overline{F}(D^2 u) = 0$ with the same boundary behavior.

Although technically involved, this setting is closer to the linear elliptic one. Indeed we show that the effective equation $\overline{F}_L$ of any (suitably regularized) uniformly elliptic (with constants independent of L) periodization of F converges almost surely to $\overline{F}$. The proof relies on a combination of homogenization and an Alexandroff–Bakelman–Pucci (ABP) estimate-type argument. We also give an error estimate for the difference $|\overline{F}_L(P, \omega) - \overline{F}(P)|$, again provided we know rates for the stochastic homogenization like the one's established in [10] and [4].

1.1. Organization of the paper

In the remainder of the introduction we introduce the notations and some of the terminology needed for the rest of the paper, discuss the general random setting and record the properties of an auxiliary cut-off function we will be using to ensure the regularity of the approximations. The next two sections are devoted to the viscous HJB equations. In Section 2 we introduce the basic assumptions and state and prove the approximation result. Section 3 is about the rate of convergence. The last two sections are about the elliptic problem. In Section 4 we discuss the assumptions and state and prove the approximation result. The rate of convergence is the topic of the last section of the paper.

1.2. Notation and conventions

The symbols C and c denote positive constants which may vary from line to line and, unless otherwise indicated, depend only on the assumptions for A, H and other appropriate parameters. We denote the d-dimensional Euclidean space by $\mathbb{R}^d$, $\mathbb{N}$ is the set of natural numbers, $\mathcal{S}^d$ is the space of $d \times d$ real valued symmetric matrices, and I_d is the $d \times d$ identity matrix. For each $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, $|y|$ denotes the Euclidean length of y, $|y|_\infty = \max_i |y_i|$ its l^∞-length, $\|X\|$ is the usual l^2-norm of $X \in \mathcal{S}^d$ and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^d$. If $E \subseteq \mathbb{R}^d$, then $|E|$ is the Lebesgue measure of E and $\mathrm{Int}(E)$, $\overline{E}$ and $\mathrm{conv}\, E$ are respectively the interior, the closure and the closure of the convex hull of E. We abbreviate almost everywhere to a.e. We use $\sharp(\mathcal{K})$ for the number of points of a finite set $\mathcal{K}$. For $r > 0$, we set $B(y, r) := \{x \in \mathbb{R}^d\colon\ |x - y| < r\}$ and $B_r := B(0, r)$. For each $z \in \mathbb{R}^d$ and $R > 0$, $Q_R(z) := z + [-R/2, R/2)^d$ in $\mathbb{R}^d$ and $Q_R := Q_R(0)$. We say that a map is 1-periodic, if it is periodic in Q_1. The distance between two subsets $U, V \subseteq \mathbb{R}^d$ is $\mathrm{dist}(U, V) = \inf\{|x - y|\colon\ x \in U,\ y \in V\}$. If $f : E \to \mathbb{R}$ then $\mathrm{osc}_E\, f := \sup_E f - \inf_E f$. The sets of functions on a set $U \subseteq \mathbb{R}^d$ which are Lipschitz, have Lipschitz continuous derivatives and are smooth functions that are written respectively as $C^{0,1}(U)$, $C^{1,1}(U)$ and $C^\infty(U)$. The set of α-Hölder continuous functions on $\mathbb{R}^d$ is $C^{0,\alpha}$ and $\|u\|$ and $[u_L]_{0,\alpha}$ denote respectively the sup-norm and α-Hölder seminorm. When we need to denote the dependence of these last quantities on a particular domain U we write $\|u\|_U$ and $[u_L]_{0,\alpha;U}$. The Borel σ-field on a metric space M is $\mathcal{B}(M)$. If $M = \mathbb{R}^d$, then $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we write a.s. or $\mathbb{P}$-a.s. to abbreviate almost surely.

Throughout the paper, all differential inequalities are taken to hold in the viscosity sense. Readers not familiar with the fundamentals of the theory of viscosity solutions may consult standard references such as [7,15].

1.3. The general probability setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a group $(\tau_y)_{y \in \mathbb{R}^d}$ of $\mathcal{F}$-measurable, measure-preserving transformations $\tau_y : \Omega \to \Omega$. That is, we assume that, for every $x, y \in \mathbb{R}^d$ and $A \in \mathcal{F}$,

$$\mathbb{P}\big[\tau_y(A)\big] = \mathbb{P}[A] \quad \text{and} \quad \tau_{x+y} = \tau_x \circ \tau_y. \tag{1.6}$$

Moreover, the group $(\tau_y)_{y \in \mathbb{R}^d}$ is assumed to be ergodic, that is, if for $A \in \mathcal{F}$,

$$\tau_y(A) = A \quad \text{for all } y \in \mathbb{R}^d, \quad \text{then either } \mathbb{P}[A] = 0 \text{ or } \mathbb{P}[A] = 1. \tag{1.7}$$

A map $f : M \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, with M either $\mathbb{R}^d$ or $\mathcal{S}^d$, which is measurable with respect to $\mathcal{B}(M) \otimes \mathcal{B} \otimes \mathcal{F}$ is called stationary if, for every $m \in M$, $y, z \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$f(m, y, \tau_z \omega) = f(m, y + z, \omega).$$

Given a random variable $f : M \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, for each $E \in \mathcal{B}$, let $\mathcal{G}(E)$ be the σ-field on Ω generated by the $f(m, x, \cdot)$ for $x \in E$ and $m \in M$. We say that the environment is "i.i.d.", if there exists $D > 0$ such that, for all $V, W \in \mathcal{B}$,

$$\text{if } \mathrm{dist}(V, W) \geqslant D \quad \text{then } \mathcal{G}(V) \text{ and } \mathcal{G}(W) \text{ are independent.} \tag{1.8}$$

We say that the environment is strongly mixing with rate $\phi : [0, \infty) \to [0, \infty)$, if $\lim_{r \to \infty} \phi(r) = 0$ and

$$\text{if } \mathrm{dist}(V, W) \geqslant r \quad \text{then} \quad \sup_{A \in \mathcal{G}(V),\, B \in \mathcal{G}(W)} \big|\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B]\big| \leqslant \phi(r). \tag{1.9}$$

Recall that "i.i.d." and strongly mixing environments are ergodic.

1.4. An auxiliary function

To avoid repetition we summarize here the properties of an auxiliary cut-off function we use in all sections to construct the periodic approximations. We fix $\eta \in (0, 1/4)$ and choose a 1-periodic smooth $\zeta_\eta : \mathbb{R}^d \to [0, 1]$ so that

$$\zeta_\eta = 0 \quad \text{in } Q_{1-2\eta}, \qquad \zeta_\eta = 1 \quad \text{in } Q_1 \backslash Q_{1-\eta}, \qquad \|D\zeta_\eta\| \leqslant c/\eta \quad \text{and} \quad \big\|D^2\zeta_\eta\big\| \leqslant c/\eta^2. \tag{1.10}$$

To simplify the notation we often omit the dependence of ζ_η on η.

2. Approximations for viscous HJB equations

We introduce the hypotheses and we state and prove the approximation result.

2.1. The hypotheses

The Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is assumed to be measurable with respect to $\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{F}$. We write $H = H(p, y, \omega)$ and we require that, for every $p, y, z \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$H(p, y, \tau_z\omega) = H(p, y + z, \omega). \tag{2.1}$$

We continue with the structural hypotheses on H. We assume that H is, uniformly in (y, ω), coercive in p, that is there exist constants $C_1 > 0$ and $\gamma > 1$ such that

$$C_1^{-1}|p|^\gamma - C_1 \leqslant H(p, x, \omega) \leqslant C_1|p|^\gamma + C_1, \tag{2.2}$$

and, for all (y, ω),

$$\text{the map } p \to H(p, y, \omega) \text{ is convex.} \tag{2.3}$$

The last assumption can be relaxed to level-set convexity at the expense of some technicalities but we are not pursuing this here.

The required regularity of H is that, for all $x, y, p, q \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\big|H(p, x, \omega) - H(p, y, \omega)\big| \leqslant C_1\big(|p|^\gamma + 1\big)|x - y| \tag{2.4}$$

and

$$\big|H(p, x, \omega) - H(q, x, \omega)\big| \leqslant C_1\big(|p|^{\gamma-1} + |q|^{\gamma-1} + 1\big)|p - q|. \tag{2.5}$$

Next we discuss the hypotheses on $A : \mathbb{R}^d \times \Omega \to \mathcal{S}^d$. We assume that, for each $(y, \omega) \in \mathbb{R}^d \times \Omega$, there exists a $d \times k$ matrix $\Sigma = \Sigma(y, \omega)$ (for some integer $k \in \mathbb{N}$) such that

$$A(y, \omega) = \Sigma(y, \omega)\Sigma^T(y, \omega). \tag{2.6}$$

The matrix Σ is supposed to be measurable with respect to $\mathcal{B} \otimes \mathcal{F}$ and stationary, that is for any $y, z \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\Sigma(y, \tau_z\omega) = \Sigma(y + z, \omega). \tag{2.7}$$

It is clear that (2.6) and (2.7) yield that A is degenerate elliptic and stationary.

We also assume that Σ is Lipschitz continuous with respect to the space variable, i.e., for all $x, y \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\big|\Sigma(x, \omega) - \Sigma(y, \omega)\big| \leqslant C_1|x - y|. \tag{2.8}$$

To simplify statements, we write

$$\text{(2.1), (2.2), (2.3), (2.4) and (2.5) hold,} \tag{2.9}$$

and

$$\text{(2.6), (2.7) and (2.8) hold.} \tag{2.10}$$

We denote by $\overline{H} = \overline{H}(p)$ the averaged Hamiltonian corresponding to the homogenization problem for H and A. We recall from the discussion in the introduction that $\overline{H}(p)$ is the a.s. limit, as $\delta \to 0$, of $-\delta v^\delta(0, \omega)$, where v^δ is the solution to (1.2). Note (see [25]) that, in view of (2.3), $\overline{H}$ is convex. Moreover (2.2) yields (again see [25])

$$C_1^{-1}|p|^\gamma - C_1 \leqslant \overline{H}(p) \leqslant C_1|p|^\gamma + C_1. \tag{2.11}$$

2.2. The periodic approximation

Fix $\eta > 0$ and let ζ_η be a smooth cut-off function satisfying (1.10). For $(p, x, \omega) \in \mathbb{R}^d \times Q_L \times \Omega$ we set

$$A_{L,\eta}(x,\omega) = \left(1 - \zeta_\eta\left(\frac{x}{L}\right)\right)A(x,\omega)$$

and

$$H_{L,\eta}(p,x,\omega) = \left(1 - \zeta_\eta\left(\frac{x}{L}\right)\right)H(p,x,\omega) + \zeta_\eta\left(\frac{x}{L}\right)H_0(p),$$

where, for a constant $C_2 > 0$ to be defined below,

$$H_0(p) = C_2^{-1}|p|^\gamma - C_2.$$

Then we extend $A_{L,\eta}$ and $H_{L,\eta}$ to $\mathbb{R}^d \times \mathbb{R}^d \times \Omega$ by periodicity, i.e., for all $(x, p) \in \mathbb{R}^d$, $\omega \in \Omega$ and $\xi \in \mathbb{Z}^d$,

$$H_{L,\eta}(p, x + L\xi, \omega) = H_{L,\eta}(p, x, \omega) \quad \text{and} \quad A_{L,\eta}(x + L\xi, \omega) = A_{L,\eta}(x, \omega).$$

To define C_2, let us recall (see [25]) that (2.9) yields a constant $C_3 \geqslant 1$ such that, for any $\omega \in \Omega$, $p \in \mathbb{R}^d$ and $\delta \in (0, 1)$, the solution v^δ of (1.2) satisfies $\|Dv^\delta + p\|_\infty \leqslant C_3(|p| + 1)$. Then we choose C_2 so large that, for all $p \in \mathbb{R}^d$,

$$C_2^{-1}\big(C_3\big(|p| + 1\big)\big)^\gamma - C_2 \leqslant C_1^{-1}|p|^\gamma - C_1.$$

In view of (2.2), (2.11) and the previous discussion on v^δ, we have

$$H_0\big(Dv^\delta + p\big) \leqslant \overline{H}(p) \quad \text{and} \quad H_0(p) \leqslant \overline{H}(p), \tag{2.12}$$

and, in addition, uniformly in (y, ω),

$$H_{L,\eta}(p, y, \omega) \text{ is coercive in } p \text{ with a constant that depends only on } C_1. \tag{2.13}$$

2.3. The approximation result

Let $\overline{H}_{L,\eta} = \overline{H}_{L,\eta}(p, \omega)$ be the averaged Hamiltonian corresponding to the homogenization problem for $H_{L,\eta}$ and $A_{L,\eta}$. We claim that, as $L \to +\infty$, $\overline{H}_{L,\eta}$ is a good a.s. approximation of $\overline{H}$.

Theorem 2.1. *Fix $\eta > 0$ and assume (1.7), (2.9) and (2.10). There exists a constant $C > 0$ such that, for all p and a.s.,*

$$\limsup_{L\to+\infty} \overline{H}_{L,\eta}(p,\omega) \leqslant \overline{H}(p) \leqslant \liminf_{L\to+\infty} \overline{H}_{L,\eta}(p,\omega) + C\big(|p|^\gamma + 1\big)\eta.$$

In general it does not seem possible to let $\eta \to 0$ simultaneously with $L \to +\infty$ in the above statement. However, it is a simple application of the discussion of Section 3, that under suitable assumptions on the random media, it is possible to choose $\eta = \eta_n \to 0$ as $L_n \to +\infty$ in such a way that

$$\lim_{n\to+\infty} \overline{H}_{L_n,\eta_n}(p) = \overline{H}(p).$$

2.4. The proof of Theorem 2.1

From now on, to simplify the notation, we drop the dependence of the various quantities with respect to η and simply write H_L, A_L and $\overline{H}_L$ for $H_{L,\eta}$, $A_{L,\eta}$ and $\overline{H}_{L,\eta}$.

The proof is divided into two parts which are stated as two separate lemmata. The first is the upper bound, which relies on homogenization and only uses the fact that $H = H_L$ in $Q_{L(1-2\eta)}$. The second is the lower bound. Here the specific construction of H_L plays a key role.

Lemma 2.2 *(The upper bound). Assume (1.7), (2.9) and (2.10). There exists $C > 0$ that depends only on C_1 such that, for all $p \in \mathbb{R}^d$ and a.s.,*

$$\overline{H}(p) \leqslant \liminf_{L\to+\infty} \overline{H}_L(p,\omega) + C\big(|p|^\gamma + 1\big)\eta.$$

Proof. Choose $\omega \in \Omega$ for which homogenization holds (recall that this is the case for almost all ω) and fix $p \in \mathbb{R}^d$. Let χ_L^p be a corrector for the L-periodic problem, i.e., a continuous, L-periodic solution of (1.4). Without loss of generality we assume that $\chi_L^p(0,\omega) = 0$. Moreover, since H_L is coercive, there exists (see [25]) a constant C_3, which depends only on C_1, such that $\|D\chi_L^p + p\|_\infty \leqslant C_3(|p|+1)$.

Define $\Phi_L^p(x,\omega) := L^{-1}\chi_L^p(Lx,\omega)$. It follows that the Φ_L^p's are 1-periodic, Lipschitz continuous uniformly in L, since $\|D\Phi_L^p + p\|_\infty \leqslant C_3(|p|+1)$, and uniformly bounded in $\mathbb{R}^d$, since $\Phi_L^p(0,\omega) = 0$. Moreover,

$$-L^{-1}\operatorname{tr}\big(A_L(Lx,\omega)D^2\Phi^p\big) + H_L\big(D\Phi_L^p + p, Lx, \omega\big) = \overline{H}_L(p,\omega) \quad \text{in } \mathbb{R}^d.$$

Since $H_L = H$ and $A_L = A$ in $Q_{L(1-2\eta)}$, we also have

$$-L^{-1}\operatorname{tr}\big(A(Lx,\omega)D^2\Phi_L^p\big) + H\big(D\Phi_L^p + p, Lx, \omega\big) = \overline{H}_L(p,\omega) \quad \text{in } \operatorname{Int}(Q_{1-2\eta}).$$

Let $L_n \to +\infty$ be such that $H_{L_n}(p,\omega) \to \liminf_{L\to+\infty} H_L(p,\omega)$. The equicontinuity and equiboundedness of the Φ_L^p's yield a further subsequence, which for notational simplicity we still denote by L_n, such that the $\Phi_{L_n}^p$'s converge uniformly in $\mathbb{R}^d$ to a Lipschitz continuous, 1-periodic map $\Phi^p : \mathbb{R}^d \to \mathbb{R}$. Note that, by periodicity

$$\int_{Q_1} D\Phi^p = 0. \tag{2.14}$$

Since homogenization holds for the ω at hand, by the choice of the subsequence, we have both in the viscosity and a.e. sense that

$$\overline{H}\big(D\Phi^p + p\big) = \liminf_{L\to+\infty} \overline{H}_L(p,\omega) \quad \text{in } \operatorname{Int}(Q_{1-2\eta}). \tag{2.15}$$

It then follows from (2.14), the convexity of $\overline{H}$ and Jensen's inequality that

$$\overline{H}(p) \leqslant \int_{Q_1} \overline{H}\big(D\Phi^p + p\big).$$

Using the bound on $\|D\Phi^p\|$ together with (2.11) and (2.15), we get

$$\begin{aligned}\int_{Q_1} \overline{H}\big(D\Phi^p + p\big) &\leqslant \int_{Q_{1-2\eta}} \overline{H}\big(D\Phi^p + p\big) + C_1\big(\big\|D\Phi^p + p\big\|_\infty^\gamma + 1\big)|Q_1 \backslash Q_{1-2\eta}| \\ &\leqslant (1-2\eta)^d \liminf_{L\to+\infty} \overline{H}_L(p,\omega) + C\big(|p|^\gamma + 1\big)\eta,\end{aligned}$$

and, after employing (2.11) once more,

$$\overline{H}(p) \leqslant \liminf_{L\to+\infty} \overline{H}_L(p,\omega) + C\big(|p|^\gamma + 1\big)\eta. \qquad \square$$

To state the next lemma recall that, for any $\delta > 0$ and $p \in \mathbb{R}^d$, $v^\delta(\cdot,\omega;p)$ solves (1.2).

Lemma 2.3 *(The lower bound). Assume (2.9) and (2.10). For any $K > 0$, there exists $C > 0$ such that, for all $p \in B_K$, $L = 1/\delta \geqslant 1$ and $\varepsilon > 0$,*

$$\Big\{\omega \in \Omega\colon \sup_{y\in Q_{1/\delta}} \big|\delta v^\delta(y,\omega;p) + \overline{H}(p)\big| \leqslant \varepsilon\Big\} \subset \Big\{\omega \in \Omega\colon \overline{H}_L(p,\omega) - \overline{H}(p) \leqslant \frac{C\varepsilon}{\eta}\Big\}.$$

Proof. Fix $\omega \in \Omega$ such that

$$\sup_{y \in Q_{1/\delta}} \left|\delta v^\delta(y,\omega;p) + \overline{H}(p)\right| \leqslant \varepsilon, \tag{2.16}$$

and let $\xi : \mathbb{R}^d \to \mathbb{R}$ be a smooth 1-periodic map such that

$$\xi = 1 \quad \text{in } Q_{1-\eta}, \qquad \xi = 0 \quad \text{in } Q_1 \backslash Q_{1-\eta/2}, \qquad \|D\xi\|_\infty \leqslant C\eta^{-1} \quad \text{and} \quad \left\|D^2\xi\right\|_\infty \leqslant C\eta^{-2}.$$

Recall that $L = 1/\delta$, define

$$\Psi_L(x,\omega) := \xi\left(\frac{x}{L}\right) v^\delta(x,\omega;p) - \left(1 - \xi\left(\frac{x}{L}\right)\right)\frac{\overline{H}(p)}{\delta} \quad \text{in } Q_L$$

and extend $\Psi_L(\cdot,\omega)$ periodically (with period L) over $\mathbb{R}^d$.

The goal is to estimate the quantity $-\mathrm{tr}(A_L(x,\omega)D^2\Psi_L) + H_L(D\Psi_L + p, x, \omega)$. In what follows we argue as if Ψ_L were smooth, the computation being actually correct in the viscosity sense.

Observe that, if $x \in Q_L$, then

$$D\Psi_L(x,\omega) = \xi\left(\frac{x}{L}\right) Dv^\delta(x,\omega;p) + \frac{1}{L} D\xi\left(\frac{x}{L}\right)\left(v^\delta(x,\omega;p) + \frac{\overline{H}(p)}{\delta}\right).$$

Hence, in view of (2.16),

$$\left|D\Psi_L(x,\omega) - \xi\left(\frac{x}{L}\right) Dv^\delta(x,\omega;p)\right| \leqslant \frac{C\varepsilon}{\eta L\delta} = \frac{C\varepsilon}{\eta}. \tag{2.17}$$

Note that $\Psi_L = v^\delta$ in $Q_{L(1-\eta)}$ while $\zeta(\cdot/L) = 1$ in $Q_L \backslash Q_{L(1-\eta)}$. It then follows from the definition of A_L and H_L that, when $x \in Q_L$,

$$\begin{aligned}
&-\mathrm{tr}\big(A_L(x,\omega)D^2\Psi_L\big) + H_L(D\Psi_L + p, x, \omega) \\
&\quad = \left(1 - \zeta\left(\frac{x}{L}\right)\right)\big[-\mathrm{tr}\big(A(x,\omega)D^2\Psi_L\big) + H(D\Psi_L + p, x, \omega)\big] + \zeta\left(\frac{x}{L}\right) H_0(D\Psi_L + p) \\
&\quad = \left(1 - \zeta\left(\frac{x}{L}\right)\right)\big[-\mathrm{tr}\big(A(x,\omega)D^2 v^\delta\big) + H\big(Dv^\delta + p, x, \omega\big)\big] + \zeta\left(\frac{x}{L}\right) H_0(D\Psi_L + p).
\end{aligned} \tag{2.18}$$

We now estimate each term in the right-hand side of (2.18) separately. For the first, in view of (1.2) and (2.16), we have

$$-\mathrm{tr}\big(A(x,\omega)D^2 v^\delta\big) + H\big(Dv^\delta + p, x, \omega\big) = -\delta v^\delta(x,\omega) \leqslant \overline{H}(p) + \varepsilon,$$

while for the second we use (2.17), the convexity of H_0 and the Lipschitz bound on v^δ to find

$$\begin{aligned}
H_0(D\Psi_L + p) &\leqslant H_0\left(\xi\left(\frac{x}{L}\right) Dv^\delta + p\right) + \frac{C\varepsilon}{\eta} \\
&\leqslant \xi\left(\frac{x}{L}\right) H_0\big(Dv^\delta + p\big) + \left(1 - \xi\left(\frac{x}{L}\right)\right) H_0(p) + \frac{C\varepsilon}{\eta},
\end{aligned}$$

and, in view of (2.12), deduce that

$$H_0(D\Psi_L + p) \leqslant \overline{H}(p) + \frac{C\varepsilon}{\eta}.$$

Combining the above estimates we find (recall that $\eta \in (0,1)$)

$$-\mathrm{tr}\big(A_L(x,\omega)D^2\Psi_L\big) + H_L(D\Psi_L + p, x, \omega) \leqslant \overline{H}(p) + \frac{C\varepsilon}{\eta}. \tag{2.19}$$

Since Ψ_L is L-periodic subsolution for the corrector equation associated to A_L and H_L, the classical comparison of viscosity solutions [7] yields

$$\overline{H}_L(p,\omega) \leqslant \overline{H}(p) + C\varepsilon/\eta. \qquad \square$$

We are now ready to present

Proof of Theorem 2.1. In view of Lemma 2.2, we only have to show that

$$\limsup_{L\to+\infty} \overline{H}_L(p,\omega) \leqslant \overline{H}(p). \tag{2.20}$$

Fix $p \in \mathbb{R}^d$, let v^δ be the solution to (1.2), and recall that, in view of (1.3), a.s. in $\omega \in \Omega$, for any $\varepsilon > 0$, there exists $\bar{\delta} = \delta(\omega)$ such that, if $\delta \in (0, \bar{\delta})$,

$$\sup_{y\in Q_{1/\delta}} \left|\delta v^\delta(y,\omega;p) + \overline{H}(p)\right| \leqslant \varepsilon.$$

For such an ω, Lemma 2.3 implies that, for $L = 1/\delta$,

$$\overline{H}_L(p,\omega) \leqslant \overline{H}(p) + C\varepsilon/\eta.$$

Letting first $L \to +\infty$ and then $\varepsilon \to 0$ yields (2.20). □

3. Error estimate for viscous HJB equations

Here we show that it is possible to quantify the convergence of $\overline{H}_{L,\eta}(\cdot,\omega)$ to $\overline{H}$. For this we assume that we have an algebraic rate of convergence for the solution $v^\delta = v^\delta(x,\omega;p)$ of (1.2) towards the ergodic constant $\overline{H}(p)$, that is we suppose that there exists $\bar{a} \in (0,1)$ and, for each $K > 0$ and $m > 0$, a map $\delta \to c_1^{K,m}(\delta)$ with $\lim_{\delta\to 0} c_1^{K,m}(\delta) = 0$ such that

$$\mathbb{P}\Big[\sup_{(y,p)\in Q_{\delta^{-m}}\times B_K} \left|\delta v^\delta(y,\cdot;p) + \overline{H}(p)\right| > \delta^{\bar{a}}\Big] \leqslant c_1^{K,m}(\delta); \tag{3.1}$$

notice that (3.1) implies that the convergence of $\delta v^\delta(y,\cdot;p)$ in balls of radius δ^{-m} is not slower than $\delta^{\bar{a}}$.

A rate of convergence like (3.1) is shown to hold for Hamilton–Jacobi equations in [1] and for viscous HJB in [2] under some additional assumptions on H and the environment. The first assumption, which is about the shape of the level sets of H, is that, for every $p, y \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$H(p,y,\omega) \geqslant H(0,0,\omega).$$

As explained in [1], from the point of view of control theory, the fact that there is a common p_0 for all ω at which $H(\cdot,0,\omega)$ has a minimum provides "some controllability". No generality is lost by assuming $p_0 = 0$ and $\operatorname{ess\,sup}_{\omega\in\Omega} H(0,0,\omega) = 0$, which in turn implies that $\min \overline{H} = 0$.

As far as the environment is concerned, (3.1) is known to hold for "i.i.d." environments and under an additional condition on H. Indeed it was shown in [1] that the δv^δ's may converge arbitrarily slowly for p's in the flat zone of $\overline{H}$, that is the set $\{p \in \mathbb{R}^d\colon \overline{H}(p) = \min \overline{H}\}$. For Hamilton–Jacobi equations, that it is when $A \equiv 0$, a sufficient condition (see [1]) for (3.1) is the existence of constants $\theta > 0$ and $c > 0$ such that

$$\mathbb{P}\big[H(0,0,\cdot) > -\lambda\big] \geqslant c\lambda^\theta,$$

while (see [2]) for viscous HJB equations the above condition has to be strengthened in the following way: there exist $\theta > 0$ and $c > 0$ such that, for $(p,x,\omega) \in \mathbb{R}^d \times \mathbb{R}^d \times \Omega$ and $p \in B_1$,

$$H(p,x,\omega) \geqslant c|p|^\theta.$$

In both cases, the function $c_1^{K,m}$ is of the form

$$c_1^{K,m}(\delta) = C\delta^{-r}\exp\big(-C\delta^{-b}\big),$$

where r, C and b are positive constants depending on A, H, K and m.

The periodic approximation $A_{L,\eta}$ and $H_{L,\eta}$ is the same as in the previous section except that now $\eta \to 0$ as $L \to +\infty$. The rate at which $\eta \to 0$ depends on the assumption on the medium. As before the smooth 1-periodic ζ_η satisfies (1.10).

We choose $\eta_L \to 0$ and, to simplify the notation, we write

$$\zeta_L(x) := \zeta_{\eta_L}\left(\frac{x}{L}\right).$$

Note that ζ_L is now L-periodic. We also use the notation A_L, H_L and $\overline{H}_L$ for A_{L,η_L}, H_{L,η_L} and $\overline{H}_{L,\eta_L}$ respectively.

The result is:

Theorem 3.1. *Assume (2.9), (2.10) and (3.1) and set $\eta_L = L^{-\frac{\bar{a}}{4(\bar{a}+1)}}$. For any $K > 0$, there exists a constant $C > 0$ such that, for $L \geqslant 1$,*

$$\mathbb{P}\Big[\sup_{|p|\leqslant K}\big|\overline{H}(p) - \overline{H}_L(p,\cdot)\big| > CL^{-\frac{\bar{a}}{4(\bar{a}+1)}}\Big] \leqslant 2c_1^{C,2(\bar{a}+1)}\big(L^{-\frac{1}{2(\bar{a}+1)}}\big).$$

The main idea of the proof, which is reminiscent to the approach of Capuzzo Dolcetta and Ishii [13] and [1], is that $|\overline{H}(p) - \overline{H}_L(p,\cdot)|$ can be controlled by $|\delta v^\delta(z,\omega;p') + \overline{H}(p')|$. The later one is estimated by the convergence rate assumption (3.1). As in the proof of Theorem 2.1 we prove separately the bounds from below and above. Since the former is a straightforward application of Lemma 2.3, here we only present the details for the latter.

3.1. Estimate for the upper bound

We state the upper bound in the following proposition.

Proposition 3.2. *For any $K > 0$, there exists $C > 0$ such that, for any $L \geqslant 1$, $\lambda \in (0,1]$ and $\delta > 0$,*

$$\begin{aligned}&\Big\{\omega \in \Omega\colon \sup_{p\in B_K}\big[\overline{H}(p) - \overline{H}_L(p,\omega)\big] > \lambda + C\big(\eta_L + L^{-\frac{1}{4}}\delta^{-\frac{1}{2}}\big)\Big\}\\ &\quad\subset \Big\{\omega\in\Omega\colon \inf_{(z,p')\in Q_L\times B_C}\big[-\delta v^\delta(z,\omega;p') - \overline{H}(p')\big] < -\lambda\Big\}.\end{aligned}$$

Proof. Fix $p \in B_K$ and let χ_L^p be a continuous, L-periodic solution of the corrector equation (1.4); recall that χ_L^p is Lipschitz continuous with Lipschitz constant $\overline{L} = \overline{L}(K)$.

Set $\varepsilon = 1/L$ and consider $w^\varepsilon(x) := \varepsilon\chi_L(\frac{x}{\varepsilon})$ which solves

$$-\varepsilon\,\mathrm{tr}\left(A_L\left(\frac{x}{\varepsilon},\omega\right)D^2w^\varepsilon\right) + H_L\left(Dw^\varepsilon + p, \frac{x}{\varepsilon},\omega\right) = \overline{H}_L(p,\omega) \quad \text{in } \mathbb{R}^d.$$

Note that w^ε is 1-periodic and Lipschitz continuous with constant $\overline{L}$. Moreover, in view of the definition of A_L and H_L,

$$-\varepsilon\,\mathrm{tr}\left(A\left(\frac{x}{\varepsilon},\omega\right)D^2w^\varepsilon\right) + H\left(Dw^\varepsilon + p, \frac{x}{\varepsilon},\omega\right) = \overline{H}_L(p,\omega) \quad \text{in } Q_{1-2\eta_L}. \tag{3.2}$$

Fix $\gamma > 0$ to be chosen later and consider the sup-convolution $w^{\varepsilon,\gamma}$ of w^ε which is given by

$$w^{\varepsilon,\gamma}(x,\omega) := \sup_{y\in\mathbb{R}^d}\left(w^\varepsilon(y,\omega) - \frac{1}{2\gamma}|y-x|^2\right).$$

Note that $w^{\varepsilon,\gamma}$ is also 1-periodic and, by the standard properties of the sup-convolution [7,15], $\|Dw^{\varepsilon,\gamma}\|_\infty \leqslant \|Dw^\varepsilon\|_\infty \leqslant \overline{L}$.

The main step of the proof is the following lemma which we prove after the end of the ongoing proof.

Lemma 3.3. *There exists a sufficiently large constant $C > 0$ with the property that, for any $\kappa > 0$ and any $\omega \in \Omega$, if*

$$\inf_{(z,p')\in Q_L\times B_{2\overline{L}}}\big[-\delta v^\delta(z,\omega;p') - \overline{H}(p')\big] \geqslant -\lambda, \tag{3.3}$$

then, for a.e. $x \in Q_r$ *with* $r = 1 - 2\eta_L - (C + \overline{L})(\gamma + (\frac{\varepsilon}{\delta\kappa})^{\frac{1}{2}})$,

$$\overline{H}\big(Dw^{\varepsilon,\gamma}(x) + p\big) \leqslant \overline{H}_L(p,\omega) + \lambda + C\left(\frac{1}{\gamma} + \kappa\right)\left(\varepsilon + \left(\frac{\varepsilon}{\delta\kappa}\right)^{\frac{1}{2}}\right). \tag{3.4}$$

We complete the ongoing proof. Jensen's inequality yields, after integrating (3.4) over Q_r,

$$\begin{aligned}\overline{H}\left(r^{-d}\int_{Q_r} Dw^{\varepsilon,\gamma}(x)\,dx + p\right) &\leqslant r^{-d}\int_{Q_r} \overline{H}\big(Dw^{\varepsilon,\gamma}(x)\,dx + p\big)\\ &\leqslant \overline{H}_L(p,\omega) + \lambda + C\left(\frac{1}{\gamma} + \kappa\right)\left(\varepsilon + \left(\frac{\varepsilon}{\delta\kappa}\right)^{\frac{1}{2}}\right).\end{aligned}$$

The 1-periodicity of $w^{\varepsilon,\gamma}$ yields $\int_{Q_1} Dw^{\varepsilon,\gamma} = 0$, and therefore

$$\begin{aligned}\left|\frac{1}{r^d}\int_{Q_r} Dw^{\varepsilon,\gamma}\right| &\leqslant \frac{1}{r^d}\left(\left|\int_{Q_1} Dw^{\varepsilon,\gamma}\right| + \|Dw^{\varepsilon,\gamma}\|_\infty |Q_1\backslash Q_r|\right)\\ &\leqslant C(1-r) = C\left(\eta_L + \gamma + \left(\frac{\varepsilon}{\delta\kappa}\right)^{\frac{1}{2}}\right),\end{aligned}$$

with the last equality following from the choice of r. Hence

$$\overline{H}(p) \leqslant \overline{H}_L(p,\omega) + \lambda + C\left(\eta_L + \gamma + \frac{\varepsilon}{\gamma} + \kappa\varepsilon + \left(\frac{\varepsilon}{\delta\kappa}\right)^{\frac{1}{2}}\left(\frac{1}{\gamma} + \kappa + 1\right)\right).$$

Choosing $\kappa = (\varepsilon\delta)^{-\frac{1}{3}}\gamma^{-\frac{2}{3}}$ and $\gamma = \varepsilon^{\frac{1}{4}}\delta^{-\frac{1}{2}}$ and recalling that $\varepsilon = L^{-1}$, we find

$$\overline{H}(p) \leqslant \overline{H}_L(p,\omega) + \lambda + C\big(\eta_L + L^{-\frac{1}{4}}\delta^{-\frac{1}{2}}\big). \tag{3.5}$$

The claim now follows. □

Next we present

Proof of Lemma 3.3. Let $\bar{x} \in \mathrm{Int}(Q_r)$ be a differentiability point of $w^{\varepsilon,\gamma}$. Recall that $w^{\varepsilon,\gamma}$ is Lipschitz continuous with Lipschitz constant $\overline{L}$ and, hence, a.e. differentiable. Then there exists a unique $\bar{y} \in \mathbb{R}^d$ such that

$$y \to w^\varepsilon(y,\omega) - \frac{1}{2\gamma}|y - \bar{x}|^2 \text{ has a maximum at } \bar{y} \tag{3.6}$$

and

$$Dw^{\varepsilon,\gamma}(\bar{x}) = \frac{\bar{y} - \bar{x}}{\gamma} \quad \text{and} \quad |\bar{y} - \bar{x}| \leqslant \overline{L}\gamma. \tag{3.7}$$

Recall that $\kappa > 0$ is fixed. For $\sigma > 0$ small, consider the map $\Phi : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$

$$\Phi(y,z,\omega) := w^\varepsilon(y,\omega) - \varepsilon v^\delta\left(\frac{z}{\varepsilon},\omega;\frac{\bar{y} - \bar{x}}{\gamma} + p\right) - \frac{|y - \bar{x}|^2}{2\gamma} - \frac{\kappa}{2}|y - \bar{y}|^2 - \frac{|y - z|^2}{2\sigma}$$

and fix a maximum point (y_σ, z_σ) of Φ.

Note that $w^{\varepsilon,\gamma}$ as well as all the special points chosen above depend on ω. Since this plays no role in what follows, to keep the notation simple we omit this dependence.

Next we derive some estimates on $|y_\sigma - z_\sigma|$ and $|y_\sigma - \bar{y}|$. The Lipschitz continuity of $v^\delta(\cdot,\omega;\frac{\bar{y}-\bar{x}}{\gamma} + p)$ yields

$$|y_\sigma - z_\sigma| \leqslant C\sigma. \tag{3.8}$$

Using this last observation as well as the fact that (y_σ, z_σ) is a maximum point of Φ and $\|v^\delta\|_\infty \leqslant C/\delta$, we get

$$w^\varepsilon(\bar{y},\omega) - C\frac{\varepsilon}{\delta} - \frac{|\bar{y}-\bar{x}|^2}{2\gamma} \leqslant \Phi(\bar{y},\bar{y}) \leqslant \Phi(y_\sigma, z_\sigma) \leqslant w^\varepsilon(y_\sigma,\omega) + C\frac{\varepsilon}{\delta} - \frac{|y_\sigma-\bar{x}|^2}{2\gamma} - \frac{\kappa}{2}|y_\sigma-\bar{y}|^2,$$

while, in view of (3.6), we also have

$$w^\varepsilon(y_\sigma,\omega) - \frac{|y_\sigma-\bar{x}|^2}{2\gamma} \leqslant w^\varepsilon(\bar{y},\omega) - \frac{|\bar{y}-\bar{x}|^2}{2\gamma}.$$

Putting together the above inequalities yields

$$|y_\sigma - \bar{y}| \leqslant C\left(\frac{\varepsilon}{\delta\kappa}\right)^{\frac{1}{2}}. \tag{3.9}$$

In particular, (3.7), the choice of r and the fact that $\bar{x} \in \mathrm{Int}(Q_r)$ imply that

$$\bar{y} \in \mathrm{Int}(Q_{1-2\eta_L - C(\frac{\varepsilon}{\delta\kappa})^{\frac{1}{2}}}) \quad \text{and} \quad y_\sigma \in Q_{1-2\eta_L}.$$

At this point, for the convenience of the reader, it is necessary to recall some basic notation and terminology from the theory of viscosity solutions (see [15]). Given a viscosity upper semicontinuous (resp. lower semicontinuous) sub-solution (resp. super-solution) u of $F(D^2u, Du, u, x) = 0$ in some open subset U of $\mathbb{R}^d$, the lower-jet (resp. upper-jet) $\bar{\mathcal{J}}^{2,+}u(x)$ (resp. $\bar{\mathcal{J}}^{2,-}u(x)$) at some $x \in U$ consists of $(X, p) \in \mathcal{S}^d \times \mathbb{R}^d$ that can be used to evaluate the equation with the appropriate inequality. For example if u is a sub-solution of $F(D^2u, Du, u, x) = 0$ in U and $(X, p) \in \bar{\mathcal{J}}^{2,+}u(x)$, then $F(X, p, u(x), x) \leqslant 0$.

Now we use the maximum principle for semicontinuous functions (see [15]). Since (y_σ, z_σ) is a maximum point of Φ, for any $\eta > 0$, there exist $Y_{\sigma,\eta}, Z_{\sigma,\eta} \in \mathcal{S}^d$ such that

$$\left(Y_{\sigma,\eta}, \frac{y_\sigma-\bar{x}}{\gamma} + \frac{y_\sigma - z_\sigma}{\sigma} + \kappa(y_\sigma-\bar{y})\right) \in \bar{\mathcal{J}}^{2,+}w^\varepsilon(y_\sigma,\omega), \qquad \left(Z_{\sigma,\eta}, \frac{y_\sigma-z_\sigma}{\sigma}\right) \in \bar{\mathcal{J}}^{2,-}v^\delta\left(\frac{z_\sigma}{\varepsilon},\omega\right),$$

and

$$\begin{pmatrix} Y_\sigma & 0 \\ 0 & \frac{1}{\varepsilon}Z_{\sigma,\eta} \end{pmatrix} \leqslant M_{\sigma,\eta} + \eta M_{\sigma,\eta}^2, \tag{3.10}$$

where

$$M_{\sigma,\eta} = \begin{pmatrix} \frac{1}{\sigma}I_d & -\frac{1}{\sigma}I_d \\ -\frac{1}{\sigma}I_d & \frac{1}{\sigma}I_d \end{pmatrix} + \begin{pmatrix} (\frac{1}{\gamma}+\kappa)I_d & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.11}$$

Evaluating the equations for w^ε and v^δ at $y_\sigma \in Q_{1-2\eta_L}$ and $z_\sigma \in \mathbb{R}^d$ respectively we find

$$-\varepsilon\,\mathrm{tr}\left(A\left(\frac{y_\sigma}{\varepsilon},\omega\right)Y_{\sigma,\eta}\right) + H\left(\frac{y_\sigma-\bar{x}}{\gamma} + \frac{y_\sigma-z_\sigma}{\sigma} + \kappa(y_\sigma-\bar{y}) + p, \frac{y_\sigma}{\varepsilon},\omega\right) \leqslant \overline{H}_L(p,\omega) \tag{3.12}$$

and

$$\delta v^\delta\left(\frac{z_\sigma}{\varepsilon},\omega; \frac{\bar{y}-\bar{x}}{\gamma} + p\right) - \mathrm{tr}\left(A\left(\frac{z_\sigma}{\varepsilon},\omega\right)Z_{\sigma,\eta}\right) + H\left(\frac{y_\sigma-z_\sigma}{\sigma} + \frac{\bar{y}-\bar{x}}{\gamma} + p, \frac{z_\sigma}{\varepsilon},\omega\right) \geqslant 0. \tag{3.13}$$

Multiplying (3.10) by the positive matrix

$$\begin{pmatrix} \Sigma(\frac{y_\sigma}{\varepsilon},\omega) \\ \Sigma(\frac{z_\sigma}{\varepsilon},\omega) \end{pmatrix}\begin{pmatrix} \Sigma(\frac{y_\sigma}{\varepsilon},\omega) \\ \Sigma(\frac{z_\sigma}{\varepsilon},\omega) \end{pmatrix}^T,$$

and taking the trace, in view of (3.11), we obtain

$$\begin{aligned} &\mathrm{tr}\left(A\left(\frac{y_\sigma}{\varepsilon},\omega\right)Y_{\sigma,\eta}\right) - \frac{1}{\varepsilon}\mathrm{tr}\left(A\left(\frac{z_\sigma}{\varepsilon},\omega\right)Z_{\sigma,\eta}\right) \\ &\quad\leqslant \frac{1}{\sigma}\left\|\Sigma\left(\frac{y_\sigma}{\varepsilon},\omega\right) - \Sigma\left(\frac{z_\sigma}{\varepsilon},\omega\right)\right\|^2 + \left(\frac{1}{\gamma}+\kappa\right)\left\|\Sigma\left(\frac{y_\sigma}{\varepsilon},\omega\right)\right\|^2 + \eta\,\mathrm{tr}\left(M_{\sigma,\eta}^2\begin{pmatrix} \Sigma(\frac{y_\sigma}{\varepsilon},\omega) \\ \Sigma(\frac{z_\sigma}{\varepsilon},\omega) \end{pmatrix}\begin{pmatrix} \Sigma(\frac{y_\sigma}{\varepsilon},\omega) \\ \Sigma(\frac{z_\sigma}{\varepsilon},\omega) \end{pmatrix}^T\right). \end{aligned}$$

Recalling that Σ satisfies (2.8) and using (3.8), we get

$$\begin{aligned}&\operatorname{tr}\left(A\left(\frac{y_\sigma}{\varepsilon},\omega\right)X_{\sigma,\eta}\right)-\frac{1}{\varepsilon}\operatorname{tr}\left(A\left(\frac{z_\sigma}{\varepsilon},\omega\right)Y_{\sigma,\eta}\right)\\&\quad\leqslant\frac{C}{\sigma\varepsilon^2}|y_\sigma-z_\sigma|^2+C\left(\frac{1}{\gamma}+\kappa\right)+\eta C(\sigma)\leqslant C\left(\frac{\sigma}{\varepsilon^2}+\frac{1}{\gamma}+\kappa\right)+\eta C(\sigma).\end{aligned}\tag{3.14}$$

Note that $C(\sigma)$ actually depends also on all the other parameters of the problem but, and this is important, is independent of η.

Next we use that H satisfies (2.4), (2.5). From (3.8) and (3.9) it follows

$$\begin{aligned}&H\left(\frac{y_\sigma-\bar{x}}{\gamma}+\frac{y_\sigma-z_\sigma}{\sigma}+\kappa(y_\sigma-\bar{y})+p,\frac{y_\sigma}{\varepsilon},\omega\right)-H\left(\frac{y_\sigma-z_\sigma}{\sigma}+\frac{\bar{y}-\bar{x}}{\gamma}+p,\frac{z_\sigma}{\varepsilon},\omega\right)\\&\quad\geqslant-C\left(\frac{|y_\sigma-\bar{y}|}{\gamma}+\kappa|y_\sigma-\bar{y}|+\frac{|y_\sigma-z_\sigma|}{\varepsilon}\right)\geqslant-C\left(\left(\frac{\varepsilon}{\delta\kappa}\right)^{\frac{1}{2}}\left(\frac{1}{\gamma}+\kappa\right)+\frac{\sigma}{\varepsilon}\right).\end{aligned}\tag{3.15}$$

We estimate the difference between (3.12) and (3.13), using (3.14) and (3.15), to find

$$-\delta v^\delta\left(\frac{z_\sigma}{\varepsilon},\omega;\frac{\bar{y}-\bar{x}}{\gamma}+p\right)\leqslant\overline{H}_L(p,\omega)+C\left(\frac{\sigma}{\varepsilon}+\frac{\varepsilon}{\gamma}+\kappa\varepsilon+\left(\frac{\varepsilon}{\delta\kappa}\right)^{\frac{1}{2}}\left(\frac{1}{\gamma}+\kappa\right)+\frac{\sigma}{\varepsilon}\right)+\eta\varepsilon C(\sigma).$$

Finally the choice of ω (recall (3.3)) implies

$$\overline{H}\left(\frac{\bar{y}-\bar{x}}{\gamma}+p\right)\leqslant\overline{H}_L(p,\omega)+\lambda+C\left(\frac{\sigma}{\varepsilon}+\frac{\varepsilon}{\gamma}+\kappa\varepsilon+\left(\frac{\varepsilon}{\delta\kappa}\right)^{\frac{1}{2}}\left(\frac{1}{\gamma}+\kappa\right)+\frac{\sigma}{\varepsilon}\right)+\eta\varepsilon C(\sigma).$$

We now let $\eta\to 0$ and then $\sigma\to 0$ to obtain

$$\overline{H}\left(\frac{\bar{y}-\bar{x}}{\gamma}+p\right)\leqslant\overline{H}_L(p,\omega)+\lambda+C\left(\frac{\varepsilon}{\gamma}+\kappa\varepsilon+\left(\frac{\varepsilon}{\delta\kappa}\right)^{\frac{1}{2}}\left(\frac{1}{\gamma}+\kappa\right)\right).$$

Using (3.7), we may now conclude (3.4) holds. □

3.2. *The full estimate*

Combining Proposition 3.2 and Lemma 2.3 yields the full estimate.

Proof of Theorem 3.1. Recall that $\eta_L=L^{-\frac{\bar{a}}{4(\bar{a}+1)}}$ and choose $\delta=L^{-\frac{1}{2(\bar{a}+1)}}$ and $\lambda=\delta^{\bar{a}}=L^{-\frac{\bar{a}}{2(\bar{a}+1)}}$. Then Proposition 3.2 implies that, for $L\geqslant 1$,

$$\mathbb{P}\Big[\sup_{p\in B_K}\big(\overline{H}(p)-\overline{H}_L(p,\cdot)\big)>CL^{-\frac{\bar{a}}{4(\bar{a}+1)}}\Big]\leqslant\mathbb{P}\Big[\inf_{(z,p')\in Q_L\times B_C}\big(-\delta v^\delta(z,\cdot;p')-\overline{H}(p')\big)<-\delta^{\bar{a}}\Big].$$

Then (3.1) gives

$$\mathbb{P}\Big[\sup_{p\in B_K}\big(\overline{H}(p)-\overline{H}_L(p,\cdot)\big)>CL^{-\frac{\bar{a}}{4(\bar{a}+1)}}\Big]\leqslant c_1^{C,2(\bar{a}+1)}\big(L^{-\frac{1}{2(\bar{a}+1)}}\big).$$

Similarly Lemma 2.3 implies, for $L\geqslant 1$, that

$$\left\{\omega\in\Omega:\ \sup_{p\in B_K}\big(\overline{H}_L(p,\omega)-\overline{H}(p)\big)>\frac{C\lambda}{\eta_L}\right\}\subset\left\{\omega\in\Omega:\ \sup_{(y,p)\in Q_L\times B_K}\big|\delta v^\delta(y,\omega;p)+\overline{H}(p)\big|>\lambda\right\}.$$

Since $\lambda/\eta_L=L^{-\frac{\bar{a}}{4(\bar{a}+1)}}$, using (3.1) we get

$$\begin{aligned}&\mathbb{P}\Big[\sup_{p\in B_K}\big(\overline{H}_L(p,\cdot)-\overline{H}(p)\big)>CL^{-\frac{\bar{a}}{4(\bar{a}+1)}}\Big]\\&\quad\leqslant\mathbb{P}\Big[\sup_{(y,p)\in Q_L\times B_K}\big|\delta v^\delta(y,\cdot;p)+\overline{H}(p)\big|>\delta^{\bar{a}}\Big]\leqslant c_1^{C,2(\bar{a}+1)}\big(L^{-\frac{1}{2(\bar{a}+1)}}\big).\end{aligned}$$

Combining both estimates gives the result. □

4. Approximations of fully nonlinear uniformly elliptic equations

We introduce the hypotheses and state and prove the approximation result. We remark that our arguments extend to nonlinear elliptic equations which include gradient dependence at the expense of some additional technicalities.

4.1. The hypotheses

The map $F : \mathcal{S}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is $\mathcal{B}(\mathcal{S}^d) \otimes \mathcal{B} \otimes \mathcal{F}$ is measurable and stationary, that is, for all $X \in \mathcal{S}^d$, $x, y \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$F(X, y, \tau_z\omega) = F(X, y + z, \omega). \tag{4.1}$$

We continue with the structural hypotheses on F. We assume that it is uniformly elliptic uniformly in ω, that is there exist constants $0 < \bar{\lambda} < \overline{\Lambda}$ such that, for all $X, Y \in \mathcal{S}^d$ with $Y \geqslant 0$, $x \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$-\overline{\Lambda}\|Y\| \leqslant F(X + Y, x, \omega) - F(X, x, \omega) \leqslant -\bar{\lambda}\|Y\|, \tag{4.2}$$

and bounded, that is there exists $\overline{C} > 0$ such that

$$\sup_{\omega\in\Omega} \big|F(0, 0, \omega)\big| \leqslant \overline{C}. \tag{4.3}$$

Note that, in view of (4.1) and (4.2), for each $R > 0$, there exists $C = C(R, \bar{\lambda}, \overline{\Lambda}, \overline{C}) > 0$ such that

$$\sup_{\|X\|\leqslant R,\ y\in\mathbb{R}^d,\ \omega\in\Omega} \big|F(X, y, \omega)\big| \leqslant C. \tag{4.4}$$

The required regularity on F is that there exists $\rho : [0, \infty) \to [0, \infty)$ such that $\lim_{r\to 0} \rho(r) = 0$ and, for all $x, y \in \mathbb{R}^d$, $\sigma > 0$, $P, X, Y \in \mathcal{S}^d$ satisfying

$$\begin{aligned} &\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{3}{\sigma}\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}, \\ &F(X + P, x, \omega) - F(Y + P, y, \omega) \leqslant \rho\left(\frac{|x - y|^2}{\sigma} + |x - y|\right). \end{aligned} \tag{4.5}$$

To simplify the statements, we write

(4.1), (4.2), (4.3) and (4.5) hold. (4.6)

4.2. The periodic approximation

Let $\zeta_\eta : \mathbb{R}^d \to [0, 1]$ be a smooth, 1-periodic satisfying (1.10), choose $\eta_L \to 0$ and, to simplify the notation, write $\zeta_L(x) := \zeta_{\eta_L}(\frac{x}{L})$.

For $(X, x, \omega) \in \mathcal{S}^d \times Q_L \times \Omega$ we set

$$F_L(X, x, \omega) = \big(1 - \zeta_L(x)\big)F(X, x, \omega) + \zeta_L(x)F_0(X),$$

where $F_0 \in C(\mathcal{S}^d)$ is a space independent and uniformly elliptic map with the same ellipticity constants as F. Then we extend F_L to be an L-periodic map in x, that is, for all $(X, x, \omega) \in \mathcal{S}^d \times \mathbb{R}^d \times \Omega$ and all $\xi \in \mathbb{Z}^d$,

$$F_L(X, x + L\xi, \omega) = F_L(X, x, \omega).$$

Note that, in view of the choice of F_0,

F_L satisfies (4.6). (4.7)

4.3. The approximation result

Let $\overline{F} = \overline{F}(\cdot)$ and $\overline{F}_L = \overline{F}_L(\cdot, \omega)$ be the averaged nonlinearities (ergodic constants) that correspond to the homogenization problem for F and $F_L(\cdot, \omega)$. We claim that, as $L \to \infty$, $\overline{F}_L(\cdot, \omega)$ is an a.s. good approximation of $\overline{F}$.

Theorem 4.1. *Assume (1.7) and (4.6). For any $P \in \mathcal{S}(\mathbb{R}^d)$ and a.s. in ω,*

$$\lim_{L\to+\infty} \overline{F}_L(P,\omega) = \overline{F}(P). \tag{4.8}$$

Proof. Fix $P \in \mathcal{S}^d$ with $\|P\| \leqslant K$, $\omega \in \Omega$ for which the homogenization holds and let χ_L be an L-periodic corrector for $F_L(P,\omega)$, that is a continuous solution to

$$F_L\big(D^2\chi_L + P, x, \omega\big) = \overline{F}_L(P,\omega) \quad \text{in } \mathbb{R}^d.$$

Without loss of generality we assume that $\chi_L(0) = 0$.

The rescaled function $v_L(x) := L^{-2}\chi_L(Lx)$ is 1-periodic and solves

$$F_L\big(D^2 v_L + P, Lx, \omega\big) = \overline{F}_L(P,\omega) \quad \text{in } \mathbb{R}^d.$$

Lemma 4.2 below (its proof is presented after the end of the ongoing one) implies the existence of $\alpha \in (0,1)$ and $C > 0$ depending only $\bar{\lambda}$, $\overline{\Lambda}$, d and $\overline{C}$ in (4.3) so that

$$\operatorname{osc}(v_L) \leqslant C \quad \text{and} \quad [v_L]_{0,\alpha} \leqslant C. \tag{4.9}$$

Note that, by the definition of $F_L(P,y,\omega)$,

$$F\big(D^2 v_L + P, Lx, \omega\big) = \overline{F}_L(P) \quad \text{in } Q_{1-2\eta_L},$$

while, since F_L is uniformly elliptic,

$$-M^+\big(D^2 v_L + P\big) \leqslant F(P, Lx, \omega) - \overline{F}_L(P) \quad \text{in } \mathbb{R}^d$$

and

$$-M^-\big(D^2 v_L + P\big) \geqslant F(P, Lx, \omega) - \overline{F}_L(P) \quad \text{in } \mathbb{R}^d,$$

where M^+ and M^- are the classical Pucci extremal operators associated with the uniform ellipticity constants $\bar{\lambda}/d$ and $\overline{\Lambda}$—see [9] for the exact definitions.

Let $L_n \to \infty$ be such that $\overline{F}_{L_n}(P,\omega) \to \limsup_{L\to+\infty} \overline{F}_L(P,\omega)$. Using (4.9) we find a further subsequence (still denoted in the same way) and a 1-periodic $v \in C^{0,\alpha}(\mathbb{R}^d)$ such that the v_{L_n}'s converge uniformly to v. In view of the results of [12] about stochastic homogenization, v solves

$$\overline{F}\big(D^2 v + P\big) = \limsup_{L\to+\infty} \overline{F}_L(P,\omega) \quad \text{in } \operatorname{Int}(Q_1), \tag{4.10}$$

while the stability of solutions also gives, for $C_P = \sup_{x\in\mathbb{R}^d} |F(P,x,\omega)| + \operatorname{ess\,sup}_{\omega\in\Omega} \limsup_{L\to+\infty} |\overline{F}_L(P,\omega)|$,

$$-M^+\big(D^2 v\big) \leqslant C_P \quad \text{and} \quad -M^-\big(D^2 v\big) \geqslant -C_P \quad \text{in } \mathbb{R}^d.$$

Let $\mathcal{B} := \bigcup_{z\in\mathbb{Z}^d} \partial Q_1(z)$. In view of (4.10), the periodicity of v implies that

$$\overline{F}\big(D^2 v + P\big) = \limsup_{L\to+\infty} \overline{F}_L(P,\omega) \quad \text{in } \mathbb{R}^d \backslash \mathcal{B}. \tag{4.11}$$

Let $\bar{x}$ be a minimum point of v and, for $\sigma \in (0,1)$ fixed, consider the map $w(x) := v(x) + \sigma|x - \bar{x}|^2$ and its convex envelope $\Gamma(w)$. Since w is a subsolution to

$$-M^+\big(D^2 w\big) = -M^+\big(D^2 v + 2\sigma I_d\big) \leqslant C'_P := C_P + 2\sigma\Lambda,$$

it follows (see [9]) that $\Gamma(w)$ is $C^{1,1}$ with $\|D^2\Gamma(w)\|_\infty \leqslant C$.

Let E be the contact set between w and its convex envelope, i.e.,

$$E := \big\{x \in \operatorname{Int}(B_{1/4}) \colon\ w(x) = \Gamma(w)(x)\big\}.$$

Note that, if $p \in B_{\sigma/4}$, then there exists $x \in E$ such that $D\Gamma(w)(x) = p$. Indeed, if $y \notin B_{1/4}(\bar{x})$ and $p \in B_{\sigma/4}$, then

$$\begin{aligned} w(y) - \langle p, y\rangle &\geqslant v(y) + \sigma|y-\bar{x}|^2 - \langle p, \bar{x}\rangle - |p||y-\bar{x}| \\ &> v(\bar{x}) - \langle p, \bar{x}\rangle + |y-\bar{x}|\big(\sigma/4 - |p|\big) \geqslant w(\bar{x}) - \langle p, \bar{x}\rangle. \end{aligned}$$

Hence any minimum point x of $w - \langle p, \cdot\rangle$ must belong to $B_{1/4}(\bar{x})$. Then, since $w(y) \geqslant w(x) + \langle p, y-x\rangle$ for any $y \in \mathbb{R}^d$, it follows that $w(x) = \Gamma(w)(x)$ and $D\Gamma(w)(x) = p$. As a consequence we have

$$|B_{\sigma/4}| \leqslant \big|D\Gamma(w)(E)\big| \leqslant \int_E \det\big(D^2\Gamma(w)\big) \leqslant C|E|.$$

Since $\mathcal{B}$ has zero measure, the above estimate shows that there exists $x \in E\backslash\mathcal{B}$ such that $w(x) = \Gamma(w)(x)$. Then, for any $y \in \mathbb{R}^d$,

$$v(y) \geqslant \Gamma(w)(y) - \sigma|y-\bar{x}|^2 \geqslant w(x) + \big\langle D\Gamma(w)(x), y-x\big\rangle - \sigma|y-\bar{x}|^2,$$

with an equality at $y = x$.

Using $\phi(y) := w(x) + \langle D\Gamma(w)(x), y-x\rangle - \sigma|y-\bar{x}|^2$ as a test function in (4.11) and the fact that $x \notin \mathcal{B}$, we get

$$\overline{F}(-2\sigma I_d + P) \geqslant \limsup_{L\to+\infty} \overline{F}_L(P, \omega).$$

Letting $\sigma \to 0$ gives one side of the equality (4.8). The proof of the reverse one follows in a symmetrical way. □

To complete the proof, it remains to explain (4.9). For this we note that v_L is 1-periodic and, in view of (4.4), belongs to the class $S^*(\bar{\lambda}/d, \overline{\Lambda}, C_0)$, where $C_0 = \sup_x |F(P, x, \omega)|$ (see [9] for the definition of $S^*(\bar{\lambda}/d, \overline{\Lambda}, C_0)$). Then (4.9) is a consequence of the classical Krylov–Safonov result about the continuity of solutions of uniformly elliptic pde. Since we do not know an exact reference for (4.9), we present below its proof.

Lemma 4.2. *Let $0 < \lambda < \Lambda$ and $C_0 > 0$ be constants. There exist $C = C(d, \lambda, \Lambda, d, C_0) > 0$ and $\alpha = \alpha(d, \lambda, \Lambda, d, C_0) \in (0, 1]$ such that, any 1-periodic $u \in S^*(\lambda, \Lambda, C_0)$ satisfies $\operatorname{osc}(u) \leqslant C$ and $[u]_{C^{0,\alpha}} \leqslant C$.*

Proof. Without loss of generality, we assume that $u(0) = 0$. For $M \geqslant 1$, let $u_M(x) := M^{-2}u(Mx)$. Note that u_M is M^{-1}-periodic and still belongs to $S^*(\lambda, \Lambda, d, C_0)$. The Krylov–Safonov result yields $C = C(\lambda, \Lambda, d, C_0) > 0$ and $\alpha = \alpha(d, \lambda, \Lambda, d, C_0) \in (0, 1]$ with

$$[u_M]_{0,\alpha;Q_1} \leqslant C\big(\|u_M\|_{Q_2} + 1\big).$$

It follows from the 1-periodicity of u and $u(0) = 0$ that

$$[u_M]_{0,\alpha;Q_1} \geqslant M^{\alpha-2}[u]_{0,\alpha} \quad \text{and} \quad \|u_M\|_{Q_2} = M^{-2}\|u\| \leqslant M^{-2}d^{\frac{1}{2}}[u]_{0,\alpha}.$$

Hence,

$$[u]_{0,\alpha} \leqslant CM^{2-\alpha}\big(M^{-2}d^{\frac{1}{2}}[u]_{0,\alpha} + 1\big).$$

Choosing M so that $CM^{-\alpha}d^{\frac{1}{2}} = \frac{1}{2}$ gives a bound on $[u]_{C^{0,\alpha}}$, from which we derive a sup bound on u. □

We remark that the proof of Theorem 4.1 shows the following fact, which we state as a separate proposition, since it may be of independent interest.

Proposition 4.3. *Let $\Sigma \subset \mathbb{R}^d$ be a set of zero measure and assume that $u \in C(\mathbb{R}^d)$ is a viscosity solution of the uniformly elliptic equation $F(D^2u, x) = 0$ in $\mathbb{R}^d\backslash\Sigma$. If, in addition, $u \in S^*(\lambda, \Lambda, d, C)$ in $\mathbb{R}^d$, for some $0 < \lambda < \Lambda$, then u is a viscosity solution of $F(D^2u, x) = 0$ in $\mathbb{R}^d$.*

5. The convergence rate for nonlinear elliptic equations

Here we show that it is possible to quantify the rate of convergence of $\overline{F}_L(\cdot, \omega)$ to $\overline{F}$. As in the viscous HJB problem, we assume that we know a rate for the convergence of the solution to the approximate cell problem to the ergodic constant $\overline{F}$.

For the sake of the presentation below and to simplify the argument it is more convenient to consider, for $L \geqslant 1$ and $P \in \mathcal{S}^d$, the solution $v^L = v^L(\cdot; P, \omega)$ of

$$v^L + F\big(D^2 v^L + P, Lx, \omega\big) = 0 \quad \text{in } \mathbb{R}^d. \tag{5.1}$$

Note that $v_L(x) := L^2 v^L(L^{-1}x)$ solves the auxiliary problem

$$L^{-2} v_L + F\big(D^2 v_L + P, x, \omega\big) = 0 \quad \text{in } \mathbb{R}^d.$$

In view of the stochastic homogenization, it is known that the v^L's converge locally uniformly and a.s. to the unique solution $\bar{v} = -\overline{F}(P)$ of

$$\bar{v} + \overline{F}\big(D^2 \bar{v} + P\big) = 0 \quad \text{in } \mathbb{R}^d.$$

We assume that there exist nonincreasing rate maps $L \to \lambda(L)$ and $L \to c_2(L)$, which tend to 0 as $L \to +\infty$, such that

$$\mathbb{P}\Big[\sup_{x \in B_5} \big|v^L(x, \cdot) + \overline{F}(P)\big| > \lambda(L)\Big] \leqslant c_2(L). \tag{5.2}$$

Recall that such a rate was obtained in [10] under a strong mixing assumption on the random media—see at the beginning of the paper for the meaning of this. The recent contribution [4] shows that for "i.i.d." environments the rate is at least algebraic, that is $\lambda(L) = L^{-\bar{a}}$ for some $\bar{a} \in (0, 1)$.

In addition to the above assumption on the convergence rate, it also necessary to enforce the regularity condition (4.5) on F. Indeed we assume that there exists a constant $C > 0$ such that

$$\text{(4.5) holds with } \rho(r) = Cr. \tag{5.3}$$

The periodic approximation of F is exactly the same as in the previous section and the result is:

Theorem 5.1. *Assume (4.6), (5.2) and (5.3) and set $\eta_L(L) = \lambda(L)^{\frac{d}{2d+1}}$. There exists a constant $C > 0$ such that, for $L \geqslant 1$,*

$$\mathbb{P}\big[\big|-\overline{F}(P) + F_L(P, \cdot)\big| > C\lambda(L)^{\frac{1}{2d+1}}\big] \leqslant c_2(L).$$

Proof. For any $L \geqslant 1$, let v^L be the solution to (5.1) and χ_L an L-periodic corrector for $F_L(P, \omega)$, that is a solution to

$$F_L\big(D^2 \chi_L + P, x, \omega\big) = \overline{F}_L(P, \omega) \quad \text{in } \mathbb{R}^d.$$

Without loss of generality we assume that $\chi_L(0, \omega) = 0$. Set $w_L(x, \omega) := L^{-2}\chi_L(Lx, \omega)$ and note that w_L is 1-periodic and solves

$$F_L\big(D^2 w_L + P, Lx, \omega\big) = \overline{F}_L(P, \omega) \quad \text{in } \mathbb{R}^d. \tag{5.4}$$

It also follows from (4.9) that $\|w_L\| \leqslant C$ and $[w_L]_{0,\alpha} \leqslant C$ and we note that the v^L's are bounded in $C^{0,\alpha}$ uniformly with respect to L.

The main part of the proof consists in showing that, given $\lambda > 0$ and ω such that

$$\sup_{x \in B_5} \big|v^L(x, \omega) + \overline{F}(P)\big| \leqslant \lambda, \tag{5.5}$$

one has

$$\big|\overline{F}(P) - F_L(P, \omega)\big| \leqslant \lambda + C\big(\eta_L^{\frac{1}{d}} + \lambda \eta_L^{-2}\big). \tag{5.6}$$

The conclusion follows by assumption (5.2) and a suitable choice of the constant $\lambda = \lambda(L)$.

To show (5.6), we assume that (5.5) holds and follow the proof of the convergence quantifying each step in an appropriate way. To simplify the expressions we suppress the dependence on ω which is fixed throughout the argument. Moreover, since the proof is long, we organize it in separate subsections.

Construction of the minimum points $\bar{x}_0$ and $\widehat{x}_0$ Let $\bar{x}_0$ be a minimum point of w_L in $\overline{Q}_1$; note that since w_L is 1-periodic, $\bar{x}_0$ is actually a minimum point of w_L in $\mathbb{R}^d$. For $a, r, r_0 \in (0,1)$ to be chosen below and $\xi \in \mathbb{R}^d$, we consider the map

$$\Phi^0_\xi(x) := w^L(x) + \frac{a}{2}|x - \bar{x}_0|^2 + \langle \xi, x - \bar{x}_0\rangle.$$

The claim is that, if $\widehat{x}_0$ is a minimum point of Φ^0_ξ with $\xi \in B_{r_0}$ and

$$2r_0 \leqslant ar, \tag{5.7}$$

then $\widehat{x}_0 \in B_r(\bar{x}_0)$. Indeed, by the definition of $\widehat{x}_0$ and $\bar{x}_0$,

$$\Phi^0_\xi(\widehat{x}_0) \leqslant \Phi^0_\xi(\bar{x}_0) = w^L(\bar{x}_0) \leqslant w^L(\widehat{x}_0),$$

hence,

$$\frac{a}{2}|\widehat{x}_0 - \bar{x}_0|^2 + \langle \xi, \widehat{x}_0 - \bar{x}_0\rangle \leqslant 0,$$

and, in view of (5.7),

$$|\widehat{x}_0 - \bar{x}_0| \leqslant 2|\xi|/a \leqslant 2r_0/a \leqslant r.$$

Next we consider two cases depending on whether $\widehat{x}_0 \in Q_1$ or not and we let E^0 denote the collection of points $\widehat{x}_0$ which belong to Q_1:

$$E^0 := \big\{\widehat{x}_0 \in Q_1 \cap B_r(\bar{x}_0)\colon \text{there exists } \xi \in B_{r_0} \text{ such that } \Phi^0_\xi \text{ has a minimum at } \widehat{x}_0\big\}.$$

Case 1: $\widehat{x}_0 \in Q_1$ Note that, by the definition of $\widehat{x}_0$, $\widehat{x}_0 \in E^0$ and w_L is touched from below at $\widehat{x}_0$ by a parabola of opening $a \in (0,1)$. It then follows from the Harnack inequality that w_L is touched from above at $\widehat{x}_0$ by a parabola of opening C. This is a classical fact about uniformly elliptic second-order equations and we refer to [11] for more details. It follows that w_L is differentiable at $\widehat{x}_0$ and, in view of the choice of $\widehat{x}_0$ for Φ^0_ξ, $Dw^L(\widehat{x}_0) + a(\widehat{x}_0 - \bar{x}_0) + \xi = 0$.

Hence ξ is determined from $\widehat{x}_0$ by the relation $\xi = \Psi^0(y) := -(Dw^L(\widehat{x}_0) + a(\widehat{x}_0 - \bar{x}_0))$. Moreover, in view of the above remark on the parabolas touching w_L from above and below, Ψ^0 is Lipschitz continuous on E^0 with a Lipschitz constant bounded by C. We refer the reader to [11] for the details of this argument.

Case 2: $\widehat{x}_0 \notin Q_1$ If $\widehat{x}_0 \notin Q_1$, then there exists $z \in \mathbb{Z}^d$ such that $\widehat{x}_0 \in Q_1(z)$. Since $\widehat{x}_0 \in Q_r(\bar{x}_0)$ with $\bar{x}_0 \in Q_1$ and $r < 1$, it follows that $|z|_\infty = 1$. Set $\bar{x}_z := \bar{x}_0 - z$, $\widehat{x}_z := \widehat{x}_0 - z \in Q_1$ (note that $\widehat{x}_z \in B_r(\bar{x}_z)$) and

$$\Phi^z_\xi(x) := w^L(x) + \frac{a}{2}|x - \bar{x}_z|^2 + \langle \xi, x - \bar{x}_z\rangle.$$

In view of the periodicity of w_L, $\widehat{x}_z$ is a minimum point of Φ^z_ξ.

Let $\mathcal{Z} := \{z \in \mathbb{Z}^d;\ |z|_\infty \leqslant 1\}$. It is clear that $\mathcal{Z}$ is a finite set and, if $z \in \mathcal{Z}$, either $|z|_\infty = 1$ or $z = 0$. Also set E^z to be the set of points $\widehat{x}_z \in Q_1 \cap B_r(\bar{x}_z)$ for which there exists $\xi \in B_{r_0}$ such that Φ^z_ξ has a minimum at $\widehat{x}_z$.

Arguing as in the previous case, we see that there is a Lipschitz map Ψ^z on E^z with Lipschitz constant independent of z such that, if $\widehat{x}_z \in E^z$ and $\xi = \Psi^z(\widehat{x}_z)$, then $\widehat{x}_z$ is a minimum of Φ^z_ξ.

The existence of interior minima It follows from the previous two steps that, for any $\xi \in B_{r_0}$, there exist $z \in \mathcal{Z}$ and $\widehat{x}_z \in E^z$ such that $\xi = \Psi^z(\widehat{x}_z)$. Hence, using that the Ψ^z's are Lipschitz continuous uniformly for $z \in \mathcal{Z}$, we find

$$|B_{r_0}| = \left|\bigcup_{z\in\mathcal{Z}} \Psi^z\big(E^z\big)\right| \leqslant \sharp(\mathcal{Z}) \max_z \big|\Psi^z\big(E^z\big)\big| \leqslant C \max_z \big|E^z\big|.$$

Therefore there must exist some $z \in \mathcal{Z}$, which we fix from now on, such that

$$\big|E^z\big| \geqslant r_0^d/C. \tag{5.8}$$

We now show that, for a suitable choice of the constants, the sets E^z and $Q_{1-3\eta_L}$ have a nonempty intersection. For this we note that, since $E^z \subset B_r(\bar{x}_z) \cap Q_1$, the claim holds true as soon as

$$\big|E^z\big| + \big|B_r(\bar{x}_z) \cap Q_{1-3\eta_L}\big| > \big|B_r(\bar{x}_z) \cap Q_1\big|.$$

As

$$\left|B_r(\bar{x}_z)\cap Q_1\right| - \left|B_r(\bar{x}_z)\cap Q_{1-3\eta_L}\right| = \left|B_r(\bar{x}_z)\cap (Q_1\backslash Q_{1-3\eta_L})\right| \leqslant Cr^{d-1}\eta_L,$$

provided $r \geqslant C\eta_L$, we conclude from (5.8), that if

$$r_0^d \geqslant Cr^{d-1}\eta_L, \tag{5.9}$$

then $E^z \cap Q_{1-3\eta_L} \neq \emptyset$.

The perturbed problem From now on we assume that (5.7) and (5.9) hold and, therefore, there exist (fixed) $z \in \mathcal{Z}$, $\xi \in B_{r_0}$ and $\widehat{x}_z \in Q_{1-3\eta_L} \cap B_r(\bar{x}_z)$ such that Φ_ξ^z has a minimum at $\widehat{x}_z$. For $b, \sigma \in (0,1)$ to be chosen below, set

$$\begin{aligned}\Phi_\sigma(x,y) &:= v^L(x) - w_L(y) - \frac{a}{2}|y-\bar{x}_z|^2 - \frac{b}{2}|y-\widehat{x}_z|^2 - \langle \xi, y-\bar{x}_z\rangle - \frac{|x-y|^2}{2\sigma}\\ &= v^L(x) - \Phi_\xi^z(y) - \frac{b}{2}|y-\widehat{x}_z|^2 - \frac{|x-y|^2}{2\sigma},\end{aligned}$$

and let $(\widetilde{x},\widetilde{y})$ be a maximum point of Φ_σ over $Q_5 \times \mathbb{R}^d$. We claim that

$$|\widetilde{y}-\widehat{x}_z| \leqslant 2\big(\lambda b^{-1}\big)^{\frac{1}{2}} \quad \text{and} \quad |\widetilde{y}-\widetilde{x}| \leqslant C\sigma^{\frac{1}{2-\alpha}}. \tag{5.10}$$

Indeed, since $\Phi_\sigma(\widetilde{x},\widetilde{y}) \geqslant \Phi_\sigma(\widehat{x}_z,\widehat{x}_z)$ and $\Phi_\xi^z(\widehat{x}_z) \leqslant \Phi_\xi^z(\widetilde{y})$, in view of (5.5), we find

$$\begin{aligned}\Phi_\sigma(\widetilde{x},\widetilde{y}) &\geqslant v^L(\widehat{x}_z) - \Phi_\xi^z(\widehat{x}_z) \geqslant -\overline{F}(P) - \lambda - \Phi_\xi^z(\widetilde{y})\\ &\geqslant v^L(\widetilde{x}) - 2\lambda - \Phi_\xi^z(\widetilde{y}),\end{aligned}$$

and, therefore,

$$\frac{b}{2}|\widetilde{y}-\widehat{x}_z|^2 + \frac{|\widetilde{x}-\widetilde{y}|^2}{2\sigma} \leqslant 2\lambda.$$

This gives the first inequality in (5.10). The maximality of $\widetilde{x}$ in $\Phi_\sigma(\cdot,\widetilde{y})$ and the Hölder regularity of v^L gives the second inequality.

If we assume that

$$2\big(\lambda b^{-1}\big)^{\frac{1}{2}} \leqslant \eta_L, \tag{5.11}$$

then, since $\widehat{x}_z \in Q_{1-3\eta_L}$ and (5.10) holds, it follows that $\widetilde{y}$ belongs to $Q_{1-2\eta_L}$. Moreover, for σ small enough, we still have by (5.10) that $\widetilde{x} \in Q_2$. In particular, $(\widetilde{x},\widetilde{y})$ is an interior maximum of Φ_σ in $Q_5 \times \mathbb{R}^d$.

The maximum principle Since $(\widetilde{x},\widetilde{y})$ is an interior maximum of Φ_σ, the maximum principle argument already used earlier yields $X, Y \in \mathcal{S}^d$ and $p_x, p_y \in \mathbb{R}^d$ such that

$$(p_x, X) \in \overline{\mathcal{J}}^{2,+}v^L(\widetilde{x}), \qquad (p_y, Y) \in \overline{\mathcal{J}}^{2,-}w_L(\widetilde{y})$$

and

$$\begin{pmatrix} X & 0\\ 0 & -Y-(a+b)I_d\end{pmatrix} \leqslant \frac{3}{\sigma}\begin{pmatrix} I_d & -I_d\\ -I_d & I_d\end{pmatrix}.$$

In view of (5.1) and (5.4) and since $\widetilde{y} \in Q_{1-2\eta_L}$, which yields that $F_L(\cdot,\widetilde{y}) = F(\cdot,\widetilde{y})$, evaluating the equations satisfied by v^L and w_L at $\widetilde{x}$ and $\widetilde{y}$ respectively we find

$$v^L(\widetilde{x}) + F(X+P, L\widetilde{x}) \leqslant 0 \tag{5.12}$$

and

$$F(Y+P, L\widetilde{y}) \geqslant \overline{F}_L(P,\omega).$$

From the uniform in x and ω Lipschitz continuity of F with respect to P, we get

$$F\big(Y + (a+b)I_d + P, L\widetilde{y}\big) \geqslant \overline{F}_L(P,\omega) - C(a+b).$$

Using (5.3) to estimate the difference between (5.12) and the above inequality, we obtain

$$v^L(\widetilde{x}) + \overline{F}_L(P,\omega) \leqslant C\left(\frac{L^2|\widetilde{x}-\widetilde{y}|^2}{\sigma} + L|\widetilde{x}-\widetilde{y}| + a + b\right).$$

Finally we use (5.5) and (5.10) to conclude that

$$-\overline{F}(P) + \overline{F}_L(P,\omega) \leqslant \lambda + C\big(L^2\sigma^{\alpha/(2-\alpha)} + L\sigma^{1/(2-\alpha)} + a + b\big).$$

Letting $\sigma \to 0$, we get

$$-\overline{F}(P) + F_L(P,\omega) \leqslant \lambda + C(a+b) \tag{5.13}$$

provided (5.5), (5.7), (5.9) and (5.11) hold.

The choice of the constants In order for (5.11), (5.7) and (5.9) to hold, we choose respectively $b = 4\lambda\eta_L^{-2}$, $r = 1/2$, $r_0 = a/4$ and $a = C\eta_L^{\frac{1}{d}}$. We then have

$$-\overline{F}(P) + F_L(P,\omega) \leqslant \lambda + C\big(\eta_L^{\frac{1}{d}} + \lambda\eta_L^{-2}\big).$$

Arguing in a similar way, we can check that, under (5.5), we also have

$$-\overline{F}(P) + F_L(P,\omega) \geqslant -\lambda - C\big(\eta_L^{\frac{1}{d}} + \lambda\eta_L^{-2}\big),$$

which yields (5.6).

Conclusion Combining (5.6) with the assumption (5.2) on the convergence rate, we find

$$\mathbb{P}\big[\big|-\overline{F}(P) + F_L(P,\cdot)\big| > \lambda(L) + C\big(\eta_L^{\frac{1}{d}} + \lambda(L)\eta_L^{-2}\big)\big] \leqslant \mathbb{P}\Big[\sup_{x\in B_5}\big|v^L(x,\cdot) + \overline{F}(P)\big| > \lambda(L)\Big] \leqslant c_2(L).$$

The choice of $\eta_L(L) = (\lambda(L))^{\frac{d}{2d+1}}$ gives the claim. □

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