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Optimal regularity for phase transition problems with convection

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Abstract

In this paper we consider a steady state phase transition problem with given convection $\mathbf{v}$. We prove, among other things, that the weak solution is locally Lipschitz continuous provided that $\mathbf{v} = D\xi$ and ξ is a harmonic function. Moreover, for continuous casting problem, i.e. when $\mathbf{v}$ is constant vector, we show that Lipschitz free boundaries are C^1 regular surfaces.

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1. Introduction

In this paper we study a stationary phase transition problem where the liquid phase is in motion. For given convection $\mathbf{v}$, the problem is of determining the temperature T from the equation

$$\Delta T = \operatorname{div}\left[\mathbf{v}\beta(T - T_S)\right] + f. \tag{1.1}$$

Here $\beta(s) = as + \ell H(s)$ is the enthalpy, a is the specific heat constant, ℓ is the latent heat constant, H is the Heaviside function, T_S is the solidification temperature and f is a given function that accounts for heat sources or sinks. As one can see, (1.1) is the heat balance equation written for the enthalpy β [21]. (1.1) is also known as Stefan problem with convection.

It is well known that (1.1) portrays various phase transition models. For instance, if $\mathbf{v}$ is constant then we have the so-called *continuous casting problem*, which is a practical example of a free boundary problem appearing in industry [1,6,13,22]. It models a metal fabrication technique whereby molten metal is poured into an open mold and subsequently cooled by a stream of water and extracted at continuous velocity. This method is used most frequently to cast steel, aluminum and copper, because it allows low cost production of metal sections of good quality [25]. Another

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example of this sort is phase change in saturated porous media. For more details concerning the physical background of this equation, see [1,6] and references therein.

If we suppose that $T_S = 0$ then (1.1) transforms into

$$\Delta T = \operatorname{div}\left[\mathbf{v}\beta(T)\right] + f.$$

Notice that β has abrupt behavior at $s = 0$. Typically $\beta(T) = \ell + \int_{T_S}^{T} a(\tau)\,d\tau$ for $T > T_S$ and $\beta(T) = \int_{T_S}^{T} a(\tau)\,d\tau$ if $T < T_S$. Here a is the specific heat which in this paper is assumed to be constant. Hence the solid region is characterized by $T < T_S$ and the liquid region by $T > T_S$. Any region where $T = T_S$ and $0 < \beta(T) < \ell$ is called mushy region [1,19]. The presence of mushy region means that we do not have sharp separation of phases. There are several boundary conditions guaranteeing that the mushy region is empty, see [13,19]. However, our primary interest here is the boundary of the sets $T > T_S$ or $T < T_S$. To fix the ideas we consider $\Gamma = \partial\{T < T_S\}$ which we call the *free boundary* and study its properties. Our methods can be equally applied to the set $\partial\{T > T_S\}$.

The objective of this paper is to prove that weak solutions of (1.1) are locally Lipschitz continuous. Moreover the Lipschitz free boundary must be C^1 smooth, see Theorems A, B and C below.

The phase transition problem with convections has been studied by several authors, see [21] and references therein. The existence of $W^{1,2}$ weak solutions to various boundary value problems for (1.1) can be established by penalization method [13,16,21]. In this way one obtains a bounded Hölder continuous solution for a suitable boundary data. Our first result, Theorem A, strengthens this result up to log-Lipschitz continuity under some weak conditions on the boundary of the domain and the boundary data. However the optimal regularity of the solutions is Lipschitz as the free boundary condition (7.5) indicates. One of our main results in this paper is the local Lipschitz continuity of weak solutions for one phase and two phase problems, see Theorem B. It should be noted that Theorem B does not follow from Theorem A in [5], since we do not assume that the free boundary is given by the graph of a Lipschitz continuous function.

Having proven the optimal regularity of weak solution, we address the free boundary regularity which is a very delicate problem. To tackle it, we apply the free boundary regularity theory for viscosity solutions. The latter is yet another notion of generalized solution, which utilizes the maximum principle at the regular (in some weak sense) free boundary points via a Hopf type lemma. This method was developed by L. Caffarelli for the pure Laplace operator in the series of papers [7,8]. Extension to more general class of operators is proven by M. Feldman in [15]. In view of these results the regularity problem reduces to the equivalence of weak solution to the viscosity solution which is contained in Theorem C.

2. Outline

The paper is organized as follows. In Section 3 we introduce some notations used throughout the paper. In Section 4 we state the phase change problem with convection and give its weak formulation. It is worthwhile to point out that one phase problem is linked to obstacle problem as the computation (4.4) shows. In particular we get that the positive part of weak solutions to continuous casting problem, i.e. when $\mathbf{v} = \mathbf{e}_N$, are locally non-degenerate, see Proposition 2.

The main results of this paper, Theorems A, B and C, are formulated in Section 5. First we show that the weak solution is locally log-Lipschitz continuous. This improves the known result that $u = T_S - T$ is α-Hölder continuous for any $\alpha < 1$. Under further assumption that the lateral boundary $\Sigma = \partial\Omega \times (0, L)$ is Liapunov–Dini surface and the Dirichlet data prescribed on Σ is $C^{0,1}$ we show that the log-Lipschitz estimate holds in $\overline{\mathcal{C}}_L = \overline{\Omega \times (0, L)}$. As a result one obtains that the free boundary is a log-Lipschitz graph over $\overline{\Omega} \subset \mathbb{R}^{N-1}$. This is contained in Theorem A and the proof is given in Section 6.

If we have sharp separation of solid and liquid phases, i.e. the interface does not have thickness, then one can deduce the free boundary condition for smooth solutions directly from Eq. (4.5). This is carried out in Section 7.

In Section 8, we prove that the weak solutions to one phase problem are Lipschitz continuous on any subdomain of $\mathcal{C}_L = \Omega \times (0, L)$. According to the free boundary condition (7.5) the Lipschitz regularity of free boundary is optimal. The rest of Section 8 deals with the one phase continuous casting problem, i.e. when $u \geqslant 0$. Using a strong connection with the obstacle problem we show that u is non-degenerate at free boundary points. This implies that the free boundary $\partial\{u > 0\}$ is locally a set of finite perimeter. Moreover, it is $N - 1$ rectifiable. In Section 9 by employing Alt–Caffarelli–Friedman monotonicity theorem we prove optimal regularity for the solutions of the two phase problem. Note that the proofs of Lipschitz continuity for one and two phase problems differ considerably.

Measure theoretic properties of $\partial\{u > 0\}$ for two phase continuous casting problem are discussed in Section 10 where we extend the results from Section 8 under some assumption on the Lebesgue density of positivity set near $X_0 \in \partial\{u > 0\}$. We point out that the same argument works for $\partial\{u < 0\}$.

Finally in Section 11 we prove Theorem C stating that the weak solution is also viscosity solution. The proof utilizes Lemmas 2.2 and 2.3 from [11] and a careful analysis of blow-up limits. As a result we obtain that Lipschitz free boundaries are C^1 from Theorem 1 of [15] and conclude the proof of Theorem C.

3. Notations

$C_0, C_1, C_D, \ldots$	generic constants
χ_D	the characteristic function of a set $D \subset \mathbb{R}^N$, $N \geqslant 2$
$\overline{\Omega}$	the closure of Ω
$\partial\Omega$	the boundary of Ω
ν	outer unit normal
$X = (x, z) \in \mathbb{R}^N$	$x = (x_1, \ldots, x_{N-1}, 0)$
Du	the gradient of u, $Du = (\partial_{x_1}u, \partial_{x_2}u, \ldots, \partial_z u)$, $\partial_{X_i} = \frac{\partial}{\partial X_i}$, $i = 1, \ldots, N-1$, $\partial_z = \frac{\partial}{\partial z}$
$\mathcal{C}_L$	the cylinder $\mathcal{C}_L = \Omega \times (0, L)$, $L > 0$ for some $\Omega \subset \mathbb{R}^{N-1}$
$B_r(X)$	$\{Y \in \mathbb{R}^N: \lvert Y - X\rvert < r\}$
B_r	$B_r(0)$
$\Gamma, \Gamma(u)$	$\partial\{u > 0\}$ – the free boundary of u
$\Omega^+(u)$	$\Omega^+(u) = \{x: u(x) > 0\}$
$\Omega^-(u)$	$\Omega^-(u) = \{x: u(x) \leqslant 0\}^\circ$
v^+	$\max(v, 0) = -\min(-v, 0) \geqslant 0$
v^-	$\max(-v, 0) = -\min(v, 0) \geqslant 0$

4. Statement of problem

Given a bounded domain $\Omega \subset \mathbb{R}^{N-1}$. For fixed $L > 0$ we denote $\mathcal{C}_L = \Omega \times (0, L)$. Let $X \in \mathcal{C}_L \subset \mathbb{R}^N$, then the notation $X = (x, z)$ is used throughout the paper, where $x \in \Omega$, $z \in (0, L)$. Our starting point is to rewrite Eq. (1.1) for $u = T_S - T$. Notice that various but equivalent forms of Eq. (1.1) are considered in [13,21,22].

4.1. Consistent mathematical model

In this section we will slightly transform (1.1). Let $T(X)$ be the temperature at a point $X \in \mathcal{C}_L$ and T_S be the solidification temperature. Thus the liquid phase is characterized by $T(X) > T_S$. Introduce the normalized temperature $\widetilde{T}(X) = T(X) - T_S$ and put

$$u(X) = -\widetilde{T}(X) = T_S - T(X).$$

As T solves the heat balance equation (1.1), it follows that the normalized $\widetilde{T}$ solves the equation

$$\Delta\widetilde{T} = \operatorname{div}\big[\mathbf{v}\beta(\widetilde{T})\big] + f.$$

If we take $u = -\widetilde{T}$ then from the previous equation we get

$$\begin{aligned} \Delta u &= \operatorname{div}\big[-\mathbf{v}\beta(-u)\big] - f \\ &= \operatorname{div}\big[\mathbf{v}\{au + \beta_0(u)\}\big] - f - \ell \operatorname{div}\mathbf{v}, \end{aligned} \tag{4.1}$$

where

$$\beta_0(s) = \begin{cases} 0 & \text{if } s < 0, \\ \in [0, \ell] & \text{if } s = 0, \\ \ell & \text{if } s > 0. \end{cases} \tag{4.2}$$

Redefining $f = -f + \ell \operatorname{div}\mathbf{v}$ and $\beta = as + \beta_0$ we infer that u solves the following equation

$$\Delta u = \operatorname{div}\big[\mathbf{v}\beta(u)\big] + f. \tag{4.3}$$

For the one phase problem, i.e. when $u \geqslant 0$ in $\mathcal{C}_L$, Eq. (4.3) can be linked to the obstacle problem [9]. To see this we suppose that u vanishes in the strip $\overline{\Omega} \times [0, \delta]$ for some positive δ. We assume that f is a function of X and u. Let us take $\beta_0(t) = \ell\chi\{t > 0\}$ and $\mathbf{v} = \mathbf{e}_N$, i.e. the unit direction of z-axis. Then, under these assertions, u is the solution of $\Delta u(X) = \partial_z \beta(u(X)) + f(X, u(X))$.

Next we introduce the Baiocchi transformation $w(x, z) = \int_0^z u(x, s)\, ds$ and compute

$$\begin{aligned}\Delta w(X) &= \int_0^z \left[\sum_{i=1}^{N-1} \partial_{x_i x_i} u(x, s)\right] ds + \partial_z u(X)\\ &= \int_0^z \left[\partial_z \beta\big(u(x, s)\big) + f\big(x, s, u(x, s)\big) - \partial_z^2 u(x, s)\right] ds + \partial_z u(X)\\ &= \beta\big(u(X)\big) - \beta\big(u(x, 0)\big) + \int_0^z f\big(x, s, u(x.s)\big)\, ds + \partial_z u(x, 0).\end{aligned}$$

By assumption $u = 0$ in $\overline{\Omega} \times [0, \delta]$, thus $u(x, 0) = \partial_z u(x, 0) = 0$, therefore from the definition of $\beta(t) = at + \ell\chi\{t > 0\}$ we get

$$\begin{aligned}\Delta w(X) &= \beta\big(u(X)\big) + \int_0^z f\big(x, s, u(x.s)\big)\, ds\\ &= au(X) + \ell\chi\big\{u(X) > 0\big\} + \overline{f}(X).\end{aligned} \tag{4.4}$$

Here $\overline{f}(X) = \int_0^z f(x, s, u(x.s))\, ds$.

This observation accounts for a strong link between the one-phase continuous casting problem and the inhomogeneous obstacle problem $\Delta w(X) = aw_z(X) + \ell\chi\{w(X) > 0\} + \overline{f}(X)$. In particular for the Stefan problem with $f = 0$ and $a = 0$, (4.4) is the classical obstacle problem provided that $\chi\{u > 0\} = \chi\{w > 0\}$. In Section 8 we will utilize (4.4) to prove the non-degeneracy of u for the one phase continuous casting problem.

4.2. *Weak formulation*

The main objective of this paper is to study the optimal regularity of weak solutions to the equation

$$\Delta u(X) = \operatorname{div}\big[\mathbf{v}(X)\beta\big(u(X)\big)\big] + f\big(X, u(X)\big) \tag{4.5}$$

and the smoothness of the free boundary $\partial\{u > 0\}$. Here $f = f(X, u)$ is a given function of variables $X \in \mathbb{R}^N$ and $u \in \mathbb{R}$, measurable in X for any $u \in \mathbb{R}$, and $\mathbf{v}$ is a given vector field defined in $\mathcal{C}_L$.

In order to formulate this equation in weak sense we shall require that $\mathbf{v} \in L^\infty(\mathcal{C}_L, \mathbb{R}^N)$ and $\mathbf{v}$ is weakly divergence free, i.e.

$$\int_{\mathcal{C}_L} \mathbf{v} \cdot D\varphi = 0, \quad \forall \varphi \in H_0^1(\mathcal{C}_L). \tag{4.6}$$

Furthermore, we shall assume that

$$f(X, u) \text{ is continuous in } u \text{ and there is } f^\star \in L^2(\mathcal{C}_L) \text{ such that } \sup_{u\in\mathbb{R}} \big|f(X, u)\big| \leqslant f^\star(X), \text{ a.e. } X \in \mathcal{C}_L, \tag{4.7}$$

see [21, p. 189].

As in Section 4.1 we interpret $u(X)$ as the normalized temperature at a point $X \in \mathcal{C}_L$ whereas f accounts for sources and $\mathbf{v}(X)$ is the velocity of convection. Recall that (4.5) manifests the heat conservation of thermodynamical system with enthalpy $\beta = \beta(u)$ defined as follows

$$\beta(s) = \begin{cases} as & \text{if } s < 0,\\ \in [0, \ell] & \text{if } s = 0,\\ as + \ell & \text{if } s > 0.\end{cases} \tag{4.8}$$

Here $a > 0$ is a constant. An equivalent definition $\beta(s) = as + \beta_0(s)$, with β_0 given by (4.2), will be used as well.

First we formulate the Dirichlet problem for (4.5) in $\mathcal{C}_L$. For $\varphi \in C^\infty(\overline{\mathcal{C}_L})$ we multiply Eq. (4.5) by φ and integrate by parts. This yields the identity

$$\int_{\partial\mathcal{C}_L} \varphi Du\cdot\nu - \int_{\mathcal{C}_L} DuD\varphi = \int_{\partial\mathcal{C}_L} \varphi\beta(u)\mathbf{v}\cdot\nu - \int_{\mathcal{C}_L} \beta(u)\mathbf{v}\cdot D\varphi + \int_{\mathcal{C}_L} f\varphi. \tag{4.9}$$

Upon taking $\varphi \in H_0^1(\mathcal{C}_L)$ in the last identity, the boundary integrals vanish. Thereby we get the first integral identity, used in the weak formulation of (4.5):

$$\int_{\mathcal{C}_L} \beta(u)\mathbf{v}\cdot D\varphi - \int_{\mathcal{C}_L} DuD\varphi = \int_{\mathcal{C}_L} f\varphi, \quad \forall\varphi \in H_0^1(\mathcal{C}_L). \tag{4.10}$$

Definition 1. Let $\mathbf{v} \in L^\infty(\mathcal{C}_L, \mathbb{R}^N)$ and (4.6)–(4.7) hold. Then $u \in H^1(\mathcal{C}_L)$ is said to be a weak solution of (4.5) if (4.10) is satisfied. Here β is the maximal monotone graph given by (4.8).

For a function $\widetilde{g} \in C(\partial\mathcal{C}_L) \cap H^1(\mathcal{C}_L)$ it is convenient to introduce the functions

$$h_0(x) = \widetilde{g}(x,0), \qquad h_L(x) = \widetilde{g}(x,L), \quad \text{and} \quad g(X) = \widetilde{g}(X) \quad \text{if } X \in \partial\Omega\times(0,L). \tag{4.11}$$

In other words $h_0(x)$ (resp. $h_L(x)$) is the restriction of the trace of $\widetilde{g} \in C(\mathcal{C}_L)\cap H^1(\mathcal{C}_L)$ on $\Omega\times\{0\}$ (resp. on $\Omega\times\{L\}$). Now consider the weak solutions to Dirichlet's problem

$$\begin{cases} \Delta u = \operatorname{div}[\mathbf{v}\beta(u)] + f & \text{in } \mathcal{C}_L, \\ u(x,0) = h_0(x), & x\in\Omega, \\ u(x,L) = h_L(x), & x\in\Omega, \\ u = g(X) & \text{on } \Sigma = \partial\Omega\times(0,L). \end{cases} \tag{DP}$$

Definition 2. Let $\mathbf{v} \in L^\infty(\mathcal{C}_L, \mathbb{R}^N)$ and (4.6) and (4.7) hold with $f^\star \in L^\infty(\mathcal{C}_L)$. A pair (u,η) is said to be a weak solution to (**DP**) if $u \in H^1(\mathcal{C}_L)$, $\eta \in \beta(u)$, $u = g$ on $\Sigma = \partial\Omega\times(0,L)$, $u(x,0) = h_0(x)$, $u(x,L) = h_L(x)$, $x\in\Omega$ and for any $\varphi \in H_0^1(\mathcal{C}_L)$

$$\int_{\mathcal{C}_L} \eta\mathbf{v}\cdot D\varphi - \int_{\mathcal{C}_L} DuD\varphi = \int_{\mathcal{C}_L} f\varphi. \tag{4.12}$$

Remark 1. It is well known that if $\mathbf{v}$ and f satisfy the conditions in Definition 2 then a weak solution (u,η) of (**DP**) exists (see [21, Theorem 4.14]). Moreover $u \in C^\alpha(\overline{\mathcal{C}_L})$ provided that $\widetilde{g} \in C^\alpha(\partial\mathcal{C}_L)\cap H^1(\mathcal{C}_L)$, see [22].

The theorem to follow is a simple comparison principle for the weak solutions of (**DP**) (see [21, Proposition 4.17]):

Proposition 1. *Let* $\mathbf{v}$ *be Lipschitz continuous in* $\overline{\mathcal{C}_L}$. *Assume that* $f(X,u)$ *is monotone decreasing in* u, *continuous in* X *and Lipschitz continuous in* u. *Let* (u,η) *be a weak solution to* (**DP**) *and* (u^*,η^*) *be a supersolution to* (**DP**): *that is* (u^*,η^*) *satisfies* $u^* \in H^1(\mathcal{C}_L)$, $u^* \geqslant \widetilde{g}$ *on* $\partial\mathcal{C}_L$, $\eta^* \in \beta(u^*)$

$$\int_{\mathcal{C}_L} \eta^*\mathbf{v}\cdot D\varphi - \int_{\mathcal{C}_L} Du^*D\varphi \leqslant \int_{\mathcal{C}_L} f\varphi, \quad \varphi \in H_0^1(\mathcal{C}_L),\ \varphi \geqslant 0.$$

If $|u| + |u^*| \geqslant \rho$, *in* $\mathscr{S}_\rho = \Omega\times(L-\rho, L)$, *for some positive small* $\rho > 0$, *then*

$$u \leqslant u^*, \qquad \eta \leqslant \eta^*.$$

Remark 2. If $\widetilde{g} \in H^1(\mathcal{C}_L)$ and for some constant $\rho > 0$, $\widetilde{g} \geqslant \rho$ (or $g \leqslant -\rho$) in $\mathscr{S}_\rho \cap \partial\mathcal{C}_L$ then there exists a unique weak solution to (**DP**), see [22, Remarks 3–4]. The assumption $|u| + |u^*| \geqslant \rho$, in the strip $\mathscr{S}_\rho = \Omega\times(L-\rho, L)$, is called "sufficient condition for stability". It is not known if the Comparison Principle holds without assuming it. However for a suitable data $\widetilde{g}$, large on $\Omega\times L$ (or respectively small on $\Omega\times\{0\}$) this assumption holds, see [22, p. 265].

5. Main results

Definition 3. Let $K \subset \mathbb{R}^N$ be a compact set.

- The class of log-Lipschitz continuous functions defined on K is denoted by $LC^{0,1}(K)$. Thus $u \in LC^{0,1}(K)$ if and only if

$$\sup_{X_1, X_2 \in K} \frac{|u(X_1) - u(X_2)|}{|X_1 - X_2| \log \frac{1}{|X_1 - X_2|}} < \infty.$$

- We say that u is locally log-Lipschitz continuous in the domain $D \subset \mathbb{R}^N$ if $u \in LC^{0,1}(K)$ for any $K \Subset D$. The class of locally log-Lipschitz continuous functions in D is denoted by $LC^{0,1}_{\text{loc}}(D)$.
- The class of Lipschitz (resp. locally Lipschitz) continuous functions is denoted by $C^{0,1}(D)$ (resp. $(C^{0,1}_{\text{loc}}(D))$).

Theorem A. *Let $\mathbf{v} \in L^\infty(\mathcal{C}_L, \mathbb{R}^N)$, $f(X, u)$ be bounded on $\mathcal{C}_L \times I$ for any finite interval $I \subset \mathbb{R}$ and (u, η) be a weak solution to* (**DP**) *in the sense of Definition 2. Suppose that $\tilde{g} \in C^{0,1}(\overline{\mathcal{C}_L})$ and $h_0(X) = -m^- < 0$ and $h_L(X) = m^+ > 0$ are constants, see (4.11). Then*

1° *u is log-Lipschitz continuous in $\overline{\mathcal{C}_L}$, i.e.*

$$\sup_{X_1, X_2 \in \overline{\mathcal{C}_L}} \frac{|u(X_1) - u(X_2)|}{|X_1 - X_2| \log \frac{1}{|X_1 - X_2|}} < \infty; \tag{5.1}$$

2° *If, in addition, $\mathbf{v} = \mathbf{e}_N$, $f = 0$ and there exists a positive constant $c_0 > 0$ such that*

$$\liminf_{z \to z_0} \frac{g(x, z) - g(x, z_0)}{z - z_0} \geqslant c_0 > 0, \quad \forall x \in \partial\Omega, \tag{5.2}$$

then the free boundary is a graph of a log-Lipschitz continuous function over $\overline{\Omega}$. Here g is the restriction of boundary data $\tilde{g}$ on the lateral boundary $\Sigma = \Omega \times (0, L)$, see (4.11).

Remark 3. The $LC^{0,1}(\overline{\mathcal{C}_L})$ estimate for u, under the assumption $\tilde{g} \in C^{0,1}$ (as in Theorem A) and $\Sigma = \partial\Omega \times (0, L)$ being a Liapunov–Dini surface, cannot be improved. Indeed, it is known that if w is harmonic in a domain D with C^2 smooth boundary, $w = \varphi$ on ∂D with $\varphi \in C^{0,1}(\overline{D})$, then near ∂D we have $|Dw(x)| = O(\log \frac{1}{\operatorname{dist}(x, \partial D)})$, see [17]. Clearly, if one takes $a = 0$, $f = 0$ then u will be harmonic away from the free boundary, so the gradient Du may not be bounded.

Next we would like to analyze the local regularity of the weak solution to (**DP**). The conditions imposed on $\mathbf{v}$ for the one phase case are weaker than those for two phase problem, namely we assume that $\mathbf{v}$ is a gradient of a harmonic function, whereas for nonnegative solutions u, $\mathbf{v}$ can be any Lipschitz vector field.

Theorem B. *Let u be a bounded weak solution of (4.5) (see Definition 1) with $f(X, u)$ being bounded on $\mathcal{C}_L \times I$ for any finite interval $I \subset \mathbb{R}$.*

1° *If $u \geqslant 0$ in $\mathcal{C}_L$ and $\mathbf{v} \in C^{0,1}(\mathcal{C}_L, \mathbb{R}^N)$ then $u \in C^{0,1}_{\text{loc}}(\mathcal{C}_L)$. Moreover, if $\mathbf{v} = e_n$ and $f(X, u) \leqslant \sigma_0$ for some $\sigma_0 < \frac{\ell}{2L}$ then u is locally non-degenerate.*

2° *Let u be a weak solution, $\mathbf{v} = D\xi$ and ξ is harmonic. Then any continuous weak solution of (4.5) is locally Lipschitz continuous.*

3° *Under assumptions above the free boundary $\partial\{u > 0\}$ is countably $N - 1$ rectifiable provided that u^+ is non-degenerate.*

Remark 4. The last statement of Theorem B can be extended to $\partial\{u < 0\}$ under the assumption that u^- is non-degenerate.

It is worthwhile to point out that $C^{0,1}$ is the best regularity for continuous weak solutions. This can be seen from the free boundary condition (see (7.5)). Notice that the linearly scaled solutions

$$u_{r_j}(X)=\frac{u(X_0+r_jX)}{r_j},\qquad X_0\in\partial\{u>0\},\ r_j>0,\ r_j\downarrow 0$$

are Lipschitz continuous by Theorem A, because $D({r_j}^{-1}u(X_0+r_jX))=(Du)(X_0+r_jX)$. Furthermore, employing a customary compactness argument we can see that $u_{r_j}\to u_0$, at least for a subsequence, locally uniformly and weakly in H^1 so that $\Delta u_0=\operatorname{div}[\mathbf{v}(X_0)\beta_0(u_0)]$, where β_0 is given by (4.2). The function u_0 is called a **blow-up limit** of u at X_0. This observation allows us to study the regularity of free boundary for the weak solutions by showing that, in fact, u is also a *viscosity solution*, see Definition 5.

Theorem C. *Let u be a weak solution of two phase continuous casting problem and suppose that $f=0$.*

1° *Then u is a viscosity solution in the sense of Definition 5.*
2° *If the free boundary is Lipschitz then it is smooth.*

Theorem C allows us to utilize the free boundary regularity theory of L. Caffarelli [7,8,12], developed for the viscosity solutions. In particular the second part of Theorem C follows from Theorem 1 in [15].

6. Log-Lipschitz estimates

The proof of Theorem A is tailored from two lemmas below. The first one deals with the interior $LC^{0,1}(\mathcal{C}_L)$ estimate.

Lemma 1. *Let u be as in Theorem A, then for any compact set $K\Subset\mathcal{C}_L$ there exists a positive constant $C=C(N,a,\ell,\sup_{\mathcal{C}_L}|u|,\sup_{X\in\mathcal{C}_L,|\tau|\leqslant\sup_{\mathcal{C}_L}|u|}|f(X,\tau)|,\operatorname{dist}(K,\partial\mathcal{C}_L))$ such that the following estimate holds*

$$\sup_{X_1,X_2\in K}\frac{|u(X_1)-u(X_2)|}{|X_1-X_2|\log\frac{1}{|X_1-X_2|}}\leqslant C. \tag{6.1}$$

Proof. To fix the ideas we assume that $B=B_1(X_0)$ is an open ball and $B\Subset\mathcal{C}_L$. By Green's representation formula

$$u(X)=v(X)+\int_B\Delta u(Y)G(X,Y)\,dY,$$

where $G(X,Y)$ is the Green function of B with pole X and v is the harmonic lifting of u in B, i.e. $\Delta v=0$ in B and $u=v$ on ∂B. Since $\Delta u=\operatorname{div}[\mathbf{v}\eta]+f$ then after partial integration Green's representation transforms into

$$u(X)=v(X)-\int_B\eta\mathbf{v}\cdot D_YG(X,Y)\,dY+\int_B f\big(Y,u(Y)\big)G(X,Y)\,dY. \tag{6.2}$$

By definition of weak solutions $\eta\in\beta(u)$ hence $|\eta|\leqslant C_0$, for $|u|\leqslant M$ is bounded by Remark 1, and $|\beta|\leqslant aM+\ell:=C_0$. In particular it follows that $|v|\leqslant M$ and it is smooth in half ball $B_{\frac12}$ whereas the log-Lipschitz estimate in $\frac12 B$ for the first integral follows from [20, Theorem 2.5.1]. As for the Green potential of f, it is enough to recall that $|u|\leqslant M$, so by assumption $f(X,u(X))$ is bounded. Thus the second integral is C^1 smooth function of $Y\in\frac12 B$. □

Next we estimate u near the lateral boundary $\partial\Omega\times(0,L)=\Sigma$.

Lemma 2. *Assume that $X_0=(x_0,z_0)\in\partial\Omega\times(0,L)$ and $R_0<\min(z_0,L-z_0)$. Then there exists a positive constant $C=C(N,a,\ell,\sup_{\mathcal{C}_L}|u|,\sup_{X\in\mathcal{C}_L,|\tau|\leqslant\sup_{\mathcal{C}_L}|u|}|f(X,\tau)|,R_0,\|g\|_{C^{0,1}})$ such that for any $X_0\in\partial\Omega\times(0,L)$ the following estimate holds*

$$\big|u(X_1)-u(X_2)\big|\leqslant C|X_1-X_2|\log\frac{1}{|X_1-X_2|},\qquad\forall X_1,X_2\in\overline{B_{R_0}(X_0)\cap\mathcal{C}_L}.$$

Proof. We first flatten out a piece of the lateral boundary $\Sigma = \partial\Omega \times (0, L)$ by a Liapunov–Dini map $Y = \mathbf{T}(X)$. Then $X = \mathbf{S}(Y)$, $\mathbf{S} = \mathbf{T}^{-1}$ and in Y-coordinates $\mathbf{T}(B_\rho(X_0) \cap \Sigma) \subset \{Y_1 = 0\}$. Then this change of variables preserves the structure of the equation because for $\widetilde{u} = u(\mathbf{S}(Y))$, $\varphi(X) \in C_0^\infty(\mathcal{C}_L)$, $\widetilde{\varphi}(Y) = \varphi(\mathbf{S}(Y))$ we have

$$\int\limits_{B_{R_0}(X_0)\cap\mathcal{C}_L} D_{X_i} u(X) D_{X_i}\varphi(X)\, dX = \int\limits_{\mathbf{T}(B_{R_0}(X_0)\cap\mathcal{C}_L)} D_{Y_m}\widetilde{u}(Y) S^m_{X_i}(X) D_{Y_k}\widetilde{\varphi}(Y) S^k_{X_i}(X)\, dY.$$

Notice that the matrix $\mathcal{A}^{mk}(Y) = S^m_{X_i}(\mathbf{S}(Y)) S^k_{X_i}(\mathbf{S}(Y))$ is a uniformly elliptic with Dini-continuous entries.

As for the first integral on the left hand side of (4.10), it transforms into

$$\int\limits_{D\cap\mathcal{C}_L} \widetilde{\mathbf{v}} S^m_{X_n}\big(\mathbf{S}(Y)\big)\beta\big(\widetilde{u}(Y)\big) D_{Y_m}\widetilde{\varphi}(Y),$$

where $\widetilde{\mathbf{v}}(Y) = \mathbf{v}(\mathbf{S}(Y))$. Therefore $\widetilde{u}$ satisfies the equation $\mathcal{L}\widetilde{u} = \operatorname{div}[\widetilde{\mathbf{v}}(D\mathbf{S})^t\beta(\widetilde{u})] + \widetilde{f}$ with $\widetilde{f}(Y) = f(\mathbf{S}(Y), u(\mathbf{S}(Y)))$, $\widetilde{u} = g_1 \equiv g(\mathbf{S}(Y))$ on $Y_1 = 0$ and

$$\mathcal{L}\widetilde{u} \equiv \operatorname{div}\mathcal{A}D\widetilde{u}.$$

In particular $\widetilde{u} \in W^{1,2}(\mathbf{T}(B_{R_0}(X_0) \cap \mathcal{C}_L))$.

Put $D = \mathbf{T}(B_{R_0}(X_0) \cap \mathcal{C}_L)$. Without loss of generality we may assume that the upper half ball $B_1^+ = \{Y, |Y| < 1, y_1 > 0\} \subset D$, $0 = \mathbf{T}(X_0)$. We want to use a reflection method and put $v = \widetilde{u} - g_1$. Clearly v satisfies the equation

$$\mathcal{L}v = \operatorname{div}\mathbf{F} + \widetilde{f} \tag{6.3}$$

with $\mathbf{F} = [\widetilde{\mathbf{v}}(D\mathbf{S})^t\beta(\widetilde{u}) - \mathcal{A}Dg_1]$. Notice that $v = 0$ on $\{y_1 = 0\} \cap D$.

Let v^* be the odd reflection of v, that is

$$v^* = \begin{cases} v(y_1, y_2, \dots, y_N), & y_1 > 0, \\ -v(-y_1, y_2, \dots, y_N), & y_1 < 0, \end{cases}$$

then v^* solves the equation

$$\mathcal{L}v^* = \operatorname{div}\mathbf{F}^* + f^*, \quad \text{in } B_1$$

where

$$f^*(Y) = \begin{cases} \widetilde{f}(y_1, y_2, \dots, y_n), & y_1 > 0, \\ -\widetilde{f}(-y_1, y_2, \dots, y_n), & y_1 < 0, \end{cases} \qquad \mathbf{F}^* = \begin{cases} \mathbf{F}(y_1, y_2, \dots, y_N), & y_1 > 0, \\ -\mathbf{F}(-y_1, y_2, \dots, y_N), & y_1 < 0. \end{cases}$$

Now we take w to be the solution of $\mathcal{L}w = 0$ in B_1 and $v^* - w = 0$ on ∂B_1. Then using Green's representation formula with Green function $G_\mathcal{A}(Y_0, Y)$ of operator $\mathcal{L}u = \operatorname{div}(\mathcal{A}Du)$ with pole Y_0 (see [24, Theorem 1.1]) and after integration by parts, we obtain

$$\begin{aligned} v^*(Y_0) - w(Y_0) &= \int\limits_{B_1} \big[f^*(Y) + \operatorname{div}\mathbf{F}^*\big] G_\mathcal{A}(Y_0, Y)\, dY \\ &= \int\limits_{B_1} f^*(Y) G_\mathcal{A}(Y_0, Y)\, dY - \int\limits_{B_1} \mathbf{F} \cdot DG_\mathcal{A}(Y_0, Y)\, dY \\ &= J_1(Y_0) + J_2(Y_0) \end{aligned}$$

where we set

$$J_1(Y_0) = \int\limits_{B_1} f^*(Y) G_\mathcal{A}(Y_0, Y)\, dY,$$

$$J_2(Y_0) = -\int\limits_{B_1} \mathbf{F} \cdot DG_\mathcal{A}(Y_0, Y)\, dY.$$

It follows from [24, Theorem 3.3] that $J_1(Y_0)\in C^{1,\gamma}(B_{\frac{1}{2}})$, for any $\gamma\in(0,1)$, because f^* is bounded.

To deal with J_2, we take small $\rho<\frac{1}{2}$ and set $J_{2,\rho}(Y')=\int_{B_1\setminus B_\rho(Y')}\mathbf{F}\cdot DG_{\mathcal{A}}(Y',Y)\,dY$. Notice that

$$\left|J_2(Y')-J_{2,\rho}(Y')\right|\leqslant C\sup|\beta|\rho,$$

again by [24, Theorem 3.3].

Differentiating $J_{2,\rho}(Y')$ we get

$$DJ_{2,\rho}(Y')=\int\limits_{\partial B_\rho(Y')}\mathbf{F}(Y)\cdot DG(Y',Y)\,dY-\int\limits_{B_1\setminus B_\rho(Y')}\mathbf{F}(Y)D^2G(Y',Y)\,dY$$

and using the estimates of [24, Theorem 3.3] (by definitions the entries $\mathcal{A}^{ij}$ are Dini continuous) we conclude

$$\left|DJ_{2,\rho}(Y')\right|\leqslant C\left[1+\log\left(1+\frac{1}{\rho}\right)\right],$$

with some tame constant C.

Now the above estimates and

$$\left|J_2(Y')-J_2(Y'')\right|\leqslant\left|J_2(Y')-J_{2,\rho}(Y'')\right|+\left|J_2(Y'')-J_{2,\rho}(Y'')\right|+\left|J_{2,\rho}(Y')-J_{2,\rho}(Y'')\right|,$$

with $\rho=|Y'-Y''|$ yield

$$\left|v^*(Y')-v^*(Y'')\right|\leqslant\left|w(Y')-w(Y'')\right|+C\left|Y'-Y''\right|\left[1+\log\left(1+\frac{1}{|Y'-Y''|}\right)\right].$$

Finally we recall that the standard elliptic theory [24] yields that w is C^1 regular in $\frac{1}{4}B$. Hence for $\rho\leqslant\frac{1}{4}$ we get that v^* has modulus of continuity $\widetilde{\sigma}^*(t)=t\log\frac{1}{t}$. Returning to X variable the result follows. □

Lemma 3. *Let u be as in Theorem A. Then $u\in LC^{0,1}(\overline{\Omega\times(0,\delta)}\cup\overline{\Omega\times(L-\delta,L)})$.*

Recall (4.11) and that $h_0(X)=-m^-<0$ and $h_L(X)=m^+>0$. For small $\delta>0$, u solves the equation $\Delta u=\operatorname{div}[(au+\ell)\mathbf{v}]+f$ in $\Omega\times(L-\delta,L)$. Then for $v=m^+-u$ we have $\Delta v=\operatorname{div}[(av+\ell)\mathbf{v}]-f(X,m^+-v(X))$ and $v=0$ on $\Omega\times\{L\}$. Thus the odd reflection of v solves the same equation in $\Omega\times(L-\delta,L+\delta)$ with a $C^{0,1}$ continuous data on the lateral boundary. Thus we can apply Lemma 2. Analogously $w=u+m^-\geqslant 0$ can be reflected across $\Omega\times\{0\}$, hence from Lemma 1 and 2 we infer that $u\in LC^{0,1}(\overline{\mathcal{C}_L})$. □

Next we want to prove the second statement of Theorem A. The first step is to show that u is monotone in z variable.

Lemma 4. *Let u be as in Theorem A* **2°**. *Let $X_i=(x_i,z_i)\in\mathcal{C}_L, i=1,2$ such that $z_2-z_1\geqslant\frac{C}{c_0}|x_1-x_2|\log\frac{1}{|x_1-x_2|}$ then*

$$u(x_1,z_1)\leqslant u(x_2,z_2).$$

Here c_0 is the constant from (5.2).

Proof. We use the domain shift argument discussed in [13]. Let us consider the cylinder

$$\mathcal{C}_{a,b}=\left\{(x,z)\in\mathcal{C}_L\colon (x+a,z+b)\in\mathcal{C}_L,\ a\in\mathbb{R}^{N-1},\ b\in(0,L)\right\}.$$

In other words $\mathcal{C}_{a,b}=(\mathcal{C}_L+(-a,-b))\cap\mathcal{C}_L$, where $(\mathcal{C}_L-(a,b))$ is the translation of $\mathcal{C}_L$ by vector $(-a,-b)\in\mathbb{R}^N$.

Let us compare $u(x,z)$ and $u(x+a,z+b)$ on the boundary of $\mathcal{C}_{a,b}$. If $X=(x,z)$ is on the top portion of $\partial\mathcal{C}_{a,b}$ then $(x+a,z+b)\in\Omega\times\{L\}$, which yields

$$u(x,z)\leqslant m^+=u(x+a,z+b)$$

since by comparison principle $\max_{\mathcal{C}_L}u=m^+$ and $\min_{\mathcal{C}_L}u=-m^-$.

On the bottom of $\partial\mathcal{C}_{a,b}$ we have

$$-m^- = u(x,z) \leqslant u(X)|_{\Omega\times\{b\}}$$

thus $u(x,z) \leqslant u(x+a, z+b)$ on the top and bottom of $\partial\mathcal{C}_{a,b}$.

Next we compare $u(x,z)$ and $u(x+a,z+b)$ on the lateral boundary. If $x \in \partial\Omega$

$$\begin{aligned} u(x+a,z+b)-u(x,z) &= \big[u(x+a,z+b)-u(x,z+b)\big]+u(x,z+b)-u(x,z) \\ &= \big[u(x+a,z+b)-u(x,z+b)\big]+g(x,z+b)-g(x,z) \\ &\geqslant -\sigma\big(|a|\big)+c_0 b. \end{aligned}$$

Here $\sigma(t) = Ct\log\frac{1}{t}$. If $x+a \in \partial\Omega$ then

$$\begin{aligned} u(x+a,z+b)-u(x,z) &= \big[u(x+a,z+b)-u(x+a,z)\big]+u(x+a,z)-u(x,z) \\ &= \big[g(x+a,z+b)-g(x+a,z)\big]+u(x+a,z)-u(x,z) \\ &\geqslant c_0 b-\sigma\big(|a|\big). \end{aligned}$$

Thus choosing $b \geqslant \frac{\sigma(|a|)}{c_0}$ the proof follows from Proposition 1. □

As a simple consequence from Lemma 4 we have the following

Corollary 1. *Let u be as in Lemma 4, then $\partial_z u \geqslant 0$.*

The monotonicity in z variable allows us to define two semicontinuous functions

$$h^+(x) = \inf\{z, u(x,z) > 0\}, \tag{6.4}$$

$$h^-(x) = \sup\{z, u(x,z) < 0\}. \tag{6.5}$$

Clearly $h^{\pm}$ are the height functions of respectively $\partial\Omega^+(u)$ and $\partial\Omega^-(u)$ measured from hyperplane $z = 0$.

Now we shall prove the log-Lipschitz continuity of $h^{\pm}$.

Lemma 5. *Let h^+ and h^- be defined by (6.4) and (6.5), then*

$$\|h^{\pm}\|_{LC^{0,1}(\overline{\Omega})} \leqslant C < \infty. \tag{6.6}$$

Moreover $h^+ = h^-$.

Proof. We shall prove the lemma for h^+. Let $\varepsilon > 0$ and take

$$z_2 = h^+(x_1) + \varepsilon + \frac{\sigma(|x_1 - x_2|)}{c_0}.$$

Then $u(x_2, z_2) \geqslant u(x_1, h^+(x_1) + \varepsilon) > 0$, implying that

$$h^+(x_2) < z_2 = h^+(x_1) + \varepsilon + \frac{\sigma(|x_1 - x_2|)}{c_0}.$$

Sending ε to zero and swapping x_1 with x_2 the first result follows.

It remains to check that $h^+ = h^-$. Indeed if there exists a point $x_0 \in \Omega$ such that $h^-(x_0) < h^+(x_0)$ then, by (6.6) there is $r > 0$ such that

$$h^-(x) < z_0 < h^+(x), \quad |x - x_0| < r, \ x \in \Omega \tag{6.7}$$

where $z_0 = \frac{1}{2}(h^+(x_0) + h^-(x_0))$.

Let w be the harmonic lifting of u in the cylinder $Q_r(z_0) = \{x \in \Omega : |x - x_0| < r\} \times (0, z_0)$. From (6.7) we see that $w \leqslant 0$ on $\partial Q_r(z_0)$. Notice that on the bottom of $Q_r(x_0)$, $\{x \in \Omega : |x - x_0| < r\} \times \{0\}$, we have $w = -m^- < 0$ hence w cannot be identically zero. Hence by maximum principle w is strictly negative in $Q_r(z_0)$.

Next we claim that $\Delta w - \partial_z \beta(w) \leqslant 0$ in $Q_r(z_0)$. Notice that on the lateral boundary $\{x \in \Omega : |x - x_0| = r\} \times (0, z_0)$ of $Q_r(x_0)$ the inequality $\partial_z u \geqslant 0$ holds by Corollary 1. This translates to w. Clearly $\partial_z w \geqslant 0$ on the top and bottom of $\partial Q_r(z_0)$. Hence applying minimum principle to harmonic function $\partial_z w$ in $Q_r(z_0)$ we conclude that $\partial_z w \geqslant 0$ in $Q_r(z_0)$.

Therefore we can compute

$$\int_{Q_r(z_0)} \beta(w)\partial_z \varphi - \int_{Q_r(z_0)} Dw D\varphi = \int_{Q_r(z_0)} \beta(w)\partial_z \varphi, \quad \forall \varphi \in H_0^1\big(Q_r(z_0)\big),\ \varphi \geqslant 0 \tag{6.8}$$

where we used $\Delta w = 0$ in $Q_r(z_0)$. On the other hand $w < 0$ in $Q_r(z_0)$. Thus $\beta(w) = aw$, see (4.8). Returning to (6.8) we get

$$\begin{aligned}\int_{Q_r(z_0)} \beta(w)\partial_z \varphi - \int_{Q_r(z_0)} Dw D\varphi &= \int_{Q_r(z_0)} aw\partial_z \varphi \\ &= -\int_{Q_r(z_0)} a\partial_z w\varphi \leqslant 0, \quad \forall \varphi \in H_0^1\big(Q_r(z_0)\big),\ \varphi \geqslant 0\end{aligned}$$

where the last line follows from $\partial_z w \geqslant 0$ in $Q_r(z_0)$.

Therefore w is a supersolution in $Q_r(z_0)$ of the free boundary problem and hence we may now apply the comparison principle (see Proposition 1) to the functions w and u to infer that $0 > w \geqslant u$ in $Q_r(z_0)$ which contradicts the first inequality in (6.7). □

7. Free boundary condition

For fixed, small $\varepsilon > 0$ and $\zeta \in C_0^\infty(\mathcal{C}_L)$, we use the equation for u^+ to obtain

$$\int_{\{u>\varepsilon\}} \big(Du^+ - \mathbf{v}\beta\big(u^+\big)\big)D\zeta = \int_{\partial\{u>\varepsilon\}} \big(D_{\nu^+}u^+ - \mathbf{v}\cdot\nu^+\beta\big(u^+\big)\big)\zeta. \tag{7.1}$$

Now take a small $\delta > 0$ and use the equation for u^- satisfied in $\Omega^-(u)$ to obtain

$$\int_{\{u<-\delta\}} \big(Du^- - \mathbf{v}\beta\big(u^-\big)\big)D\zeta = \int_{\partial\{u<-\delta\}} \big(D_{\nu^-}u^- - \mathbf{v}\cdot\nu^-\beta\big(u^-\big)\big)\zeta. \tag{7.2}$$

Substracing off the second integral from the first one, and after having sent $\varepsilon \downarrow 0$, $\delta \downarrow 0$ we infer

$$\lim_{\varepsilon\downarrow 0} \int_{\partial\{u>\varepsilon\}} \big(D_{\nu^+}u^+ - \mathbf{v}\cdot\nu^+\beta\big(u^+\big)\big)\zeta = \lim_{\delta\downarrow 0} \int_{\partial\{u<-\delta\}} \big(D_{\nu^-}u^- - \mathbf{v}\cdot\nu^-\beta\big(u^-\big)\big)\zeta. \tag{7.3}$$

From the definition of β we know that $\beta(0^-) = 0$, $\beta(0^+) = \ell$ hence

$$\lim_{\varepsilon\downarrow 0} \int_{\partial\{u>\varepsilon\}} \big(D_{\nu^+}u^+ - \mathbf{v}\cdot\nu^+\ell\big)\zeta = \lim_{\delta\downarrow 0} \int_{\partial\{u<-\delta\}} D_{\nu^-}u^-\zeta. \tag{7.4}$$

Therefore the formal free boundary condition follows

$$\boxed{D_{\nu^+}u^+ - \mathbf{v}\cdot\nu^+\ell = D_{\nu^-}u^-.} \tag{7.5}$$

Remark 5. As (7.5) suggests the best regularity of u is the Lipschitz continuity.

8. Proof of Theorem B, for $u \geqslant 0$

8.1. Linear growth

In this section we deal with the one-phase continuous casting problem i.e. when $u \geqslant 0$ in $\overline{\mathcal{C}_L}$ and $\mathbf{v} = \mathbf{e}_N$. Our first goal here is to prove that for any compact $K \Subset \mathcal{C}_L$

$$\sup_{X_1, X_2 \in K} \frac{|u(X_1) - u(X_2)|}{|X_1 - X_2|} < \infty.$$

Thereby from Remark 5 we will obtain the best local regularity of u.

Remark 6. It is enough to show that if $\mathbf{v}$ is Lipschitz continuous and $f(X, u)$ is bounded then $u(X) \leqslant C|X - X_0|$ in K for some positive constant C and $X_0 \in \partial\{u > 0\} \cap K$.

Next theorem is quite general and can be applied to one-phase problems with convection $\mathbf{v} \in C^{0,1}(\mathcal{C}_L)$.

Theorem 7. *Let $0 \leqslant u \leqslant M$ be a bounded weak solution of (4.5). Then u is locally Lipschitz continuous provided that $\mathbf{v} \in C^{0,1}(\mathcal{C}_L, \mathbb{R}^N)$ and $f(X, u)$ is bounded on $\mathcal{C}_L \times [0, M]$.*

Proof. Take a compact set $K \Subset \mathcal{C}_L$. There exists a tame constant C such that

$$\sup_{B_{2^{-k-1}}(X)} u \leqslant \max\left(C2^{-k}, \frac{1}{2} \sup_{B_{2^{-k}}(X)} u\right), \qquad \forall X \in K \cap \Gamma. \tag{8.1}$$

Clearly (8.1) implies the linear growth of u as indicated in Remark 6.

Suppose that (8.1) fails, then there exist $k_j \in \mathbb{N}$, $k_j \uparrow \infty$, $X_j \in K \cap \Gamma_j$ and weak solution u_j to (4.5) with free boundary $\Gamma_j = \partial\{u_j > 0\}$, such that $0 \leqslant u_j \leqslant M$ and

$$\sup_{B_{2^{-k_j-1}}(X)} u_j \geqslant \max\left(j2^{-k_j}, \frac{1}{2} \sup_{B_{2^{-k_j}}(X_j)} u_j\right). \tag{8.2}$$

Put

$$v_j(X) = \frac{u_j(X_j + 2^{-k_j} X)}{S_j} \quad \text{with } S_j = \sup_{B_{2^{-k_j-1}}(X_j)} u.$$

It follows from (8.2) that

$$v_j(0) = 0, \qquad \sup_{B_{\frac{1}{2}}} v_j \geqslant \frac{1}{2}, \quad 0 \leqslant v_j(X) \leqslant 2,\ X \in B_1. \tag{8.3}$$

Since the functions u_j are bounded it follows from (8.2) that $M > j2^{-k_j}$ implying that $k_j \to \infty$.

By scale invariance of Eq. (4.5) we get

$$\begin{aligned} \Delta v_j &= \frac{2^{-2k_j}}{S_j}(\Delta u_j)\big(X_j + 2^{-k_j} X\big) \\ &= \frac{2^{-k_j}}{S_j} \operatorname{div}\big[\big(aS_j v_j + \ell H(v_j)\big)\mathbf{v}\big(X_j + 2^{-k_j} X\big)\big] + \frac{2^{-2k_j}}{S_j} f\big(X_j + 2^{-k_j} X\big) \\ &\equiv \operatorname{div} \mathbf{F}_j + \frac{2^{-2k_j}}{S_j} f\big(X_j + 2^{-k_j} X\big), \end{aligned} \tag{8.4}$$

where H is the Heaviside function and

$$\mathbf{F}_j = \frac{2^{-k_j}}{S_j}\big[\big(aS_j v_j + \ell H(v_j)\big)\mathbf{v}\big(X_j + 2^{-k_j} X\big)\big].$$

Since by assumption $\mathbf{v} \in C^{0,1}(\mathcal{C}_L, \mathbb{R}^N)$ we get from (8.2) and (4.8), the decay estimate

$$|\mathbf{F}_j| \leqslant \frac{2^{-k_j}}{S_j}\beta(2)\sup|\mathbf{v}| \leqslant \frac{M}{j}\beta(2)\sup|\mathbf{v}| \to 0. \tag{8.5}$$

Similarly $\frac{2^{-2k_j}}{S_j}|f(X_j + 2^{-k_j}X)| \leqslant \frac{2^{-k_j}}{S_j}\sup|f| \leqslant \frac{1}{j}\sup|f|$ where the last inequality follows from (8.2) and the definition of S_j.

Let $\eta \in C_0^\infty(B_1)$ such that $\eta \equiv 1$ in $B_{\frac{3}{4}}$ and $\varphi = v_j\eta^2 \in H_0^1(B_1)$. From the weak formulation of the equation we have

$$\int_{B_1} |Dv_j|^2\eta^2 = -\int_{B_1} 2\eta v_j D\eta Dv_j + \int_{B_1} \mathbf{F}_j\big(Dv_j\eta^2 + 2\eta v_j D\eta\big) - \int_{B_1} f_j v_j \eta^2.$$

Employing Cauchy–Schwarz inequality and estimating the left hand side we get $\int_{B_{\frac{3}{4}}} |Dv_j|^2 \leqslant C\int_{B_1}[v_j^2 + |\mathbf{F}_j|^2 + |f_j|]$ with some dimensional constant C independent of j. Utilizing (8.3) we obtain from DeGiorgi's theorem that v_j's are uniformly γ-Hölder continuous in $B_{3/4}$ for some $\gamma \in (0,1)$. Then using a customary compactness argument and the decay estimate (8.5) for $\mathbf{F}_j$ we have, at least for a subsequence $j(m)$, $v_{j(m)} \to v_0$ uniformly in $B_{\frac{3}{4}}$ and weakly in $H^1(B_{\frac{3}{4}})$ and

$$\int Dv_0 D\varphi \leftarrow \int Dv_{j(m)} D\varphi = \int \mathbf{F}_{j(m)} \cdot D\varphi - \int f_j\varphi \to 0, \quad \forall \varphi \in C_0^\infty(B_{\frac{3}{4}}).$$

Thus v_0 is a nonnegative continuous harmonic function in $B_{\frac{3}{4}}$ such that $v_0(0) = 0$ and $\sup_{B_{\frac{1}{2}}} v_0 = \frac{1}{2}$ in view of (8.3). However this contradicts the strong maximum principle and the proof follows. □

8.2. Non-degeneracy

Let w be the Baiocchi transformation of u given by

$$w(X) = \int_0^z u(x,s)\,ds.$$

Note that if $\partial_z u \geqslant 0$ then $\Omega^+(w) = \Omega^+(u)$ otherwise the inclusion

$$\Omega^+(u) \subseteq \Omega^+(w) \tag{8.6}$$

always holds.

Proposition 2. *Let $w \geqslant 0$ be a bounded solution of (4.4) in $\mathcal{C}_L$ and $f \equiv 0$. Then for any compact set $K \Subset \mathcal{C}_L$ and any $X_0 \in \overline{\Omega^+(w)} \cap K$ we have*

$$\sup_{\partial B_r(X_0)} w \geqslant r^2\frac{\ell/2 - Mr}{2N}$$

with $M = \|Du\|_{L^\infty(K)}$. In particular for any $X_0 \in \Omega^+(u) \cap K$, $B_r(X_0) \subset K$ we infer

$$\sup_{B_r(X_0)} u \geqslant r\frac{\ell/2 - Mr}{2N}. \tag{8.7}$$

Proof. We use an argument from [9]. Notice that it is enough to consider the case $X_0 \in \Omega^+(u)$ since by continuity of u it extends to $X_0 \in \partial\{u > 0\}$. Let $X_0 \in \Omega^+(w)$. Put $w_r(X) = \frac{w(X_0+rX)}{r^2}$, then by (4.4) we have

$$\Delta w_r = (\Delta w)(X_0 + rX) = au(X_0 + rX) + \ell\chi\big\{u(X_0 + rX) > 0\big\}.$$

If Proposition 2 fails then $\sup_{\partial B_1} w_r < \frac{\ell/2 - Mr}{2N}$ for some $r > 0$. Let $q(X) = w_r(X) - \frac{\ell/2 - Mr}{2N}|X - X_0|^2$. Then taking into account (8.6), in $B_1 \cap \Omega^+(w_r)$ we have

$$\Delta q = \ell + au(X_0 + rX) - \frac{\ell/2 - Mr}{2N}\Delta|X - X_0|^2 \geqslant \ell - Mr - 2N\frac{\ell/2 - Mr}{2N} = \ell/2 > 0.$$

This in conjuncture with the boundary conditions gives

$$\begin{cases} \Delta q(x) > 0 \quad x \in B_1, \\ q|_{\partial B_1 \cap \Omega^+(w)} < 0, \\ q|_{B_1 \cap \partial\Omega^+(w)} < 0. \end{cases} \tag{8.8}$$

Thus by maximum principle $q < 0$ in $B_1 \cap \Omega^+(w)$ implying that $w(X_0) < 0$ which contradicts $X_0 \in \Omega^+(w)$. Recalling the definition of $w(X) = \int_0^z u(x, s)\,ds$ and (8.6) we conclude that

$$\sup_{\partial B_r(X_0)} w \leqslant r \sup_{B_r(X_0)} u. \qquad \square$$

Remark 8. If $f(X, u) < \frac{\ell}{2L}$ near the free boundary then Proposition 2 still holds. Indeed, take $q(X) = w_r(X) - \frac{\ell/2-Mr}{2N}|X - X_0|^2$ and argue as above. The boundary conditions in (8.8) still hold. Notice that $\overline{f}(X) = \int_0^z f(x, s, u(x, s))\,ds < \frac{z\ell}{2L}$. As for the Laplacian we recall (4.4) and compute $\Delta q = \frac{\ell}{2} - \overline{f} > \frac{\ell}{2}(1 - \frac{z}{L}) \geqslant 0$ if $r < r_0$ for a sufficiently small r_0. Then applying the strong maximum principle we arrive at the desired result.

8.3. Rectifiability of the free boundary

We now study the measure theoretic properties of the free boundary for continuous casting problem, i.e. when $\mathbf{v} = \mathbf{e}_N$. First we let $w(X) = e^{-\frac{az}{2}}u(X)$ and consider the measure Δw. Here a is the constant appearing in the definition of enthalpy β (4.8). Notice that

$$\partial\{w > 0\} = \partial\{u > 0\}. \tag{8.9}$$

Next by product rule we have

$$\Delta w = \Delta\big[e^{-\frac{az}{2}}u\big] = \left[\frac{a^2}{4}u - a\partial_z u + \Delta u\right]e^{-\frac{az}{2}}. \tag{8.10}$$

Observe that

$$\Delta u^+ = a\partial_z u^+ + f, \quad \text{in } \Omega^+(u). \tag{8.11}$$

Combining (8.10) and (8.11) we obtain

$$\Delta w^\pm = \frac{a^2}{4}w^\pm + fe^{-\frac{az}{2}} \quad \text{in } \Omega^\pm(u). \tag{8.12}$$

Lemma 6. *Let u be the weak solution of* **(DP)** *and $\mathbf{v} = \mathbf{e}_N$. Then*

1° *If $f = 0$ then $\mu = \Delta w$ is a nonnegative Radon measure. If $f \neq 0$ and $C \geqslant \frac{\|f\|_\infty}{2N}$ then $u + C|X|^2$ is subharmonic in each $D \Subset \mathcal{C}_L$ and $d\mu + C\,dX$ is a nonnegative Radon measure.*

2° *If $f = 0$, $D \Subset \mathcal{C}_L$ then there exist positive constants c_D, C_D such that*

$$c_D r^{N-1} \leqslant \int\limits_{B_r(X_0)} d\mu \leqslant C_D r^{N-1}, \quad \forall B_r(X_0) \subset D,\ X_0 \in \partial\{u > 0\}.$$

3° *If $f = 0$, $D \Subset \mathcal{C}_L$ then $\mathcal{H}^{N-1}(\partial\{u > 0\} \cap D) < \infty$ and hence the free boundary is a set of locally finite perimeter in $\mathcal{C}_L$. Moreover*

$$\mathcal{H}^{N-1}\big(\partial\{u > 0\} \setminus \partial_{\mathrm{red}}\{u > 0\}\big) = 0.$$

Proof. The first assertion follows from (8.11), see also the proof of Lemma 9. Notice that w is Lipschitz continuous in D since so is u. By divergence theorem $\int_{B_r(X_0)} \Delta w = \int_{\partial B_r(X_0)} Du \cdot \nu \leqslant C_D r^{N-1}$ which proves the second inequality in **2**°. The proof of the first one is by contradiction. Suppose that there are $X_k \in \partial\{u>0\}\cap D$ and $0<r_k \downarrow 0$ such that $\int_{B_{r_k}(X_k)} d\mu \leqslant \frac{1}{k} r_k^{N-1}$. Put $w_k(X)=\frac{w(X_k+r_kX)}{r_k}$ and recall that by Theorem A and Proposition 2 w_k's are Lipschitz continuous and non-degenerate in the unit ball B_1. Moreover

$$0 \leqslant \frac{1}{r_k^{N-1}} \int\limits_{B_{r_k}(X_k)} d\mu = \int\limits_{B_1} d\mu_k \leqslant \frac{1}{k}, \qquad \mu_k = \Delta w_k.$$

Using a customary compactness argument we can extract a subsequence $j=j(k)$ such that $w_{j(k)}$ converge uniformly in $C^{0,1}(B_1)$ and weakly in $H^1(B_1)$ to a non-zero (by non-degeneracy of w_k's), harmonic function $w_0 \geqslant 0$ defined in B_1 since $\mu_k \rightharpoonup 0$ as measures. By uniform convergence $w_0(0)=0$ and this contradicts the strong maximum principle for w_0 is non-zero.

It remains to prove **3**°. Fix a $\delta>0$. Let $B_{r_i}(X_i)$ be a ball a covering of $E\subset D\cap\partial\{u>0\}$ such that $r_i\leqslant\delta$. Let $Y_i \in E\cap B_{r_i}(X_i)$ and for each i consider the Besicovitch type covering $B_{2r_i}(Y_i)$ of E. Note that

$$\bigcup_i B_{r_i}(X_i) \subset \bigcup_i B_{2r_i}(Y_i).$$

From Besicovitch's covering lemma we can extract a subcovering $\mathcal{F}=\bigcup_{k=1}^{m(N)}\mathcal{G}_k$ of balls $B_i=B_{2r_i}(Y_i)$ such that $\sum_i \chi_{B_i} \leqslant C$ for some dimensional constant C and

$$E \subset \bigcup_{k=1}^{m(N)} \bigcup_{B_i\in\mathcal{G}_k} B_i,$$

where the balls B_i in each $\mathcal{G}_k$ are disjoint and $\mathcal{G}_k$ are countable. Hence

$$\begin{aligned} c_D \sum_{B_i\in\mathcal{F}} r_i^{N-1} &\leqslant \sum_{B_i\in\mathcal{F}} \int\limits_{B_i} d\mu \\ &= \sum_{k=1}^{m(N)} \sum_{B_i\in\mathcal{G}_k} \int\limits_{B_i} d\mu \\ &\leqslant m(N) \int\limits_{B_{8\delta}(E)} d\mu. \end{aligned} \tag{8.13}$$

Thus for the δ-premeasure we get

$$\mathcal{H}_\delta^{N-1}(E) \leqslant \frac{m(N)}{c_D} \int\limits_{B_{8\delta}(E)} d\mu < \infty$$

and letting $\delta\to 0$ we arrive at the desired result.

Let $K=(\partial\{u>0\}\setminus\partial_{\mathrm{red}}\{u>0\})\cap D$. By [14, Theorem 4.5.6(3)], there exists $K_0\subset K$ such that $\mathcal{H}^{N-1}(K)=\mathcal{H}^{N-1}(K_0)$ and for each $X_0\in K_0$

$$\mu\big(B_r(X_0)\big)=o\big(r^{N-1}\big).$$

Take $0<r_k\to 0$ and consider the sequence $u_k(X)=\frac{u(X_0+r_kX)}{r_k}$. Clearly u_k's are non-degenerate and Lipschitz. As in the proof of part **2**°, by a customary compactness argument we can extract a subsequence $j=j(k)$ such that $u_{j(k)}$ converges uniformly in $C^{0,1}(B_1)$ and weakly in $H^1(B_1)$ to a non-zero, harmonic function $u_0\geqslant 0$, $u_0(0)=0$. This contradicts the strong maximum principle. Therefore $K_0=\emptyset$ and $\mathcal{H}^{N-1}(K)=0$. □

Corollary 2. *The free boundary $\partial\{u>0\}$ is countably $N-1$ rectifiable, i.e. for any $D \Subset \mathcal{C}_L$*

$$\partial\{u>0\}\cap D \subset M_0 \bigcup \left(\bigcup_{j=1}^{\infty} M_j\right)$$

such that $\mathcal{H}^{N-1}(M_0)=0$ and M_j, $j\geqslant 1$ is an $N-1$ dimensional embedded C^1 submanifold of $\mathbb{R}^N$.

Proof. By Lemma 6, **3°** the free boundary $\partial\{u>0\}$ is a set of locally finite perimeter and $\mathcal{H}^{N-1}(\partial\{u>0\}\setminus\partial_{\mathrm{red}}\{u>0\})=0$. The rest follows from Lemma 11.1 and Theorem 14.3 of [23]. □

9. Local Lipschitz estimate for two phase problem

In this section we prove the optimal local regularity of the solution for two phase problem.

9.1. Technicalities

We begin with the following useful observation. If $w=e^{-\frac{a\xi(X)}{2}}u(X)$ then

$$\begin{aligned}\Delta w&=e^{-\frac{a\xi}{2}}\left\{\Delta u-aDu\cdot D\xi+u\left[\frac{a^2|D\xi|^2}{4}-\frac{a\Delta\xi}{2}\right]\right\}\\&=e^{-\frac{a\xi}{2}}\left\{\operatorname{div}\big(\beta(u)D\xi\big)+f-aDu\cdot D\xi+u\left[\frac{a^2|D\xi|^2}{4}-\frac{a\Delta\xi}{2}\right]\right\}.\end{aligned}\tag{9.1}$$

Thus, taking into account that $\Delta\xi=0$, it follows that the positive and negative parts of $w=w^+-w^-$ satisfy the equations

$$\begin{aligned}\Delta w^+&=e^{-\frac{a\xi}{2}}\left\{\big(au^++\ell\big)\Delta\xi+f+u^+\left[\frac{a^2|D\xi|^2}{4}-\frac{a\Delta\xi}{2}\right]\right\}=fe^{-\frac{a\xi}{2}}+w^+\frac{a^2|D\xi|^2}{4},\\\Delta w^-&=e^{-\frac{a\xi}{2}}\left\{au^-\Delta\xi-f+u^-\left[\frac{a^2|D\xi|^2}{4}-\frac{a\Delta\xi}{2}\right]\right\}=-fe^{-\frac{a\xi}{2}}+w^-\frac{a^2|D\xi|^2}{4}.\end{aligned}\tag{9.2}$$

Therefore w^+, w^- are continuous, nonnegative functions in $\mathcal{C}_L$ and

$$\Delta w^\pm\geqslant-\left[\sup_{\mathcal{C}_L}|f|+\sup_{\mathcal{C}_L}|u|\frac{a^2\|\xi\|^2_{C^1}}{4}\right]e^{-\frac{a\|\xi\|_\infty}{2}}\equiv-\gamma_0,\quad\text{in }\{u>0\}\cup\{u<0\}.\tag{9.3}$$

This observation together with Lemma 8 and 9 will allow us to employ the monotonicity formula of [10] and show that u is locally Lipschitz continuous in $\mathcal{C}_L$.

Lemma 7. *For any compact set $K\Subset\mathcal{C}_L$ there exists a positive tame constant $C=C(K)$ such that for any $B_\rho(X_0)\subset K$, $X_0\in\Gamma(u)\cap K$ the following estimate holds*

$$\left|\int_{B_\rho(X_0)}\Delta w\right|\leqslant C\rho^{N-1}.\tag{9.4}$$

Proof. We employ the identity (9.1) and use Green's formula to obtain

$$\begin{aligned}\int_{B_\rho(X_0)}\Delta w&=\int_{B_\rho(X_0)}e^{-\frac{a\xi}{2}}\left\{\operatorname{div}\big(\beta(u)D\xi\big)+f-aDu\cdot D\xi+u\left[\frac{a^2|D\xi|^2}{4}-\frac{a\Delta\xi}{2}\right]\right\}\\&=\int_{B_\rho(X_0)}e^{-\frac{a\xi}{2}}\left\{f+u\frac{a^2|D\xi|^2}{4}\right\}+\int_{\partial B_\rho(X_0)}(D\xi\cdot\nu)e^{-\frac{a\xi}{2}}\big\{\beta(u)-au\big\}\\&\quad-\int_{B_\rho(X_0)}\big\{\beta(u)-au\big\}D\xi\cdot De^{-\frac{a\xi}{2}}\end{aligned}\tag{9.5}$$

which yields

$$\left|\int_{B_\rho(X_0)}\Delta w\right|\leqslant C\rho^{N-1}.\qquad\square$$

Lemma 8. *For any compact set $K \Subset \mathcal{C}_L$ there exists a positive number ρ_0 depending only on* $\operatorname{dist}(K, \partial\mathcal{C}_L)$, N *and a positive tame constant $C = C(K)$ such that for any $B_\rho(X_0) \subset K$, $X_0 \in \Gamma(u) \cap K$ the following estimate holds*

$$\left| \fint_{\partial B_\rho(X_0)} w \right| \leqslant C\rho, \quad \rho < \rho_0. \tag{9.6}$$

Proof. From Green's representation formula

$$w(X_0) = \int_{\partial B_\rho(X_0)} w(Y) P(Y, X_0)\, dH^{N-1} - \int_{B_\rho(X_0)} G(X, X_0) \Delta w(X)\, dX, \tag{9.7}$$

where $P(Y, X_0)$ is the kernel of Poisson and $G(X, X_0)$ is Green's function of $B_\rho(X_0)$ with pole at X_0. At $X_0 \in \Gamma(u)$, $w(X_0) = 0$ implying that

$$\begin{aligned} \fint_{\partial B_\rho(X_0)} w(Y)\, dH^{N-1} &= \int_{B_\rho(X_0)} G(X, X_0) \Delta w(X)\, dX \\ &= \int_0^\rho G(s) \frac{d}{ds} \left(\int_0^s t^{N-1} \int_{\partial B_1} \Delta w(t\xi)\, dH^{N-1}(\xi)\, dt \right) ds \\ &= G(s) \int_{B_s(X_0)} \Delta w \bigg|_0^\rho - \int_0^\rho G'(s) \int_{B_s(X_0)} \Delta w. \end{aligned} \tag{9.8}$$

Now the result follows from (9.5) and the estimate $G(s) \leqslant Cs^{2-N}$. □

Next crucial step in our approach is to employ Alt–Caffarelli–Friedman type monotonicity theorem, see [11, Lemmas 2.2 and 2.3].

Theorem 9. *Let w^+, w^- be two continuous, nonnegative subharmonic functions in $B_1(X_0)$, $w^- w^+ = 0$, $w^+(X_0) = w^-(X_0) = 0$. Then*

$$\Phi(R, X_0, w_1, w_2) = \frac{1}{R^4} \int_{B_R(X_0)} \frac{|\nabla w_1(X)|^2}{|X - X_0|^{N-2}}\, dX \int_{B_R(X_0)} \frac{|\nabla w_2(X)|^2}{|X - X_0|^{N-2}}\, dX$$

is monotone increasing function of $R < 1$.

Moreover if $\Phi(R) = \gamma > 0$, $\forall R \in (0, 1)$ then $\operatorname{supp} w^+ \cap \partial B_R(X_0)$ *and* $\operatorname{supp} w^- \cap \partial B_R(X_0)$ *are half spheres.*

We will also need the "almost monotonicity" result from [10, Theorem 1.3 and Remark 1.5].

Theorem 10. *Let w^+, w^- be nonnegative, continuous functions on $B_2(X_0)$. Suppose that $\Delta w^\pm > -1$ in the sense of distributions and $w^+(X_0) = w^-(X_0) = 0$, $w^+(X) w^-(X) = 0$ for all $X \in B_1$. Then there is a dimensional constant C such that*

$$\Phi(R) \leqslant C \left(1 + \int_{B_2(X_0)} (w^+)^2 + \int_{B_2(X_0)} (w^-)^2 \right), \quad R < 1.$$

Lemma 9. *Let $w(X) = u(X) e^{-\frac{a\xi(X)}{2}}$ and γ_0 be defined by (9.3). Then for $C > \frac{\gamma_0}{2N}$, $w^\pm(X) + C|X|^2$ are subharmonic in $\mathcal{C}_L$.*

Moreover w^+ and w^- satisfy the assumptions of Theorem 10.

Proof. As it is pointed out in [18, Chapter 1.5, p. 54] the subharmonicity has local nature, thus it is enough to show that for each $X \in \mathcal{C}_L$ there exists $r(X) > 0$ such that

$$v(X) \leqslant \fint_{B_r(X)} v, \quad r < r(X),$$

where $v = w^+ + C|X|^2$. Since in $\Omega^+(u)$, $\Delta v \geqslant 0$ by (9.2), and we may take $r(X) = \operatorname{dist}(X, \Gamma)$. If $X \in \{u \leqslant 0\}$ then we use the subharmonicity of $C|X|^2$ to get

$$v(X) = C|X|^2 \leqslant \fint_{B_r(X)} C|Y|^2\,dY \leqslant \fint_{B_r(X)} \big(w^+(Y) + C|Y|^2\big)\,dY = \fint_{B_r(X)} v(Y)\,dY.$$

Similarly we can prove that $\bar{v} = w^- + C|X|^2$ is subharmonic. Hence we conclude that $\Delta w^+ \geqslant -C$ ($\Delta w^- \geqslant -C$) in $\mathcal{C}_L$ in the sense of distributions. □

Remark 11. In view of Lemma 9 and Hölder continuity of u, see Remark 1, the pair w^+, w^- satisfies the requirements of Theorem 10

9.2. Proof of Theorem B

We will show that w is Lipschitz continuous. This is clearly enough to conclude that $u \in C^{0,1}(\mathcal{C}_L)$ since $Dw = De^{-\frac{a\xi(X)}{2}}w + e^{-\frac{a\xi(X)}{2}}Du$. For $X \in \mathcal{C}_L$ let $X_0 \in \Gamma = \partial\{u > 0\}$ be the closest point to X and let $\rho = |X - X_0| = \operatorname{dist}(X, \Gamma)$. To fix the ideas we assume that $B_1(X_0) \subset \mathcal{C}_L$.

Now suppose that $w(X) \geqslant M\rho > 0$ for some large $M > 0$. We have $w(X) \geqslant M\rho = \frac{M}{\rho}\rho^2 \gg M\rho^2$ and $|\Delta w^+| \leqslant C_0$ for some tame constant $C_0 > 0$, see (9.2). Then it follows from [10, Lemma 4.6] that there is a tame constant $C > 0$ such that

$$\max_{B_{\frac{\rho}{2}}(X)} w \leqslant C \min_{B_{\frac{\rho}{2}}(X)} w.$$

Thus we obtain the inequality

$$\inf_{B_{\frac{\rho}{2}}(X)} w^+ \geqslant \frac{M\rho}{C} \geqslant \frac{M\rho}{2}$$

provided that M is large enough and ρ is small. Therefore

$$\fint_{\partial B_\rho(X_0)} w^+ \geqslant c_1 \fint_{S_\rho} w^+ \geqslant c_1 \frac{M\rho}{2},$$

where $S_\rho = \partial B_\rho(X_0) \cap B_{\frac{\rho}{2}}(X)$ and c_1 depends only on the dimension N. By Lemma 8

$$\fint_{\partial B_\rho(X)} w^- \geqslant \fint_{\partial B_\rho(X)} w^+ - C\rho \geqslant \left(\frac{c_1 M}{2} - C\right)\rho > \frac{Mc_1}{4}\rho$$

if M is sufficiently large.

Let $Y \in \overrightarrow{XX_0} \cap B_{\frac{\rho}{2}}(X_0)$. We use polar coordinates (r, ω) about Y. Let E be the set of $\omega \in \partial B_1(Y)$ such that if $(r, \omega) \in \partial B_\rho(X_0)$ then $w(r, \omega) < 0$. Applying the estimate (5.16) from [4, p. 443] we get

$$\frac{Mc_1}{4}\rho \leqslant \frac{1}{\rho} \fint_{\partial B_\rho(X_0)} w^- \leqslant |E|^{\frac{1}{2}} \frac{1}{\rho}\left[\int_{B_\rho(X_0)} \frac{|Dw^-|^2}{|Z - X_0|^{n-2}}\,dZ\right]^{\frac{1}{2}},$$

and integrating $\partial_r w^+(r, \omega)$ on the set $(r, \omega) \in B_\rho(X_0) \setminus B_{\frac{\rho}{4}}(Y)$, $\omega \in E$ we get (see inequality (5.17) in [4, p. 443])

$$c_1\frac{M}{2}|E|\rho^n \leqslant \iint\limits_{(r,\omega)\in B_\rho(X_0)\setminus B_{\frac{\rho}{4}}(Y),\omega\in E} \partial_r w^+(r,\omega) \leqslant |E|^{\frac{1}{2}}\rho^{n-1}\left[\int\limits_{B_\rho(X_0)}\frac{|Dw^+|^2}{|Z-X_0|^{n-2}}\,dZ\right]^{\frac{1}{2}}.$$

Thus by Theorem 10 and Remark 11 it follows that

$$M^2 \leqslant \frac{8}{c_1^2}\left[\Phi(\rho)\right]^{\frac{1}{2}} \leqslant \frac{8}{c_1^2}C\left(1+\int\limits_{B_1(X_0)}\left(w^+\right)^2+\int\limits_{B_1(X_0)}\left(w^-\right)^2\right)^{\frac{1}{2}}$$

with some tame constant $c_1>0$ and the proof follows. □

Lemma 10. *Let u be a weak solution of* **(DP)**. *Put $u_m(x)=\frac{u(X_0+r_mX)}{r_m}$, where $r_m\downarrow 0, r_m>0$ and $X_0\in\partial\{u>0\}$. u_m is called blow-up sequence at $X_0\in\partial\{u>0\}$. There exists a subsequence $r_{m_j}\downarrow 0$ and a limit $u_0\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, called a blow-up limit of u at $x_0\in\partial\{u>0\}$, such that for each compact set $K\subset\mathbb{R}^N$*

$$Du_{m_j} \rightharpoonup Du_0 \quad \text{weakly-star in } K, \tag{9.9}$$

$$Du_{m_j} \longrightarrow Du_0 \quad \text{a.e. in } K, \tag{9.10}$$

$$u_{m_j} \longrightarrow u_0 \quad \text{strongly in } H^1_{\mathrm{loc}}\left(\mathbb{R}^N\right) \text{ and } C^\alpha_{\mathrm{loc}}\left(\mathbb{R}^N\right),\ \forall\alpha\in(0,1) \text{ as } m_j\longrightarrow\infty, \tag{9.11}$$

$$\partial\{u_{m_j}>0\} \longrightarrow \partial\{u_0>0\} \quad \text{in Hausdorff distance in } K, \tag{9.12}$$

$$\chi_{\{u_{m_j}>0\}} \longrightarrow \chi_{\{u_0>0\}} \quad \text{in } L^1(K). \tag{9.13}$$

Furthermore the limit u_0 solves the equation

$$\Delta u_0 = \operatorname{div}\left(\mathbf{v}(X_0)\beta_0(u_0)\right).$$

Here $\beta_0(t)=\ell$ is the Heaviside function given by (4.2).

Proof. The proof is quite standard and we refer to Section 4.7 of [2] and pages 19–20 of [3]. □

10. Non-degeneracy of u and rectifiability of $\partial\{u>0\}$

The goal of this section is to discuss the measure theoretic properties of free boundary for two-phase continuous casting problem. As we have seen, the non-degeneracy of u^+ is crucial in the proof of countably $(N-1)$-rectifiability of $\partial\{u>0\}$. For the one phase case this follows from the corresponding result for obstacle problem and Baiocchi transformation, see Proposition 2. However the Baiocchi transformation does not work for two phase case since (8.6) fails.

10.1. Non-degeneracy of u^+

Definition 4. Let u be a solution to **(DP)**. Then u^+ is said to be non-degenerate at $X_0\in\partial\{u>0\}$ if there exists a constant $c_0>0$ such that

$$\frac{1}{r^{N-1}}\int\limits_{\partial B_r(X_0)} u^+ \geqslant c_0 r$$

for small $r>0$. If $D\Subset\mathcal{C}_L$ then u^+ is said to be non-degenerate on D if u^+ is non-degenerate at each point $X\in D$ with the same constant $c_0>0$.

By means of (4.4) we were able to link the one-phase continuous casting problem to the obstacle problem for w and retrieve the non-degeneracy from the corresponding result for w.

Unfortunately this technique does not apply to the two-phase problem. Although the non-degeneracy property of u^+ is not vital for the remaining 2 sections, however for the completeness we would like to indicate how the measure theoretic properties of the free boundary follow, similar to Lemma 6, for the two-phase problem once u^+ is non-degenerate. There are various conditions, imposed on the boundary data, guaranteeing that u^+ is non-degenerate (see [19] and references therein). Below we give one in terms of the Lebesgue density of free boundary, which states that if at $X_0 \in \partial\{u > 0\}$ the free boundary is not tangent to $\mathbf{e}_N$, in some measure theoretic sense, then u^+ is non-degenerate.

Lemma 11. *Let u be a weak solution to* (**DP**)*, $\mathbf{v} = \mathbf{e}_N$ and $f = 0$. Let*

$$\mathscr{A}(u) = \{X \in \Omega^+(u),\ (X - X_0)\cdot \mathbf{e}_N > 0\}, \qquad \mathscr{B}(u) = \{X \in \Omega^+(u),\ (X - X_0)\cdot \mathbf{e}_N < 0\},$$
$$\widetilde{\mathscr{B}}(u) = \{X = (x,z) \in \mathcal{C}_L,\ (x,-z) \in \mathscr{B}(u)\}.$$

If $\liminf_{\rho\downarrow 0} \frac{|(\widetilde{\mathscr{B}}\setminus\mathscr{A})\cap B_\rho(X_0)|}{|B_\rho|} > 0$ *then u^+ is non-degenerate at X_0.*

Proof. For $\rho > 0$, $B_\rho(X_0) \subset \mathcal{C}_L$ we set $u_\rho(X) = \frac{u(X_0+\rho X)}{\rho}$. Then $\Delta u_\rho^+(X) = \rho(\Delta u^+)(X_0 + \rho X) = \rho a D_z u_\rho^+ \to 0$ as $\rho \to 0$, because u is Lipschitz continuous and in view of Lemma 10.

Moreover $\Delta u_\rho = \operatorname{div}(\mathbf{v}\beta_\rho(u_\rho))$ in B_1, where

$$\beta_\rho(s) = \begin{cases} as\rho & \text{if } s < 0,\\ \in [0,\ell] & \text{if } s = 0,\\ as\rho + \ell & \text{if } s > 0,\end{cases} \tag{10.1}$$

and $\beta_\rho \to \beta_0 \in \ell H(s)$, where H is the Heaviside function given by (4.2).

Suppose that there exists $\rho_k \downarrow 0$ such that

$$\frac{1}{\rho_k} \fint_{\partial B_{\rho_k}(X_0)} u^+ = \int_{\partial B_1} u_{\rho_k}^+ \to 0.$$

Without loss of generality we may assume that ρ_k is a subsequence for which u_{ρ_k} converges to u_0 as stated in Lemma 10. It follows from Green's representation

$$\begin{aligned}\int_{\partial B_1} u_{\rho_k} &= \int_{B_1} \operatorname{div}\big[\beta_{\rho_k}(u_{\rho_k})\mathbf{v}\big] G(X,0)\\ &= -\frac{1}{N\omega_N}\int_{B_1} \beta_{\rho_k}(u_{\rho_k})\frac{z}{|X|^n}\\ &= \frac{1}{N\omega_N}\int_{\widetilde{\mathscr{B}}(u_{\rho_k})} \frac{\ell z}{|X|^n} - \frac{1}{N\omega_N}\int_{\mathscr{A}(u_{\rho_k})} \frac{\ell z}{|X|^n} + o(\rho_k) \longrightarrow \lim_{k\to\infty} \int_{\widetilde{\mathscr{B}}(u_{\rho_k})\setminus\mathscr{A}(u_{\rho_k})} \frac{\ell z}{|X|^n} \equiv c_0 > 0.\end{aligned}$$

Here $\omega_N = |B_1|$. On the other hand $\lim_{k\to 0}\int_{\partial B_1} u_{\rho_k} = -\int_{\partial B_1} u_0^- \leqslant 0$ implying that $c_0 \leqslant 0$ which is a contradiction. This completes the proof. □

10.2. Measure theoretic properties of free boundary

Let $w = e^{-\frac{a\xi}{2}}u$. Since by Theorem A and Lemma 9 w^+ and w^- are continuous and $\Delta w^\pm \geqslant -C$, then $d\mu^+ = \Delta w^+$ and $d\mu^- = \Delta w^-$ are Radon measures supported on $\Omega^+(u)$ and $\Omega^-(u)$, respectively. Lemma to follow summarizes some properties of $\partial\{u > 0\}$ under assumption that u^+ is non-degenerate.

Lemma 12. *Let u be an weak solution of (4.5). Then if*

1° *For any $D \Subset \mathcal{C}_L$, there exists a positive tame constant C depending on data such that for any $B_R(X_0) \subset D$, $X_0 \in \partial\{u > 0\}$*

$$-CR^N \leqslant \int_{B_R(X_0)} d\mu^\pm \leqslant CR^{N-1}.$$

2° *If in addition u^+ is non degenerate in D then there exist tame constants c_D, C_D such that*

$$c_D R^{N-1} \leqslant \int_{B_R(X_0)} d\mu^+ \leqslant C_D R^{N-1}.$$

3° *Let u^+ be non-degenerate. Then for any $D \Subset \mathcal{C}_L$ we have*

$$\mathcal{H}^{N-1}\big(\partial\{u>0\} \cap D\big) < \infty.$$

Furthermore $\partial\{u>0\}$ is a set of locally finite perimeter and

$$\mathcal{H}^{N-1}\big(\partial\{u>0\} \setminus \partial_{\text{red}}\{u>0\}\big) = 0.$$

In particular $\partial\{u>0\}$ is countably $N-1$ rectifiable.

The proofs are the same as that of Lemma 6 and Corollary 2.

11. Viscosity solutions

Viscosity solution is yet another notion of generalized solution for the free boundary problems. We begin with the definition of viscosity subsolutions [7,8,12] and [15]. Notice that the free boundary condition (7.5) can be rewritten as

$$\partial_e u^+ - \partial_{-e} u^- = \ell\big\langle \mathbf{v}(X), e\big\rangle.$$

In what follows e denotes the interior normal.

Remark 12. It is convenient to write the free boundary condition by means of the relation

$$\mathbb{S} = \mathcal{G}(\mathbb{T}, e, X), \tag{11.1}$$

where $\mathcal{G}(\mathbb{T}, e, X) = \mathbb{T} + \ell\langle \mathbf{v}(X), e\rangle$. Note that $G(\mathbb{S}, \mathbb{T}, e, X) = \mathbb{S} - \mathcal{G}(\mathbb{T}, e, X) = \mathbb{S} - \mathbb{T} - \ell\langle \mathbf{v}(X), e\rangle$ is increasing in $\mathbb{S}$, decreasing in $\mathbb{T}$ and continuous in e and X and hence is an elliptic free boundary relation (see [12, p. 6]).

Now we give the definition of viscosity solutions.

Definition 5. Let u be a continuous function in $D \subset \mathcal{C}_L$. Then u is said to be a viscosity solution of the free boundary problem **(DP)**, if one of the following is true

1° u solves the equation $\Delta u = a\partial_z u$ in $\Omega^+(u) \cup \Omega^-(u)$, $a > 0$.

2° Along $\Gamma = \partial\{u>0\}$, u satisfies the free boundary condition in the following sense

2°a If $X_0 \in \Gamma = \partial\{u>0\}$ is a regular point from the right, with touching ball $B \subset \Omega^+(u)$, in B

$$u^+ \geqslant \mathbb{S}\langle X - X_0, \nu\rangle^+ + o\big(|X - X_0|\big), \quad \mathbb{S} > 0,$$

in $\complement B$

$$u^- \leqslant \mathbb{T}\langle X - X_0, \nu\rangle^- + o\big(|X - X_0|\big), \quad \mathbb{T} \geqslant 0,$$

with equality in every non-tangential domain in both cases, then

$$G(\mathbb{S}, \mathbb{T}, e, X) \leqslant 0.$$

2°b If $X_0 \in \Gamma = \partial\{u>0\}$ is a regular point from the left, with touching ball $B \subset \Omega^-(u)$, in B

$$u^- \geqslant \mathbb{T}\langle X - X_0, \nu\rangle^+ + o\big(|X - X_0|\big), \quad b > 0,$$

in $\complement B$

$$u^+ \leqslant \mathbb{S}\langle X - X_0, \nu\rangle^- + o\big(|X - X_0|\big), \quad \mathbb{S} \geqslant 0,$$

with equality in every non-tangential domain in both cases, then

$$G(\mathbb{S}, \mathbb{T}, e, X) \geqslant 0.$$

The regularity theory of the free boundary for viscosity solutions with $a = 0$ (i.e. when $\Delta u = 0$ in $\Omega^+(u) \cup \Omega^-(u)$) can be found in [7,8] and [12]. Subsequently, these results have been extended to more general class of elliptic operators (see [15] and references therein). In [15] it is shown that the Lipschitz free boundaries are smooth. In order to apply this result to our problem we need to prove that continuous weak solutions to **(DP)** are also viscosity solutions. To do so we will need an asymptotic development estimate for the solutions of $\Delta u = a\partial_z u$.

Definition 6. For $X_0 \in \partial D$, we say that X_0 is a regular point from the right (left) if there exists a ball $B \subset D$ ($B \subset \complement D$) and $X_0 \in B \cap \partial D$.

Lemma 13. *Let $u > 0$ be a continuous solution of $\Delta u = a\partial_z u$ in a domain Ω with touching ball B at $X_0 \in \partial\Omega$. Assume that u vanishes on $B_1(X_0) \cap \partial\Omega$. Then the following is true.*

1°a *If X_0 is regular from the right, with touching ball B, either near X_0, in B, u grows more than any linear function or it has the asymptotic development*

$$u(x) \geqslant \mathbb{S}\langle X - X_0, e\rangle + o\big(|X - X_0|\big) \tag{11.2}$$

with $\mathbb{S} > 0$, where e is the unit normal to ∂B at X_0, inward to Ω.

1°b *Moreover, if u is Lipschitz continuous in $B_1(X_0)$ then the equality holds in every non-tangential region.*

2°a *If X_0 is regular from the left, near X_0, then*

$$u(x) \leqslant \mathbb{T}\langle X - X_0, e\rangle^+ + o\big(|X - X_0|\big) \tag{11.3}$$

with $\mathbb{T} \geqslant 0$.

2°b *Moreover, equality holds in every non-tangential region.*

For the proof see Appendix A.

The proof of part **2**° of Theorem C follows from part **1**° and Theorem 1 in [15]. Thus it is enough to prove the following theorem.

Theorem 13. *If u is a continuous weak solution of (4.5) with $\mathbf{v} = \mathbf{e}_N$, $f = 0$ then u is also a viscosity solution in the sense of Definition 5.*

Proof. We need to verify the free boundary condition at the points regular either from the right or from the left. Let X_0 be a free boundary point and $B \subset \Omega^+(u)$ a touching ball at X_0. By Lemma 13, $\mathbb{S} > 0$.

First we suppose that u^- is non-degenerate, then the blow-up sequence of u at X_0, $u_k(x) = \frac{u(X_0 + r_k X)}{r_k}$, for any sequence $r_k \downarrow 0$, has a subsequence $j = j(k)$, that converges to a function u_0. Moreover, $\Delta u_0 = \operatorname{div}[\mathbf{v}(X_0)\beta_0(u_0)]$, by Lemma 10. In particular it follows that the blow-up limit u_0 is harmonic in $\{u_0 > 0\} \cup \{u_0 < 0\}$.

Since $B \subset \Omega^+(u)$, it follows $\{X \in \mathbb{R}^N, \langle e, X\rangle \geqslant 0\} \subset \Omega^+(u_0)$. On the other hand X_0 is regular from the right and $\mathbb{S} > 0$, thus it follows that u^+, u^- are non-degenerate. Then, by (9.2) with $\xi(X) = z$, we infer that $w^\pm = u^\pm e^{-\frac{az}{2}}$ are subharmonic functions. Furthermore in view of Lemma 10 $w_k(X) = e^{-\frac{a(z_0 + r_k z)}{2}} u_k(X)$ converges to $w_0(X) = e^{-\frac{az_0}{2}} u_0$. Thus by Theorem in [11] and Remark 11 the following limit exists

$$\lim_{r \to 0} \Phi\big(X_0, r_k, w^+, w^-\big) = \gamma > 0.$$

Since $u^\pm$ (and hence $w^\pm$) are non-degenerate we get $\gamma > 0$.

Because of the scale invariance of Φ we have

$$\Phi\big(sr_k, X_0, w^+, w^-\big) = \Phi\big(s, 0, w_k^+, w_k^-\big).$$

Letting $r_k \to 0$ we infer that $\Phi(s, 0, w_0^+, w_0^-) = \gamma$, for any $s > 0$. Thus by Theorem 9, $\operatorname{supp}(w^\pm) \cap B_r$ are spherical caps. Since $\{X \in \mathbb{R}^N, \langle e, X\rangle \geqslant 0\} \subset \Omega^+(u_0)$ it follows that the spherical caps are fixed half spheres modulo scaling and that the free boundary of w_0 is the hyperplane $\Pi = \{X \in \mathbb{R}^N, \langle e, X\rangle = 0\}$. Clearly the free boundary of u_0 is the same hyperplane Π, because $w_0 = u_0 e^{-\frac{az_0}{2}}$. Finally recalling that u_0 solves the equation $\Delta u_0 = \operatorname{div}[\mathbf{v}(X_0)\beta_0(u_0)]$ we conclude that the free boundary condition (7.5) is satisfied in the classical sense.

Now suppose that u^- is degenerate. Let $u_0 \geqslant 0$ be a blow-up at X_0. Since by Lemma 13 the equality $u^+ = \mathbb{S}\langle X - X_0, e\rangle^+ + o(|X - X_0|)$ holds in any non-tangential region, then in $B_{\frac{\sqrt{2}}{2}}(e)$, we have $u_0 = \mathbb{S}\langle X, e\rangle$ and u_0 is harmonic in $\{u_0 > 0\}$. Let $U(X) = u_0(X) - \mathbb{S}\langle X, e\rangle$ then $U = 0$ in $B_{\frac{\sqrt{2}}{2}}(e)$ and U is harmonic in the half space $\{X \in \mathbb{R}^N, \langle X, e\rangle \geqslant 0\}$. This implies that $U \equiv 0$ in $\{X \in \mathbb{R}^N, \langle X, e\rangle \geqslant 0\}$ and hence $u_0 = \mathbb{S}\langle X, e\rangle$ in $\{X \in \mathbb{R}^N, \langle X, e\rangle \geqslant 0\}$.

If $\partial\{u_0 > 0\} = \Pi$ then we are done. Otherwise let $Y_0 \in \Pi = \{X \in \mathbb{R}^N, \langle X, e\rangle = 0\}$ and $Y_0 \neq 0$. Choose $r > 0$ so that $B_r(Y_0) \cap \partial\{u_0 > 0\} \subset \Pi$. Then writing the equation $\Delta u_0 = \operatorname{div}[\mathbf{e}_N \beta_0(u_0)]$ in weak form we get

$$\begin{aligned}
\int\limits_{B_r(Y_0)} Du_0 \cdot D\varphi &= \int\limits_{B_r(Y_0)} \beta_0(u_0)\partial_z\varphi \\
&= \int\limits_{B_r(Y_0)\cap\{X\in\mathbb{R}^N, \langle X,e\rangle>0\}} \ell\partial_z\varphi + \int\limits_{B_r(Y_0)\cap\{X\in\mathbb{R}^N, \langle X,e\rangle<0\}} \ell\partial_z\varphi \\
&= \int\limits_{B_r(Y_0)\cap\Pi} \ell\varphi\langle e, \mathbf{e}_N\rangle - \int\limits_{B_r(Y_0)\cap\Pi} \ell\varphi\langle e, \mathbf{e}_N\rangle \\
&= 0,
\end{aligned} \tag{11.4}$$

for all $\varphi \in C_0^\infty(B_r(Y_0))$. Therefore u_0 is harmonic in $B_r(Y_0)$ and the strong maximum principle gives $u_0 = 0$ in $B_r(Y_0)$. We see that $\partial\{u_0 > 0\}$ must be the hyperplane Π and hence the free boundary condition (7.5) for u_0 holds in the classical sense. In other words the blow-up limit at X_0 is unique and it is $\mathbb{S}\langle Y, e\rangle^+ - \mathbb{T}\langle Y, e\rangle^-$ with $\mathbb{S}, \mathbb{T}$ satisfying the free boundary condition (11.1). In fact we get that $\partial\{u > 0\}$ is flat at X_0.

Returning to u, we conclude that $u(X) = \mathbb{S}\langle (X - X_0), e\rangle^+ - \mathbb{T}\langle (X - X_0), e\rangle^- + o(|X - X_0|)$ near X_0. □

Conflict of interest statement

I declare that there is no conflict of interests.

Appendix A

Here we prove Lemma 13 which is a mild generalization of Lemma 11.17 in [12]. We decided to provide it for the sake of completeness.

First we establish the inequality (11.2). Without loss of generality we assume that $X_0 = 0$ and $e = \mathbf{e}_N$. Let $0 \in \partial\{u > 0\}$ and $B_R(Y_0) \subset \Omega$ be a touching ball at 0. For $C, \tau > 0$ we define

$$h(X) = C\big[\exp\big(-\tau|X|^2\big) - \exp\big(-\tau R^2\big)\big] = C\exp\big(-\tau R^2\big)\big[\exp\big(\tau\big(R^2 - |X|^2\big)\big) - 1\big].$$

Suppose that

$$\tau > \max\left(\frac{4N}{R^2}, \frac{4a}{R}\right), \qquad C < \frac{u(Re)}{c_0[\exp(-\frac{\tau R^2}{4}) - \exp(-\tau R^2)]} \tag{A.1}$$

where $c_0 > 0$ is the constant from Harnack's inequality (A.2). Then $h(X)$ can be used as a barrier to control u from above in $B_R(Y_0) \setminus B_{\frac{R}{2}}(Y_0)$. Indeed, we have

$$\begin{aligned}
\Delta h - a\partial_z h &= 2\tau C\exp\big(-\tau|X|^2\big)\big[2\tau|X|^2 - N + az\big] \\
&\geqslant 2\tau C\exp\big(-\tau|X|^2\big)\left[\frac{\tau R^2}{2} - N - aR\right] \\
&= 2\tau C\exp\big(-\tau|X|^2\big)\left[\frac{\tau R^2}{4} - N + R\left(\frac{\tau R}{4} - a\right)\right] \\
&> 0
\end{aligned}$$

provided that the first inequality in (A.1) holds.

On the other hand $h(X)=0\leqslant u(X)$ if $X\in\partial B_R(Y_0)$. From Harnack's inequality we have that

$$u(Re)\leqslant \max_{B_{\frac{R}{2}}(Y_0)} u\leqslant c_0 \min_{B_{\frac{R}{2}}(Y_0)} u. \tag{A.2}$$

In particular $u(Re)\leqslant c_0u(X)$ for any $X\in\overline{B_{\frac{R}{2}}(Y_0)}$. Thus for $X\in\partial B_{\frac{R}{2}}$ we have $h(X)=C\exp(-\frac{\tau R^2}{4})-\exp(-\tau R^2)<u(x)$ if the second inequality in (A.1) is satisfied. Therefore we infer from comparison principle that $u(X)\geqslant h(X)$ in $B_R(Y_0)\setminus B_{\frac{R}{2}}(Y_0)$. Notice that near the origin

$$h(X)=C(R)z+o\big(|X|\big) \quad \text{with } C(R)>0. \tag{A.3}$$

Let k_0 be the smallest positive integer such that $\frac{1}{2^{k_0}}\leqslant\frac{R}{2}$ and introduce

$$\alpha_0=\sup\big\{m\colon u(X)\geqslant mh(X) \text{ in } B_{2^{-k_0}}\cap B_R(Y_0)\big\}.$$

For $k=1,2,3,\dots$ we let

$$\alpha_k=\sup\big\{m\colon u(X)\geqslant mh(X) \text{ in } B_{2^{-(k_0+k)}}\cap B_R(Y_0)\big\}.$$

Note that $\{\alpha_k\}$ increases and put $\alpha=\sup\alpha_k$. From $u(X)\geqslant h(X)$, $X\in\overline{B_R(Y_0)\setminus B_{\frac{R}{2}}(Y_0)}$ it follows that $\alpha_0>0$ hence $\alpha>0$. If $\sup\alpha_k=\infty$ then u grows faster than any linear function. Otherwise taking $\widetilde{\alpha}=\alpha C(R)$ and recalling (A.3) we get (11.2) with $\mathbb{S}=\widetilde{\alpha}$.

Now we prove part **1°b** of Lemma 13. Clearly if u is Lipschitz continuous then $\alpha<\infty$. To show the equality in non-tangential domains we argue towards a contradiction. Suppose that there is a sequence $X^k\in B_R(Y_0)$ and $\delta_0>0$ such that

$$u\big(X^k\big)>\widetilde{\alpha}z^k+\delta_0\big|X^k\big|, \quad \big|X^k\big|=r_k\sim \operatorname{dist}\big(X^k,\partial B_R(Y_0)\big). \tag{A.4}$$

From Harnack's inequality we have that $u(X)-\widetilde{\alpha}z\geqslant c_0\delta_0$ on some fixed portion of $\partial B_{r_k}\subset B_R(Y_0)$ since X^k approaches $\partial B_R(Y_0)$ in non-tangential fashion.

Consider the scaled function $u_k(X)=\frac{u(r_kX)}{r_k}$. Since u is Lipschitz continuous it follows that, for a subsequence k_j,

$$u_{k_j}\to u_0 \tag{A.5}$$

uniformly to some non-negative harmonic function $u_0\geqslant 0$ defined in the half space $\{X\in\mathbb{R}^N\colon e\cdot X=z\geqslant 0\}$. By construction we have

$$u_0(X)-\widetilde{\alpha}z\geqslant 0, \quad \text{in } \langle X,e\rangle\geqslant 0. \tag{A.6}$$

Furthermore, from (A.4) we conclude that there is $\overline{X}=(\bar{x},\bar{z})\in\partial B_1$ such that $\bar{z}>0$ and $u_0(\overline{X})-\widetilde{\alpha}\bar{z}\geqslant\frac{c_0\delta_0}{2}$. This in conjunction with (A.6) and Harnack's inequality implies that there is a small $s>0$ such that $\overline{B_s(e)}\subset\{X\in\mathbb{R}^N\colon z>0\}$ and

$$u_0(X)-\widetilde{\alpha}z\geqslant\frac{c_0\delta_0}{100} \quad \text{in } B_s(e). \tag{A.7}$$

Let w be the solution of

$$\begin{cases} \Delta w=0 & \text{in } B_1(e)\setminus B_s(e),\\ h=\frac{c_0\delta_0}{200} & \text{on } \partial B_s(e),\\ h=0 & \text{on } \partial B_1(e). \end{cases} \tag{A.8}$$

and

$$\begin{cases} \Delta w_k=r_ka\|Du\|_\infty & \text{in } B_1(e)\setminus B_s(e),\\ w_k=-(u_k-\widetilde{\alpha}z)^-+w & \text{on } \partial(B_1(e)\setminus B_s(e)). \end{cases} \tag{A.9}$$

It is easy to check that $u_k-\widetilde{\alpha}z\geqslant w_k$ on $\partial(B_1(e)\setminus B_s(e))$ for sufficiently large k. To see this it is enough to show that $(u_k-\widetilde{\alpha}z)^+\geqslant w$ on $\partial(B_1(e)\setminus B_s(e))$. From (A.7) and the uniform convergence $u_k\to u_0$ we infer that $u_k(X)-\widetilde{\alpha}z\geqslant\frac{c_0\delta_0}{200}$ in $B_s(e)$ for any sufficiently large k. Hence on $\partial B_s(e)$ we have that $(u_k-\widetilde{\alpha}z)^+=(u_k-\widetilde{\alpha}z)\geqslant\frac{c_0\delta_0}{200}=w$. As for $\partial B_1(e)$ we see that there $(u_k-\widetilde{\alpha}z)^+\geqslant 0=w$. Now we can apply the comparison principle to conclude $u_k-\widetilde{\alpha}z\geqslant w_k$ in $B_1(e)\setminus B_s(e)$.

From the regularity theory of elliptic PDEs we know that $w, w_k \in C^3(\overline{B_1(e) \setminus B_s(e)})$. Furthermore, by strong maximum principle $0 \leqslant w \leqslant \frac{c_0\delta_0}{200}$. Note that because w is C^3 smooth near 0 we have

$$w(X) = C_1 z + o(|X|) \geqslant \frac{C_1}{2} z \tag{A.10}$$

where C_1 is a tame constant. Notice that $C_1 > 0$ which follows from Hopf's lemma.

Finally we show that on $B_1(e) \setminus B_s(e)$, $w(X) - w_k(X)$ converges to zero uniformly in the Lipschitz norm. Combining (A.5), (A.6), (A.8) and (A.9) we conclude that $w_k \to w$ uniformly in $\overline{B_1(e) \setminus B_s(e)}$. Recalling that $B_R(Y_0)$ is a touching ball at $X_0 = 0$ and (A.6), it follows that there is a small $t > 0$ such that $w_k = w = 0$ on $\partial B_1(e) \cap B_t$ for sufficiently large k. Thereby we conclude that

$$Dw_k \to Dw, \quad \text{uniformly in } B_t \cap B_1(e).$$

On the other hand from (A.10) we obtain

$$u_k(X) - \widetilde{\alpha} z \geqslant w_k = w + (w_k - w) \geqslant \frac{C_1}{4} z$$

if k is large. This is in contradiction with the definition of α since returning to u we get $u(Y) \geqslant (\widetilde{\alpha} + \frac{C_1}{4})z$ in B_{r_k}. This finishes the proof of part **1**° of Lemma 13.

Now we turn to part **2**°, i.e. when $B_R(Y_0)$ touches X_0 from outside. Let $\eta(X)$ be the solution of the Dirichlet problem $\Delta\eta(X) = a\partial_z\eta(X)$ in $B_{2R}(Y_0) \setminus B_R(Y_0)$ such that $\eta = 0$ on $\partial B_R(Y_0)$ and $\eta = \max_{\partial B_{2R}(y_0)} u$ on $\partial B_{2R}(Y_0)$. From comparison principle we have that $u(X) \leqslant \eta(X)$ in $B_{2R}(Y_0) \cap \Omega$. Since $\eta \in C^3(\overline{B_{2R}(Y_0) \setminus B_R(Y_0)})$ it follows from Hopf's principle that $\eta(X) = C(R)z + o(|X|)$ near the origin with $C(R) > 0$.

If k_0 is the smallest positive integer such that $\frac{1}{2^{k_0}} < \frac{R}{2}$ we can define

$$\gamma_0 = \inf\{m: m\eta(X) \geqslant u(X) \text{ in } \complement B_R(Y_0) \cap B_{2^{-k_0}}\}.$$

Now for $k = 1, 2, 3, \ldots$ we define

$$\gamma_k = \inf\{m: m\eta(X) \geqslant u(X) \text{ in } \complement B_R(Y_0) \cap B_{2^{-(k_0+k)}}\}.$$

Clearly $\{\gamma_k\}$ decreases. Let $\gamma = \inf \gamma_k$. Then $\gamma \geqslant 0$ and if $\mathbb{T} = \gamma C(R)$ we have near 0,

$$u(X) \leqslant \mathbb{T} z^+ + o(|X|). \tag{A.11}$$

For the proof that equality holds in (A.11) inside every non-tangential region one can proceed as for the equality (11.2). □

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