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On some large global solutions to 3-D density-dependent Navier–Stokes system with slow variable: Well-prepared data

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Abstract

In this paper, we consider the global wellposedness of 3-D incompressible inhomogeneous Navier–Stokes equations with initial data slowly varying in the vertical variable, that is, initial data of the form $(1+\varepsilon^\sigma a_0(x_h,\varepsilon x_3), (\varepsilon u_0^h(x_h,\varepsilon x_3), u_0^3(x_h,\varepsilon x_3)))$ for some $\sigma>0$ and ε being sufficiently small. We remark that initial data of this type does not satisfy the smallness conditions in [11,18] no matter how small ε is.

MSC: 35Q30; 76D03

Keywords: Inhomogeneous Navier–Stokes equations; Littlewood–Paley theory; Wellposedness

1. Introduction

In this paper, we investigate the existence of some large global solutions to the following 3-D incompressible inhomogeneous Navier–Stokes equations with initial data slowly varying in one space variable:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) - \Delta u + \nabla\Pi = 0, \\ \operatorname{div} u = 0, \\ \rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = m_0, \end{cases} \tag{1.1}$$

where $\rho, u=(u_1,u_2,u_3)$ stand for the density and velocity of the fluid respectively, Π is a scalar pressure function. Such a system describes a fluid which is obtained by mixing two immiscible fluids that are incompressible and that

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have different densities. It may also describe a fluid containing a melted substance. One may check [20] for more background of this system.

When the initial density has a positive lower bound, Ladyženskaja and Solonnikov [19] first established the unique resolvability of (1.1) in bounded domain Ω with homogeneous Dirichlet boundary condition for u. Similar results were obtained by Danchin [14] in $\mathbb{R}^d$ with initial data in the almost critical Sobolev spaces. Simon [24] proved the global existence of weak solutions of (1.1) with finite energy (see also the book by Lions [20] and the references therein). Abidi, Gui and Zhang [3] investigated the large time decay and stability to any given global smooth solutions of (1.1).

When the initial density is away from zero, we denote $a \stackrel{\text{def}}{=} \frac{1}{\rho} - 1$, and then (1.1) can be equivalently reformulated as

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u + (1+a)(\nabla \Pi - \Delta u) = 0, \\ \operatorname{div} u = 0, \\ (a,u)|_{t=0} = (a_0, u_0). \end{cases} \tag{1.2}$$

Notice that just as the classical Navier–Stokes system (which corresponds to the case when $a = 0$ in (1.2)), the inhomogeneous Navier–Stokes system (1.2) also has a scaling. Indeed if (a,u) solves (1.2) with initial data (a_0, u_0), then for $\forall \ell > 0$,

$$(a,u)_\ell \stackrel{\text{def}}{=} \big(a(\ell^2 \cdot, \ell \cdot), \ell u(\ell^2 \cdot, \ell \cdot)\big) \quad \text{and} \quad (a_0, u_0)_\ell \stackrel{\text{def}}{=} \big(a_0(\ell \cdot), \ell u_0(\ell \cdot)\big) \tag{1.3}$$

$(a,u)_\ell$ is also a solution of (1.2) with initial data $(a_0, u_0)_\ell$.

In [13], Danchin studied in general space dimension d the unique solvability of the system (1.2) with initial data being small in the scaling invariant (or critical) homogeneous Besov spaces. This result was extended to more general Besov spaces by Abidi in [1], and by Abidi, Paicu in [2]. The smallness assumption on the initial density was removed in [4,5].

Very recently, Danchin and Mucha [16] noticed that it was possible to establish existence *and* uniqueness of a solution to (1.1) in the case of a small discontinuity for the initial density and in a critical functional framework. More precisely, the global existence and uniqueness was established for any data (ρ_0, u_0) such that for some $p \in [1, 2d)$ and small enough constant c, we have

$$\|\rho_0 - 1\|_{\mathcal{M}(B_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d))} + \|u_0\|_{B_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d)} \le c. \tag{1.4}$$

Above, $\mathcal{M}(B_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d))$ denotes the multiplier space of $B_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d)$, which is the set of distributions a such that ψa is in $B_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ whenever ψ is in $B_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d)$, and the norm of which is determined by

$$\|a\|_{\mathcal{M}(B_{p,1}^{-1+\frac{d}{p}})} \stackrel{\text{def}}{=} \sup_{\|\psi\|_{B_{p,1}^{-1+\frac{d}{p}}}=1} \|\psi a\|_{B_{p,1}^{-1+\frac{d}{p}}}.$$

On the other hand, motivated by results concerning the global wellposedness of 3-D incompressible anisotropic Navier–Stokes system with the third component of the initial velocity field being large (see for instance [22]), we [23] relaxed the smallness condition in [2] so that (1.2) still has a unique global solution provided that

$$\Big(\|a_0\|_{B_{p,1}^{\frac{3}{p}}} + \big\|u_0^h\big\|_{B_{p,1}^{-1+\frac{3}{p}}}\Big) \exp\Big(C_0 \big\|u_0^3\big\|_{B_{p,1}^{-1+\frac{3}{p}}}^2\Big) \le c_0 \tag{1.5}$$

for some c_0 sufficiently small and $p \in (1,6)$. This smallness condition (1.5) was improved by Huang and the authors of this paper [18] to

$$\Big(\|a_0\|_{L^\infty} + \big\|u_0^h\big\|_{B_{p,r}^{-1+\frac{d}{p}}}\Big) \exp\Big(C_r \big\|u_0^d\big\|_{B_{p,r}^{-1+\frac{d}{p}}}^{2r}\Big) \le c_0 \tag{1.6}$$

for some $p \in]1, d[$, $r \in]1, \infty[$ and in general d space dimension.

Before going further, we recall the functional space framework we are going to use in what follows. As in [8,12,21], the definitions of the spaces we are going to work with requires anisotropic dyadic decomposition of the Fourier variables. Let us recall from [6] that

$$\begin{aligned}
&\Delta_k^h a=\mathcal{F}^{-1}\big(\varphi\big(2^{-k}|\xi_h|\big)\widehat{a}\big), &&\Delta_\ell^v a=\mathcal{F}^{-1}\big(\varphi\big(2^{-\ell}|\xi_3|\big)\widehat{a}\big),\\
&S_k^h a=\mathcal{F}^{-1}\big(\chi\big(2^{-k}|\xi_h|\big)\widehat{a}\big), &&S_\ell^v a=\mathcal{F}^{-1}\big(\chi\big(2^{-\ell}|\xi_3|\big)\widehat{a}\big)\quad\text{and}\\
&\Delta_j a=\mathcal{F}^{-1}\big(\varphi\big(2^{-j}|\xi|\big)\widehat{a}\big), &&S_j a=\mathcal{F}^{-1}\big(\chi\big(2^{-j}|\xi|\big)\widehat{a}\big),
\end{aligned}\tag{1.7}$$

where $\xi_h=(\xi_1,\xi_2)$, $\mathcal{F}a$ and $\widehat{a}$ denote the Fourier transform of the distribution a, $\chi(\tau)$ and $\varphi(\tau)$ are smooth functions such that

$$\operatorname{Supp}\varphi\subset\left\{\tau\in\mathbb{R}/\frac{3}{4}\le|\tau|\le\frac{8}{3}\right\}\quad\text{and}\quad\forall\tau>0,\quad\sum_{j\in\mathbb{Z}}\varphi\big(2^{-j}\tau\big)=1,$$

$$\operatorname{Supp}\chi\subset\left\{\tau\in\mathbb{R}/|\tau|\le\frac{4}{3}\right\}\quad\text{and}\quad\chi(\tau)+\sum_{j\ge 0}\varphi\big(2^{-j}\tau\big)=1.$$

Definition 1.1. Let $(p,r)\in[1,+\infty]^2$, $s\in\mathbb{R}$ and $u\in\mathcal{S}_h'(\mathbb{R}^3)$, which means that $u\in\mathcal{S}'(\mathbb{R}^3)$ and $\lim_{j\to-\infty}\|S_j u\|_{L^\infty}=0$, we set

$$\|u\|_{B^s_{p,r}}\stackrel{\text{def}}{=}\left(\sum_{j\in\mathbb{Z}}2^{rjs}\|\Delta_j u\|_{L^p}^r\right)^{\frac{1}{r}}.$$

- For $s<\frac{3}{p}$ (or $s=\frac{3}{p}$ if $r=1$), we define $B^s_{p,r}(\mathbb{R}^3)\stackrel{\text{def}}{=}\{u\in\mathcal{S}_h'(\mathbb{R}^3)\mid\|u\|_{B^s_{p,r}}<\infty\}$.
- If $k\in\mathbb{N}$ and $\frac{3}{p}+k\le s<\frac{3}{p}+k+1$ (or $s=\frac{3}{p}+k+1$ if $r=1$), then $B^s_{p,r}(\mathbb{R}^3)$ is defined as the subset of distributions $u\in\mathcal{S}_h'(\mathbb{R}^3)$ such that $\partial^\beta u\in B^{s-k}_{p,r}(\mathbb{R}^3)$ whenever $|\beta|=k$.

Notations. In all that follows, we shall denote

$$\mathcal{B}^s_p\stackrel{\text{def}}{=}B^s_{p,1}\big(\mathbb{R}^3\big).$$

Definition 1.2. Let p be in $[1,+\infty]$, $s_1\le\frac{2}{p}$, $s_2\le\frac{1}{p}$ and u in $\mathcal{S}_h'(\mathbb{R}^3)$, we set

$$\|u\|_{\mathfrak{B}_p^{s_1,s_2}}\stackrel{\text{def}}{=}\sum_{k,\ell\in\mathbb{Z}^2}2^{ks_1}2^{\ell s_2}\big\|\Delta_k^h\Delta_\ell^v u\big\|_{L^p}.$$

The case when $s_1>\frac{2}{p}$ or $s_2>\frac{1}{p}$ can be similarly modified as that in Definition 1.1.

Motivated by the study to the global wellposedness of 3-D classical Navier–Stokes system by Chemin, Gallagher [9] and by Chemin, Gallagher and the first author of this paper [10] with some large initial data slowly varying in one direction, Chemin and the authors of this paper [11] proved the same type of global wellposedness result as that in [23] but with the smallness condition being formulated in critical anisotropic Besov spaces for the initial velocity field and initial density in the usual isotropic Besov spaces. We should point out that there is tremendous difficulty in propagating anisotropic regularity for the transport equation.

More precisely, the following theorems were proved in [11]:

Theorem 1.1. *(See Theorem 1.2 of [11].) Let p be in $]3,4[$ and r in $[p,6[$. Let us consider an initial data (a_0,u_0) in the space $\mathcal{B}_p^{\frac{3}{p}}\times(\mathfrak{B}_p^{-1+\frac{2}{p},\frac{1}{p}}\cap\mathcal{B}_r^{-1+\frac{3}{r}})$. Then there exist positive constants c_0 and C_0 such that if*

$$\eta\stackrel{\text{def}}{=}\big(\|a_0\|_{\mathcal{B}_p^{\frac{3}{p}}}+\big\|u_0^h\big\|_{\mathfrak{B}_p^{-1+\frac{2}{p},\frac{1}{p}}}\big)\exp\big(C_0\big\|u_0^3\big\|^2_{\mathfrak{B}_p^{-1+\frac{2}{p},\frac{1}{p}}}\big)\le c_0,\tag{1.8}$$

the system (1.2) has a unique global solution

$$a \in \mathcal{C}_b\big([0,\infty); \mathcal{B}_p^{\frac{3}{p}}(\mathbb{R}^3)\big) \quad \text{and} \quad u \in \mathcal{C}_b\big([0,\infty); \mathcal{B}_r^{-1+\frac{3}{r}}(\mathbb{R}^3)\big) \cap L^1\big(\mathbb{R}^+; \mathcal{B}_r^{1+\frac{3}{r}}(\mathbb{R}^3)\big). \tag{1.9}$$

Moreover, there hold

$$\begin{aligned}
&\|u^h\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathfrak{B}_p^{-1+\frac{2}{p},\frac{1}{p}})} + \|a\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathcal{B}_p^{\frac{3}{p}})} + \sum_{i,j=1}^{3} \|\partial_i\partial_j u^h\|_{L^1(\mathbb{R}^+;\mathfrak{B}_p^{-1+\frac{2}{p},\frac{1}{p}})} \leq C\eta, \\
&\|u^3\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathfrak{B}_p^{-1+\frac{2}{p},\frac{1}{p}})} + \sum_{i,j=1}^{3} \|\partial_i\partial_j u^3\|_{L^1(\mathbb{R}^+;\mathfrak{B}_p^{-1+\frac{2}{p},\frac{1}{p}})} \leq 2\|u_0^3\|_{\mathfrak{B}_p^{-1+\frac{2}{p},\frac{1}{p}}} + c_2.
\end{aligned} \tag{1.10}$$

Theorem 1.2. *(See Theorem 1.3 of [11].) Let σ be a real number greater than $1/4$ and a_0 a function of $\mathcal{B}_p^{\frac{3}{p}} \cap \mathcal{B}_q^{-1+\frac{3}{q}}$ for some p in $]3,4[$ and q in $]\frac{3}{2},2[$. Let $v_0^h = (v_0^1, v_0^2)$ be a horizontal, smooth divergence free vector field on $\mathbb{R}^3$, belonging, as well as all its derivatives, to $L^2(\mathbb{R}_{x_3}; \dot{H}^{-1}(\mathbb{R}^2))$. Furthermore, we assume that for any α in $\mathbb{N}^3$, $\partial^\alpha\partial_3 v_0^h$ belongs to $\mathfrak{B}_2^{-1,\frac{1}{2}}(\mathbb{R}^3)$. Then there exists a positive ε_0 such that if $\varepsilon \leq \varepsilon_0$, the initial data*

$$a_0^\varepsilon(x) = \varepsilon^\sigma a_0(x_h, \varepsilon x_3), \qquad u_0^\varepsilon(x) = \big(v_0^h(x_h, \varepsilon x_3), 0\big) \tag{1.11}$$

generates a unique global solution $(a^\varepsilon, u^\varepsilon)$ of (1.2).

Let us remark that Theorem 1.1 implies the global wellposedness of (1.2) with initial data of the form:

$$\big(a_0(x_h, x_3), \big(\varepsilon u_0^h(x_h, \varepsilon x_3), u_0^3(x_h, \varepsilon x_3)\big)\big)$$

for any smooth divergence free vector field $u_0 = (u_0^h, u_0^3)$ and with ε, $\|a_0\|_{\mathcal{B}_p^{\frac{3}{p}}}$, for some p in $]3,4[$, being sufficiently small.

We also mention that Gui, Huang and the second author of this paper [17] proved that: given $a_0 \in W^{1,p} \cap H^2$ for some $p \in (1,2)$ and $\sigma > \frac{1}{p}$, v_0^h as in Theorem 1.2 and some smooth divergence free vector field $w_0 = (w_0^h, w_0^3)$, (1.2) has a unique global solution with initial data

$$a_0^\varepsilon(x) = \varepsilon^\sigma a_0(x_h, \varepsilon x_3) \quad \text{and} \quad u_0^\varepsilon(x) = \big(v_0^h + \varepsilon w_0^h, w_0^3\big)(x_h, \varepsilon x_3) \tag{1.12}$$

provided that ε is sufficiently small.

The purpose of this paper is to decrease the value of σ in (1.11) and (1.12) when the initial velocity is a sort of well-prepared data with one slow variable.

Our main result in this paper can be stated as follows:

Theorem 1.3. *Let $a_0 \in H^2$ and $u_0 \in \mathfrak{B}_2^{0,\frac{1}{2}} \cap H^2$ with $\operatorname{div} u_0 = 0$. Then for any $\sigma \in]0, 1/2[$, there exist some positive constants C_0 and $c_0 > 0$ so that if*

$$\varepsilon \exp\big(C_0\|u_0\|^2_{\mathfrak{B}_2^{0,\frac{1}{2}}}\big) + \varepsilon^{\frac{\sigma}{2}}\|a_0\|_{\mathcal{B}_2^{\frac{3}{2}}} \exp\big(C_0 \exp\big(C_0\|u_0\|^2_{\mathfrak{B}_2^{0,\frac{1}{2}}}\big)\big) \leq c_0, \tag{1.13}$$

(1.2) with initial data

$$a_0^\varepsilon(x) = \varepsilon^\sigma a_0(x_h, \varepsilon x_3), \qquad u_0^\varepsilon(x) = \big(\varepsilon u_0^h(x_h, \varepsilon x_3), u_0^3(x_h, \varepsilon x_3)\big) \tag{1.14}$$

has a unique global solution

$$a \in \mathcal{C}\big([0,\infty); H^2(\mathbb{R}^3)\big) \qquad u \in \mathcal{C}_b\big([0,\infty); H^2(\mathbb{R}^3)\big) \cap L^2\big(\mathbb{R}^+; \dot{H}^3(\mathbb{R}^3)\big). \tag{1.15}$$

Moreover, there hold

$$\|u^h\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathfrak{B}_2^{0,\frac{1}{2}})}+\sum_{i,j=1}^3\|\partial_i\partial_j u^h\|_{L^1(\mathbb{R}^+;\mathfrak{B}_2^{0,\frac{1}{2}})}\le C\varepsilon\exp\bigl(C\|u_0\|^2_{\mathfrak{B}_2^{0,\frac{1}{2}}}\bigr),$$
$$\|u^3\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathfrak{B}_2^{0,\frac{1}{2}})}+\sum_{i,j=1}^3\|\partial_i\partial_j u^3\|_{L^1(\mathbb{R}^+;\mathfrak{B}_2^{0,\frac{1}{2}})}\le 4\bigl(\varepsilon+\|u_0\|_{\mathfrak{B}_2^{0,\frac{1}{2}}}\bigr). \tag{1.16}$$

Let us point out that as

$$\|u_0(x_h,\varepsilon x_3)\|_{B_{p,r}^{-1+\frac{3}{p}}}\ge C\varepsilon^{-\frac{1}{p}}\quad\text{and}\quad \varepsilon^\sigma\|a_0(x_h,\varepsilon x_3)\|_{\mathcal{B}_q^{\frac{3}{q}}}\ge C\varepsilon^{\sigma-\frac{1}{q}},$$

initial data like (1.14) does not satisfy either the smallness condition (1.6) nor that in (1.8).

We also mention that by combining the method in this paper with that in [4], we can improve the regularity of the initial data in Theorem 1.3 to be the critical one. For simplicity, we shall not pursue this direction here.

The organization of this paper is as follows:

In the second section, we first prove the estimate for the free transport equation with convection velocity in some anisotropic Besov type space, we then prove the related estimates for the pressure term.

In the third section, we prove Theorem 1.3.

In Appendix A, we collect some basic facts on Littlewood–Paley theory which has been used throughout this paper.

1.1. Scheme of the proof and organization of the paper

Again due to the difficulty of propagating anisotropic regularity for the transport equation as we mentioned before, motivated by [10], we re-scale the unknowns as

$$a(t,x)\stackrel{\text{def}}{=}\varepsilon^\sigma b(t,x_h,\varepsilon x_3),\qquad u^h(t,x)\stackrel{\text{def}}{=}\varepsilon v^h(t,x_h,\varepsilon x_3),\qquad u^3(t,x)\stackrel{\text{def}}{=}v^3(t,x_h,\varepsilon x_3). \tag{1.17}$$

Let us denote

$$\nabla_\varepsilon\stackrel{\text{def}}{=}(\nabla_h,\varepsilon\partial_3)\quad\text{and}\quad\Delta_\varepsilon=\Delta_h+\varepsilon^2\partial_3^2.$$

Then (b,v) verifies

$$\begin{cases}\partial_t b+\varepsilon v\cdot\nabla b=0,\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\ \partial_t v^h+\varepsilon v\cdot\nabla v^h+\bigl(1+\varepsilon^\sigma b\bigr)\bigl(\nabla_h\Pi-\Delta_\varepsilon v^h\bigr)=0,\\ \partial_t v^3+\varepsilon v\cdot\nabla v^3+\bigl(1+\varepsilon^\sigma b\bigr)\bigl(\varepsilon^2\partial_3\Pi-\Delta_\varepsilon v^3\bigr)=0,\\ \operatorname{div} v=0,\\ (b,v)|_{t=0}=(a_0,u_0).\end{cases} \tag{1.18}$$

The advantage of this formulation is that there is no slow variable for a_0. However, the estimate to the pressure function turns out to be a big difficulty. As a matter of fact, by taking space divergence to (1.18) and using $\operatorname{div} v=0$, we get

$$\begin{aligned}\nabla_\varepsilon\Pi&=\varepsilon^\sigma\nabla_\varepsilon(-\Delta_\varepsilon)^{-1}\bigl(\operatorname{div}_h(b\nabla_h\Pi)+\varepsilon\partial_3(b\varepsilon\partial_3\Pi)\bigr)+\varepsilon\nabla_\varepsilon(-\Delta_\varepsilon)^{-1}\operatorname{div}_h\operatorname{div}_h\bigl(v^h\otimes v^h\bigr)\\&\quad+\varepsilon\nabla_\varepsilon(-\Delta_\varepsilon)^{-1}\operatorname{div}_h\partial_3\bigl(v^3v^h\bigr)-2\varepsilon\nabla_\varepsilon(-\Delta_\varepsilon)^{-1}\partial_3\bigl(v^3\operatorname{div}_h v^h\bigr)\\&\quad-\varepsilon^\sigma\nabla_\varepsilon(-\Delta_\varepsilon)^{-1}\operatorname{div}_h\bigl(b\Delta_\varepsilon v^h\bigr)-\varepsilon^\sigma\nabla_\varepsilon(-\Delta_\varepsilon)^{-1}\partial_3\bigl(b\Delta_\varepsilon v^3\bigr).\end{aligned} \tag{1.19}$$

Notice that the last term above is of order $\varepsilon^{\sigma-1}$, which looks like the problem of the classical Navier–Stokes system with ill-prepared data slowly varying in one direction, where the authors [11] require the analyticity assumption for the third variable of u_0 in order to prove the global wellposedness result. Our new observation here is to write

$$\varepsilon^\sigma\nabla_\varepsilon(-\Delta_\varepsilon)^{-1}\partial_3\bigl(b\Delta_\varepsilon v^3\bigr)=\varepsilon^\sigma\nabla_\varepsilon(-\Delta_\varepsilon)^{-1}\bigl(\partial_3 b\Delta_\varepsilon v^3\bigr)-\varepsilon^\sigma\nabla_\varepsilon(-\Delta_\varepsilon)^{-1}\bigl(b\Delta_\varepsilon\operatorname{div}_h v^h\bigr),$$

which together with the law of product forces us to consider the propagation of regularity for the transport equation of (1.18) in $\mathcal{B}_q^s$ for some $s>1$. To overcome this difficulty, we will use a completely different argument than that used in the proof of Theorem 1.1 in [11], where we only need to propagate $\mathcal{B}_q^s$ with $s\in\,]0,1[$ for the transport equation.

Considering the strong anisotropic property of the system (1.18), we shall first estimate v in some critical anisotropic Besov spaces. Then by carefully choosing the norm for the convection velocity in the transport equation of (1.18), we succeed in propagating the isentropic regularity in $\mathcal{B}_2^{\frac{3}{2}}$ for b.

Let us complete this section by the notations of the paper: Let A, B be two operators, we denote $[A; B] = AB - BA$, the commutator between A and B. For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \le Cb$, $L_T^r(L_h^p(L_v^q))$ stands for the space $L^r([0,T]; L^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}; L^q(\mathbb{R}_{x_3})))$. We denote by $(a|b)$ the $L^2(\mathbb{R}^3)$ inner product of a and b, $(d_j)_{j\in\mathbb{Z}}$ (resp. $(d_{j,k})_{j,k\in\mathbb{Z}^2}$) will be a generic element of $\ell^1(\mathbb{Z})$ (resp. $\ell^1(\mathbb{Z}^2)$) so that $\sum_{j\in\mathbb{Z}} d_j = 1$ (resp. $\sum_{j,k\in\mathbb{Z}^2} d_{j,k} = 1$).

For X a Banach space and I an interval of $\mathbb{R}$, we denote by $\mathcal{C}(I; X)$ the set of continuous functions on I with values in X, and $L^q(I; X)$ stands for the set of measurable functions on I with values in X, such that $t \longmapsto \|f(t)\|_X$ belongs to $L^q(I)$.

2. Preliminary estimates

In this section, we shall first present a result concerning the propagating of isentropic regularity for the transport equation with the convection velocity in some appropriate anisotropic space. We shall also provide the estimates for the pressure function in (1.18).

Lemma 2.1. *Let $b_0 \in \mathcal{B}_2^{\frac{3}{2}}$, v satisfy $\int_0^T \|\nabla v(t)\|_{\mathbf{L}}\, dt < \infty$ with $v = v_{\mathfrak{a}} + v_{\mathfrak{b}}$ and*

$$\|\nabla v(t)\|_{\mathbf{L}} \overset{\text{def}}{=} \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \big(\|\Delta_j \nabla_h v_{\mathfrak{a}}(t)\|_{L_v^2(L_h^\infty)} + \|\Delta_j \nabla_h v_{\mathfrak{b}}(t)\|_{L_h^4(L_v^\infty)} + \|\Delta_j \partial_3 v(t)\|_{L_v^2(L_h^\infty)} \big). \tag{2.1}$$

Then the following transport equation

$$\begin{cases} \partial_t b + v \cdot \nabla b = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ b|_{t=0} = b_0 \end{cases} \tag{2.2}$$

has a unique solution $b \in \mathcal{C}([0,T]; \mathcal{B}_2^{\frac{3}{2}})$, which satisfies for $0 \le t \le T$

$$\|b\|_{\widetilde{L}_t^\infty(\mathcal{B}_2^{\frac{3}{2}})} \le \|b_0\|_{\mathcal{B}_2^{\frac{3}{2}}} \exp\bigg(C \int_0^t \|\nabla v(t')\|_{\mathbf{L}}\, dt' \bigg), \tag{2.3}$$

where the norm $\|\cdot\|_{\widetilde{L}_t^\infty(\mathcal{B}_2^{\frac{3}{2}})}$ is given by Definition A.1 in Appendix A.

Remark 2.1. It is easy to observe from (2.1) and Lemma A.1 that for all $f \in \mathcal{B}_2^{\frac{3}{2}}(\mathbb{R}^3)$, one has

$$\|\nabla f\|_{L^\infty} \lesssim \|\nabla f\|_{\mathbf{L}} \lesssim \|\nabla f\|_{\mathcal{B}_2^{\frac{3}{2}}}.$$

Since the diffusion terms in (1.18) is Δ_ε, which is anisotropic in the horizontal and vertical variables, we shall derive the *a priori* estimate of the convection velocity v in (2.2) in the anisotropic Besov spaces, which prevents us from getting the $L^1(\mathbb{R}^+; \mathcal{B}_2^{\frac{3}{2}}(\mathbb{R}^3))$ estimate for the gradient of the velocity v. That is the main reason why we choose to work with the more complicated norm $\|\cdot\|_{\mathbf{L}}$ given by (2.1).

Proof. For simplicity, we just prove the *a priori* Estimate (2.3) for smooth enough solutions of (2.2). The existence part of Lemma 2.1 follows from constructing appropriate approximate solutions and then performing the uniform estimate of the type (2.3) for such approximate solutions. The uniqueness part of Lemma 2.1 is a direct consequence of (2.3).

We first get by taking ∂_i, $i = 1, 2, 3$, to (2.2) that

$$\partial_t \partial_i b + v \cdot \nabla \partial_i b + \partial_i v \cdot \nabla b = 0 \quad \text{for } i = 1, 2, 3.$$

Applying Δ_j to the above equation and then taking the L^2 inner product of the resulting equation with $\Delta_j\partial_i b$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j\partial_i b(t)\|_{L^2}^2+\big(\Delta_j(v\cdot\nabla\partial_i b)\mid\Delta_j\partial_i b\big)+\big(\Delta_j(\partial_i v\cdot\nabla b)\mid\Delta_j\partial_i b\big)=0. \tag{2.4}$$

Using Bony's decomposition (A.1) for $v\cdot\nabla\partial_i b$ gives

$$v\cdot\nabla\partial_i b=T(v,\nabla\partial_i b)+\mathcal{R}(v,\nabla\partial_i b).$$

It follows from a standard commutator's argument that

$$\begin{aligned}\big(\Delta_j\big(T(v,\nabla\partial_i b)\big)\mid\Delta_j\partial_i b\big)=\sum_{|j'-j|\le 4}&\Big(\big([\Delta_j;S_{j'-1}v]\cdot\nabla\Delta_{j'}\partial_i b\mid\Delta_j\partial_i b\big)\\&+\big((S_{j'-1}v-S_{j-1}v)\cdot\nabla\Delta_j\Delta_{j'}\partial_i b\mid\Delta_j\partial_i b\big)\Big)\\&-\frac{1}{2}\int_{\mathbb{R}^3}S_{j-1}\operatorname{div}v|\Delta_j\partial_i b|^2\,dx.\end{aligned}$$

So that we deduce from (2.4) that

$$\begin{aligned}\|\Delta_j\partial_i b(t)\|_{L^2}\lesssim 2^j\|\Delta_j b_0\|_{L^2}+\sum_{|j'-j|\le 4}\int_0^t&\Big(\big\|[\Delta_j;S_{j'-1}v]\cdot\nabla\Delta_{j'}\partial_i b\big\|_{L^2}\\&+\big\|(S_{j'-1}v-S_{j-1}v)\cdot\nabla\Delta_j\Delta_{j'}\partial_i b\big\|_{L^2}\Big)(t')\,dt'\\&+\int_0^t\Big(\|S_{j-1}\operatorname{div}v\|_{L^\infty}\|\Delta_j\partial_i b\|_{L^2}+\big\|\Delta_j\mathcal{R}(v,\nabla\partial_i b)\big\|_{L^2}\\&+\big\|\Delta_j(\partial_i v\cdot\nabla b)\big\|_{L^2}\Big)(t')\,dt'.\end{aligned} \tag{2.5}$$

Applying classical commutator's estimate (see [6] for instance) and Lemma A.1 yields

$$\begin{aligned}\sum_{|j'-j|\le 4}\int_0^t\big\|[\Delta_j;S_{j'-1}v]\cdot\nabla\Delta_{j'}\partial_i b(t')\big\|_{L^2}\,dt'&\lesssim 2^{-j}\sum_{|j'-j|\le 4}2^{j'}\int_0^t\big\|S_{j'-1}\nabla v(t')\big\|_{L^\infty}\big\|\Delta_{j'}\partial_i b(t')\big\|_{L^2}\,dt'\\&\lesssim d_j2^{-\frac{j}{2}}\int_0^t\big\|\nabla v(t')\big\|_{L^\infty}\big\|\nabla b(t')\big\|_{\mathcal{B}_2^{\frac{1}{2}}}\,dt'.\end{aligned}$$

The same estimate holds for the second line of (2.5) and $\int_0^t\|S_{j-1}\operatorname{div}v(t')\|_{L^\infty}\|\Delta_j\partial_i b(t')\|_{L^2}\,dt'$.

On the other hand, we deduce from Lemma A.1 that

$$\begin{aligned}\big\|\Delta_j\mathcal{R}(v,\nabla\partial_i b)(t')\big\|_{L^2}&\lesssim\sum_{j'\ge j-N_0}\big\|\Delta_{j'}v(t')\big\|_{L^\infty}\big\|S_{j'+2}\nabla\partial_i b(t')\big\|_{L^2}\\&\lesssim\sum_{j'\ge j-N_0}d_{j'}(t)2^{-\frac{j'}{2}}\big\|\nabla v(t')\big\|_{L^\infty}\big\|\nabla b(t')\big\|_{\mathcal{B}_2^{\frac{1}{2}}}\\&\lesssim d_j(t)2^{-\frac{j}{2}}\big\|\nabla v(t')\big\|_{L^\infty}\big\|\nabla b(t')\big\|_{\mathcal{B}_2^{\frac{1}{2}}}.\end{aligned}$$

Hence by virtue of (2.5), we obtain

$$\|\Delta_j\partial_i b\|_{L_t^\infty(L^2)}\lesssim 2^j\|\Delta_j b_0\|_{L^2}+\int_0^t\Big(d_j2^{-\frac{j}{2}}\|\nabla v\|_{L^\infty}\|\partial_i b\|_{\mathcal{B}_2^{\frac{1}{2}}}+\big\|\Delta_j(\partial_i v\cdot\nabla b)\big\|_{L^2}\Big)(t')\,dt'. \tag{2.6}$$

To deal with the last term in (2.6), we get, by applying again Bony's decomposition (A.1) to $\partial v\cdot\nabla b$, that

$$\partial_i v\cdot\nabla b=T_{\partial_i v}\nabla b+\mathcal{R}(\partial_i v,\nabla b).$$

Applying Lemma A.1 gives

$$\big\|\Delta_j(T_{\partial_i v}\nabla b)\big\|_{L^2}\lesssim\sum_{|j'-j|\le 5}\|S_{j'-1}\partial_i v\|_{L^\infty}\|\Delta_{j'}\nabla b\|_{L^2}\lesssim d_j(t)2^{-\frac{j}{2}}\|\nabla v\|_{L^\infty}\|\nabla b\|_{\mathcal{B}_2^{\frac12}},$$

and

$$\|S_{j'+2}\nabla b\|_{L_v^\infty(L_h^2)}+\|S_{j'+2}\nabla b\|_{L_v^2(L_h^4)}\lesssim\sum_{\ell\le j'+1}2^{\frac{\ell}{2}}\|\Delta_\ell\nabla b\|_{L^2}\lesssim\|\nabla b\|_{\mathcal{B}_2^{\frac12}},$$

so that

$$\begin{aligned}\big\|\Delta_j\big(\mathcal{R}(\nabla_h v,\nabla b)\big)\big\|_{L^2}&\lesssim\sum_{j'\ge j-N_0}\big(\|\Delta_{j'}\nabla_h v_{\mathfrak{a}}\|_{L_v^2(L_h^\infty)}\|S_{j'+2}\nabla b\|_{L_v^\infty(L_h^2)}\\&\qquad+\|\Delta_{j'}\nabla_h v_{\mathfrak{b}}\|_{L_h^4(L_v^\infty)}\|S_{j'+2}\nabla b\|_{L_v^2(L_h^4)}\big)\\&\lesssim d_j(t)2^{-\frac{j}{2}}\|\nabla v\|_{\mathbf{L}}\|\nabla b\|_{\mathcal{B}_2^{\frac12}},\end{aligned}$$

the same estimate holds for $\|\Delta_j(\mathcal{R}(\partial_3 v,\nabla b))\|_{L^2}$. As a consequence, we obtain

$$\big\|\Delta_j(\partial_i v\cdot\nabla b)\big\|_{L_t^1(L^2)}\lesssim d_j2^{-\frac{j}{2}}\int_0^t\big(\|\nabla v\|_{L^\infty}+\|\nabla v\|_{\mathbf{L}}\big)\|\nabla b\|_{\mathcal{B}_2^{\frac12}}\,dt',$$

which along with the fact: $\|\nabla v\|_{L_t^1(L^\infty)}\lesssim\|\nabla v\|_{L_t^1(\mathbf{L})}$, and (2.6) implies that

$$\|b\|_{\widetilde{L}_t^\infty(\mathcal{B}_2^{\frac32})}\le\|b_0\|_{\mathcal{B}_2^{\frac32}}+C\int_0^t\big\|\nabla v(t')\big\|_{\mathbf{L}}\big\|b(t')\big\|_{\mathcal{B}_2^{\frac32}}\,dt'.$$

Applying Gronwall's inequality to the above inequality yields (2.3). This completes the proof of the lemma. □

To estimate the pressure function Π in (1.18), we get, by taking space divergence to the momentum equation of (1.18), that

$$\begin{aligned}-\Delta_\varepsilon\Pi&=\varepsilon^\sigma\operatorname{div}_h(b\nabla_h\Pi)+\varepsilon^{2+\sigma}\partial_3(b\partial_3\Pi)+\varepsilon\operatorname{div}_h\operatorname{div}_h\big(v^h\otimes v^h\big)\\&\quad+\varepsilon\operatorname{div}_h\partial_3\big(v^3v^h\big)+\varepsilon\partial_3^2\big(v^3\big)^2-\varepsilon^\sigma\operatorname{div}_h\big(b\Delta_\varepsilon v^h\big)-\varepsilon^\sigma\partial_3\big(b\Delta_\varepsilon v^3\big).\end{aligned}\tag{2.7}$$

Proposition 2.1. *Let (b,v,Π) be a smooth enough solution of (1.18) on $[0,T]$ and $g(t)\stackrel{\text{def}}{=}\|v^3(t)\|^2_{\mathfrak{B}_2^{1,\frac12}}$. We denote*

$$\Pi_\lambda\stackrel{\text{def}}{=}\Pi\exp\bigg(-\lambda\int_0^t g(t')\,dt'\bigg)\quad\text{and}\quad v_\lambda\stackrel{\text{def}}{=}v\exp\bigg(-\lambda\int_0^t g(t')\,dt'\bigg).\tag{2.8}$$

Then for $t\in[0,T]$ and $\|b\|_{\widetilde{L}_T^\infty(\mathcal{B}_2^{\frac32})}\le\frac{1}{2C\varepsilon^\sigma}$, one has

$$\begin{aligned}\|\nabla_\varepsilon\Pi_\lambda\|_{L_t^1(\mathfrak{B}_2^{0,\frac12})}&\le\frac{C}{1-C\varepsilon^\sigma\|b\|_{\widetilde{L}_t^\infty(\mathcal{B}_2^{\frac32})}}\Big(\varepsilon\big\|v^h\big\|_{L_t^\infty(\mathfrak{B}_2^{0,\frac12})}\big\|v_\lambda^h\big\|_{L_t^1(\mathfrak{B}_2^{2,\frac12})}+\big\|v_\lambda^h\big\|^{\frac12}_{L_{t,g}^1(\mathfrak{B}_2^{0,\frac12})}\big\|v_\lambda^h\big\|^{\frac12}_{L_t^1(\mathfrak{B}_2^{2,\frac12})}\\&\quad+\varepsilon^{\frac{\sigma}{2}}\|b\|_{\widetilde{L}_t^\infty(\mathcal{B}_2^{\frac32})}\|\Delta_\varepsilon v_\lambda\|_{L_t^1(\mathfrak{B}_2^{0,\frac12})}\Big),\end{aligned}\tag{2.9}$$

where the norm $\|\cdot\|_{L_{t,g}^1}$ is given by Definition A.2 in Appendix A.

Proof. Thanks to (2.7) and $\operatorname{div} v = 0$, one has (1.19), from which, we infer

$$\begin{aligned}\|\nabla_\varepsilon \Pi_\lambda\|_{L^1_t(\mathfrak{B}^{0,\frac12}_2)} &\lesssim \varepsilon^\sigma \|b\nabla_\varepsilon \Pi_\lambda\|_{L^1_t(\mathfrak{B}^{0,\frac12}_2)} + \varepsilon \big\|v^h \otimes v^h_\lambda\big\|_{L^1_t(\mathfrak{B}^{1,\frac12}_2)} + \big\|v^3 v^h_\lambda\big\|_{L^1_t(\mathfrak{B}^{1,\frac12}_2)} \\ &\quad + \big\|v^3 \operatorname{div}_h v^h_\lambda\big\|_{L^1_t(\mathfrak{B}^{0,\frac12}_2)} + \varepsilon^\sigma \big\|(-\Delta_\varepsilon)^{-1}\nabla_\varepsilon(\nabla b\cdot \Delta_\varepsilon v_\lambda)\big\|_{L^1_t(\mathfrak{B}^{0,\frac12}_2)}. \end{aligned} \tag{2.10}$$

Next we estimate term by term above.

Notice that the operator $(\varepsilon|D_3|)^{\frac{\sigma}{2}}(-\Delta_h)^{\frac12-\frac{\sigma}{4}}(-\Delta_\varepsilon)^{-1}\nabla_\varepsilon$ is a Fourier multiplier with symbol $(\varepsilon|\xi_3|)^{\frac{\sigma}{2}} \times |\xi_h|^{1-\frac{\sigma}{2}}(|\xi_h|^2+\varepsilon^2\xi_3^2)^{-1}(i\xi_h, i\varepsilon\xi_3)$, which is bounded by 1. Therefore, by virtue of Definition 1.2, the operator $(\varepsilon|D_3|)^{\frac{\sigma}{2}}(-\Delta_h)^{\frac12-\frac{\sigma}{4}}(-\Delta_\varepsilon)^{-1}\nabla_\varepsilon$ maps uniformly bounded from $\mathfrak{B}^{0,\frac12}_2$ to $\mathfrak{B}^{0,\frac12}_2$, we write

$$\begin{aligned}&\varepsilon^\sigma \big\|(-\Delta_\varepsilon)^{-1}\nabla_\varepsilon(\nabla b\cdot\Delta_\varepsilon v_\lambda)\big\|_{L^1_t(\mathfrak{B}^{0,\frac12}_2)} \\ &\quad = \varepsilon^{\frac{\sigma}{2}} \Big\|\big(\varepsilon|D_3|\big)^{\frac{\sigma}{2}}(-\Delta_h)^{\frac12-\frac{\sigma}{4}}(-\Delta_\varepsilon)^{-1}\nabla_\varepsilon(-\Delta_h)^{-\frac12+\frac{\sigma}{4}}\big(|D_3|\big)^{-\frac{\sigma}{2}}(\nabla b\cdot\Delta_\varepsilon v_\lambda)\Big\|_{L^1_t(\mathfrak{B}^{0,\frac12}_2)} \\ &\quad \lesssim \varepsilon^{\frac{\sigma}{2}}\|\nabla b\cdot\Delta_\varepsilon v_\lambda\|_{L^1_t(\mathfrak{B}^{\frac{\sigma}{2}-1,\frac{1-\sigma}{2}}_2)},\end{aligned}$$

which together with the law of product of Lemma A.2 and Proposition A.1 ensures that

$$\begin{aligned}\varepsilon^\sigma \big\|(-\Delta_\varepsilon)^{-1}\nabla_\varepsilon(\nabla b\cdot\Delta_\varepsilon v_\lambda)\big\|_{L^1_t(\mathfrak{B}^{0,\frac12}_2)} &\lesssim \varepsilon^{\frac{\sigma}{2}}\|\nabla b\|_{L^\infty_t(\mathfrak{B}^{\frac{\sigma}{2},\frac{1-\sigma}{2}}_2)}\|\Delta_\varepsilon v_\lambda\|_{L^1_t(\mathfrak{B}^{0,\frac12}_2)} \\ &\lesssim \varepsilon^{\frac{\sigma}{2}}\|\nabla b\|_{L^\infty_t(\mathcal{B}^{\frac12}_2)}\|\Delta_\varepsilon v_\lambda\|_{L^1_t(\mathfrak{B}^{0,\frac12}_2)}.\end{aligned}$$

Whereas applying the law of product of Lemma A.2 to $v^h\otimes v^h$ gives

$$\big\|v^h\otimes v^h\big\|_{\mathfrak{B}^{1,\frac12}_2} \lesssim \big\|v^h\big\|^2_{\mathfrak{B}^{1,\frac12}_2} \lesssim \big\|v^h\big\|_{\mathfrak{B}^{0,\frac12}_2}\big\|v^h\big\|_{\mathfrak{B}^{2,\frac12}_2},$$

which together (2.8) ensures

$$\big\|v^h\otimes v^h_\lambda\big\|_{L^1_t(\mathfrak{B}^{1,\frac12}_2)} \lesssim \big\|v^h\big\|_{L^\infty_t(\mathfrak{B}^{0,\frac12}_2)}\big\|v^h_\lambda\big\|_{L^1_t(\mathfrak{B}^{2,\frac12}_2)}.$$

Along the same line, applying the law of product of Lemma A.2 to the remaining terms in (2.10), we readily get

$$\begin{aligned}\|\nabla_\varepsilon\Pi_\lambda\|_{L^1_t(\mathfrak{B}^{0,\frac12}_2)} &\le C\bigg(\varepsilon^\sigma\|b\|_{\widetilde{L}^\infty_t(\mathfrak{B}^{1,\frac12}_2)}\|\nabla_\varepsilon\Pi_\lambda\|_{L^1_t(\mathfrak{B}^{0,\frac12}_2)} + \varepsilon\big\|v^h\big\|_{L^\infty_t(\mathfrak{B}^{0,\frac12}_2)}\big\|v^h_\lambda\big\|_{L^1_t(\mathfrak{B}^{2,\frac12}_2)} \\ &\quad + \int_0^t \big\|v^3\big\|_{\mathfrak{B}^{1,\frac12}_2}\big\|v^h_\lambda\big\|_{\mathfrak{B}^{1,\frac12}_2}\,dt' + \varepsilon^{\frac{\sigma}{2}}\|\nabla b\|_{L^\infty_t(\mathcal{B}^{\frac12}_2)}\|\Delta_\varepsilon v_\lambda\|_{L^1_t(\mathfrak{B}^{0,\frac12}_2)}\bigg),\end{aligned} \tag{2.11}$$

which together with the fact that

$$\int_0^t \big\|v^3\big\|_{\mathfrak{B}^{1,\frac12}_2}\big\|v^h_\lambda\big\|_{\mathfrak{B}^{1,\frac12}_2}\,dt' \lesssim \bigg(\int_0^t \big\|v^3\big\|^2_{\mathfrak{B}^{1,\frac12}_2}\big\|v^h_\lambda\big\|_{\mathfrak{B}^{0,\frac12}_2}\,dt'\bigg)^{\frac12}\big\|v^h_\lambda\big\|^{\frac12}_{L^1_t(\mathfrak{B}^{2,\frac12}_2)} \lesssim \big\|v^h_\lambda\big\|^{\frac12}_{L^1_{t,g}(\mathfrak{B}^{0,\frac12}_2)}\big\|v^h_\lambda\big\|^{\frac12}_{L^1_t(\mathfrak{B}^{2,\frac12}_2)},$$

implies (2.9). This completes the proof of the proposition. □

Taking $\lambda = 0$ in (2.11) leads to the following corollary:

Corollary 2.1. *Under the assumption of Proposition 2.1, one has*

$$\|\nabla_\varepsilon \Pi\|_{L^1_t(\mathfrak{B}^{0,\frac{1}{2}}_2)} \le \frac{C}{1 - C\varepsilon^\sigma \|b\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac{3}{2}}_2)}} \Big(\varepsilon \|v^h\|_{L^\infty_t(\mathfrak{B}^{0,\frac{1}{2}}_2)} \|v^h\|_{L^1_t(\mathfrak{B}^{2,\frac{1}{2}}_2)} + \|v^h\|_{L^2_t(\mathfrak{B}^{1,\frac{1}{2}}_2)} \|v^3\|_{L^2_t(\mathfrak{B}^{1,\frac{1}{2}}_2)} + \varepsilon^{\frac{\sigma}{2}} \|b\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac{3}{2}}_2)} \|\Delta_\varepsilon v\|_{L^1_t(\mathfrak{B}^{0,\frac{1}{2}}_2)}\Big), \tag{2.12}$$

whenever $\|b\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac{3}{2}}_2)} \le \frac{1}{2C\varepsilon^\sigma}$.

3. The proof of Theorem 1.3

The purpose of this section is to present the proof of Theorem 1.3.

Proof of Theorem 1.3. Given initial data $(a_0^\varepsilon, u_0^\varepsilon)$ by (1.14), Theorem 0.2 of [14] ensures that the system (1.2) has a unique solution on $]0, T^*[$ so that

$$a \in \mathcal{C}\big([0, T^*[; H^2(\mathbb{R}^3)\big) \qquad u \in \mathcal{C}\big([0, T^*[; H^2(\mathbb{R}^3)\big) \in L^2\big([0, T^*[; H^3(\mathbb{R}^3)\big).$$

We may assume that T^* is the lifespan to this solution (a, u). Then to complete the proof of Theorem 1.3, it amounts to prove that $T^* = \infty$ and there holds (1.16).

In order to do so, we denote (b, v) to be given by (1.17). Then (b, v) verifies the system (1.18) with initial data (a_0, u_0). We shall first investigate the $L^1_T([0, T^*[; Lip(\mathbb{R}^3))$ estimate of v, and then use the relation (1.17) to derive the $L^1_T([0, T^*[; Lip(\mathbb{R}^3))$ of u, finally we establish the $L^\infty([0, T^*[; H^2(\mathbb{R}^3))$ estimate for (a, u).

In fact, motivated by [23,22], we shall deal with the L^2 type energy estimate for each dyadic block of v^h and v^3 separately.

3.1. *The anisotropic estimate of v^h*

Let $g, v_\lambda, \Pi_\lambda$ be given by (2.8). Then by virtue of (1.18), we have

$$\partial_t v^h_\lambda + \lambda g(t) v^h_\lambda - \Delta_\varepsilon v^h_\lambda = -\varepsilon v \cdot \nabla v^h_\lambda - \big(1 + \varepsilon^\sigma b\big)\nabla_h \Pi_\lambda + \varepsilon^\sigma b \Delta_\varepsilon v^h_\lambda.$$

Applying the operator $\Delta^h_j \Delta^v_k$ to the above equation and then taking the L^2 inner production of the resulting equation with $\Delta^h_j \Delta^v_k v^h_\lambda$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta^h_j \Delta^v_k u^h_\lambda(t)\|^2_{L^2} + \lambda g(t)\|\Delta^h_j \Delta^v_k v^h_\lambda(t)\|^2_{L^2} - \int_{\mathbb{R}^3} \Delta_\varepsilon \Delta^h_j \Delta^v_k v^h_\lambda \cdot \Delta^h_j \Delta^v_k v^h_\lambda \, dx$$
$$= -\int_{\mathbb{R}^3} \Delta^h_j \Delta^v_k \big(\varepsilon v \cdot \nabla v^h_\lambda + \big(1 + \varepsilon^\sigma b\big)\nabla_h \Pi_\lambda - \varepsilon^\sigma b \Delta_\varepsilon v^h_\lambda\big) \cdot \Delta^h_j \Delta^v_k v^h_\lambda \, dx.$$

However by using integration by parts and Lemma A.1, we get

$$-\int_{\mathbb{R}^3} \Delta_\varepsilon \Delta^h_j \Delta^v_k v^h_\lambda \cdot \Delta^h_j \Delta^v_k v^h_\lambda \, dx = \|\nabla_h \Delta^h_j \Delta^v_k v^h_\lambda\|^2_{L^2} + \varepsilon^2 \|\partial_3 \Delta^h_j \Delta^v_k v^h_\lambda\|^2_{L^2} \ge c\big(2^{2j} + \varepsilon^2 2^{2k}\big)\|\Delta^h_j \Delta^v_k v^h_\lambda\|^2_{L^2},$$

whence a similar argument as that in [11,15,23] gives rise to

$$\frac{d}{dt}\|\Delta^h_j \Delta^v_k v^h_\lambda(t)\|_{L^2} + \lambda g(t)\|\Delta^h_j \Delta^v_k v^h_\lambda(t)\|_{L^2} + c\big(2^{2j} + \varepsilon^2 2^{2k}\big)\|\Delta^h_j \Delta^v_k v^h_\lambda(t)\|_{L^2}$$
$$\le \varepsilon\|\Delta^h_j \Delta^v_k \big(v \cdot \nabla v^h_\lambda\big)\|_{L^2} + \|\Delta^h_j \Delta^v_k \big(\big(1 + \varepsilon^\sigma b\big)\nabla_h \Pi_\lambda\big)\|_{L^2} + \varepsilon^\sigma \|\Delta^h_j \Delta^v_k \big(b \Delta_\varepsilon v^h\big)\|_{L^2}.$$

Integrating the above inequality over $[0,t]$ and using Definition A.2, we write

$$\begin{aligned}&\|v_\lambda^h\|_{\widetilde{L}_t^\infty(\mathfrak{B}_2^{0,\frac12})}+\lambda\|v_\lambda^h\|_{L^1_{t,g}(\mathfrak{B}_2^{0,\frac12})}+c\big(\|v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})}+\varepsilon^2\|v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac52})}\big)\\&\quad\le\|u_0^h\|_{\mathfrak{B}_2^{0,\frac12}}+C\big(\varepsilon\|v\cdot\nabla v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}+\|(1+\varepsilon^\sigma b)\nabla_h\Pi_\lambda\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}+\varepsilon^\sigma\|b\Delta_\varepsilon v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}\big).\end{aligned}$$

Notice that

$$\|v^h(t)\|_{\mathfrak{B}_2^{0,\frac32}}\le\|v^h(t)\|^{\frac12}_{\mathfrak{B}_2^{0,\frac12}}\|v^h(t)\|^{\frac12}_{\mathfrak{B}_2^{0,\frac52}},$$

applying Lemma A.2 and then Definition A.2 yields

$$\begin{aligned}\varepsilon\|v^3\partial_3v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}&\lesssim\int_0^t\|v^3(t')\|_{\mathfrak{B}_2^{1,\frac12}}\varepsilon\|v_\lambda^h(t')\|_{\mathfrak{B}_2^{0,\frac32}}\,dt'\\&\lesssim\bigg(\int_0^t\|v^3(t')\|^2_{\mathfrak{B}_2^{1,\frac12}}\|v^h(t')\|_{\mathfrak{B}_2^{0,\frac12}}\,dt'\bigg)^{\frac12}\big(\varepsilon^2\|v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac52})}\big)^{\frac12},\end{aligned}$$

and

$$\begin{aligned}\varepsilon\|v\cdot\nabla v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}&\le\varepsilon\|v^h\cdot\nabla_hv_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}+\varepsilon\|v^3\partial_3v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}\\&\le\frac{c}{4}\varepsilon^2\|v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac52})}+C\big(\varepsilon\|v^h\|_{L^\infty_t(\mathfrak{B}_2^{0,\frac12})}\|v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})}+\|v_\lambda^h\|_{L^1_{t,g}(\mathfrak{B}_2^{0,\frac12})}\big).\end{aligned}$$

While we get, by applying again the law of product of Lemma A.2 and Proposition A.1, that

$$\|b\nabla_h\Pi_\lambda\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}\lesssim\|b\|_{L^\infty_t(\mathfrak{B}_2^{1,\frac12})}\|\nabla_h\Pi_\lambda\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}\lesssim\|b\|_{L^\infty_t(\mathcal{B}_2^{\frac32})}\|\nabla_h\Pi_\lambda\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})},$$

and

$$\|b\Delta_\varepsilon v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}\lesssim\|b\|_{L^\infty_t(\mathcal{B}_2^{\frac32})}\|\Delta_\varepsilon v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}.$$

As a consequence, we obtain

$$\begin{aligned}&\|v_\lambda^h\|_{\widetilde{L}_t^\infty(\mathfrak{B}_2^{0,\frac12})}+\lambda\|v_\lambda^h\|_{L^1_{t,g}(\mathfrak{B}_2^{0,\frac12})}+\frac{3c}{4}\big(\|v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})}+\varepsilon^2\|v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac52})}\big)\\&\quad\le\|u_0^h\|_{\mathfrak{B}_2^{0,\frac12}}+C\big(\varepsilon\|v^h\|_{L^\infty_t(\mathfrak{B}_2^{0,\frac12})}\|v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})}+\|v_\lambda^h\|_{L^1_{t,g}(\mathfrak{B}_2^{0,\frac12})}\\&\qquad+\big(1+\varepsilon^\sigma\|b\|_{\widetilde{L}_t^\infty(\mathcal{B}_2^{\frac32})}\big)\|\nabla_h\Pi_\lambda\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}+\varepsilon^\sigma\|b\|_{\widetilde{L}_t^\infty(\mathcal{B}_2^{\frac32})}\|\Delta_\varepsilon v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}\big).\end{aligned}\tag{3.1}$$

Now for some c_1 sufficiently small, we denote T_0 to be determined by

$$T_0\overset{\text{def}}{=}\sup\big\{t\in[0,T^*[,\ \|b\|_{\widetilde{L}_t^\infty(\mathcal{B}_2^{\frac32})}\le c_1/\varepsilon^{\frac{\sigma}{2}}\big\}.\tag{3.2}$$

Then by plugging Estimate (2.9) into (3.1), we get for $t\le T_0$ that

$$\begin{aligned}&\|v_\lambda^h\|_{\widetilde{L}_t^\infty(\mathfrak{B}_2^{0,\frac12})}+\lambda\|v_\lambda^h\|_{L^1_{t,g}(\mathfrak{B}_2^{0,\frac12})}+\frac{c}{2}\big(\|v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})}+\varepsilon^2\|v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac52})}\big)\\&\quad\le\|u_0^h\|_{\mathfrak{B}_2^{0,\frac12}}+\frac{c}{8}\|v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})}\\&\qquad+C\big(\varepsilon\|v^h\|_{L^\infty_t(\mathfrak{B}_2^{0,\frac12})}\|v_\lambda^h\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})}+\|v_\lambda^h\|_{L^1_{t,g}(\mathfrak{B}_2^{0,\frac12})}+\varepsilon^{\frac{\sigma}{2}}\|b\|_{\widetilde{L}_t^\infty(\mathcal{B}_2^{\frac32})}\|\Delta_\varepsilon v_\lambda\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}\big).\end{aligned}$$

Then taking $\lambda \geq C$ and c_1 being small enough in (3.2) results in

$$\begin{aligned}&\|v_\lambda^h\|_{\widetilde{L}_t^\infty(\mathfrak{B}_2^{0,\frac{1}{2}})}+\frac{c}{4}\big(\|v_\lambda^h\|_{L_t^1(\mathfrak{B}_2^{2,\frac{1}{2}})}+\varepsilon^2\|v_\lambda^h\|_{L_t^1(\mathfrak{B}_2^{0,\frac{5}{2}})}\big)\\&\quad\leq\|u_0^h\|_{\mathfrak{B}_2^{0,\frac{1}{2}}}+C\big(\varepsilon\|v^h\|_{L_t^\infty(\mathfrak{B}_2^{0,\frac{1}{2}})}\|v_\lambda^h\|_{L_t^1(\mathfrak{B}_2^{2,\frac{1}{2}})}+c_1\|\Delta_\varepsilon v^3\|_{L_t^1(\mathfrak{B}_2^{2,\frac{1}{2}})}\big)\end{aligned}\tag{3.3}$$

for $t \leq T_0$.

3.2. The anisotropic estimate of v^3

By virtue of the v^3 equation of (1.18), we get, by a similar argument of Subsection 3.1, that

$$\begin{aligned}&\|\Delta_j^h\Delta_k^v v^3\|_{L_t^\infty(L^2)}+c\big(2^{2j}+\varepsilon^2 2^{2k}\big)\|\Delta_j^h\Delta_k^v v^3\|_{L_t^1(L^2)}\\&\quad\leq\|\Delta_j^h\Delta_k^v u_0^3\|_{L^2}+C\big(\varepsilon\|\Delta_j^h\Delta_k^v(v\cdot\nabla v^3)\|_{L_t^1(L^2)}\\&\qquad+\varepsilon\|\Delta_j^h\Delta_k^v\big((1+\varepsilon^\sigma b)\varepsilon\partial_3\Pi\big)\|_{L_t^1(L^2)}+\varepsilon^\sigma\|\Delta_j^h\Delta_k^v(b\Delta_\varepsilon v^3)\|_{L_t^1(L^2)}\big),\end{aligned}$$

from which, we infer

$$\begin{aligned}&\|v^3\|_{\widetilde{L}_t^\infty(\mathfrak{B}_2^{0,\frac{1}{2}})}+c\big(\|v^3\|_{L_t^1(\mathfrak{B}_2^{2,\frac{1}{2}})}+\varepsilon^2\|v^3\|_{L_t^1(\mathfrak{B}_2^{0,\frac{5}{2}})}\big)\\&\quad\leq\|u_0^3\|_{\mathfrak{B}_2^{0,\frac{1}{2}}}+C\big(\varepsilon\|v\cdot\nabla v^3\|_{L_t^1(\mathfrak{B}_2^{0,\frac{1}{2}})}+\varepsilon\|(1+\varepsilon^\sigma b)\varepsilon\partial_3\Pi\|_{L_t^1(\mathfrak{B}_2^{0,\frac{1}{2}})}+\varepsilon^\sigma\|b\Delta_\varepsilon v^3\|_{L_t^1(\mathfrak{B}_2^{0,\frac{1}{2}})}\big).\end{aligned}$$

As $\operatorname{div} v = 0$, applying the law of product of Lemma A.2 yields

$$\begin{aligned}\|v\cdot\nabla v^3\|_{L_t^1(\mathfrak{B}_2^{0,\frac{1}{2}})}&\leq\|v^h\cdot\nabla_h v^3\|_{L_t^1(\mathfrak{B}_2^{0,\frac{1}{2}})}+\|v^3\operatorname{div}_h v^h\|_{L_t^1(\mathfrak{B}_2^{0,\frac{1}{2}})}\\&\lesssim\|v^h\|_{\widetilde{L}_t^\infty(\mathfrak{B}_2^{0,\frac{1}{2}})}\|v^3\|_{L_t^1(\mathfrak{B}_2^{2,\frac{1}{2}})}+\|v^3\|_{\widetilde{L}_t^\infty(\mathfrak{B}_2^{0,\frac{1}{2}})}\|v^h\|_{L_t^1(\mathfrak{B}_2^{2,\frac{1}{2}})}.\end{aligned}$$

Then thanks to (2.12), we deduce that

$$\begin{aligned}&\|v^3\|_{\widetilde{L}_t^\infty(\mathfrak{B}_2^{0,\frac{1}{2}})}+\frac{c}{2}\big(\|v^3\|_{L_t^1(\mathfrak{B}_2^{2,\frac{1}{2}})}+\varepsilon^2\|v^3\|_{L_t^1(\mathfrak{B}_2^{0,\frac{5}{2}})}\big)\\&\quad\leq\|u_0^3\|_{\mathfrak{B}_2^{0,\frac{1}{2}}}+C\big(\varepsilon\|v^h\|_{\widetilde{L}_t^\infty(\mathfrak{B}_2^{0,\frac{1}{2}})}\big(\varepsilon\|v^h\|_{L_t^1(\mathfrak{B}_2^{2,\frac{1}{2}})}+\|v^3\|_{L_t^1(\mathfrak{B}_2^{2,\frac{1}{2}})}\big)\\&\qquad+\varepsilon\|v^3\|_{\widetilde{L}_t^\infty(\mathfrak{B}_2^{0,\frac{1}{2}})}\|v^h\|_{L_t^1(\mathfrak{B}_2^{2,\frac{1}{2}})}+\varepsilon^{1+\frac{\sigma}{2}}\|b\|_{\widetilde{L}_t^\infty(B_2^{\frac{3}{2}})}\|\Delta_\varepsilon v^h\|_{L_t^1(\mathfrak{B}_2^{0,\frac{1}{2}})}\big)\end{aligned}\tag{3.4}$$

for $t \leq T_0$ determined by (3.2).

3.3. The closure of the anisotropic estimate

Let $\mathfrak{T}$ be determined by

$$\mathfrak{T}\stackrel{\text{def}}{=}\max\Big\{t\in\,]0,T_0[:\ \|v^h\|_{\widetilde{L}_t^\infty(\mathfrak{B}_2^{0,\frac{1}{2}})}+c\big(\|v^h\|_{L_t^1(\mathfrak{B}_2^{2,\frac{1}{2}})}+\varepsilon^2\|v^h\|_{L_t^1(\mathfrak{B}_2^{0,\frac{5}{2}})}\big)\leq C_1\big(\varepsilon+\|u_0\|_{\mathfrak{B}_2^{0,\frac{1}{2}}}\big)\Big\}\tag{3.5}$$

for some sufficiently large constant C_1, which should be chosen later on. We shall prove that $\mathfrak{T} = T_0 = \infty$ whenever there holds (1.13). Otherwise, taking

$$\varepsilon\leq\frac{c}{8CC_1(\varepsilon+\|u_0\|_{\mathfrak{B}_2^{0,\frac{1}{2}}})},$$

we deduce from (2.8) and (3.3) that for $t \leq \mathfrak{T}$

$$\Big(\|v^h\|_{\widetilde{L}^\infty_t(\mathfrak{B}_2^{0,\frac12})} + \frac{c}{4}\big(\|v^h\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})} + \varepsilon^2\|v^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac52})}\big)\Big)\exp\Big(-C\int_0^t \|v^3(t')\|^2_{\mathfrak{B}_2^{1,\frac12}}\,dt'\Big)$$
$$\le \|v^h_\lambda\|_{\widetilde{L}^\infty_t(\mathfrak{B}_2^{0,\frac12})} + \frac{c}{4}\big(\|v^h_\lambda\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})} + \varepsilon^2\|v^h_\lambda\|_{L^1_t(\mathfrak{B}_2^{0,\frac52})}\big)$$
$$\le 2\big(\|u^h_0\|_{\mathfrak{B}_2^{0,\frac12}} + Cc_1\|\Delta_\varepsilon v^3\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}\big),$$

which leads to

$$\|v^h\|_{\widetilde{L}^\infty_t(\mathfrak{B}_2^{0,\frac12})} + \frac{c}{4}\big(\|v^h\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})} + \varepsilon^2\|v^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac52})}\big)$$
$$\le 2\big(\|u^h_0\|_{\mathfrak{B}_2^{0,\frac12}} + Cc_1\|\Delta_\varepsilon v^3\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})}\big)\exp\Big(C\int_0^t \|v^3(t')\|^2_{\mathfrak{B}_2^{1,\frac12}}\,dt'\Big). \tag{3.6}$$

Let us take ε is so small that $\varepsilon CC_1c_1 \le c$. Then for $t \le \mathfrak{T}$, one has

$$\varepsilon^{1+\frac{\sigma}{2}}\|b\|_{\widetilde{L}^\infty_t(\mathcal{B}_2^{\frac32})}\|\Delta_\varepsilon v^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac12})} \le \frac{\varepsilon CC_1c_1}{c}\big(\varepsilon + \|u_0\|_{\mathfrak{B}_2^{0,\frac12}}\big) \le \varepsilon + \|u_0\|_{\mathfrak{B}_2^{0,\frac12}}.$$

Hence if we take ε so small that

$$CC_1\varepsilon\big(\varepsilon + \|u_0\|_{\mathfrak{B}_2^{0,\frac12}}\big) \le \frac{c}{4} \quad \text{and} \quad CC_1^2\varepsilon\big(\varepsilon + \|u_0\|_{\mathfrak{B}_2^{0,\frac12}}\big)^2 \le c,$$

we deduce from (3.4) that for $t \le \mathfrak{T}$

$$\|v^3\|_{\widetilde{L}^\infty_t(\mathfrak{B}_2^{0,\frac12})} + \frac{c}{2}\big(\|v^3\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})} + \varepsilon^2\|v^3\|_{L^1_t(\mathfrak{B}_2^{0,\frac52})}\big) \le 4\big(\varepsilon + \|u_0\|_{\mathfrak{B}_2^{0,\frac12}}\big). \tag{3.7}$$

Therefore combining (3.6) with (3.7), we conclude that there exits a small enough positive constant η_0 so that if

$$\varepsilon \le \eta_0 C_1^{-2}\big(\varepsilon + \|u_0\|_{\mathfrak{B}_2^{0,\frac12}}\big)^{-2}$$

one has

$$\|v^h\|_{\widetilde{L}^\infty_t(\mathfrak{B}_2^{0,\frac12})} + \frac{c}{4}\big(\|v^h\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})} + \varepsilon^2\|v^h\|_{L^1_t(\mathfrak{B}_2^{0,\frac52})}\big)$$
$$\le 4\big(\varepsilon + \|u_0\|_{\mathfrak{B}_2^{0,\frac12}}\big)\exp\Big(\frac{32C}{c}\big(\varepsilon + \|u_0\|_{\mathfrak{B}_2^{0,\frac12}}\big)^2\Big) \quad \text{for } t \le \mathfrak{T}. \tag{3.8}$$

On the other hand, it is easy to observe that when $k > \ell$, the support to the Fourier transform of $\Delta^h_k\Delta^v_\ell v$ is included in $2^k\widetilde{\mathcal{C}}$ for some annulus $\widetilde{\mathcal{C}}$ in $\mathbb{R}^3$, so that the operator $\Delta_j\Delta^h_k\Delta^v_\ell$ is identically zero if $|j-k| > N_0$ for some fixed integer N_0. Along the same line, when $k \le \ell$, the operator $\Delta_j\Delta^h_k\Delta^v_\ell$ is identically zero if $j > \ell + N_0$. Hence it follows from Lemma A.1 that

$$\varepsilon\|\Delta_j\partial_3 v\|_{L^1_t(L^2_v(L^\infty_h))} \lesssim \sum_{\substack{k>\ell\\ |k-j|\le N_0}} \varepsilon 2^k 2^\ell \|\Delta^h_k\Delta^v_\ell v\|_{L^1_t(L^2)} + \sum_{\substack{k\le\ell\\ \ell\ge j-N_0}} \varepsilon 2^k\|\Delta^h_k\Delta^v_\ell\partial_3 v\|_{L^1_t(L^2)}$$
$$\lesssim \varepsilon\|v\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})}\sum_{\substack{k>\ell\\ |k-j|\le N_0}} d_{k,\ell}2^{-\frac{k}{2}} + \varepsilon\|\partial_3 v\|_{L^1_t(\mathfrak{B}_2^{1,\frac12})}\sum_{\substack{k\le\ell\\ \ell\ge j-N_0}} d_{k,\ell}2^{-\frac{\ell}{2}},$$

for some $(d_{k,\ell})_{k,\ell\in\mathbb{Z}^2} \in \ell^1(\mathbb{Z}^2)$. This yields

$$\varepsilon\|\Delta_j\partial_3 v\|_{L^1_t(L^2_v(L^\infty_h))} \lesssim d_j 2^{-\frac{j}{2}}\big(\|v\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})} + \varepsilon\|\partial_3 v\|_{L^1_t(\mathfrak{B}_2^{1,\frac12})}\big). \tag{3.9}$$

Applying Lemma A.1 again gives

$$\sum_{\substack{k>\ell\\|k-j|\leq N_0}} \|\Delta_j\Delta_k^h\Delta_\ell^v\nabla_h v\|_{L^1_t(L^4_h(L^\infty_v))} \lesssim \sum_{\substack{k>\ell\\|k-j|\leq N_0}} 2^{\frac{3k}{2}}2^{\frac{\ell}{2}}\|\Delta_k^h\Delta_\ell^v v\|_{L^1_t(L^2)}$$
$$\lesssim \|v\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})}\sum_{\substack{k>\ell\\|k-j|\leq N_0}} d_{k,\ell}2^{-\frac{k}{2}} \lesssim d_j 2^{-\frac{j}{2}}\|v\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})},$$

and

$$\sum_{\substack{k\leq\ell\\ \ell\geq j-N_0}} \|\Delta_j\Delta_k^h\Delta_\ell^v\nabla_h v\|_{L^1_t(L^2_v(L^\infty_h))} \lesssim \sum_{\substack{k\leq\ell\\ \ell\geq j-N_0}} 2^{2k}\|\Delta_k^h\Delta_\ell^v v\|_{L^1_t(L^2)}$$
$$\lesssim \|v\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})}\sum_{\substack{k\leq\ell\\ \ell\geq j-N_0}} d_{k,\ell}2^{-\frac{\ell}{2}} \lesssim d_j 2^{-\frac{j}{2}}\|v\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})}.$$

Let us denote

$$v_{\mathfrak{a}} \stackrel{\text{def}}{=} \sum_{k\leq\ell}\Delta_k^h\Delta_\ell^v v \quad\text{and}\quad v_{\mathfrak{b}} \stackrel{\text{def}}{=} \sum_{k>\ell}\Delta_k^h\Delta_\ell^v v.$$

We thus obtain

$$\sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\big(\|\Delta_j\nabla_h v_{\mathfrak{a}}\|_{L^1_t(L^2_v(L^\infty_h))} + \|\Delta_j\nabla_h v_{\mathfrak{b}}\|_{L^1_t(L^4_h(L^\infty_v))}\big) \leq C\|v\|_{L^1_t(\mathfrak{B}_2^{2,\frac12})}.$$

This together with (2.1), (3.7), (3.8) and (3.9) ensures that

$$\varepsilon\|\nabla v\|_{L^1_{\mathfrak{T}}(\mathbf{L})} \leq C\exp\big(C\|u_0\|^2_{\mathfrak{B}_2^{0,\frac12}}\big) \tag{3.10}$$

for some large enough positive constant C.

By virtue of (3.10), we get, by applying Lemma 2.1 to the first equation of (1.18), that

$$\|b\|_{\widetilde{L}^\infty_{\mathfrak{T}}(\mathcal{B}_2^{\frac32})} \leq \|a_0\|_{\mathcal{B}_2^{\frac32}}\exp\big(C\varepsilon\|\nabla v\|_{L^1_{\mathfrak{T}}(\mathbf{L})}\big) \leq \|a_0\|_{\mathcal{B}_2^{\frac32}}\exp\big(C\exp\big(C\|u_0\|^2_{\mathfrak{B}_2^{0,\frac12}}\big)\big). \tag{3.11}$$

Then taking $C_1 \stackrel{\text{def}}{=} 8\exp(\frac{32C}{c}(\varepsilon+\|u_0\|_{\mathfrak{B}_2^{0,\frac12}})^2)$ in (3.5), (3.2) together with (3.5) and (3.11) ensures that $\mathfrak{T}=T_0=T^*$ whenever there holds (1.13).

3.4. The H^2 estimate of (a, u)

Thanks to (1.17), (3.7) and (3.8), we get by applying Lemma A.1 that

$$\|\nabla u\|_{L^1_{T^*}(L^\infty)} \lesssim \|u\|_{L^1_{T^*}(\mathfrak{B}_2^{2,\frac12})} + \|u\|_{L^1_{T^*}(\mathfrak{B}_2^{0,\frac52})}$$
$$\lesssim \|v\|_{L^1_{T^*}(\mathfrak{B}_2^{2,\frac12})} + \varepsilon^2\|\partial_3^2 v\|_{L^1_{T^*}(\mathfrak{B}_2^{0,\frac12})} \lesssim C\exp\big(C\|u_0\|^2_{\mathfrak{B}_2^{0,\frac12}}\big), \quad\text{and}$$
$$\|u\|_{L^2_{T^*}(L^\infty)} \lesssim \|v\|^{\frac12}_{L^\infty_{T^*}(\mathfrak{B}_2^{0,\frac12})}\|v\|^{\frac12}_{L^1_{T^*}(\mathfrak{B}_2^{2,\frac12})} \lesssim C\exp\big(C\|u_0\|^2_{\mathfrak{B}_2^{0,\frac12}}\big). \tag{3.12}$$

Moreover, u satisfies Estimate (1.16) on $[0, T^*[$. Therefore, it suffices to prove that $T^*=\infty$ in order to complete the proof of Theorem 1.3.

With (3.12), we now turn to the propagating of H^2 regularity for (1.2) with data given by (1.14). The main difficulty in solving this problem is due to the fact that $a(t,x_h,x_3)=\varepsilon^\sigma b(t,x_h,\varepsilon x_3)$ is not small in $\widetilde{L}^\infty(\mathbb{R}^+,\mathcal{B}_2^{\frac32}(\mathbb{R}^3))$ even if ε

is very small. That is the reason why we are going to use energy method as that in [3,4]. For this, we denote $\rho \stackrel{\text{def}}{=} \frac{1}{1+a}$ and $\rho_{0,\varepsilon}(x) \stackrel{\text{def}}{=} \frac{1}{1+\varepsilon^\sigma a_0(x_h,\varepsilon x_3)}$ and $u_{0,\varepsilon}(x) = (\varepsilon u_0^h(x_h,\varepsilon x_3), u_0^3(x_h,\varepsilon x_3))$, then to solve (1.2) for (a,u) is equivalent to solve (ρ,u) via

$$\begin{cases} \partial_t\rho + u\cdot\nabla\rho = 0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \rho\partial_t u + \rho u\cdot\nabla u - \Delta u + \nabla\Pi = 0, \\ \operatorname{div} u = 0, \\ \rho|_{t=0} = \rho_{0,\varepsilon}, \quad u|_{t=0} = u_{0,\varepsilon}. \end{cases} \tag{3.13}$$

Notice that

$$m_\varepsilon \stackrel{\text{def}}{=} \frac{1}{1+\varepsilon^\sigma\|a_0\|_{L^\infty}} \le \|\rho(t)\|_{L^\infty} \le \frac{1}{1-\varepsilon^\sigma\|a_0\|_{L^\infty}} \stackrel{\text{def}}{=} M_\varepsilon \quad \text{for } t<T^*.$$

And it follows by a standard energy estimate that for any $t<T^*$

$$\frac12\|\sqrt{\rho}u\|_{L_t^\infty(L^2)}^2 + \|\nabla u\|_{L_t^2(L^2)}^2 = \frac12\|\sqrt{\rho_{0,\varepsilon}}u_{0,\varepsilon}\|_{L^2}^2 \le C_\varepsilon. \tag{3.14}$$

Taking the L^2 inner product of the momentum equation of (3.13) with $\partial_t u$ leads to

$$\frac12\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}\partial_t u\|_{L^2}^2 \le \left|(\rho u\cdot\nabla u \mid \partial_t u)_{L^2}\right| \le \frac12\|\sqrt{\rho}\partial_t u\|_{L^2}^2 + C\|\rho\|_{L^\infty}\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2,$$

from which and (3.12), we deduce that for any $t<T^*$

$$\|\nabla u\|_{L_t^\infty(L^2)}^2 + \|\sqrt{\rho}\partial_t u\|_{L_t^2(L^2)}^2 \le 2\|\nabla u_{0,\varepsilon}\|_{L^2}\exp\bigl(C_\varepsilon\|u\|_{L_t^2(L^\infty)}^2\bigr) \le C_\varepsilon. \tag{3.15}$$

Note that we can also write the momentum equation of (3.13) as

$$\begin{cases} \Delta u - \nabla\Pi = \rho\partial_t u + \rho u\cdot\nabla u, \\ \operatorname{div} u = 0. \end{cases} \tag{3.16}$$

Then by virtue of (3.12) and (3.15), we get, by using classical estimates on linear Stokes operator, that for $t<T^*$

$$\begin{aligned} \|\nabla^2 u\|_{L_t^2(L^2)} + \|\nabla\Pi\|_{L_t^2(L^2)} &\le \|\rho\partial_t u\|_{L_t^2(L^2)} + \|\rho u\cdot\nabla u\|_{L_t^2(L^2)} \\ &\le C_\varepsilon\bigl(\|\sqrt{\rho}\partial_t u\|_{L_t^2(L^2)} + \|u\|_{L_t^2(L^\infty)}\|\nabla u\|_{L_t^\infty(L^2)}\bigr) \le C_\varepsilon. \end{aligned} \tag{3.17}$$

To estimate $\|\nabla^2 u\|_{L_t^\infty(L^2)}$, following [3,4], we get, by first acting ∂_t to the momentum equation of (3.13) and then taking the L^2 inner product of the resulting equation with $\partial_t u$, that

$$\begin{aligned} &\frac12\frac{d}{dt}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|\nabla\partial_t u(t)\|_{L^2}^2 \\ &\quad = -\int_{\mathbb{R}^3}\Bigl(\partial_t\rho\partial_t u\cdot(u\cdot\nabla u) + \rho\partial_t u\cdot(\partial_t u\cdot\nabla u) - \frac32\operatorname{div}(\rho u)|\partial_t u|^2\Bigr)\,dx, \end{aligned} \tag{3.18}$$

where we used the transport equation $\partial_t\rho = -\operatorname{div}(\rho u)$. Next we estimate term by term above. Using once again the transport equation $\partial_t\rho = -\operatorname{div}(\rho u)$ and integration by parts, one has

$$\begin{aligned} &\int_{\mathbb{R}^3}\partial_t\rho\partial_t u\cdot(u\cdot\nabla u)\,dx \\ &\quad = \sum_{j=1}^3\Bigl(\int_{\mathbb{R}^3}\rho u^j\partial_j\partial_t u(u\cdot\nabla u)\,dx + \int_{\mathbb{R}^3}\rho u^j\partial_t u(\partial_j u\cdot\nabla u)\,dx + \int_{\mathbb{R}^3}\rho u^j\partial_t u(u\cdot\nabla\partial_j u)\,dx\Bigr). \end{aligned}$$

Notice that

$$\|u\|_{L^\infty} \le C\|\nabla u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac12},$$

we get, by applying Hölder inequality, that

$$\sum_{j=1}^{3}\left|\int_{\mathbb{R}^3}\rho u^j\partial_j\partial_t u(u\cdot\nabla u)\,dx\right|\le\|\rho\|_{L^\infty}\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}\|\nabla\partial_t u\|_{L^2}$$
$$\le C\|\rho\|_{L^\infty}\|\nabla u\|_{L^2}^2\left\|\nabla^2 u\right\|_{L^2}\|\nabla\partial_t u\|_{L^2}$$
$$\le C_\varepsilon\|\nabla u\|_{L^2}^4\left\|\nabla^2 u\right\|_{L^2}^2+\frac{1}{8}\|\nabla\partial_t u\|_{L^2}^2.$$

Along the same line, we have

$$\sum_{j=1}^{3}\left|\int_{\mathbb{R}^3}\rho u^j\partial_t u(\partial_j u\cdot\nabla u)\,dx\right|\le\|\rho\|_{L^\infty}\|u\|_{L^\infty}\|\partial_t u\|_{L^6}\|\nabla u\|_{L^3}\|\nabla u\|_{L^2}$$
$$\le\|\rho\|_{L^\infty}\|\nabla u\|_{L^2}^2\left\|\nabla^2 u\right\|_{L^2}\|\nabla\partial_t u\|_{L^2}$$
$$\le C_\varepsilon\|\nabla u\|_{L^2}^4\left\|\nabla^2 u\right\|_{L^2}^2+\frac{1}{8}\|\nabla\partial_t u\|_{L^2}^2,$$

and

$$\sum_{j=1}^{3}\left|\int_{\mathbb{R}^3}\rho u^j\partial_t u(u\cdot\nabla\partial_j u)\,dx\right|\le\|\rho\|_{L^\infty}^{\frac{1}{2}}\|u\|_{L^\infty}^2\|\sqrt{\rho}\partial_t u\|_{L^2}\left\|\nabla^2 u\right\|_{L^2}$$
$$\le C_\varepsilon\|\nabla u\|_{L^2}^2\left\|\nabla^2 u\right\|_{L^2}^2+\left\|\nabla^2 u\right\|_{L^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}^2.$$

This yields

$$\left|\int_{\mathbb{R}^3}\partial_t\rho\partial_t u\cdot(u\cdot\nabla u)\right|\le\left\|\nabla^2 u\right\|_{L^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}^2$$
$$+C_\varepsilon\left(1+\|\nabla u\|_{L^2}^2\right)\|\nabla u\|_{L^2}^2\left\|\nabla^2 u\right\|_{L^2}^2+\frac{1}{4}\|\nabla\partial_t u\|_{L^2}^2. \tag{3.19}$$

Similar to the derivation of (3.19), we get

$$\left|\int_{\mathbb{R}^3}\rho\partial_t u\cdot(\partial_t u\cdot\nabla u)\,dx\right|\le\|\nabla u\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}^2, \tag{3.20}$$

and

$$\left|\int_{\mathbb{R}^3}\operatorname{div}(\rho u)|\partial_t u|^2\,dx\right|=2\left|\int_{\mathbb{R}^3}\rho u\cdot(\partial_t u\cdot\nabla\partial_t u)\,dx\right|$$
$$\le\|\rho\|_{L^\infty}^{\frac{1}{2}}\|u\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|\nabla\partial_t u\|_{L^2}$$
$$\le C_\varepsilon\|u\|_{L^\infty}^2\|\sqrt{\rho}\partial_t u\|_{L^2}^2+\frac{1}{4}\|\nabla\partial_t u\|_{L^2}^2. \tag{3.21}$$

Plugging Estimates (3.19)–(3.21) into (3.18), we obtain

$$\frac{d}{dt}\left\|\sqrt{\rho}\partial_t u(t)\right\|_{L^2}^2+\left\|\nabla\partial_t u(t)\right\|_{L^2}^2\le C_\varepsilon\left(1+\left\|\nabla u(t)\right\|_{L^2}^2\right)\left\|\nabla u(t)\right\|_{L^2}^2\left\|\nabla^2 u(t)\right\|_{L^2}^2$$
$$+C_\varepsilon\left(\left\|u(t)\right\|_{L^\infty}^2+\left\|\nabla u(t)\right\|_{L^\infty}+\left\|\nabla^2 u(t)\right\|_{L^2}^2\right)\left\|\sqrt{\rho}\partial_t u(t)\right\|_{L^2}^2. \tag{3.22}$$

Notice that by taking L^2 inner product of the momentum equation of (3.13) at time $t=0$ with $\partial_t u(0,x)$, we get

$$\left\|\sqrt{\rho_0}\partial_t u(0)\right\|_{L^2}^2+\int_{\mathbb{R}^3}(\rho_0 u_0\cdot\nabla u_0-\Delta u_0)\cdot\partial_t u(0)\,dx=0,$$

which gives rise to

$$\begin{aligned}\big\|\sqrt{\rho}\partial_t u(0)\big\|_{L^2} &\le C\big(\big\|\rho u\cdot\nabla u(0)\big\|_{L^2}+\big\|\Delta u(0)\big\|_{L^2}\big)\\ &\le C\big(\|\rho_{0,\varepsilon}\|_{L^\infty}\|u_{0,\varepsilon}\|_{L^\infty}\|\nabla u_{0,\varepsilon}\|_{L^2}+\|\Delta u_{0,\varepsilon}\|_{L^2}\big)\le C_\varepsilon.\end{aligned}$$

Hence by virtue of (3.12), (3.15) and (3.17), we get, by applying Gronwall Lemma to (3.22) that

$$\begin{aligned}&\|\sqrt{\rho}\partial_t u\|_{L^\infty_t(L^2)}+\|\nabla\partial_t u\|_{L^2_t(L^2)}\\ &\quad\le C_\varepsilon\big(\big\|\sqrt{\rho}\partial_t u(0)\big\|_{L^2}+\big(1+\|\nabla u\|^2_{L^\infty_t(L^2)}\big)\|\nabla u\|^2_{L^\infty(L^2)}\big\|\nabla^2 u\big\|^2_{L^2_t(L^2)}\big)\\ &\qquad\times\exp\big(\|u\|^2_{L^2_t(L^\infty)}+\|\nabla u\|_{L^1_t(L^\infty)}+\big\|\nabla^2 u\big\|^2_{L^2_t(L^2)}\big)\le C_\varepsilon\end{aligned}\tag{3.23}$$

for $t<T^*$.

On the other hand, we deduce from (3.16) that

$$\begin{aligned}\big\|\nabla^2 u\big\|_{L^\infty_t(L^2)}+\|\nabla\Pi\|_{L^\infty_t(L^2)} &\le C\big(\|\rho\partial_t u\|_{L^\infty_t(L^2)}+\|\rho u\cdot\nabla u\|_{L^\infty_t(L^2)}\big)\\ &\le C_\varepsilon\big(\|\sqrt{\rho}\partial_t u\|_{L^\infty_t(L^2)}+\|\nabla u\|^2_{L^\infty_t(L^2)}\big)+\frac{1}{2}\big\|\nabla^2 u\big\|_{L^\infty_t(L^2)},\end{aligned}$$

which along with (3.15) and (3.23) implies that

$$\big\|\nabla^2 u\big\|_{L^\infty_t(L^2)}+\|\nabla\Pi\|_{L^\infty_t(L^2)}\le C_\varepsilon\quad\text{for } t<T^*.\tag{3.24}$$

It follows the same line that

$$\begin{aligned}&\|\Delta u\|_{L^2_t(\dot H^1)}+\|\nabla\Pi\|_{L^2_t(\dot H^1)}\le\big\|\nabla(\rho\partial_t u)\big\|_{L^2_t(L^2)}+\big\|\nabla(\rho u\cdot\nabla u)\big\|_{L^2_t(L^2)}\\ &\quad\le C\big(\|\rho\|_{L^\infty_t(L^\infty)}+\|\nabla\rho\|_{L^\infty_t(L^3)}\big)\big(\|\partial_t u\|_{L^2_t(\dot H^1)}+\big\|\nabla(u\cdot\nabla u)\big\|_{L^2_t(L^2)}\big),\end{aligned}$$

from which, we infer that for $t<T^*$

$$\begin{aligned}\|\Delta u\|_{L^2_t(\dot H^1)}+\|\nabla\Pi\|_{L^2_t(\dot H^1)} &\le C_\varepsilon\big(1+\|b\|_{L^\infty_t(\mathcal{B}^{\frac32}_2)}\big)\big(\|\partial_t u\|_{L^2_t(\dot H^1)}\\ &\quad+\big(\|\nabla u\|^{\frac12}_{L^2_t(L^2)}\big\|\nabla^2 u\big\|^{\frac12}_{L^2_t(L^2)}+\|u\|_{L^2_t(L^\infty)}\big)\big\|\nabla^2 u\big\|_{L^\infty_t(L^2)}\big)\le C_\varepsilon,\end{aligned}\tag{3.25}$$

where we used (1.17) so that

$$\|\nabla\rho\|_{L^\infty_t(L^3)}\le C_\varepsilon\|\nabla b\|_{L^\infty_t(L^3)}\le C_\varepsilon\|b\|_{L^\infty_t(\mathcal{B}^{\frac32}_2)}.$$

Summing up (3.14), (3.15), (3.17) and (3.23)–(3.25), we arrive at

$$\|u\|_{L^\infty_t(H^2)}+\|\nabla u\|_{L^2_t(H^2)}+\|\partial_t u\|_{L^2_t(H^1)}+\|\nabla\Pi\|_{L^2_t(H^1)}\le C_\varepsilon\quad\text{for } t<T^*.\tag{3.26}$$

Finally thanks to (1.17) and (3.26), to derive the $L^\infty_t(H^2)$ bounds for $\rho-1$, we only need to estimate $\|\nabla^2\rho\|_{L^\infty_t(L^2)}$. Indeed taking ∇^2 to continuous equation of (3.13), and then taking the L^2 inner product of the resulting equation with $\nabla^2\rho$, we obtain

$$\|\rho\|_{L^\infty_t(\dot H^2)}\le\|\rho_0\|_{\dot H^2}\exp\left(C\int_0^t\|\nabla u\|_{H^2}\,dt'\right)\quad\text{for } t<T^*.\tag{3.27}$$

(3.26) together with (3.27) ensures that $T^*=\infty$. The uniqueness of this class of solution for (1.2) is classical (see [4,14] for instance), and we omit the details here. This completes the proof of Theorem 1.3. □

Conflict of interest statement

None declared.

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Appendix A. Littlewood–Paley analysis

In this section, we collect some basic facts on Littlewood–Paley theory, which we used throughout the paper. For the convenience of the readers, we first recall the following Bernstein type lemma from [12,21]:

Lemma A.1. *Let $\mathcal{B}_h$ (resp. $\mathcal{B}_v$) a ball of $\mathbb{R}^2_h$ (resp. $\mathbb{R}_v$), and $\mathcal{C}_h$ (resp. $\mathcal{C}_v$) a ring of $\mathbb{R}^2_h$ (resp. $\mathbb{R}_v$); let $1 \le p_2 \le p_1 \le \infty$ and $1 \le q_2 \le q_1 \le \infty$. Then there hold:*

If the support of $\widehat{a}$ is included in $2^k\mathcal{B}_h$, then

$$\left\| \partial^\alpha_{x_h} a \right\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{k(|\alpha|+2(\frac{1}{p_2}-\frac{1}{p_1}))} \|a\|_{L^{p_2}_h(L^{q_1}_v)}.$$

If the support of $\widehat{a}$ is included in $2^\ell\mathcal{B}_v$, then

$$\left\| \partial^\beta_{x_3} a \right\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{\ell(\beta+(\frac{1}{q_2}-\frac{1}{q_1}))} \|a\|_{L^{p_1}_h(L^{q_2}_v)}.$$

If the support of $\widehat{a}$ is included in $2^k\mathcal{C}_h$, then

$$\|a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{-kN} \sup_{|\alpha|=N} \left\| \partial^\alpha_{x_h} a \right\|_{L^{p_1}_h(L^{q_1}_v)}.$$

If the support of $\widehat{a}$ is included in $2^\ell\mathcal{C}_v$, then

$$\|a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{-\ell N} \left\| \partial^N_{x_3} a \right\|_{L^{p_1}_h(L^{q_1}_v)}.$$

To consider the product of a distribution in the isentropic Besov space with a distribution in the anisotropic Besov space, we need the following result which allows to embed isotropic Besov spaces into the anisotropic ones.

Proposition A.1. *(See Lemma 2.2 of [11].) Let s and t be positive real numbers. Then for any $p \in [1,\infty]$, one has*

$$\|f\|_{\mathfrak{B}^{s,t}_p} \lesssim \|f\|_{\mathcal{B}^{s+t}_p}.$$

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we will use Chemin–Lerner type spaces $\widetilde{L}^\lambda_T(B^s_{p,r}(\mathbb{R}^3))$ (see [6] for instance).

Definition A.1. Let $(r,\lambda,p) \in [1,+\infty]^3$ and $T \in]0,+\infty]$. We define $\widetilde{L}^\lambda_T(B^s_{p\,r})$ as the completion of $C([0,T];\mathcal{S})$ by the norm

$$\|f\|_{\widetilde{L}^\lambda_T(B^s_{p,r})} \stackrel{\text{def}}{=} \left(\sum_{q\in\mathbb{Z}} 2^{qrs} \left(\int_0^T \left\| \Delta_q f(t) \right\|^\lambda_{L^p} dt \right)^{\frac{r}{\lambda}} \right)^{\frac{1}{r}} < \infty,$$

with the usual change if $r = \infty$.

We also need the following form of functional framework, which corresponds to the weighted Chemin–Lerner type norm introduced in [22,23] for $r = 1$.

Definition A.2. Let $f(t) \in L^1_{loc}(\mathbb{R}^+)$, $f(t) \geq 0$ and X be a Banach space. We define

$$\|u\|_{L^1_{T,f}(X)} \stackrel{\text{def}}{=} \int_0^T f(t)\big\|u(t)\big\|_X\, dt.$$

To study product laws between distributions, we need para-differential decomposition of Bony [7]: let $a, b \in \mathcal{S}'(\mathbb{R}^3)$,

$$\begin{aligned}
&ab = T(a,b) + \mathcal{R}(a,b), \quad \text{or} \quad ab = T(a,b) + \bar{T}(a,b) + R(a,b), \quad \text{where}\\
&T(a,b) = \sum_{j\in\mathbb{Z}} S_{j-1}a\Delta_j b, \quad \bar{T}(a,b) = T(b,a), \quad \mathcal{R}(a,b) = \sum_{j\in\mathbb{Z}} \Delta_j a S_{j+2} b, \quad \text{and}\\
&R(a,b) = \sum_{j\in\mathbb{Z}} \Delta_j a\tilde{\Delta}_j b, \quad \text{with} \quad \tilde{\Delta}_j b = \sum_{\ell=j-1}^{j+1} \Delta_\ell a.
\end{aligned} \tag{A.1}$$

Finally we recall the following law of product from [11]:

Lemma A.2. *(See Lemma 2.3 of [11].) Let $p \geq q \geq 1$ with $\frac{1}{p} + \frac{1}{q} \leq 1$, and $s_1 \leq \frac{2}{q}$, $s_2 \leq \frac{2}{p}$ with $s_1 + s_2 > 0$. Let $\sigma_1 \leq \frac{1}{q}$, $\sigma_2 \leq \frac{1}{p}$ with $\sigma_1 + \sigma_2 > 0$. Then for $a \in \mathfrak{B}^{s_1,\sigma_1}_q(\mathbb{R}^3)$, $b \in \mathfrak{B}^{s_2,\sigma_2}_p(\mathbb{R}^3)$, one has $ab \in \mathfrak{B}^{s_1+s_2-\frac{2}{q},\sigma_1+\sigma_2-\frac{1}{q}}_p(\mathbb{R}^3)$, and*

$$\|ab\|_{\mathfrak{B}^{s_1+s_2-\frac{2}{q},\sigma_1+\sigma_2-\frac{1}{q}}_p} \lesssim \|a\|_{\mathfrak{B}^{s_1,\sigma_1}_q} \|b\|_{\mathfrak{B}^{s_2,\sigma_2}_p}.$$

References

[1] H. Abidi, Équation de Navier–Stokes avec densité et viscosité variables dans l'espace critique, Rev. Mat. Iberoam. 23 (2) (2007) 537–586.
[2] H. Abidi, M. Paicu, Existence globale pour un fluide inhomogéne, Ann. Inst. Fourier (Grenoble) 57 (2007) 883–917.
[3] H. Abidi, G. Gui, P. Zhang, On the decay and stability to global solutions of the 3-D inhomogeneous Navier–Stokes equations, Commun. Pure Appl. Math. 64 (2011) 832–881.
[4] H. Abidi, G. Gui, P. Zhang, On the wellposedness of 3-D inhomogeneous Navier–Stokes equations in the critical spaces, Arch. Ration. Mech. Anal. 204 (2012) 189–230.
[5] H. Abidi, G. Gui, P. Zhang, Wellposedness of 3-D inhomogeneous Navier–Stokes equations with highly oscillating initial velocity field, J. Math. Pures Appl. 100 (2013) 166–203.
[6] H. Bahouri, J.Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren Math. Wiss., vol. 343, Springer-Verlag, Berlin, Heidelberg, 2011.
[7] J.M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Sci. Éc. Norm. Super. 14 (4) (1981) 209–246.
[8] J.Y. Chemin, B. Desjardins, I. Gallagher, E. Grenier, Fluids with anisotropic viscosity, Modél. Math. Anal. Numér. 34 (2000) 315–335.
[9] J.Y. Chemin, I. Gallagher, Large, global solutions to the Navier–Stokes equations slowly varying in one direction, Trans. Am. Math. Soc. 362 (2010) 2859–2873.
[10] J.Y. Chemin, I. Gallagher, M. Paicu, Global regularity for some classes of large solutions to the Navier–Stokes equations, Ann. of Math. (2) 173 (2011) 983–1012.
[11] J.Y. Chemin, M. Paicu, P. Zhang, Global large solutions to 3-D inhomogeneous Navier–Stokes system with one slow variable, J. Differ. Equ. 256 (2014) 223–252.
[12] J.Y. Chemin, P. Zhang, On the global wellposedness to the 3-D incompressible anisotropic Navier–Stokes equations, Commun. Math. Phys. 272 (2007) 529–566.
[13] R. Danchin, Density-dependent incompressible viscous fluids in critical spaces, Proc. R. Soc. Edinb. A 133 (2003) 1311–1334.
[14] R. Danchin, Local and global well-posedness results for flows of inhomogeneous viscous fluids, Adv. Differ. Equ. 9 (2004) 353–386.
[15] R. Danchin, Local theory in critical spaces for compressible viscous and heat-conducting gases, Commun. Partial Differ. Equ. 26 (2001) 1183–1233.
[16] R. Danchin, P.B. Mucha, A Lagrangian approach for the incompressible Navier–Stokes equations with variable density, Commun. Pure Appl. Math. 65 (2012) 1458–1480.
[17] G. Gui, J. Huang, P. Zhang, Large global solutions to the 3-D inhomogeneous Navier–Stokes equations, J. Funct. Anal. 261 (2011) 3181–3210.
[18] J. Huang, M. Paicu, P. Zhang, Global wellposedness to incompressible inhomogeneous fluid system with bounded density and non-Lipschitz velocity, Arch. Ration. Mech. Anal. 209 (2013) 631–682.
[19] O.A. Ladyženskaja, V.A. Solonnikov, The unique solvability of an initial–boundary value problem for viscous incompressible inhomogeneous fluids (in Russian), in: Boundary Value Problems of Mathematical Physics, and Related Questions of the Theory of Functions, vol. 8, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 52 (1975) 52–109, 218–219.

[20] P.L. Lions, Mathematical Topics in Fluid Mechanics. Vol. 1. Incompressible Models, Oxford Lecture Ser. Math. Appl., vol. 3, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1996.
[21] M. Paicu, Équation anisotrope de Navier–Stokes dans des espaces critiques, Rev. Mat. Iberoam. 21 (2005) 179–235.
[22] M. Paicu, P. Zhang, Global solutions to the 3-D incompressible anisotropic Navier–Stokes system in the critical spaces, Commun. Math. Phys. 307 (2011) 713–759.
[23] M. Paicu, P. Zhang, Global solutions to the 3-D incompressible inhomogeneous Navier–Stokes system, J. Funct. Anal. 262 (2012) 3556–3584.
[24] J. Simon, Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure, SIAM J. Math. Anal. 21 (5) (1990) 1093–1117.