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Infinitely many new curves of the Fučík spectrum

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Dedicated to the memory of our Fathers

Abstract

In this paper we present some results on the Fučík spectrum for the Laplace operator, that give new information on its structure. In particular, these results show that, if Ω is a bounded domain of $\mathbb{R}^N$ with $N > 1$, then the Fučík spectrum has infinitely many curves asymptotic to the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$, where λ_1 denotes the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$. Notice that the situation is quite different in the case $N = 1$; in fact, in this case the Fučík spectrum may be obtained by direct computation and one can verify that it includes only two curves asymptotic to these lines.

Résumé

Nous présentons des résultats qui donnent de nouvelles informations sur la structure du spectre de Fučík pour l'opérateur de Laplace. En particulier, ces résultats montrent que, si Ω est un domaine borné de $\mathbb{R}^N$ avec $N > 1$, alors le spectre de Fučík a un nombre infini de courbes qui ont comme asymptotes les droites $\{\lambda_1\} \times \mathbb{R}$ et $\mathbb{R} \times \{\lambda_1\}$, où λ_1 est la première valeur propre de l'operateur $-\Delta$ in $H_0^1(\Omega)$. La situation est bien différente dans le cas $N = 1$; en effect, dans ce cas on peut vérifier qu'il y a seulement deux courbes dans le spectre de Fučík, qui ont ces droites comme asymptotes.

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1. Introduction

Let Ω be a bounded connected domain of $\mathbb{R}^N$ with $N \geq 1$ and set $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$. The Fučík spectrum of the Laplace operator $-\Delta$ in $H_0^1(\Omega)$ is defined as the set Σ of all the pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that the Dirichlet problem

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$$\Delta u - \alpha u^- + \beta u^+ = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

has nontrivial solutions (i.e. $u \in H_0^1(\Omega)$, $u \not\equiv 0$).

The Fučík spectrum arises, for example, in the study of problems of the type

$$\Delta u + g(x,u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

where g is a Carathéodory function in $\Omega \times \mathbb{R}$ such that

$$\lim_{t\to-\infty} \frac{g(x,t)}{t} = \alpha, \qquad \lim_{t\to+\infty} \frac{g(x,t)}{t} = \beta, \quad \forall x \in \Omega, \tag{1.3}$$

with α and β in $\mathbb{R}$. These problems may lack of compactness in the sense that the well known Palais–Smale compactness condition fails if the pair (α, β) belongs to the Fučík spectrum Σ.

After the pioneering researches in [8,1] and the first papers [12,17], many works have been devoted to study these problems (see, for example, the references in [27–29]). In [27–29] we obtained new solutions of problems of this type using a method which does not require to know whether or not $(\alpha, \beta) \in \Sigma$ and, in addition, may be useful to obtain new information on the structure of Σ (a similar method is used also in [9,10]).

Let us denote by $\lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$ the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$. It is clear that Σ contains all the pairs (λ_i, λ_i) (which are the only pairs (α, β) in Σ such that $\alpha = \beta$) and includes the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$; if $\alpha \ne \lambda_1$, $\beta \ne \lambda_1$ and $(\alpha, \beta) \in \Sigma$, then $\alpha > \lambda_1$, $\beta > \lambda_1$ and the eigenfunctions u corresponding to (α, β) are sign changing functions; moreover, $(\alpha, \beta) \in \Sigma$ if and only if $(\beta, \alpha) \in \Sigma$ because a function u solves (1.1) if and only if $-u$ solves (1.1) with (β, α) in place of (α, β).

Several papers have been devoted to study the structure of Σ and its relation with existence and multiplicity of solutions for equations with asymmetric nonlinearities (see, for example, [2–4,6,7,11–25,33–37] etc.). In [12] it is shown that the two lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ are isolated in Σ. Many results concern the curve in Σ emanating from each pair (λ_i, λ_i) (local existence and multiplicity, variational characterizations, local and global properties, etc.).

Combining these results, one can infer, in particular, that Σ contains a first curve which passes through (λ_2, λ_2) and extends to infinity. In [15] the authors prove directly the existence of such a first curve, show that it is asymptotic to the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$, give a variational characterization of it and deduce that all the corresponding eigenfunctions have exactly two nodal regions (extending the well known nodal domain theorem of Courant).

In the case $N = 1$, Σ may be obtained by direct computation. It consists of curves emanating from the pairs (λ_i, λ_i), $\forall i \in \mathbb{N}$; if i is an even positive integer, there exists only one curve while, if i is odd, there exist exactly two curves emanating from (λ_i, λ_i). All these curves are smooth, unbounded and decreasing (i.e., on each curve, α decreases as β increases); moreover, on each curve, α tends to an eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ as $\beta \to +\infty$; conversely, for every eigenvalue λ_i there exist exactly three curves asymptotic to the lines $\{\lambda_i\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_i\}$; they pass, respectively, through the pairs $(\lambda_{2i-1}, \lambda_{2i-1})$, $(\lambda_{2i}, \lambda_{2i})$ and $(\lambda_{2i+1}, \lambda_{2i+1})$. In particular, if $N = 1$, there are only two nontrivial curves of Σ, asymptotic to $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$.

On the contrary, the situation is quite different in the case $N > 1$. In fact, using the method developed in [27–29], we can show that, if Ω is a domain of $\mathbb{R}^N$ with $N > 1$, there exist infinitely many curves of the Fučík spectrum Σ, asymptotic to the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$. In the present paper we consider the case $N \ge 3$. The case $N = 2$, which requires more refined estimates, is considered in [32].

The main result of this paper may be stated as follows (this result has been first announced in [30]).

Theorem 1.1. *Let Ω be a bounded domain of $\mathbb{R}^N$ with $N \ge 3$. Then, there exists a nondecreasing sequence $(b_k)_k$ of positive numbers, having the following properties. For every positive integer k and for all $\beta > b_k$ there exists $\alpha_{k,\beta} > \lambda_1$ such that the pair $(\alpha_{k,\beta}, \beta)$ belongs to the Fučík spectrum Σ. Moreover, for every positive integer k, $\alpha_{k,\beta}$ depends continuously on β in $]b_k, +\infty[$, $\alpha_{k,\beta} < \alpha_{k+1,\beta}$ for all $\beta > b_{k+1}$ and $\lim_{\beta\to+\infty} \alpha_{k,\beta} = \lambda_1$.*

The proof follows directly from Theorem 2.1. It is clear that, if we replace (α, β) and u by (β, α) and $-u$, from Theorem 1.1 we obtain infinitely many curves of Σ asymptotic to the line $\mathbb{R} \times \{\lambda_1\}$.

Notice that, even for $k = 1$, Theorem 1.1 does not give the first curve of the Fučík spectrum (see for instance [15]) since, for all $\beta > b_k$, the pair $(\alpha_{k,\beta}, \beta)$ does not belong to the first curve (see also Remark 5.8 for more details).

The method we use for the proof is completely variational. For all $\beta > 0$, we consider the functional f_β defined by $f_\beta(u) = \int_\Omega [|Du|^2 - \beta(u^+)^2]dx$, constrained on the set $S = \{u \in H_0^1(\Omega) : \int_\Omega (u^-)^2 dx = 1\}$. For $\beta > 0$ large enough, the eigenfunction u is obtained as a constrained critical point for f_β on S, while α arises as the Lagrange multiplier with respect to the constraint S.

For every positive integer k, the eigenfunction $u_{k,\beta}$ corresponding to the pair $(\alpha_{k,\beta}, \beta)$, we obtain in this way, presents k bumps; for $\beta > 0$ large enough, the set $\{x \in \Omega : u_{k,\beta}(x) < 0\}$ is a connected open subset of Ω while the set $\{x \in \Omega : u_{k,\beta}(x) > 0\}$ has exactly k connected components. As $\beta \to +\infty$, the bumps concentrate near points. We describe the asymptotic behaviour of the concentration points and, in particular, we show that, if the distance between two concentration points tends to zero as $\beta \to +\infty$, then the approaching rate is less than the concentration rate, so that the bumps remain quite distinct; moreover, we describe the asymptotic profile of the rescaled bumps.

Finally, let us point out a natural question: where come from the curves given by Theorem 1.1? (they might come from bifurcations of the first curve of the Fučík spectrum, or from pairs (λ_i, λ_i) of higher eigenvalues, or may be they do not meet the line $\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha = \beta\}$, etc.). It is a widely open problem which perhaps might produce interesting results (see also Remark 5.9 for a more detailed discussion of this question). The paper is organized as follows. In Section 2 we state the main results which, in particular, imply and specify Theorem 1.1. In Section 3 we describe the variational framework and introduce a functional $f_{\beta,\varepsilon}$, converging to f_β as $\varepsilon \to 0$, which for all $\varepsilon > 0$ presents more manageable variational properties with respect to f_β. In Section 4 we obtain some useful asymptotic estimates as $\beta \to +\infty$. Finally, in Section 5 we let $\varepsilon \to 0$, and prove the main results. Also we discuss some generalizations, forthcoming results on related questions, open problems, etc.

2. Statement of the main results

Let us denote by e_1 the positive eigenfunction related to the first eigenvalue λ_1, normalized in $L^2(\Omega)$, i.e. $e_1 \in H_0^1(\Omega)$, $e_1 > 0$, $\Delta e_1 + \lambda_1 e_1 = 0$ in Ω and $\int_\Omega e_1^2 dx = 1$ (since Ω is a connected domain, e_1 is unique and strictly positive in Ω). For every open subset A of $\mathbb{R}^N$, we denote by $\lambda_1(A) \le \lambda_2(A) \le \lambda_3(A) \le \dots$ the eigenvalues of $-\Delta$ in $H_0^1(A)$; every function in $H_0^1(A)$ is extended outside A by the value zero. The main results presented in this paper may be gathered in the following theorem (which contains and specifies Theorem 1.1).

Theorem 2.1. *Let Ω be a bounded connected domain of $\mathbb{R}^N$ with $N \ge 3$. Then, there exists a nondecreasing sequence $(b_k)_k$ of positive numbers, having the following properties. For every positive integer k and for all $\beta > b_k$, there exist $\alpha_{k,\beta} > \lambda_1$ and $u_{k,\beta} \in H_0^1(\Omega)$, with $u_{k,\beta}^+ \not\equiv 0$ and $u_{k,\beta}^- \not\equiv 0$, such that (1.1), with $\alpha = \alpha_{k,\beta}$ and $u = u_{k,\beta}$, is satisfied for all $\beta > b_k$. Moreover, for every positive integer k, $\alpha_{k,\beta}$ depends continuously on β in $]b_k, +\infty[$, $\alpha_{k,\beta} < \alpha_{k+1,\beta}$, $\forall \beta > b_{k+1}$, $\alpha_{k,\beta} \to \lambda_1$, as $\beta \to +\infty$, while $u_{k,\beta} \to -e_1$ in $H_0^1(\Omega)$.*

In addition, there exist $r > 0$ and, for all $k \ge 1$ and $\beta > b_k$, k points $x_{1,\beta}, \dots, x_{k,\beta}$ in Ω such that

(1) $\mathrm{dist}(x_{i,\beta}, \partial\Omega) > \frac{r}{\sqrt{\beta}}$, $\forall i \in \{1, \dots, k\}$, $|x_{i,\beta} - x_{j,\beta}| > \frac{2r}{\sqrt{\beta}}$ *for* $i \ne j$;
(2) $u_{k,\beta}(x) \le 0$, $\forall x \in \Omega \setminus \bigcup_{i=1}^k B(x_{i,\beta}, \frac{r}{\sqrt{\beta}})$, *and* $u_{k,\beta}^+ \not\equiv 0$ *in* $B(x_{i,\beta}, \frac{r}{\sqrt{\beta}})$, $\forall i \in \{1, \dots, k\}$;
(3) $\lim_{\beta \to +\infty} e_1(x_{i,\beta}) = \max_\Omega e_1$, $\forall i \in \{1, \dots, k\}$, $\lim_{\beta \to +\infty} \sqrt{\beta}|x_{i,\beta} - x_{j,\beta}| = \infty$ *for* $i \ne j$;
(4) *if* $\rho_\beta > 0$, $\forall \beta > b_k$, $\lim_{\beta \to +\infty} \rho_\beta = 0$ *and* $\lim_{\beta \to +\infty} (\rho_\beta \sqrt{\beta}) = \infty$, *then*

$$\lim_{\beta \to +\infty} \sup \left\{ \left| u_{k,\beta}(x) + e_1(x) \right| : x \in \Omega \setminus \bigcup_{i=1}^k B(x_{i,\beta}, \rho_\beta) \right\} = 0;$$

(5) *if,* $\forall k \in \mathbb{N}$, $\forall i \in \{1, \dots, k\}$, $\forall \beta > b_k$ *and* $\forall x \in \sqrt{\beta}(\Omega - x_{i,\beta})$ *we set* $U_{i,k,\beta}(x) = \frac{1}{s_{i,k,\beta}} u_{k,\beta}(\frac{x}{\sqrt{\beta}} + x_{i,\beta})$ *where* $s_{i,k,\beta} = \sup\{u_{k,\beta}(x) : x \in B(x_{i,\beta}, \frac{r}{\sqrt{\beta}})\}$, *then the rescaled function* $U_{i,k,\beta}$ *converges as* $\beta \to +\infty$ *to the radial solution* U *of the problem*

$$\Delta U + U^+ = 0 \quad \text{in } \mathbb{R}^N, \qquad U(0) = 1 \tag{2.1}$$

and the convergence is uniform on the compact subsets of $\mathbb{R}^N$.

The proof will be given in Section 5. Let us point out that Theorem 2.1 holds true also for $N=2$, but in this case the proof requires more refined estimates; moreover, the asymptotic behaviour of $u_{k,\beta}$, as $\beta\to+\infty$, is quite different in the cases $N=2$ and $N>2$. In fact, if $N=2$, we have $\lim_{\beta\to+\infty} s_{i,k,\beta}=0$, $\forall k\in\mathbb{N}$, $\forall i\in\{1,\dots,k\}$, while, if $N>2$, $\lim_{\beta\to+\infty} s_{i,k,\beta}=c$, $\forall k\in\mathbb{N}$, $\forall i\in\{1,\dots,k\}$, where c is a positive constant depending only on N and $\sup_\Omega e_1$. This different behaviour is strictly related to the fact that, if U is the radial solution of problem (2.1), then $\inf_{\mathbb{R}^N} U=-\infty$ for $N=2$, while $\inf_{\mathbb{R}^N} U>-\infty$ for $N>2$. The case $N=2$ is presented in [32].

3. The variational framework

In order to prove Theorem 2.1, for every positive integer k we construct k-peaks eigenfunctions of the following type. For every $\beta>0$, let us set $r_\beta=\frac{3\bar r_1}{\sqrt{\beta}}$ where $\bar r_1$ is the radius of the balls in $\mathbb{R}^N$ for which the first eigenvalue of the Laplace operator is equal to 1, i.e.

$$\min\left\{\int_{B(0,\bar r_1)}|Du|^2dx : u\in H_0^1\big(B(0,\bar r_1)\big),\ \int_{B(0,\bar r_1)}u^2dx=1\right\}=1.$$

Let us consider the set

$$\Omega_{k,\beta}=\big\{(x_1,\dots,x_k)\in\Omega^k : |x_i-x_j|\ge 2r_\beta \text{ if } i\ne j,\ \operatorname{dist}(x_i,\partial\Omega)\ge r_\beta \text{ for } i=1,\dots,k\big\}.$$

It is clear that $\Omega_{k,\beta}\ne\emptyset$ for β large enough and that, if $(x_1,\dots,x_k)\in\Omega_{k,\beta}$, the balls $B(x_1,r_\beta),\dots,B(x_k,r_\beta)$ are pairwise disjoint and included in Ω.

We say that a function $u\in H_0^1(\Omega)$ belongs to $E^\beta_{x_1,\dots,x_k}$ (i.e. it is a k-peaks function with respect to the balls $B(x_1,r_\beta),\dots,B(x_k,r_\beta)$) if $u^+=\sum_{i=1}^k u_i^+$ where, for all $i\in\{1,\dots,k\}$, $u_i^+\in H_0^1(\Omega)$, $u_i^+\not\equiv 0$, $u_i^+\ge 0$ in Ω, $\|u_i^+\|^{-2}_{L^2(\Omega)}\int_\Omega x\cdot[u_i^+(x)]^2dx=x_i$ and $u_i^+(x)=0$, $\forall x\in\Omega\setminus B(x_i,r_\beta)$.

For all $\beta>0$ and $\varepsilon>0$, let us consider the functional $f_{\beta,\varepsilon}:H_0^1(\Omega)\to\mathbb{R}$ defined by

$$f_{\beta,\varepsilon}(u)=\int_\Omega|Du|^2dx-2\int_\Omega G_{\beta,\varepsilon}(u)dx, \tag{3.1}$$

where $G_{\beta,\varepsilon}(t)=\int_0^t g_{\beta,\varepsilon}(\tau)d\tau$, $\forall t\in\mathbb{R}$, with $g_{\beta,\varepsilon}(t)=0$, $\forall t\le\varepsilon$, and $g_{\beta,\varepsilon}(t)=\beta(t-\varepsilon)$, $\forall t\ge\varepsilon$.

Now, our aim is to find k-peaks functions that are constrained critical points for the functional $f_{\beta,\varepsilon}$ constrained on the set $S=\{u\in H_0^1(\Omega):\int_\Omega(u^-)^2dx=1\}$.

Let us consider the set $M^{\beta,\varepsilon}_{x_1,\dots,x_k}$ consisting of all the functions $u\in E^\beta_{x_1,\dots,x_k}$ such that $\|u^-\|_{L^2(\Omega)}=1$ and $f'_{\beta,\varepsilon}(u)[u_i^+]=0$ for $i=1,\dots,k$.

One can easily verify that for all $\varepsilon>0$, if a function $u\in E^\beta_{x_1,\dots,x_k}$ is a critical point for $f_{\beta,\varepsilon}$ constrained on S, then $u\in M^{\beta,\varepsilon}_{x_1,\dots,x_k}$ and, for every $i\in\{1,\dots,k\}$, $f'_{\beta,\varepsilon}(u+tu_i^+)[u_i^+]$ is positive for $t\in\,]-1,0[$ and negative for $t>0$ (since $\frac{1}{\tau}g_{\beta,\varepsilon}(\tau)$ is strictly increasing with respect to τ in $]\varepsilon,+\infty[$); so the function u is the unique maximum point for $f_{\beta,\varepsilon}$ on the set $\{u+tu_i^+ : t\in[-1,+\infty[\}$ (notice that, for $\varepsilon=0$, $f_{\beta,0}$ and $M^{\beta,0}_{x_1,\dots,x_k}$ do not have the same properties; it is the reason for which we first introduce the parameter $\varepsilon>0$ and then let $\varepsilon\to0$).

Proposition 3.1. *Let k be a positive integer, $\beta>0$ large enough so that $\Omega_{k,\beta}\ne\emptyset$ and consider a point $(x_1,\dots,x_k)\in\Omega_{k,\beta}$. Then, for all $\varepsilon>0$, $M^{\beta,\varepsilon}_{x_1,\dots,x_k}\ne\emptyset$ and the minimum of the functional $f_{\beta,\varepsilon}$ on the set $M^{\beta,\varepsilon}_{x_1,\dots,x_k}$ is achieved.*

Proof. We have $M^{\beta,\varepsilon}_{x_1,\dots,x_k}\ne\emptyset$ because of the choice of the radius r_β. In fact, taking into account that $\sqrt{\beta}r_\beta>\bar r_1$, one can find $k+1$ nonnegative functions $v_1,\dots,v_k,\bar v$ in $H_0^1(\Omega)$ such that $v_i=0$ in $\Omega\setminus B(x_i,r_\beta)$, $\int_\Omega|Dv_i|^2dx<\beta\int_\Omega v_i^2dx$, $\int_\Omega x\cdot v_i^2(x)dx=x_i\int_\Omega v_i^2dx$, $\int_\Omega\bar v v_idx=0$ for $i=1,\dots,k$ and $\int_\Omega\bar v^2dx=1$. Thus, taking into account that

$$\lim_{t\to+\infty}\frac{1}{t^2}f_{\beta,\varepsilon}(tv_i)=\int_\Omega|Dv_i|^2dx-\beta\int_\Omega v_i^2dx<0 \tag{3.2}$$

and that (since $\varepsilon > 0$)

$$\lim_{t\to 0^+} \frac{1}{t^2} f_{\beta,\varepsilon}(tv_i) = \int_\Omega |Dv_i|^2 dx > 0, \tag{3.3}$$

we infer that for all $\varepsilon > 0$ there exist k positive numbers $t_{1,\varepsilon}, \dots, t_{k,\varepsilon}$ such that the function $u = \sum_{i=1}^k t_{i,\varepsilon} v_i - \bar{v}$ belongs to $M^{\beta,\varepsilon}_{x_1,\dots,x_k}$.

Notice that $\inf\{f_{\beta,\varepsilon}(u) : u \in M^{\beta,\varepsilon}_{x_1,\dots,x_k}\} \ge \lambda_1$ for all $\beta > 0$, $\varepsilon > 0$, $(x_1, \dots, x_k) \in \Omega_{k,\beta}$. In fact, if $u \in M^{\beta,\varepsilon}_{x_1,\dots,x_k}$, we have $f_{\beta,\varepsilon}(u) = f_{\beta,\varepsilon}(-u^-) + \sum_{i=1}^k f_{\beta,\varepsilon}(u_i^+)$, where $f_{\beta,\varepsilon}(-u^-) = \int_\Omega |Du^-|^2 dx \ge \lambda_1$ (since $\|u^-\|_{L^2(\Omega)} = 1$) and $f_{\beta,\varepsilon}(u_i^+) > 0$ for $i = 1, \dots, k$ (because $u \in M^{\beta,\varepsilon}_{x_1,\dots,x_k}$ implies $f_{\beta,\varepsilon}(u_i^+) = \max\{f_{\beta,\varepsilon}(tu_i^+) : t \ge 0\} > 0$ for $\varepsilon > 0$).

Now, let us consider a minimizing sequence $(u_n)_n$ for $f_{\beta,\varepsilon}$ on $M^{\beta,\varepsilon}_{x_1,\dots,x_k}$. The same arguments as above show that (since $\sup\{f_{\beta,\varepsilon}(u_n) : n \in \mathbb{N}\} < +\infty$) we have

$$\lambda_1 \le \liminf_{n\to\infty} f_{\beta,\varepsilon}\big(-u_n^-\big) \le \limsup_{n\to\infty} f_{\beta,\varepsilon}\big(-u_n^-\big) < +\infty \tag{3.4}$$

and, for $i = 1, \dots, k$,

$$0 \le \liminf_{n\to\infty} f_{\beta,\varepsilon}\big(\big(u_n^+\big)_i\big) \le \limsup_{n\to\infty} f_{\beta,\varepsilon}\big(\big(u_n^+\big)_i\big) < +\infty. \tag{3.5}$$

Notice that $f_{\beta,\varepsilon}(-u_n^-) = \int_\Omega |Du_n^-|^2 dx$, so (3.4) implies that the sequence $(u_n^-)_n$ is bounded in $H_0^1(\Omega)$.

Now, let us prove that also the sequences $[(u_n^+)_i]_n$ are bounded in $H_0^1(\Omega)$, $\forall i \in \{1, \dots, k\}$. Taking into account that $f'_{\beta,\varepsilon}(u_n)[(u_n^+)_i] = 0$, $\forall n \in \mathbb{N}$, $\forall i \in \{1, \dots, k\}$, we have

$$\int_\Omega \big|D\big(u_n^+\big)_i\big|^2 dx = \int_\Omega g_{\beta,\varepsilon}\big(\big(u_n^+\big)_i\big)\big(u_n^+\big)_i dx \le \beta \int_\Omega \big(u_n^+\big)_i^2 dx. \tag{3.6}$$

Thus, it suffices to prove that the sequences $[(u_n^+)_i]_n$ are bounded in $L^2(\Omega)$ for $i = 1, \dots, k$. Arguing by contradiction, assume that (up to a subsequence) $\lim_{n\to\infty} \|(u_n^+)_i\|_{L^2(\Omega)} = \infty$ for some $i \in \{1, \dots, k\}$ and set $(\bar{u}_n)_i = \frac{(u_n^+)_i}{\|(u_n^+)_i\|_{L^2(\Omega)}}$. Then, (3.6) implies $\int_\Omega |D(\bar{u}_n)_i|^2 dx \le \beta$, $\forall n \in \mathbb{N}$; so (up to a subsequence) $[(\bar{u}_n)_i]_n$ converges weakly in $H_0^1(\Omega)$, in $L^2(\Omega)$ and a.e. in Ω to a function $\bar{u}_i \in H_0^1(\Omega)$. It follows that $\int_\Omega |D\bar{u}_i|^2 dx \le \beta$, $\int_\Omega \bar{u}_i^2 dx = 1$, $\bar{u}_i \ge 0$ in Ω and $\bar{u}_i = 0$ in $\Omega \setminus B(x_i, r_\beta)$. Moreover, one can verify by direct computation that the properties $f'_{\beta,\varepsilon}(u_n)[(u_n^+)_i] = 0$, $\forall n \in \mathbb{N}$, and $\lim_{n\to\infty} \|(u_n^+)_i\|_{L^2(\Omega)} = \infty$ imply $\lim_{n\to\infty} \int_\Omega |D(\bar{u}_n)_i|^2 dx = \beta$. As a consequence,

$$\lim_{n\to\infty} f'_{\beta,\varepsilon}\big(t(\bar{u}_n)_i\big)\big[(\bar{u}_n)_i\big] = 2t\beta - 2\int_\Omega g_{\beta,\varepsilon}(t\bar{u}_i)(\bar{u}_i) dx, \quad \forall t \ge 0. \tag{3.7}$$

Then, since $\int_\Omega \bar{u}_i^2 dx = 1$, we obtain for all $\varepsilon > 0$

$$\liminf_{t\to+\infty}\bigg[t\beta - \int_\Omega g_{\beta,\varepsilon}(t\bar{u}_i)\bar{u}_i dx\bigg] = \liminf_{t\to+\infty} \int_\Omega \big[\beta t\bar{u}_i - g_{\beta,\varepsilon}(t\bar{u}_i)\big]\bar{u}_i dx > 0. \tag{3.8}$$

Notice that, if we set $t_{n,i} = \|(u_n^+)_i\|_{L^2(\Omega)}$, we have $f'_{\beta,\varepsilon}(t(\bar{u}_n)_i)[(\bar{u}_n)_i] > 0$, $\forall t \in]0, t_{n,i}[$. Since $\lim_{n\to\infty} t_{n,i} = +\infty$, it follows

$$\begin{aligned} \liminf_{n\to\infty} f_{\beta,\varepsilon}\big(\big(u_n^+\big)_i\big) &= \liminf_{n\to\infty} \int_0^{t_{n,i}} f'_{\beta,\varepsilon}\big(t(\bar{u}_n)_i\big)\big[(\bar{u}_n)_i\big] dt \\ &\ge 2\int_0^\tau \bigg[t\beta - \int_\Omega g_{\beta,\varepsilon}(t\bar{u}_i)\bar{u}_i dx\bigg] dt, \quad \forall \tau > 0. \end{aligned} \tag{3.9}$$

Then, as $\tau \to +\infty$, from (3.8) we obtain $\lim_{n\to\infty} f_{\beta,\varepsilon}((u_n^+)_i) = +\infty$, in contradiction with (3.5). Thus, we can say that also the sequences $[(u_n^+)_i]_n$ are bounded in $H_0^1(\Omega)$ for $i = 1, \dots, k$. As a consequence, there exist $u^-, u_1^+, \dots, u_k^+$ in $H_0^1(\Omega)$ such that (up to a subsequence) u_n^- converges as $n \to \infty$ to u^- and $(u_n^+)_i$ converges to u_i^+, for $i = 1, \dots, k$, weakly in $H_0^1(\Omega)$, in $L^2(\Omega)$ and a.e. in Ω.

Now, let us prove that $u_i^+ \not\equiv 0$, $\forall i \in \{1, \dots, k\}$. Arguing by contradiction, assume that $u_i^+ \equiv 0$ for some $i \in \{1, \dots, k\}$. Then (because of the $L^2(\Omega)$ convergence) from (3.6) we infer that $\lim_{n\to\infty} \int_\Omega |D(u_n^+)_i|^2 dx = 0$, which implies $\lim_{n\to\infty} f_{\beta,\varepsilon}((u_n^+)_i) = 0$. Therefore, we obtain a contradiction if we prove that

$$\inf\{f_{\beta,\varepsilon}(v_i^+) : v \in M^{\beta,\varepsilon}_{x_1,\dots,x_k}\} > 0, \quad \forall \varepsilon > 0. \tag{3.10}$$

Taking into account that $f_{\beta,\varepsilon}(v_i^+) = \max\{f_{\beta,\varepsilon}(tv_i^+) : t > 0\}$, it is clear that it suffices to prove that there exist two positive constants $\rho_{\beta,\varepsilon}$ and $c_{\beta,\varepsilon}$ such that $f_{\beta,\varepsilon}(v) \ge c_{\beta,\varepsilon}$, $\forall v \in S_i(\rho_{\beta,\varepsilon})$, where

$$S_i(\rho_{\beta,\varepsilon}) = \left\{ v \in H_0^1\big(B(x_i, r_\beta)\big) : v \ge 0 \text{ in } B(x_i, r_\beta), \int_{B(x_i,r_\beta)} |Dv|^2 dx = \rho^2_{\beta,\varepsilon} \right\}. \tag{3.11}$$

In order to prove the existence of $c_{\beta,\varepsilon} > 0$ and $\rho_{\beta,\varepsilon} > 0$ with these properties, let us consider the positive integer $\tilde{j}$ such that

$$\lambda_{\tilde{j}}\big(B(x_i, 3\bar{r}_1)\big) \le 1 < \lambda_{\tilde{j}+1}\big(B(x_i, 3\bar{r}_1)\big). \tag{3.12}$$

Taking into account the choice of r_β, it follows that

$$\lambda_{\tilde{j}}\big(B(x_i, r_\beta)\big) \le \beta < \lambda_{\tilde{j}+1}\big(B(x_i, r_\beta)\big). \tag{3.13}$$

Now, let us denote by Σ^1_β and Σ^2_β the closed subspaces of $H_0^1(B(x_i, r_\beta))$ spanned by the eigenfunctions of the Laplace operator $-\Delta$ in $H_0^1(B(x_i, r_\beta))$, corresponding to eigenvalues $\lambda_j(B(x_i, r_\beta))$ with, respectively, $1 \le j \le \tilde{j}$ and $j \ge \tilde{j} + 1$.

For all $\beta > 0$ and $\varepsilon > 0$, there exists $\nu_{\beta,\varepsilon} > 0$ such that, if $v \in \Sigma^1_\beta$ and $\int_{B(x_i,r_\beta)} |Dv|^2 dx \le \nu^2_{\beta,\varepsilon}$, then $|v(x)| \le \varepsilon$, $\forall x \in B(x_i, r_\beta)$.

For all $v \in H_0^1(B(x_i, r_\beta))$ such that $\int_{B(x_i,r_\beta)} |Dv|^2 dx \le \nu^2_{\beta,\varepsilon}$, set $v = v_{1,\beta} + v_{2,\beta}$, with $v_{1,\beta} \in \Sigma^1_\beta$ and $v_{2,\beta} \in \Sigma^2_\beta$. Then, taking into account that $\int_{B(x_i,r_\beta)} |Dv_{1,\beta}|^2 dx \le \nu^2_{\beta,\varepsilon}$ and as a consequence $v_{1,\beta} \le \varepsilon$, we have

$$f_{\beta,\varepsilon}(v) = f_{\beta,\varepsilon}(v_{1,\beta} + v_{2,\beta}) = f_{\beta,\varepsilon}(v_{1,\beta} + v_{2,\beta}) - f_{\beta,\varepsilon}(v_{1,\beta}) + f_{\beta,\varepsilon}(v_{1,\beta}) \tag{3.14}$$

where $f_{\beta,\varepsilon}(v_{1,\beta}) = \int_{B(x_i,r_\beta)} |Dv_{1,\beta}|^2 dx$ and

$$\begin{aligned} f_{\beta,\varepsilon}(v_{1,\beta} + v_{2,\beta}) - f_{\beta,\varepsilon}(v_{1,\beta}) &\ge f'_{\beta,\varepsilon}(v_{1,\beta})[v_{2,\beta}] + \int_{B(x_i,r_\beta)} |Dv_{2,\beta}|^2 - \beta \int_{B(x_i,r_\beta)} v^2_{2,\beta} dx \\ &= \int_{B(x_i,r_\beta)} \big[|Dv_{2,\beta}|^2 - \beta v^2_{2,\beta}\big] dx \\ &\ge \left(1 - \frac{\beta}{\lambda_{\tilde{j}+1}(B(x_i, r_\beta))}\right) \int_{B(x_i,r_\beta)} |Dv_{2,\beta}|^2 dx \end{aligned} \tag{3.15}$$

because $f'_{\beta,\varepsilon}(v_{1,\beta})[v_{2,\beta}] = 0$ and $\int_{B(x_i,r_\beta)} |Dv_{2,\beta}|^2 dx \ge \lambda_{\tilde{i}+1}(B(x_i, r_\beta)) \int_{B(x_i,r_\beta)} v^2_{2,\beta} dx$.

It follows that, for a suitable constant $\tilde{c}_{\beta,\varepsilon} > 0$, we have $f_{\beta,\varepsilon}(v) \ge \tilde{c}_{\beta,\varepsilon} \int_{B(x_i,r_\beta)} |Dv|^2 dx$, $\forall v \in H_0^1(B(x_i, r_\beta))$ such that $\int_{B(x_i,r_\beta)} |Dv|^2 dx \le \nu^2_{\beta,\varepsilon}$. Therefore, it follows easily that there exist two constants $\rho_{\beta,\varepsilon} \in]0, \nu_{\beta,\varepsilon}[$ and $c_{\beta,\varepsilon} > 0$ satisfying all the required properties.

Thus, we can say that $u_i^+ \not\equiv 0$, $\forall i \in \{1, \dots, k\}$. Moreover, as a further consequence of the $L^2(\Omega)$ convergence, we have

$$\|u_i^+\|_{L^2(\Omega)}^{-2} \int_\Omega \left[u_i^+(x)\right]^2 x\,dx = x_i, \quad \forall i \in \{1, \dots, k\}. \tag{3.16}$$

From the weak $H_0^1(\Omega)$ convergence, it follows that $f'_{\beta,\varepsilon}(u_i^+)[u_i^+] \le 0$, $\forall i \in \{1, \dots, k\}$. Therefore, for all $i \in \{1, \dots, k\}$ there exists $t_i \in]0, 1]$ (t_i depends also on β and ε) such that the function $\tilde{u} = -u^- + \sum_{i=1}^k t_i u_i^+$ belongs to $M^{\beta,\varepsilon}_{x_1,\dots,x_k}$. Moreover, since $f_{\beta,\varepsilon}(t_i(u_n^+)_i) \le f_{\beta,\varepsilon}((u_n^+)_i)$, $\forall n \in \mathbb{N}$, we have

$$\liminf_{n\to\infty} f_{\beta,\varepsilon}\left(t_i\left(u_n^+\right)_i\right) \le \liminf_{n\to\infty} f_{\beta,\varepsilon}\left(\left(u_n^+\right)_i\right), \quad \forall i \in \{i, \dots, k\}. \tag{3.17}$$

It follows that

$$\begin{aligned} f_{\beta,\varepsilon}(\tilde{u}) &\le \liminf_{n\to\infty} f_{\beta,\varepsilon}\left(-u_n^- + \sum_{i=1}^k t_i\left(u_n^+\right)_i\right) \\ &\le \liminf_{n\to\infty} f_{\beta,\varepsilon}\left(-u_n^- + \sum_{i=1}^k \left(u_n^+\right)_i\right) = \inf\left\{f_{\beta,\varepsilon}(u) : u \in M^{\beta,\varepsilon}_{x_1,\dots,x_k}\right\}. \end{aligned} \tag{3.18}$$

Thus, we can conclude that the minimum of $f_{\beta,\varepsilon}$ on $M^{\beta,\varepsilon}_{x_1,\dots,x_k}$ is achieved and $f_{\beta,\varepsilon}(\tilde{u}) = \min\{f_{\beta,\varepsilon}(u) : u \in M^{\beta,\varepsilon}_{x_1,\dots,x_k}\}$. □

Proposition 3.1 allows us to introduce the function $\varphi_{k,\beta,\varepsilon} : \Omega_{k,\beta} \to \mathbb{R}$ defined by

$$\varphi_{k,\beta,\varepsilon}(x_1, \dots, x_k) = \min_{M^{\beta,\varepsilon}_{x_1,\dots,x_k}} f_{\beta,\varepsilon}, \quad \forall (x_1, \dots, x_k) \in \Omega_{k,\beta}, \tag{3.19}$$

for $k \in \mathbb{N}$, $\beta > 0$, $\varepsilon > 0$, with β large enough so that $\Omega_{k,\beta} \neq \emptyset$.

Proposition 3.2. *For every positive integer k, for all $\beta > 0$ and $\varepsilon > 0$ (with β large enough so that $\Omega_{k,\beta} \neq \emptyset$), there exists $(x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon}) \in \Omega_{k,\beta}$ such that $\varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon}$.*

Proof. Let us consider a sequence $(x_{1,n}, \dots, x_{k,n})$ in $\Omega_{k,\beta}$ such that

$$\lim_{n\to\infty} \varphi_{k,\beta,\varepsilon}(x_{1,n}, \dots, x_{k,n}) = \sup_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon}. \tag{3.20}$$

Then, there exists $(x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon}) \in \Omega_{k,\beta}$ such that, up to a subsequence, $(x_{1,n}, \dots, x_{k,n}) \to (x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon})$ as $n \to \infty$.

By Proposition 3.1, there exists $u_{k,\beta,\varepsilon} \in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$ such that $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \min\{f_{\beta,\varepsilon}(u) : u \in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}\}$. For every $n \in \mathbb{N}$, let us consider the function $\hat{u}_n \in M^{\beta,\varepsilon}_{x_{1,n},\dots,x_{k,n}}$ such that $(\hat{u}_n^+)_i(x) = (u_{k,\beta,\varepsilon}^+)_i(x + x_{i,\beta,\varepsilon} - x_{i,n})$ and $\hat{u}_n^-$ is the minimizing function for the minimum

$$\min\left\{\int_\Omega |Dv|^2 dx : v \in H_0^1(\Omega),\ v \ge 0 \text{ in } \Omega,\ \int_\Omega v^2 dx = 1,\ \int_\Omega v\left(\hat{u}_n^+\right)_i dx = 0 \text{ for } i = 1, \dots, k\right\}. \tag{3.21}$$

One can verify by standard arguments that $\hat{u}_n \to u_{k,\beta,\varepsilon}$ in $H_0^1(\Omega)$ and $f_{\beta,\varepsilon}(\hat{u}_n) \to f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})$ as $n \to \infty$. Moreover, we have $\min\{f_{\beta,\varepsilon}(u) : u \in M^{\beta,\varepsilon}_{x_{1,n},\dots,x_{k,n}}\} \le f_{\beta,\varepsilon}(\hat{u}_n)$ because $\hat{u}_n \in M^{\beta,\varepsilon}_{x_{1,n},\dots,x_{k,n}}$, $\forall n \in \mathbb{N}$. Thus, we obtain

$$\begin{aligned} \sup_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon} &= \lim_{n\to\infty} \varphi_{k,\beta,\varepsilon}(x_{1,n}, \dots, x_{k,n}) \\ &\le \lim_{n\to\infty} f_{\beta,\varepsilon}(\hat{u}_n) = f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon}), \end{aligned} \tag{3.22}$$

which implies $\varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon}$. □

4. Asymptotic estimates

In this section we describe the asymptotic behaviour as $\beta \to +\infty$ of the mini-max function $u_{k,\beta,\varepsilon}$ obtained in Section 3. Here we need some notion on the capacity. For every bounded domain A of $\mathbb{R}^N$, with $N \geq 3$, the capacity of A is defined by

$$\operatorname{cap} A = \min\left\{ \int_{\mathbb{R}^N} |Du|^2 dx : u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ u \geq 1 \text{ a.e. in } A \right\}. \tag{4.1}$$

It is well known that there exists a unique minimizing function u_A. Moreover, if $A_1, \ldots, A_s$, with $s > 1$, are pairwise disjoint bounded domains of $\mathbb{R}^N$, then we have

$$\operatorname{cap}\left(\bigcup_{i=1}^{s} A_i \right) < \sum_{i=1}^{s} \operatorname{cap}(A_i). \tag{4.2}$$

In fact, if we set $\check{u}(x) = \max\{u_{A_i}(x) : i = 1, \ldots, s\}$, we obtain

$$\operatorname{cap}\left(\bigcup_{i=1}^{s} A_i \right) \leq \int_{\mathbb{R}^N} |D\check{u}|^2 dx < \sum_{i=1}^{s} \int_{\mathbb{R}^N} |Du_{A_i}|^2 dx = \sum_{i=1}^{s} \operatorname{cap}(A_i). \tag{4.3}$$

Proposition 4.1. *For all positive integer k and for all sequences $(\beta_n)_n$, $(\varepsilon_n)_n$ of positive numbers, let us consider a sequence $(x_{1,\beta_n,\varepsilon_n}, \ldots, x_{k,\beta_n,\varepsilon_n})$ of points in Ω^k and a sequence of functions $(u_{k,\beta_n,\varepsilon_n})_n$ in $H_0^1(\Omega)$ such that $(x_{1,\beta_n,\varepsilon_n}, \ldots, x_{k,\beta_n,\varepsilon_n}) \in \Omega_{k,\beta_n}$, $u_{k,\beta_n,\varepsilon_n} \in M^{\beta_n,\varepsilon_n}_{x_{1,\beta_n,\varepsilon_n},\ldots,x_{k,\beta_n,\varepsilon_n}}$ and $f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n}) = \min\{f_{\beta_n,\varepsilon_n}(u) : u \in M^{\beta_n,\varepsilon_n}_{x_{1,\beta_n,\varepsilon_n},\ldots,x_{k,\beta_n,\varepsilon_n}}\}$, $\forall n \in \mathbb{N}$. Moreover assume that, as $n \to \infty$, $\beta_n \to +\infty$ and $\varepsilon_n \to \varepsilon$ such that $0 \leq \varepsilon < \frac{U(3\bar{r}_1)\max_\Omega e_1}{\lim_{|x|\to\infty} U(x) - U(3\bar{r}_1)}$. Then, $u_{k,\beta_n,\varepsilon_n} \to -e_1$ in $H_0^1(\Omega)$ and*

$$\limsup_{n\to\infty} \beta_n^{\frac{N-2}{2}} \left[f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n}) - \lambda_1 \right] \leq k \operatorname{cap}(\bar{r}_1) \left(\varepsilon + \max_\Omega e_1 \right)^2, \tag{4.4}$$

where, for short, we denote by $\operatorname{cap}(\bar{r}_1)$ the capacity of the balls of radius $\bar{r}_1$ in $\mathbb{R}^N$.

If we assume in addition that $f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n}) = \max_{\Omega_{k,\beta_n}} \varphi_{k,\beta_n,\varepsilon_n}$, $\forall n \in \mathbb{N}$, we can say that

$$\lim_{n\to\infty} \beta_n^{\frac{N-2}{2}} \left[f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n}) - \lambda_1 \right] = k \operatorname{cap}(\bar{r}_1) \left(\varepsilon + \max_\Omega e_1 \right)^2, \tag{4.5}$$

$$\lim_{n\to\infty} e_1(x_{i,\beta_n,\varepsilon_n}) = \max_\Omega e_1, \quad \forall i \in \{1, \ldots, k\}, \tag{4.6}$$

$$\lim_{n\to\infty} \sqrt{\beta_n} |x_{i,\beta_n,\varepsilon_n} - x_{j,\varepsilon_n,\beta_n}| = \infty \quad \text{for } i \neq j; \tag{4.7}$$

moreover, if we set $U_\varepsilon(x) = \varepsilon - \frac{(\varepsilon + \max_\Omega e_1)}{\lim_{|x|\to\infty} U(x)} U(x)$, $\forall x \in \mathbb{R}^N$, $\forall \varepsilon > 0$, as $n \to \infty$ we have

$$u_{k,\beta_n,\varepsilon_n}\left(\frac{x}{\sqrt{\beta_n}} + x_{i,\beta_n,\varepsilon_n} \right) \to U_\varepsilon(x), \quad \forall x \in \mathbb{R}^N,\ \forall i \in \{1, \ldots, k\}, \tag{4.8}$$

and the convergence is uniform on the compact subsets of $\mathbb{R}^N$.

Proof. In the proof, for short, let us write $x_{i,n}$ and u_n instead of $x_{i,\beta_n,\varepsilon_n}$ and $u_{k,\beta_n,\varepsilon_n}$. Taking into account that $r_{\beta_n} \to 0$, standard arguments show that $u_n^- \to e_1$ in $H_0^1(\Omega)$.

Notice that $\sup\{U_\varepsilon(x) : |x| \geq 3\bar{r}_1\} < 0$ if and only if $\varepsilon < \frac{U(3\bar{r}_1)\max_\Omega e_1}{\lim_{|x|\to\infty} U(x) - U(3\bar{r}_1)}$, as one can verify by direct computation. Then, in order to prove (4.4), we can consider the sequence $(\tilde{u}_n)_n$ in $M^{\beta_n,\varepsilon_n}_{x_{1,n},\ldots,x_{k,n}}$ defined as follows. For $i = 1, \ldots, k$,

$$(\tilde{u}_n^+)_i(x) = U_{\varepsilon_n}\left(\sqrt{\beta_n}(x - x_{i,n})\right], \quad \forall x \in B\left(x_{i,n}, \frac{\rho_{\varepsilon_n}}{\sqrt{\beta_n}} \right), \tag{4.9}$$

where ρ_{ε_n} is the radius of $\mathrm{supp}(U^+_{\varepsilon_n})$ (which, for large n, is a ball strictly contained in $B(0,3\bar r_1)$ because of the assumptions on ε) and $\tilde u_n^-$ is the function in $H^1_0(\Omega)$ such that

$$\int_\Omega (\tilde u_n^-)^2 dx = 1, \qquad \tilde u_n^-(x)=0, \quad \forall x\in \bigcup_{i=1}^k B\left(x_{i,n},\frac{\rho_{\varepsilon_n}}{\sqrt{\beta_n}}\right), \tag{4.10}$$

$$\int_\Omega |D\tilde u_n^-|^2 dx = \min\left\{\int_\Omega |Du|^2dx : u\in H^1_0(\Omega),\ \int_\Omega u^2dx=1,\ u\ge 0 \text{ in } \Omega,\right.$$
$$\left. u=0 \text{ in } \bigcup_{i=1}^k B\left(x_{i,n},\frac{\rho_{\varepsilon_n}}{\sqrt{\beta_n}}\right)\right\}. \tag{4.11}$$

It is clear that $\tilde u_n\in M^{\beta_n,\varepsilon_n}_{x_{1,n},\dots,x_{k,n}}$ (since $f'_{\beta_n,\varepsilon_n}(\tilde u_n)[(\tilde u_n^+)_i]=0$ for $i=1,\dots,k$, as one can easily verify taking into account the properties of U). It follows that

$$f_{\beta_n,\varepsilon_n}(u_n)=\min\left\{f_{\beta_n,\varepsilon_n}(u): u\in M^{\beta_n,\varepsilon_n}_{x_{1,n},\dots,x_{k,n}}\right\}$$
$$\le f_{\beta_n,\varepsilon_n}(\tilde u_n)=f_{\beta_n,\varepsilon_n}\left(-\tilde u_n^-\right)+\sum_{i=1}^k f_{\beta_n,\varepsilon_n}\left((\tilde u_n^+)_i\right). \tag{4.12}$$

A direct computation shows that

$$\lim_{n\to\infty}\beta_n^{\frac{N-2}{2}} f_{\beta_n,\varepsilon_n}\left((\tilde u_n^+)_i\right)=\int_{B(0,\rho_\varepsilon)\setminus B(0,\bar r_1)}|DU_\varepsilon|^2dx, \quad \text{for } i=1,\dots,k. \tag{4.13}$$

Moreover, $\tilde u_n^-\to e_1$ in $H^1_0(\Omega)$ and $f_{\beta_n,\varepsilon_n}(-\tilde u_n^-)=\int_\Omega |D\tilde u_n^-|^2dx\to\lambda_1$. If we set $\tilde v_n=-\tilde u_n^-+e_1$, we obtain

$$f_{\beta_n,\varepsilon_n}\left(-\tilde u_n^-\right)=\lambda_1+\int_\Omega |D\tilde v_n|^2-2\lambda_1\int_\Omega e_1\tilde v_n dx \tag{4.14}$$

and, after rescaling,

$$\beta_n^{\frac{N-2}{2}}\left[f_{\beta_n,\varepsilon_n}\left(-\tilde u_n^-\right)-\lambda_1\right]=\int_{\sqrt{\beta_n}\Omega}|D\tilde V_n|^2dx-\frac{2\lambda_1}{\beta_n}\int_{\sqrt{\beta_n}\Omega} e_1\left(\frac{x}{\sqrt{\beta_n}}\right)\tilde V_n(x)dx, \tag{4.15}$$

where $\tilde V_n(x)=\tilde v_n(\frac{x}{\sqrt{\beta_n}})$, $\forall x\in\sqrt{\beta_n}\Omega$.

Clearly, there exist $x_1,\dots,x_k$ in $\bar\Omega$ such that, up to a subsequence, $x_{i,n}\to x_i$, as $n\to\infty$, for $i=1,\dots,k$. Moreover, arguing as in [27–29], one can find h ($h\le k$) pairwise disjoint subsets $S_1,\dots,S_h$ of $\{1,\dots,k\}$ such that $\bigcup_{j=1}^h S_j=\{1,\dots,k\}$ and $\sqrt{\beta_n}|x_{i,n}-x_{j,n}|\to\infty$ if i and j belong to different subsets while it remains bounded if i and j both belong to the same subset (it is clear that in this case $x_i=x_j$). In addition, if S_j (for $j=1,\dots,h$) consists of k_j elements, these arguments allow us to say that there exist k_j pairwise disjoint balls in $\mathbb{R}^N$, $B(y_1^j,\rho_\varepsilon),\dots,B(y^j_{k_j},\rho_\varepsilon)$, such that

$$\lim_{n\to\infty}\int_{\sqrt{\beta_n}\Omega}\left[\left|D\tilde V_n(x)\right|^2-2\frac{\lambda_1}{\beta_n}e_1\left(\frac{x}{\sqrt{\beta_n}}\right)\tilde V_n(x)\right]dx=\sum_{j=1}^h m_j^2\,\mathrm{cap}\left(\bigcup_{i=1}^{k_j}B\left(y_i^j,\rho_\varepsilon\right)\right), \tag{4.16}$$

where $m_j=e_1(x_i)$ for $i\in S_j$ (it is clear that different choices of i in S_j give the same constant m_j). Thus, from (4.12), (4.13) and (4.16) we obtain

$$\limsup_{n\to\infty}\beta_n^{\frac{N-2}{2}}\left[f_{\beta_n,\varepsilon_n}(u_n)-\lambda_1\right]=k\int_{B(0,\rho_\varepsilon)\setminus B(0,\bar r_1)}|DU_\varepsilon|^2dx+\sum_{j=1}^h m_j^2\,\mathrm{cap}\left(\bigcup_{i=1}^{k_j}B\left(y_i^j,\rho_\varepsilon\right)\right). \tag{4.17}$$

Since $m_j \le \max_\Omega e_1$ for $j = 1, \dots, h$ and

$$\operatorname{cap}\left(\bigcup_{i=1}^{k_j} B\left(y_i^j, \rho_\varepsilon\right)\right) \le \sum_{i=1}^{k_j} \operatorname{cap} B\left(y_i^j, \rho_\varepsilon\right) = k_j \operatorname{cap}\left(B(0, \rho_\varepsilon)\right), \tag{4.18}$$

it follows

$$\begin{aligned}\limsup_{n\to\infty} \beta_n^{\frac{N-2}{2}} \left[f_{\beta_n,\varepsilon_n}(u_n) - \lambda_1\right] &\le k \int\limits_{B(0,\rho_\varepsilon)\setminus B(0,\bar r_1)} |DU_\varepsilon|^2 dx + \left(\max_\Omega e_1\right)^2 \operatorname{cap}\left(B(0, \rho_\varepsilon)\right) \sum_{j=1}^h k_j \\ &= k \int\limits_{\mathbb{R}^N\setminus B(0,\bar r_1)} |DU_\varepsilon|^2 dx = k \operatorname{cap}(\bar r_1)\left(\varepsilon + \max_\Omega e_1\right)^2,\end{aligned} \tag{4.19}$$

that is (4.4). Let us point out that in (4.4) we have the strict inequality if $m_j < \max_\Omega e_1$ or $k_j > 1$ for some $j \in \{1, \dots, h\}$ (because of (4.2)).

Now, let us prove that, if we assume in addition that $f_{\beta_n,\varepsilon_n}(u_n) = \max_{\Omega_{k,\beta_n}} \varphi_{k,\beta_n,\varepsilon_n}$, $\forall n \in \mathbb{N}$, then we have (4.5). In fact, in this case we can show that

$$\liminf_{n\to\infty} \beta_n^{\frac{N-2}{2}} \left[f_{\beta_n,\varepsilon_n}(u_n) - \lambda_1\right] \ge k \operatorname{cap}(\bar r_1)\left(\varepsilon + \max_\Omega e_1\right)^2. \tag{4.20}$$

In order to prove (4.20), let us choose $\bar x \in \Omega$ such that $e_1(\bar x) = \max_\Omega e_1$ and a sequence $(\bar x_{1,n}, \dots, \bar x_{k,n})_n$ in Ω^k such that $(\bar x_{1,n}, \dots, \bar x_{k,n}) \in \Omega_{k,\beta_n}$, $\forall n \in \mathbb{N}$,

$$\lim_{n\to\infty} |\bar x_{i,n} - \bar x| = 0, \quad \forall i \in \{1, \dots, k\}, \tag{4.21}$$

and

$$\lim_{n\to\infty} \sqrt{\beta_n} |\bar x_{i,n} - \bar x_{j,n}| = \infty, \quad \text{if } i \ne j. \tag{4.22}$$

Taking into account Proposition 3.1, for every $n \in \mathbb{N}$ there exists $\bar u_n \in M^{\beta_n,\varepsilon_n}_{\bar x_{1,n},\dots,\bar x_{k,n}}$ such that $f_{\beta_n,\varepsilon_n}(\bar u_n) = \varphi_{k,\beta_n,\varepsilon_n}(\bar x_{1,n}, \dots, \bar x_{k,n})$. Notice that

$$f_{\beta_n,\varepsilon_n}(\bar u_n) = f_{\beta_n,\varepsilon_n}\left(-\bar u_n^-\right) + \sum_{i=1}^k f_{\beta_n,\varepsilon_n}\left(\left(\bar u_n^+\right)_i\right) \tag{4.23}$$

where $f_{\beta_n,\varepsilon_n}((\bar u_n^+)_i) > 0$, $\forall n \in \mathbb{N}$, $\forall i \in \{1, \dots, k\}$. Moreover, since $\lim_{n\to\infty} r_{\beta_n} = 0$, we have $\bar u_n^- \to e_1$ in $H_0^1(\Omega)$ and $f_{\beta_n,\varepsilon_n}(-\bar u_n^-) \to \lambda_1$ as $n \to \infty$. If we set $\bar w_n = -\bar u_n^- + e_1$, we obtain

$$f_{\beta_n,\varepsilon_n}\left(-\bar u_n^-\right) = \lambda_1 + \int\limits_\Omega \left(|D\bar w_n|^2 - 2\lambda_1 e_1 \bar w_n\right) dx. \tag{4.24}$$

Hence, taking into account (4.4), it follows that

$$\limsup_{n\to\infty} \beta_n^{\frac{N-2}{2}} \int\limits_\Omega \left(|D\bar w_n|^2 - 2\lambda_1 e_1 \bar w_n\right) dx < +\infty, \tag{4.25}$$

namely

$$\limsup_{n\to\infty} \int\limits_{\sqrt{\beta_n}\Omega} \left[\left|D\bar W_n(x)\right|^2 - \frac{2}{\beta_n}\lambda_1 e_1\left(\frac{x}{\sqrt{\beta_n}}\right)\bar W_n(x)\right] dx < +\infty, \tag{4.26}$$

where $\bar W_n(x) = \bar w_n\left(\frac{x}{\sqrt{\beta_n}}\right)$.

As a consequence, arguing as in [27–29], one can verify that, for $i = 1, \dots, k$, there exists $\bar{W}^i \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that (up to a subsequence) $\bar{W}_n(x + \sqrt{\beta_n}\bar{x}_{i,n}) \to \bar{W}^i(x)$; moreover, the convergence is uniform on the compact subsets of $\mathbb{R}^N$ and

$$\sum_{i=1}^{k} \int_{\mathbb{R}^N} |D\bar{W}^i|^2 dx \le \liminf_{n\to\infty} \int_{\sqrt{\beta_n}\Omega} \left[|D\bar{W}_n(x)|^2 - \frac{2}{\beta_n}\lambda_1 e_1\left(\frac{x}{\beta_n}\right)\bar{W}_n(x)\right] dx. \tag{4.27}$$

Now, we examine the asymptotic behaviour of the functions $(\bar{u}_n^+)_i$ for $i = 1, \dots, k$. Let us set $\bar{V}_{i,n}(x) = c_{n,i}(\bar{u}_n^+)_i(\frac{x}{\sqrt{\beta_n}} + \bar{x}_{i,n})$, $\forall x \in \sqrt{\beta_n}\Omega$, where $c_{n,i} = \beta_n^{-\frac{N}{2}} \|(\bar{u}_n^+)_i\|^{-1}_{L^2(\Omega)}$. Then, $\bar{V}_{i,n} \in H_0^1(B(0, 3\bar{r}_1))$, $\int_{B(0,3\bar{r}_1)} \bar{V}_{i,n}^2 dx = 1$ and $\int_{B(0,\bar{r}_1)} |D\bar{V}_{i,n}|^2 dx < 1$, $\forall n \in \mathbb{N}$, $\forall i \in \{1, \dots, k\}$ (because $f'_{\beta_n,\varepsilon_n}(\bar{u}_n)[(\bar{u}_n^+)_i] = 0$). Therefore, up to a subsequence, $\bar{V}_{i,n}$ converges to a function $\bar{V}_i \in H_0^1(B(0, 3\bar{r}_1))$ in L^2, weakly in H_0^1 and a.e. in $B(0, 3\bar{r}_1)$. Thus, we have $\int_{B(0,3\bar{r}_1)} \bar{V}_i^2 dx = 1$ and $\int_{B(0,3\bar{r}_1)} |D\bar{V}_i|^2 dx \le 1$ for $i = 1, \dots, k$. As a consequence, we obtain

$$\int_{\mathbb{R}^N} |D\bar{W}^i|^2 dx \ge \operatorname{cap}(\bar{r}_1)\left(\max_{\Omega} e_1\right)^2, \quad \forall i \in \{1, \dots, k\}, \tag{4.28}$$

because the balls of radius $\bar{r}_1$ have the smallest capacity among the domains whose first eigenvalue is less than or equal to 1. Moreover, since only these balls have this property, in the case $\varepsilon = \lim_{n\to\infty} \varepsilon_n = 0$, (4.4) and (4.28) allow us to say that $\bar{W}^i = \max_\Omega e_1[1 + \frac{U^-}{\lim_{|x|\to\infty} U(x)}]$ and $\bar{V}^i = cU^+$, $\forall i \in \{1, \dots, k\}$, where $c = \|U^+\|^{-1}_{L^2}$. Furthermore, the minimality property of $\bar{u}_n$ implies that $\bar{u}_n(\frac{x}{\sqrt{\beta_n}} + x_{i,n}) \to U_0(x) = \max_\Omega e_1 |\lim_{|x|\to\infty} U(x)|^{-1} U(x)$ uniformly on the compact subsets of $\mathbb{R}^N$ (as one can verify arguing as in [27–29]). In the case $\varepsilon > 0$, arguing as in the proof of Proposition 3.1, one can verify that there exist k positive numbers $\bar{t}_1, \dots, \bar{t}_k$ such that

$$\bar{t}_i \int_{B(0,3\bar{r}_1)} |D\bar{V}_i|^2 dx = \int_{B(0,3\bar{r}_1)} g_{1,\varepsilon}(\bar{t}_i\bar{V}_i)\bar{V}_i dx, \quad \forall i \in \{1, \dots, k\}, \tag{4.29}$$

and

$$\bar{t}_i^2 \int_{B(0,3\bar{r}_1)} |D\bar{V}_i|^2 dx - 2 \int_{B(0,3\bar{r}_1)} G_{1,\varepsilon}(\bar{t}_i\bar{V}_i) dx \le \liminf_{n\to\infty} \beta_n^{\frac{N-2}{2}} f_{\beta_n,\varepsilon_n}\left((\bar{u}_n^+)_i\right). \tag{4.30}$$

Thus, taking into account (4.27), we obtain

$$\liminf_{n\to\infty} \beta_n^{\frac{N-2}{2}} \left[f_{\beta_n,\varepsilon_n}(\bar{u}_n) - \lambda_1\right] \ge \sum_{i=1}^{k} F_\varepsilon\left(\bar{W}^i + \bar{t}_i\bar{V}_i\right), \tag{4.31}$$

where $F_\varepsilon : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ is the functional defined by

$$F_\varepsilon(v) = \int_{\mathbb{R}^N} |Dv|^2 dx - 2\int_{\mathbb{R}^N} \Gamma_\varepsilon(v) dx \tag{4.32}$$

with $\Gamma_\varepsilon(t) = \int_0^t \gamma_\varepsilon(\tau)d\tau$, where $\gamma_\varepsilon(\tau) = \tau - (\varepsilon + \max_\Omega e_1)$, $\forall \tau \ge \varepsilon + \max_\Omega e_1$, and $\gamma_\varepsilon(\tau) = 0$, $\forall \tau \le \varepsilon + \max_\Omega e_1$.

Now, notice that

$$F_\varepsilon\left(W^i + \bar{t}_i\bar{V}_i\right) \ge F_\varepsilon\left(U_\varepsilon + \max_\Omega e_1\right) > 0, \quad \forall \varepsilon > 0,\ \forall i \in \{1, \dots, k\}, \tag{4.33}$$

because $F_\varepsilon(U_\varepsilon + \max_\Omega e_1)$ is the mountain pass level for the functional F_ε while $F_\varepsilon(\bar{W}^i + \bar{t}_i\bar{V}_i)$ is the maximum of F_ε on the continuous path $\Pi : [0, +\infty[\to \mathcal{D}^{1,2}(\mathbb{R}^N)$ defined by $\Pi(t) = t\bar{W}^i$ for $t \in [0, 1]$, $\Pi(t) = \bar{W}^i + (t-1)\bar{t}_i\bar{V}_i$, $\forall t \in [1, +\infty[$, which satisfies $\Pi(0) = 0$, $\lim_{t\to+\infty} \|\Pi(t)\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = +\infty$, $F_\varepsilon(\Pi(0)) = 0$ and $\lim_{t\to+\infty} F_\varepsilon(\Pi(t)) = -\infty$, as one can verify by direct computation.

Thus, we finally obtain (4.20) taking into account that $f_{\beta_n,\varepsilon_n}(u_n) = \max_{\Omega_{k,\beta_n}} \varphi_{k,\beta_n,\varepsilon_n} \ge \varphi_{k,\beta_n,\varepsilon_n}(\bar{x}_{1,n}, \dots, \bar{x}_{k,n}) = f_{\beta_n,\varepsilon_n}(\bar{u}_n)$ and that $F_\varepsilon(U_\varepsilon + \max_\Omega e_1) = (\varepsilon + \max_\Omega e_1)^2 \operatorname{cap}(\bar{r}_1)$.

Let us point out that, indeed, we must have $\bar{W}^i + \bar{t}_i \bar{V}_i = U_\varepsilon + \max_\Omega e_1$ otherwise in (4.20) we have the strict inequality, in contradiction with (4.4). In fact, the radial function $U_\varepsilon + \max_\Omega e_1$ is the unique mountain pass type critical point for F_ε (as one can show by radial symmetrization arguments) while $\bar{W}^i + \bar{t}_i \bar{V}_i$ is the maximum point for F_ε on the continuous path Π. Therefore, taking also into account the minimality properties of $\bar{u}_n$, it follows that $\bar{u}_n(\frac{x}{\sqrt{\beta_n}} + \bar{x}_{i,n}) \to U_\varepsilon(x)$, $\forall x \in \mathbb{R}^N$, $\forall i \in \{1, \dots, k\}$, and the convergence is uniform on the compact subsets of $\mathbb{R}^N$.

Thus, we can say that (4.5) is satisfied and that (4.6), (4.7) hold otherwise in (4.4) we have the strict inequality; as a consequence, arguing as before for $\bar{u}_n$, we can say that also (4.8) must hold otherwise we have the strict inequality in (4.20), in contradiction with (4.5).

Finally, notice that $u_n^+ \to 0$ in $H_0^1(\Omega)$, which implies $u_n \to -e_1$ in $H_0^1(\Omega)$; so the proof is complete. □

Proposition 4.2. *For all positive integer k, for $\beta > 0$ such that $\Omega_{k,\beta} \neq \emptyset$ and for all $\varepsilon \in]0, \bar{\varepsilon}]$ with $0 < \bar{\varepsilon} < \frac{U(3\bar{r}_1)\max_\Omega e_1}{\lim_{|x|\to\infty} U(x) - U(3\bar{r}_1)}$, let us choose $(x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon})$ in $\Omega_{k,\beta}$ and $u_{k,\beta,\varepsilon}$ in $M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$ such that $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon})$,*

$$\lim_{\beta\to+\infty} \inf\{e_1(x_{i,\beta,\varepsilon}) : i \in \{1, \dots, k\},\ \varepsilon \in]0, \bar{\varepsilon}]\} = \max_\Omega e_1, \tag{4.34}$$

$$\lim_{\beta\to+\infty} \sqrt{\beta} \inf\{|x_{i,\beta,\varepsilon} - x_{j,\beta,\varepsilon}| : \varepsilon \in]0, \bar{\varepsilon}]\} = +\infty \quad \text{for } i \neq j. \tag{4.35}$$

Then, there exists $\bar{r} \in]0, 3\bar{r}_1[$ and $\bar{\beta}_k > 0$ such that

$$\sup\left\{u_{k,\beta,\varepsilon}(x) : x \in \Omega \setminus \bigcup_{i=1}^k B\left(x_{i,\beta,\varepsilon}, \frac{\bar{r}}{\sqrt{\beta}}\right),\ \varepsilon \in]0, \bar{\varepsilon}],\ \beta \geq \bar{\beta}_k\right\} < 0. \tag{4.36}$$

Proof. By the minimality of $u_{k,\beta,\varepsilon}$, we have only to check near the spheres $\partial B(x_{i,\beta,\varepsilon}, r_\beta)$. Arguing as in the proof of Proposition 4.1, one can verify that

$$\lim_{\beta\to+\infty} \sup\left\{\left|u_{k,\beta,\varepsilon}\left(\frac{x}{\sqrt{\beta}} + x_{i,\beta,\varepsilon}\right) - U_\varepsilon(x)\right| : x \in K,\ \varepsilon \in]0, \bar{\varepsilon}]\right\} = 0, \quad \forall i \in \{1, \dots, k\}, \tag{4.37}$$

for every compact subset K of $\mathbb{R}^N$.

Therefore, in order to complete the proof, it suffices to notice that there exists $\bar{r} \in]0, 3\bar{r}_1[$ such that

$$\sup\{U_\varepsilon(x) : |x| \geq \bar{r},\ \varepsilon \in]0, \bar{\varepsilon}]\} < 0, \tag{4.38}$$

as one can easily verify taking into account the choice of $\bar{\varepsilon}$. □

Remark 4.3. Let us point out that the strict inequality (4.36) given by Proposition 4.2 is important because the condition $u \leq 0$ in $\Omega \setminus \bigcup_{i=1}^k B(x_{i,\beta,\varepsilon}, r_\beta)$ is an unilateral constraint that would give rise to a variational inequality if $u = 0$ somewhere in $\Omega \setminus \bigcup_{i=1}^k B(x_{i,\beta,\varepsilon}, r_\beta)$. On the contrary, since (4.36) holds, u satisfies the equation $\Delta u + \tilde{\alpha} u = 0$ in $\Omega \setminus \bigcup_{i=1}^k B(x_{i,\beta,\varepsilon}, r_\beta)$ for a suitable Lagrange multiplier $\tilde{\alpha} > 0$, as we show in next lemma.

Lemma 4.4. *Let us consider $k, \beta, \varepsilon, \bar{\varepsilon}, x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon}$ and $u_{k,\beta,\varepsilon}$ satisfying the same assumptions as in Proposition 4.2. Let $\bar{\beta}_k$ be the positive number given by Proposition 4.2. Then, for all $\beta > \bar{\beta}_k$ and $\varepsilon \in]0, \bar{\varepsilon}]$, there exist Lagrange multipliers $\alpha_{k,\beta,\varepsilon} \in \mathbb{R}$ and $\mu_{i,\beta,\varepsilon} \in \mathbb{R}^N$, for $i = 1, \dots, k$, such that*

$$\frac{1}{2} f'_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})[\psi] = \int_\Omega \left\{-\alpha_{k,\beta,\varepsilon} u^-_{k,\beta,\varepsilon} + \sum_{i=1}^k (u^+_{k,\beta,\varepsilon})_i [\mu_{i,\beta,\varepsilon} \cdot (x - x_{i,\beta,\varepsilon})]\right\} \psi\, dx, \quad \forall \psi \in H_0^1(\Omega). \tag{4.39}$$

Moreover, $\alpha_{k,\beta,\varepsilon} = \int_\Omega |Du^-_{k,\beta,\varepsilon}|^2 dx$, $\lim_{\beta\to+\infty} \alpha_{k,\beta,\varepsilon} = \lambda_1$, $\forall k \in \mathbb{N}$, $\forall \varepsilon \in]0, \bar{\varepsilon}]$, and

$$\lim_{\beta\to+\infty} \beta^{-\frac{3}{2}} \mu_{i,\beta,\varepsilon} = 0, \quad \forall i \in \{1, \dots, k\},\ \forall \varepsilon \in]0, \bar{\varepsilon}]. \tag{4.40}$$

Proof. Unlike the case of the smooth constraints $\int_\Omega (u^-)^2 dx = 1$ and $\int_\Omega [u_i^+(x)]^2 x dx = x_i \int_\Omega (u_i^+)^2 dx$, for which the Lagrange multipliers theorem applies, the constraints $f'_{\beta,\varepsilon}(u)[u_i^+] = 0$, for $i = 1, \dots, k$, do not satisfy suitable regularity conditions. However, they are "natural constraints", in the sense that they do not give rise to Lagrange multipliers (while the multipliers $\alpha_{k,\beta,\varepsilon}$ and $\mu_{i,\beta,\varepsilon}$ come from the other constraints).

Notice that $u_{k,\beta,\varepsilon}$ is the unique maximum point for $f_{\beta,\varepsilon}$ on the set $\{u_{k,\beta,\varepsilon} + \sum_{i=1}^k t_i (u^+_{k,\beta,\varepsilon})_i : t_i \geq -1$ for $i = 1, \dots, k\}$; moreover, $f'_{\beta,\varepsilon}(u_{k,\beta,\varepsilon} + \sum_{i=1}^k t_i (u^+_{k,\beta,\varepsilon})_i)[(u^+_{k,\beta,\varepsilon})_i]$ is positive for $t_i \in [-1, 0[$ and negative for $t_i > 0$.

In order to prove (4.39), arguing by contradiction, we assume that (4.39) is not satisfied for any choice of the multipliers $\alpha_{k,\beta,\varepsilon}$ in $\mathbb{R}$ and $\mu_{1,\beta,\varepsilon}, \dots, \mu_{k,\beta,\varepsilon}$ in $\mathbb{R}^N$. Then, it follows by standard methods that there exists a continuous map $\eta :]-1, +\infty]^k \to H_0^1(\Omega)$, such that $\eta(t_1, \dots, t_k) = u_{k,\beta,\varepsilon} + \sum_{i=1}^k t_i (u^+_{k,\beta,\varepsilon})_i$ if $(t_1, \dots, t_k) \notin [-1/2, 1/2]^k$, $\|\eta(t)^-\|_{L^2(\Omega)} = 1$, $\eta(t) \in E^\beta_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$, $\forall t \in [-1, +\infty[^k$, $f_{\beta,\varepsilon}(\eta(t)) < f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})$, $\forall t \in [-1, +\infty[^k$.

Therefore, applying Brouwer Theorem (see [5] and also [26]), we infer that there exists $t \in [-1/2, 1/2]^k$ such that $\eta(t) \in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$, which gives a contradiction because $f_{\beta,\varepsilon}(\eta(t)) < f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})$ and

$$f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon}) = \min\{f_{\beta,\varepsilon}(u) : u \in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}\}. \tag{4.41}$$

Thus, we can conclude that there exist the multipliers $\alpha_{k,\beta,\varepsilon}$ in $\mathbb{R}$ and $\mu_{i,\beta,\varepsilon}$ in $\mathbb{R}^N$ satisfying (4.39).

Now, if in (4.39) we set $\psi = u^-_{k,\beta,\varepsilon}$, we obtain $\alpha_{k,\beta,\varepsilon} = \int_\Omega |Du^-_{k,\beta,\varepsilon}|^2 dx$; then, since $r_\beta \to 0$, it follows that $\lim_{\beta\to+\infty} \alpha_{k,\beta,\varepsilon} = \lambda_1$, $\forall k \in \mathbb{N}$, $\forall \varepsilon \in]0, \bar{\varepsilon}]$.

In order to prove (4.40), for every $i \in \{1, \dots, k\}$ we set $\psi = \psi_{i,\beta,\varepsilon}(x) = \frac{1}{\beta}(u^+_{k,\beta,\varepsilon})_i(x)[\mu_{i,\beta,\varepsilon} \cdot (x - x_{i,\beta,\varepsilon})]$. Then, after rescaling, we obtain

$$\begin{aligned}\frac{1}{2}\beta^{\frac{N-2}{2}} f'_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})[\psi_{i,\beta,\varepsilon}] &= \beta^{-\frac{3}{2}} \int_{B(0,3\bar{r}_1)} (u^+_{k,\beta,\varepsilon})_i\left(\frac{x}{\sqrt{\beta}} + x_{i,\beta,\varepsilon}\right)\psi_{i,\beta,\varepsilon}\left(\frac{x}{\sqrt{\beta}} + x_{i,\beta,\varepsilon}\right)(\mu_{i,\beta,\varepsilon} \cdot x)dx \\ &= \int_{B(0,3\bar{r}_1)} \left[(u^+_{k,\beta,\varepsilon})_i\left(\frac{x}{\sqrt{\beta}} + x_{i,\beta,\varepsilon}\right)\right]^2 \left(\frac{\mu_{i,\beta,\varepsilon}}{\beta^{\frac{3}{2}}} \cdot x\right)^2 dx.\end{aligned} \tag{4.42}$$

Arguing by contradiction, assume that (up to a subsequence) $\lim_{\beta\to+\infty} \beta^{-\frac{3}{2}} |\mu_{i,\beta,\varepsilon}| > 0$. So, taking into account the properties of the function U_ε, from Proposition 4.1 we infer that

$$\lim_{\beta\to+\infty} |\mu_{i,\beta,\varepsilon}|^{-1} \beta^{\frac{N+1}{2}} f'_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})[\psi_{i,\beta,\varepsilon}] = 0, \quad \forall i \in \{1, \dots, k\},\ \forall \varepsilon \in]0, \bar{\varepsilon}], \tag{4.43}$$

while

$$\lim_{\beta\to+\infty} |\mu_{i,\beta,\varepsilon}|^{-1} \beta^{\frac{3}{2}} \int_{B(0,3\bar{r}_1)} \left[(u^+_{k,\beta,\varepsilon})_i\left(\frac{x}{\sqrt{\beta}} + x_{i,\beta,\varepsilon}\right)\right]^2 \left(\frac{\mu_{i,\beta,\varepsilon}}{\beta^{\frac{3}{2}}} \cdot x\right)^2 dx > 0. \tag{4.44}$$

Thus, we get a contradiction and (4.40) is proved. □

Lemma 4.5. *Let us consider $k, \beta_n, \varepsilon_n, (x_{1,\beta_n,\varepsilon_n}, \dots, x_{k,\beta_n,\varepsilon_n}), u_{k,\beta_n,\varepsilon_n}$ satisfying the same assumptions as in Proposition 4.1. Moreover, for all $n \in \mathbb{N}$, let us consider $(\hat{x}_{1,\beta_n,\varepsilon_n}, \dots, \hat{x}_{k,\beta_n,\varepsilon_n})$ in Ω_{k,β_n}, $\hat{u}_{k,\beta_n,\varepsilon_n}$ in $M^{\beta_n,\varepsilon_n}_{\hat{x}_{1,\beta_n,\varepsilon_n},\dots,\hat{x}_{k,\beta_n,\varepsilon_n}}$ and assume that $f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n}) = \varphi_{k,\beta_n,\varepsilon_n}(x_{1,\beta_n,\varepsilon_n}, \dots, x_{k,\beta_n,\varepsilon_n}) = \max_{\Omega_{k,\beta_n}} \varphi_{k,\beta_n,\varepsilon_n}$, $f_{\beta_n,\varepsilon_n}(\hat{u}_{k,\beta_n,\varepsilon_n}) = \varphi_{k,\beta_n,\varepsilon_n}(\hat{x}_{1,\beta_n,\varepsilon_n}, \dots, \hat{x}_{k,\beta_n,\varepsilon_n})$, $\forall n \in \mathbb{N}$,*

$$\lim_{n\to\infty} \sqrt{\beta_n}(\hat{x}_{i,\beta_n,\varepsilon_n} - x_{i,\beta_n,\varepsilon_n}) = 0, \quad \forall i \in \{1, \dots, k\}. \tag{4.45}$$

Then, we have

$$\lim_{n\to\infty} \sup_\Omega |\hat{u}_{k,\beta_n,\varepsilon_n} - u_{k,\beta_n,\varepsilon_n}| = 0. \tag{4.46}$$

If we assume in addition that $(\hat{x}_{1,\beta_n,\varepsilon_n}, \dots, \hat{x}_{k,\beta_n,\varepsilon_n}) \neq (x_{1,\beta_n,\varepsilon_n}, \dots, x_{k,\beta_n,\varepsilon_n})$, $\forall n \in \mathbb{N}$, then $\sup_\Omega |\hat{u}_{k,\beta_n,\varepsilon_n} - u_{k,\beta_n,\varepsilon_n}| > 0$ and the rescaled function $Z_{i,n}$ defined by

$$Z_{i,n}(x) = \Big(\sup_{\Omega} |\hat u_{k,\beta_n,\varepsilon_n} - u_{k,\beta_n,\varepsilon_n}|\Big)^{-1} (\hat u_{k,\beta_n,\varepsilon_n} - u_{k,\beta_n,\varepsilon_n})\Big(\frac{x}{\sqrt{\beta_n}} + x_{i,\beta_n,\varepsilon_n}\Big),$$
$$\forall x \in \sqrt{\beta_n}(\Omega - x_{i,\beta_n,\varepsilon_n}),\ \forall i \in \{1,\dots,k\}, \tag{4.47}$$

up to a subsequence, converges as $n \to \infty$ to a function Z_i which is a weak solution of the equation

$$\Delta Z + a(x)Z = 0 \quad \text{in } \mathbb{R}^N \tag{4.48}$$

where $a(x) = 1$ if $x \in B(0, \bar r_1)$ and $a(x) = 0$ otherwise; moreover, the convergence is uniform on the compact subsets of $\mathbb{R}^N$. Furthermore, there exists $i \in \{1,\dots,k\}$ such that $Z_i \not\equiv 0$.

Proof. For short, in the proof let us write $\hat u_n, u_n, \hat x_{i,n}, x_{i,n}$ instead of $\hat u_{k,\beta_n,\varepsilon_n}, u_{k,\beta_n,\varepsilon_n}, \hat x_{i,\beta_n,\varepsilon_n}, x_{i,\beta_n,\varepsilon_n}$.

From Proposition 4.1, we obtain

$$\limsup_{n\to\infty} \beta_n^{\frac{N-2}{2}} \big[f_{\beta_n,\varepsilon_n}(\hat u_n) - \lambda_1\big] \le k \operatorname{cap}(\bar r_1)\Big(\varepsilon + \max_{\Omega} e_1\Big)^2. \tag{4.49}$$

Moreover, the assumptions on $\hat x_{i,n}$ and $x_{i,n}$ imply

$$\lim_{n\to\infty} \sqrt{\beta_n}|\hat x_{i,n} - \hat x_{j,n}| = \infty \quad \text{for } i \ne j,\ i,j \in \{1,\dots,k\}. \tag{4.50}$$

Hence, arguing as in the proof of Proposition 4.1, one can show that $\hat u_n(\frac{x}{\sqrt{\beta_n}} + \hat x_{i,n}) \to U_\varepsilon(x)$, $\forall x \in \mathbb{R}^N$, $\forall i \in \{1,\dots,k\}$, and the convergence is uniform on the compact subsets of $\mathbb{R}^N$ (in fact, all the conditions we use in Proposition 4.1 to prove the similar property for u_n, are also satisfied by $\hat u_n$).

It follows that

$$\lim_{n\to\infty} \sup_{B(x_{i,n}, 2r_{\beta_n})} |\hat u_n - u_n| = 0 \quad \text{for } i = 1,\dots,k; \tag{4.51}$$

moreover, taking into account the minimality properties of $\hat u_n$ and u_n, standard arguments allow us to say that

$$\lim_{n\to\infty} \sup_{\Omega\setminus\bigcup_{i=1}^k B(x_{i,n}, 2r_{\beta_n})} |\hat u_n - u_n| = 0; \tag{4.52}$$

thus, (4.46) is proved. It is clear that $\sup_\Omega |\hat u_n - u_n| > 0$ if $x_{i,n} \ne \hat x_{i,n}$ for some $i \in \{1,\dots,k\}$, otherwise we should have $\hat x_{i,n} = x_{i,n}$, $\forall i \in \{1,\dots,k\}$. Therefore, if $(\hat x_{1,n},\dots,\hat x_{k,n}) \ne (x_{i,n},\dots,x_{k,n})$, $\forall n \in \mathbb{N}$, $Z_{i,n}$ is well defined and, up to a subsequence, it converges as $n \to \infty$ to a function $Z_i \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $\sup_{\mathbb{R}^N} |Z_i| \le 1$.

For short, in next formulas we write s_n instead of $\sup_\Omega |\hat u_n - u_n|$.

From Lemma 4.4, if we denote by $\hat\alpha_{k,\beta_n,\varepsilon_n}$ and $\hat\mu_{i,\beta_n,\varepsilon_n}$ the Lagrange multipliers corresponding to the function $\hat u_n$, we obtain

$$\begin{aligned}
\frac12 f'_{\beta_n,\varepsilon_n}(\hat u_n)[\psi] - \frac12 f'_{\beta_n,\varepsilon_n}(u_n)[\psi] &= \int_\Omega D(\hat u_n - u_n)\cdot D\psi\,dx - \int_\Omega \big[g_{\beta_n,\varepsilon_n}(\hat u_n) - g_{\beta_n,\varepsilon_n}(u_n)\big]\psi\,dx \\
&= (\alpha_{k,\beta_n,\varepsilon_n} - \hat\alpha_{k,\beta_n,\varepsilon_n})\int_\Omega \hat u_n^- \psi\,dx - \alpha_{k,\beta_n,\varepsilon_n}\int_\Omega (\hat u_n^- - u_n^-)\psi\,dx \\
&\quad + \sum_{i=1}^k \int_\Omega \big[(\hat u_n^+)_i - (u_n^+)_i\big]\big[\hat\mu_{i,\beta_n,\varepsilon_n}\cdot(x - \hat x_{i,n})\big]\psi\,dx \\
&\quad + \sum_{i=1}^k \int_\Omega (u_n^+)_i\big[(\hat\mu_{i,\beta_n,\varepsilon_n} - \mu_{i,\beta_n,\varepsilon_n})\cdot(x - \hat x_{i,n})\big]\psi\,dx \\
&\quad + \sum_{i=1}^k \int_\Omega (u_n^+)_i\big[\mu_{i,\beta_n,\varepsilon_n}\cdot(x_{i,n} - \hat x_{i,n})\big]\psi\,dx, \quad \forall \psi \in H_0^1(\Omega).
\end{aligned} \tag{4.53}$$

Taking into account the minimality properties of $\hat{u}_n$ and u_n, since $\alpha_{k,\beta_n,\varepsilon_n} = \int_\Omega |Du_n^-|^2 dx$ and $\hat{\alpha}_{k,\beta_n,\varepsilon_n} = \int_\Omega |D\hat{u}_n^-|^2 dx$, it follows that

$$\limsup_{n\to\infty} \frac{1}{s_n} |\hat{\alpha}_{k,\beta_n,\varepsilon_n} - \alpha_{k,\beta_n,\varepsilon_n}| < +\infty. \tag{4.54}$$

Moreover, since $\hat{x}_{i,n} = [\int_\Omega (\hat{u}_n^+)_i^2 dx]^{-1} \int_\Omega [(\hat{u}_n^+)_i(x)]^2 x dx$ and $x_{i,n} = [\int_\Omega (u_n^+)_i^2 dx]^{-1} \int_\Omega [(u_n^+)_i(x)]^2 x dx$, it follows by direct computation that

$$\limsup_{n\to\infty} \frac{\sqrt{\beta_n}}{s_n} |\hat{x}_{i,n} - x_{i,n}| < +\infty. \tag{4.55}$$

From Lemma 4.4 we have also $\lim_{n\to\infty} \beta_n^{-\frac{3}{2}} \hat{\mu}_{i,\beta_n,\varepsilon_n} = 0$ and $\lim_{n\to\infty} \beta_n^{-\frac{3}{2}} \mu_{i,\beta_n,\varepsilon_n} = 0$.

Now, we can prove that

$$\lim_{n\to\infty} \beta_n^{-\frac{3}{2}} s_n^{-1} |\hat{\mu}_{i,\beta_n,\varepsilon_n} - \mu_{i,\beta_n,\varepsilon_n}| = 0. \tag{4.56}$$

Arguing by contradiction, assume that (up to a subsequence) the limit (4.56) is positive for some $i \in \{1, \dots, k\}$. Then, for n large enough, we can consider the function

$$\bar{Z}_{i,n} = \beta_n^{\frac{3}{2}} s_n |\hat{\mu}_{i,\beta_n,\varepsilon_n} - \mu_{i,\beta_n,\varepsilon_n}|^{-1} Z_{i,n} \tag{4.57}$$

which, as $Z_{i,n}$, remains uniformly bounded as $n \to \infty$. Moreover, there exists $\mu_i' \in \mathbb{R}^N$, $|\mu_i'| = 1$, such that, up to a subsequence, $|\hat{\mu}_{i,\beta_n,\varepsilon_n} - \mu_{i,\beta_n,\varepsilon_n}|^{-1}(\hat{\mu}_{i,\beta_n,\varepsilon_n} - \mu_{i,\beta_n,\varepsilon_n}) \to \mu_i'$ as $n \to \infty$. Hence, after rescaling in (4.53) we infer that (up to a subsequence) $\bar{Z}_{i,n}$ converges as $n \to \infty$ to a bounded function $\bar{Z}_i \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, such that

$$\int_{\mathbb{R}^N} [D\bar{Z}_i \cdot D\psi - a(x)\bar{Z}_i \psi] dx = \int_{B(0,3\bar{r}_1)} U_\varepsilon^+(x)\psi(x)(x \cdot \mu_i') dx, \quad \forall \psi \in \mathcal{D}^{1,2}(\mathbb{R}^N). \tag{4.58}$$

Now, set $\Psi = (DU \cdot \mu_i')$. Since this function satisfies the equation $\Delta\Psi + a(x)\Psi = 0$ in $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N} [D\bar{Z}_i \cdot D\Psi - a(x)\bar{Z}_i \Psi] dx = 0, \tag{4.59}$$

while $\int_{B(0,3\bar{r}_1)} U_\varepsilon^+(x)\Psi(x)(x \cdot \mu_i') dx < 0$.

Thus, we have a contradiction and we can conclude that (4.56) holds.

Now, after rescaling, we can let $n \to \infty$ in (4.53); so, it follows by usual arguments that (up to a subsequence) $Z_{i,n}$ converges as $n \to \infty$ to a solution Z_i of Eq. (4.48) and that the convergence is uniform on the compact subsets of $\mathbb{R}^N$.

In order to prove that $Z_i \not\equiv 0$ for some $i \in \{1, \dots, k\}$, we argue by contradiction and assume that $Z_i \equiv 0$ for $i = 1, \dots, k$. In this case, $Z_{i,n} \to 0$ as $n \to \infty$, $\forall i \in \{1, \dots, k\}$, uniformly on the compact subsets of $\mathbb{R}^N$; moreover, if we set $z_n = \frac{1}{s_n}(\hat{u}_n - u_n)$, taking into account the minimality properties of $\hat{u}_n$ and u_n, we can say that (up to a subsequence) $(z_n)_n$ converges uniformly in Ω to a function z. Now we prove that $z \equiv 0$ in Ω, so we have a contradiction because $\sup_\Omega |z_n| = 1$, $\forall n \in \mathbb{N}$.

In order to prove that $z \equiv 0$ in Ω, notice that

$$\lim_{n\to\infty} \sup\left\{ |z_n(x)| : x \in \bigcup_{i=1}^k B(x_{i,n}, 2r_{\beta_n}) \right\} = 0; \tag{4.60}$$

moreover, for n large enough so that $\bigcup_{i=1}^k \bar{B}(\hat{x}_{i,n}, r_{\beta_n}) \subset \bigcup_{i=1}^k B(x_{i,n}, 2r_{\beta_n})$, the function z_n satisfies in $\Omega \setminus \bigcup_{i=1}^k \bar{B}(x_{i,n}, 2r_{\beta_n})$ the equation $\Delta z_n + \frac{1}{s_n}(\hat{\alpha}_{k,\beta_n,\varepsilon_n}\hat{u}_n - \alpha_{k,\beta_n,\varepsilon_n} u_n) = 0$. Let us consider the function $w_n \in H_0^1(\Omega)$, such that $w_n = z_n$ in $\bigcup_{i=1}^k \bar{B}(x_{i,n}, 2r_{\beta_n})$ and $\Delta w_n = 0$ in $\Omega \setminus \bigcup_{i=1}^k \bar{B}(x_{i,n}, 2r_{\beta_n})$. Since $\lim_{n\to\infty} \sup\{|z_n(x)| : x \in \bigcup_{i=1}^k \bar{B}(x_{i,n}, 2r_{\beta_n})\} = 0$, it follows that also $\lim_{n\to\infty} \sup_\Omega |w_n| = 0$. If we set $\tilde{z}_n = z_n - w_n$, we obtain

$$\Delta\tilde{z}_n + \alpha_{k,\beta_n,\varepsilon_n}\tilde{z}_n + \alpha_{k,\beta_n,\varepsilon_n} w_n + \frac{1}{s_n}(\hat{\alpha}_{k,\beta_n,\varepsilon_n} - \alpha_{k,\beta_n,\varepsilon_n})\hat{u}_n = 0 \quad \text{in } \Omega \setminus \bigcup_{i=1}^k \bar{B}(x_{i,n}, 2r_{\beta_n}). \tag{4.61}$$

Taking into account that $\lim_{n\to\infty}\alpha_{k,\beta_n,\varepsilon_n}=\lim_{n\to\infty}\hat{\alpha}_{k,\beta_n,\varepsilon_n}=\lambda_1$, that $\limsup_{n\to\infty}\frac{1}{s_n}|\hat{\alpha}_{k,\beta_n,\varepsilon_n}-\alpha_{k,\beta_n,\varepsilon_n}|<+\infty$ and that $\hat{u}_n\to -e_1$ in $H_0^1(\Omega)$, it follows that, up to a subsequence, $\alpha_{k,\beta_n,\varepsilon_n}w_n+\frac{1}{s_n}(\hat{\alpha}_{k,\beta_n,\varepsilon_n}-\alpha_{k,\beta_n,\varepsilon_n})\hat{u}_n\to ce_1$ for a suitable constant $c\in\mathbb{R}$. Now, let us set $\tilde{z}_{n,1}=e_1\int_\Omega \tilde{z}_n e_1\,dx$ and $\tilde{z}_{n,2}=\tilde{z}_n-\tilde{z}_{n,1}$. From (4.61) we obtain

$$\left(1-\frac{\alpha_{k,\beta_n,\varepsilon_n}}{\lambda_2}\right)\|\tilde{z}_{n,2}\|^2_{H_0^1(\Omega)}-c_n\|\tilde{z}_{n,2}\|_{H_0^1(\Omega)}\le 0 \tag{4.62}$$

for a suitable sequence $(c_n)_n$ in $\mathbb{R}$ such that $\lim_{n\to\infty}c_n=0$.

Since $\lim_{n\to\infty}\alpha_{k,\beta_n,\varepsilon_n}=\lambda_1<\lambda_2$, it follows that $\lim_{n\to\infty}\|\tilde{z}_{n,2}\|_{H_0^1(\Omega)}=0$.

Therefore, we can say that (up to a subsequence) $(\tilde{z}_n)_n$ and $(z_n)_n$ converge to the function $z=\bar{c}e_1$ where $\bar{c}=\lim_{n\to\infty}\int_\Omega z_ne_1\,dx$.

On the other hand, $\lim_{n\to\infty}z_n(x_{i,n})=0$, $\forall i\in\{1,\dots,k\}$. Therefore, we have

$$0=\lim_{n\to\infty}z_n(x_{i,n})=\lim_{n\to\infty}z(x_{i,n})=\bar{c}\lim_{n\to\infty}e_1(x_{i,n})=\bar{c}\max_\Omega e_1, \tag{4.63}$$

which implies $\bar{c}=0$ because $\max_\Omega e_1\neq 0$. It follows that $z\equiv 0$ in Ω, which gives a contradiction.

Thus, we can conclude that $Z_i\not\equiv 0$ for some $i\in\{1,\dots,k\}$ and the proof is complete. □

Lemma 4.6. *Let $Z_1,\dots,Z_k$ be the functions obtained in Lemma 4.5. Then, for every $i\in\{1,\dots,k\}$, there exists $\tau_i\in\mathbb{R}^N$ such that $Z_i(x)=(DU(x)\cdot\tau_i)$, $\forall x\in\mathbb{R}^N$. Moreover, there exists $i\in\{1,\dots,k\}$ such that $\tau_i\neq 0$.*

Proof. Notice that the function U is nondegenerate in the sense that, if $Z\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ is a weak solution of Eq. (4.48), then there exists $\tau\in\mathbb{R}^N$ such that $Z(x)=(DU(x)\cdot\tau)$, $\forall x\in\mathbb{R}^N$ (for the proof, see analogous results proved in [27–29]). Therefore, since the function Z_i satisfies Eq. (4.48) for $i=1,\dots,k$ as proved in Lemma 4.5, it follows that for every $i\in\{1,\dots,k\}$ there exists $\tau_i\in\mathbb{R}^N$, having the required property. Moreover, $\tau_i\neq 0$ for some $i\in\{1,\dots,k\}$ because $Z_i\not\equiv 0$ for some $i\in\{1,\dots,k\}$, as we proved in Lemma 4.5. □

Proposition 4.7. *For all positive integer k, for $\beta>0$ large enough so that $\Omega_{k,\beta}\neq\emptyset$ and for all $\varepsilon>0$, let us consider $(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon})$ in $\Omega_{k,\beta}$ and $u_{k,\beta,\varepsilon}$ in $M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$ such that*

$$f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})=\varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon})=\max_{\Omega_{k,\beta}}\varphi_{k,\beta,\varepsilon}. \tag{4.64}$$

Then, there exists $\tilde{\beta}_k>0$ such that, for all $\beta>\tilde{\beta}_k$ and $\varepsilon\in\,]0,\frac{1}{2}\frac{U(3\bar{r}_1)\max_\Omega e_1}{\lim_{|x|\to\infty}U(x)-U(3\bar{r}_1)}[$, $u_{k,\beta,\varepsilon}$ is a constrained critical point for the functional $f_{\beta,\varepsilon}$ constrained on the set $S=\{u\in H_0^1(\Omega):\|u^-\|_{L^2(\Omega)}=1\}$.

Proof. Clearly, it suffices to prove that the Lagrange multipliers $\mu_{i,\beta,\varepsilon}$ given by Lemma 4.4 vanish for β large enough, namely that there exists $\tilde{\beta}_k>0$ such that $\mu_{i,\beta,\varepsilon}=0$, $\forall\beta>\tilde{\beta}_k$, $\forall i\in\{1,\dots,k\}$, $\forall\varepsilon\in\,]0,\frac{1}{2}\frac{U(3\bar{r}_1)\max_\Omega e_1}{\lim_{|x|\to\infty}U(x)-U(3\bar{r}_1)}[$.

Arguing by contradiction, assume that there exist a sequence $(\beta_n)_n$ of positive numbers and a sequence $(\varepsilon_n)_n$ in $]0,\frac{1}{2}\frac{U(3\bar{r}_1)\max_\Omega e_1}{\lim_{|x|\to\infty}U(x)-U(3\bar{r}_1)}[$ such that $\lim_{n\to\infty}\beta_n=+\infty$ and $(\mu_{1,\beta_n,\varepsilon_n},\dots,\mu_{k,\beta_n,\varepsilon_n})\neq 0$, $\forall n\in\mathbb{N}$. Without any loss of generality, we can assume that

$$|\mu_{1,\beta_n,\varepsilon_n}|=\max\bigl\{|\mu_{i,\beta_n,\varepsilon_n}|:i=1,\dots,k\bigr\},\qquad\forall n\in\mathbb{N}. \tag{4.65}$$

Up to a subsequence, $\varepsilon_n\to\tilde{\varepsilon}$ and $\frac{\mu_{1,\beta_n,\varepsilon_n}}{|\mu_{1,\beta_n,\varepsilon_n}|}\to\tilde{\mu}$, as $n\to\infty$, for suitable $\tilde{\varepsilon}\in[0,\frac{1}{2}\frac{U(3\bar{r}_1)\max_\Omega e_1}{\lim_{|x|\to\infty}U(x)-U(3\bar{r}_1)}]$ and $\tilde{\mu}\in\mathbb{R}^N$, such that $|\tilde{\mu}|=1$.

Now, let us choose $(\hat{x}_{1,\beta_n,\varepsilon_n},\dots,\hat{x}_{k,\beta_n,\varepsilon_n})$ in Ω_{k,β_n} and $\hat{u}_{k,\beta_n,\varepsilon_n}$ in $M^{\beta_n,\varepsilon_n}_{\hat{x}_{1,\beta_n,\varepsilon_n},\dots,\hat{x}_{k,\beta_n,\varepsilon_n}}$ such that $f_{\beta_n,\varepsilon_n}(\hat{u}_{k,\beta_n,\varepsilon_n})=\varphi_{k,\beta_n,\varepsilon_n}(\hat{x}_{1,\beta_n,\varepsilon_n},\dots,\hat{x}_{k,\beta_n,\varepsilon_n})$ and $\hat{x}_{i,\beta_n,\varepsilon_n}=x_{i,\beta_n,\varepsilon_n}$ for $i=2,\dots,k$ while $\hat{x}_{1,\beta_n,\varepsilon_n}=x_{1,\beta_n,\varepsilon_n}+\frac{\delta_n}{\sqrt{\beta_n}}\tilde{\mu}$ with $\delta_n>0$, $\forall n\in\mathbb{N}$, $\lim_{n\to\infty}\delta_n=0$ and, in addition,

$$\lim_{n\to\infty}\delta_n\beta_n^{\frac{N+3}{2}}|\mu_{1,\beta_n,\varepsilon_n}|^{-1}=0. \tag{4.66}$$

Notice that this choice of $(\hat{x}_{1,\beta_n,\varepsilon_n},\dots,\hat{x}_{k,\beta_n,\varepsilon_n})$ in Ω_{k,β_n} is indeed possible because $\lim_{n\to\infty}\sqrt{\beta_n}|x_{i,\beta_n,\varepsilon_n}-x_{1,\beta_n,\varepsilon_n}|=\infty$ for $i\neq 1$, as proved in Proposition 4.1. Moreover, we have $\sup_\Omega|\hat{u}_{k,\beta_n,\varepsilon_n}-u_{k,\beta_n,\varepsilon_n}|>0$, $\forall n\in\mathbb{N}$, because $\hat{u}_{k,\beta_n,\varepsilon_n}\neq u_{k,\beta_n,\varepsilon_n}$ since $\delta_n>0$.

For short, let us write s_n instead of $\sup_\Omega|\hat{u}_{k,\beta_n,\varepsilon_n}-u_{k,\beta_n,\varepsilon_n}|$.

One can verify by direct computation that

$$f_{\beta_n,\varepsilon_n}(\hat{u}_{k,\beta_n,\varepsilon_n})=f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n})+f'_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n})[\hat{u}_{k,\beta_n,\varepsilon_n}-u_{k,\beta_n,\varepsilon_n}]+R_n \tag{4.67}$$

where

$$R_n\geq-\beta_n\int_\Omega(\hat{u}_{k,\beta_n,\varepsilon_n}-u_{k,\beta_n,\varepsilon_n})^2dx\geq-\beta_n|\Omega|s_n^2. \tag{4.68}$$

From Lemmas 4.5 and 4.6, we infer that there exist $\tau_1,\dots,\tau_k$ in $\mathbb{R}^N$ such that (up to a subsequence) the rescaled function $\frac{1}{s_n}(\hat{u}_{k,\beta_n,\varepsilon_n}-u_{k,\beta_n,\varepsilon_n})(\frac{x}{\sqrt{\beta_n}}+x_{i,\beta_n,\varepsilon_n})$ converges as $n\to\infty$ to $(DU(x)\cdot\tau_i)$, for $i=1,\dots,k$, uniformly on the compact subsets of $\mathbb{R}^N$.

We say that $\tau_1\neq 0$ and $\tau_i=0$ for $i\neq 1$. In fact, for $i=1,\dots,k$, we have

$$\begin{aligned}
&\int_{B(0,3\bar{r}_1)}\left[\hat{u}^+_{k,\beta_n,\varepsilon_n}\left(\frac{x}{\sqrt{\beta_n}}+x_{i,\beta_n,\varepsilon_n}\right)\right]^2xdx\\
&=\int_{B(0,3\bar{r}_1)}\left[u^+_{k,\beta_n,\varepsilon_n}\left(\frac{x}{\sqrt{\beta_n}}+x_{i,\beta_n,\varepsilon_n}\right)\right]^2xdx\\
&\quad+2\int_{B(0,3\bar{r}_1)}u^+_{k,\beta_n,\varepsilon_n}\left(\frac{x}{\sqrt{\beta_n}}+x_{i,\beta_n,\varepsilon_n}\right)(\hat{u}^+_{k,\beta_n,\varepsilon_n}-u^+_{k,\beta_n,\varepsilon_n})\left(\frac{x}{\sqrt{\beta_n}}+x_{i,\beta_n,\varepsilon_n}\right)xdx+o(s_n).
\end{aligned} \tag{4.69}$$

Taking into account the choice of $(\hat{x}_{1,\beta_n,\varepsilon_n},\dots,\hat{x}_{k,\beta_n,\varepsilon_n})$, if $i\neq 1$, for n large enough we obtain

$$\int_{B(0,3\bar{r}_1)}\left[\hat{u}^+_{k,\beta_n,\varepsilon_n}\left(\frac{x}{\sqrt{\beta_n}}+x_{i,\beta_n,\varepsilon_n}\right)\right]^2xdx=\int_{B(0,3\bar{r}_1)}\left[u^+_{k,\beta_n,\varepsilon_n}\left(\frac{x}{\sqrt{\beta_n}}+x_{i,\beta_n,\varepsilon_n}\right)\right]^2xdx=0. \tag{4.70}$$

Therefore, as $n\to\infty$, we get

$$\int_{B(0,3\bar{r}_1)}U^+_{\tilde{\varepsilon}}(x)\left[DU_{\tilde{\varepsilon}}(x)\cdot\tau_i\right]xdx=0\quad\text{for } i=2,\dots,k. \tag{4.71}$$

It follows that

$$\int_{B(0,3\bar{r}_1)}U^+_{\tilde{\varepsilon}}(x)\left[DU_{\tilde{\varepsilon}}(x)\cdot\tau_i\right]xdx=\frac{1}{2}\int_{B(0,\rho_{\tilde{\varepsilon}})}\left[DU^2_{\tilde{\varepsilon}}(x)\cdot\tau_i\right]xdx=-\frac{\tau_i}{2}\int_{B(0,\rho_{\tilde{\varepsilon}})}U^2_{\tilde{\varepsilon}}(x)dx=0, \tag{4.72}$$

where $\rho_{\tilde{\varepsilon}}$ denotes the radius of $\operatorname{supp}U^+_{\tilde{\varepsilon}}$ (which is a ball). Therefore, we have $\tau_i=0$ for $i=2,\dots,k$.

On the contrary, if $i=1$, for n large enough we have

$$\int_{B(0,3\bar{r}_1)}\left[u^+_{k,\beta_n,\varepsilon_n}\left(\frac{x}{\sqrt{\beta_n}}+x_{i,\beta_n,\varepsilon_n}\right)\right]^2xdx=0 \tag{4.73}$$

while

$$\int_{B(0,3\bar{r}_1)}\left[\hat{u}^+_{k,\beta_n,\varepsilon_n}\left(\frac{x}{\sqrt{\beta_n}}+x_{i,\beta_n,\varepsilon_n}\right)\right]^2xdx=\delta_n\tilde{\mu}\int_{B(0,3\bar{r}_1)}\left[\hat{u}^+_{k,\beta_n,\varepsilon_n}\left(\frac{x}{\sqrt{\beta_n}}+x_{i,\beta_n,\varepsilon_n}\right)\right]^2dx. \tag{4.74}$$

So, as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\frac{\delta_n}{s_n}\tilde\mu\int\limits_{B(0,\rho_{\bar\varepsilon})}U_{\bar\varepsilon}^2dx=2\int\limits_{B(0,\rho_{\bar\varepsilon})}U_{\bar\varepsilon}(x)\big[DU_{\bar\varepsilon}(x)\cdot\tau_1\big]x\,dx=-\tau_1\int\limits_{B(0,\rho_{\bar\varepsilon})}U_{\bar\varepsilon}^2dx \tag{4.75}$$

where, taking into account Lemma 4.6, $\tau_1\neq 0$ because $\tau_i=0$ for $i\neq 1$. As a consequence, $\lim_{n\to\infty}\frac{\delta_n}{s_n}=|\tau_1|>0$ and $\tau_1=-|\tau_1|\tilde\mu$.

From (4.67), (4.68) and Lemma 4.4, we obtain

$$\begin{aligned}&f_{\beta_n,\varepsilon_n}(\hat u_{k,\beta_n,\varepsilon_n})-f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n})\\&\quad\geq 2\int\limits_{\Omega}\left\{-\alpha_{k,\beta_n,\varepsilon_n}u^-_{k,\beta_n,\varepsilon_n}+\sum_{i=1}^k\big(u^+_{k,\beta_n,\varepsilon_n}\big)_i\big[\mu_{i,\beta_n,\varepsilon_n}\cdot(x-x_{i,\beta_n,\varepsilon_n})\big]\right\}(\hat u_{k,\beta_n,\varepsilon_n}-u_{k,\beta_n,\varepsilon_n})dx\\&\qquad-\beta_n|\Omega|s_n^2.\end{aligned} \tag{4.76}$$

Notice that, since $\int_\Omega(\hat u^-_{k,\beta_n,\varepsilon_n})^2dx=1$ and $\int_\Omega(u^-_{k,\beta_n,\varepsilon_n})^2dx=1$, we have

$$2\int\limits_\Omega u^-_{k,\beta_n,\varepsilon_n}\big(\hat u^-_{k,\beta_n,\varepsilon_n}-u^-_{k,\beta_n,\varepsilon_n}\big)dx=-\int\limits_\Omega\big(\hat u^-_{k,\beta_n,\varepsilon_n}-u^-_{k,\beta_n,\varepsilon_n}\big)^2dx,\quad\forall n\in\mathbb N; \tag{4.77}$$

moreover, $\hat u^+_{k,\beta_n,\varepsilon_n}\leq[u_{k,\beta_n,\varepsilon_n}+s_n]^+$ and $\sup_\Omega(u^-_{k,\beta_n,\varepsilon_n}[u_{k,\beta_n,\varepsilon_n}+s_n]^+)\leq s_n^2$; thus, we get

$$\begin{aligned}\int\limits_\Omega u^-_{k,\beta_n,\varepsilon_n}(\hat u_{k,\beta_n,\varepsilon_n}-u_{k,\beta_n,\varepsilon_n})dx&\leq\int\limits_\Omega u^-_{k,\beta_n,\varepsilon_n}\big(\hat u^+_{k,\beta_n,\varepsilon_n}-u^+_{k,\beta_n,\varepsilon_n}\big)dx-\int\limits_\Omega u^-_{k,\beta_n,\varepsilon_n}\big(\hat u^-_{k,\beta_n,\varepsilon_n}-u^-_{k,\beta_n,\varepsilon_n}\big)dx\\&=\int\limits_\Omega u^-_{k,\beta_n,\varepsilon_n}\hat u^+_{k,\beta_n,\varepsilon_n}dx+\frac12\int\limits_\Omega\big(\hat u^-_{k,\beta_n,\varepsilon_n}-u^-_{k,\beta_n,\varepsilon_n}\big)^2dx\\&\leq\frac32|\Omega|s_n^2,\quad\forall n\in\mathbb N.\end{aligned} \tag{4.78}$$

Therefore, after rescaling, it follows

$$\begin{aligned}&\frac{\beta_n^{\frac{N+1}{2}}}{|\mu_{1,\beta_n,\varepsilon_n}|s_n}\big[f_{\beta_n,\varepsilon_n}(\hat u_{k,\beta_n,\varepsilon_n})-f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n})\big]\\&\quad\geq\sum_{i=1}^k\int\limits_{B(0,3\bar r_1)}u^+_{k,\beta_n,\varepsilon_n}\left(\frac{x}{\sqrt{\beta_n}}+x_{i,\beta_n,\varepsilon_n}\right)\frac{1}{s_n}(\hat u_{k,\beta_n,\varepsilon_n}-u_{k,\beta_n,\varepsilon_n})\left(\frac{x}{\sqrt{\beta_n}}+x_{i,\beta_n,\varepsilon_n}\right)\left(\frac{\mu_{i,\beta_n,\varepsilon_n}}{|\mu_{1,\beta_n,\varepsilon_n}|}\cdot x\right)dx\\&\qquad-\frac{2s_n\beta_n^{\frac{N+3}{2}}}{|\mu_{1,\beta_n,\varepsilon_n}|}|\Omega|\end{aligned} \tag{4.79}$$

for n large enough. Then, as $n\to\infty$, we obtain

$$\begin{aligned}&\liminf_{n\to\infty}\frac{\beta_n^{\frac{N+1}{2}}}{s_n|\mu_{1,\beta_n,\varepsilon_n}|}\big[f_{\beta_n,\varepsilon_n}(\hat u_{k,\beta_n,\varepsilon_n})-f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n})\big]\\&\quad\geq\int\limits_{B(0,3\bar r_1)}U^+_{\bar\varepsilon}(x)\big[DU(x)\cdot\tau_1\big](\tilde\mu\cdot x)=-|\tau_1|\int\limits_{B(0,3\bar r_1)}U^+_{\varepsilon}(x)\big[DU(x)\cdot\tilde\mu\big](\tilde\mu\cdot x)dx>0,\end{aligned} \tag{4.80}$$

which is a contradiction because

$$f_{\beta_n,\varepsilon_n}(\hat u_{k,\beta_n,\varepsilon_n})=\varphi_{k,\beta_n,\varepsilon_n}(\hat x_{1,\beta_n,\varepsilon_n},\dots,\hat x_{k,\beta_n,\varepsilon_n})\leq\max_{\Omega_{k,\beta_n}}\varphi_{k,\beta_n,\varepsilon_n}=f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n}). \tag{4.81}$$

Thus, the proof is complete. □

5. Proof of the main results and final remarks

In this section we study the behaviour as $\varepsilon \to 0$ of the function $u_{k,\beta,\varepsilon}$ obtained by mini-max methods in Sections 3 and 4. In particular, our aim is to show that for all $\beta > \tilde{\beta}_k$ (see Proposition 4.7) $\alpha_{k,\beta,\varepsilon} \to \alpha_{k,\beta}$, $u_{k,\beta,\varepsilon} \to u_{k,\beta}$ as $\varepsilon \to 0$ (up to a subsequence) for suitable $\alpha_{k,\beta} \in \mathbb{R}$, $u_{k,\beta} \in H_0^1(\Omega)$ and that $u_{k,\beta}$ is an eigenfunction for the Fučík spectrum, corresponding to the pair $(\alpha_{k,\beta}, \beta)$, namely $u_{k,\beta}$ solves the problem

$$\Delta u - \alpha_{k,\beta} u^- + \beta u^+ = 0 \quad \text{in } \Omega,\ u \in H_0^1(\Omega), \qquad u \not\equiv 0 \text{ in } \Omega. \tag{5.1}$$

Lemma 5.1. *For all $\beta > 0$ and $\varepsilon > 0$, let us consider a point $(x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon})$ in $\Omega_{k,\beta}$ and a function $u_{k,\beta,\varepsilon}$ in $M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$ such that $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon})$. Moreover, assume that (up to a subsequence) $(x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon})$ converges as $\varepsilon \to 0$ to a point $(x_{1,\beta}, \dots, x_{k,\beta})$.*

Then, up to a subsequence, $-u^-_{k,\beta,\varepsilon} + \sum_{i=1}^k (u^+_{k,\beta,\varepsilon})_i \|(u^+_{k,\beta,\varepsilon})_i\|^{-1}_{L^2(\Omega)}$ converges in $H_0^1(\Omega)$, as $\varepsilon \to 0$, to a function $\bar{u}_{k,\beta} \in E^{\beta}_{x_{1,\beta},\dots,x_{k,\beta}}$; moreover,

$$\int_\Omega \left|D\left(\bar{u}^+_{k,\beta}\right)_i\right|^2 dx = \beta, \quad \forall i \in \{1, \dots, k\}, \tag{5.2}$$

and

$$\int_\Omega \left|D\bar{u}^-_{k,\beta}\right|^2 dx = \min\left\{\int_\Omega \left|Du^-\right|^2 dx : u \in E^{\beta}_{x_{1,\beta},\dots,x_{k,\beta}},\ \left\|u^-\right\|_{L^2(\Omega)} = 1,\right.$$
$$\left.\int_\Omega \left|Du_i^+\right|^2 dx = \beta,\ \int_\Omega \left(u_i^+\right)^2 dx = 1 \textit{ for } i = 1, \dots, k\right\}. \tag{5.3}$$

Proof. Notice that, since $u_{k,\beta,\varepsilon} \in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$, we have

$$\int_\Omega \left|D\left(u^+_{k,\beta,\varepsilon}\right)_i\right|^2 dx = \int_\Omega g_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})\left(u^+_{k,\beta,\varepsilon}\right)_i dx < \beta \int_\Omega \left(u^+_{k,\beta,\varepsilon}\right)_i^2 dx, \quad \forall \varepsilon > 0,\ \forall i \in \{1, \dots, k\}. \tag{5.4}$$

Let us set $(\bar{u}^+_{k,\beta,\varepsilon})_i = \|(u^+_{k,\beta,\varepsilon})_i\|^{-1}_{L^2(\Omega)} (u^+_{k,\beta,\varepsilon})_i$. Then, we have

$$\int_\Omega \left|D\left(\bar{u}^+_{k,\beta,\varepsilon}\right)_i\right|^2 dx < \beta, \quad \forall \varepsilon > 0,\ \forall i \in \{1, \dots, k\}. \tag{5.5}$$

It follows that, up to a subsequence, $(\bar{u}^+_{k,\beta,\varepsilon})_i$ converges as $\varepsilon \to 0$ to a function $(\bar{u}^+_{k,\beta})_i$ in $L^2(\Omega)$, weakly in $H_0^1(\Omega)$ and a.e. in Ω. Moreover, since

$$\limsup_{\varepsilon \to 0} \left\|u^-_{k,\beta,\varepsilon}\right\|_{H_0^1(\Omega)} < +\infty, \tag{5.6}$$

also $u^-_{k,\beta,\varepsilon}$ converges as $\varepsilon \to 0$ to a function $u^-_{k,\beta}$ in $L^2(\Omega)$, weakly in $H_0^1(\Omega)$ and a.e. in Ω. As a consequence, the function $\bar{u}_{k,\beta} = -u^-_{k,\beta} + \sum_{i=1}^k (\bar{u}^+_{k,\beta})_i$ belongs to $E^{\beta}_{x_{1,\beta},\dots,x_{k,\beta}}$ and $\|\bar{u}^-_{k,\beta}\|_{L^2(\Omega)} = 1$. Notice that, indeed, $(\bar{u}^+_{k,\beta,\varepsilon})_i \to (\bar{u}^+_{k,\beta})_i$, $\forall i \in \{1, \dots, k\}$, and $u^-_{k,\beta,\varepsilon} \to u^-_{k,\beta}$ strongly in $H_0^1(\Omega)$ as $\varepsilon \to 0$. In fact, we have

$$\int_\Omega \left|D\left(\bar{u}^+_{k,\beta}\right)_i\right|^2 dx = \beta, \quad \forall i \in \{1, \dots, k\} \tag{5.7}$$

and

$$\lim_{\varepsilon \to 0} \int_\Omega \left|Du^-_{k,\beta,\varepsilon}\right|^2 dx = \int_\Omega \left|Du^-_{k,\beta}\right|^2 dx. \tag{5.8}$$

For the proof, we argue by contradiction and assume that

$$\int_\Omega \left|Du^-_{k,\beta}\right|^2 dx < \liminf_{\varepsilon\to 0}\int_\Omega \left|Du^-_{k,\beta,\varepsilon}\right|^2 dx \tag{5.9}$$

or

$$\int_\Omega \left|D\left(\bar u^+_{k,\beta}\right)_i\right|^2 dx < \beta \quad \text{for some } i\in\{1,\dots,k\}. \tag{5.10}$$

In this case, by slight modifications of the supports of $u^-_{k,\beta}$ and $(\bar u^+_{k,\beta})_i$, one can construct a function $\tilde u_{k,\beta}\in E^\beta_{x_{1,\beta},\dots,x_{k,\beta}}$ such that $\|\tilde u^-_{k,\beta}\|_{L^2(\Omega)}=1$, $\|(\tilde u^+_{k,\beta})_i\|_{L^2(\Omega)}=1$,

$$\int_\Omega \left|D\tilde u^-_{k,\beta}\right|^2 dx < \liminf_{\varepsilon\to 0}\int_\Omega \left|Du^-_{k,\beta,\varepsilon}\right|^2 dx \tag{5.11}$$

and

$$\int_\Omega \left|D\left(\tilde u^+_{k,\beta}\right)_i\right|^2 dx < \beta \quad \text{for some } i\in\{1,\dots,k\}. \tag{5.12}$$

Without any loss of generality, we can assume that (5.12) is satisfied for $i=1$.

Then, for all $\varepsilon>0$, let us consider the function $\tilde u_{k,\beta,\varepsilon}\in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$ such that $(\tilde u^+_{k,\beta,\varepsilon})_i=(u^+_{k,\beta,\varepsilon})_i$ for $i=2,\dots,k$, $(\tilde u^+_{k,\beta,\varepsilon})_1(x)=t_\varepsilon(\tilde u^+_{k,\beta})_1(x-x_{1,\beta,\varepsilon}+x_{1,\beta})$, $\forall x\in\Omega$, where t_ε is the positive number such that $f'_{\beta,\varepsilon}(t_\varepsilon(\tilde u^+_{k,\beta})_1)[(\tilde u^+_{k,\beta})_1]=0$ and $\tilde u^-_{k,\beta,\varepsilon}$ is the nonnegative function in $H^1_0(\Omega)$ such that $\tilde u^-_{k,\beta,\varepsilon}(x)=0$, $\forall x\in\bigcup_{i=1}^k \operatorname{supp}(\tilde u^+_{k,\beta,\varepsilon})_i$, $\|\tilde u^-_{k,\beta,\varepsilon}\|_{L^2(\Omega)}=1$ and

$$\int_\Omega \left|D\tilde u^-_{k,\beta,\varepsilon}\right|^2 dx=\min\left\{\int_\Omega |Du|^2dx : u\in H^1_0(\Omega),\ u\ge 0 \text{ in } \Omega,\right.$$
$$\left. u(x)=0,\ \forall x\in\bigcup_{i=1}^k \operatorname{supp}\left(\tilde u^+_{k,\beta,\varepsilon}\right)_i,\ \|u\|_{L^2(\Omega)}=1\right\}. \tag{5.13}$$

Then, we have

$$f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})-f_{\beta,\varepsilon}(\tilde u_{k,\beta,\varepsilon})=f_{\beta,\varepsilon}\left(\left(u^+_{k,\beta,\varepsilon}\right)_1\right)-f_{\beta,\varepsilon}\left(\left(\tilde u^+_{k,\beta,\varepsilon}\right)_1\right)+f_{\beta,\varepsilon}\left(u^-_{k,\beta,\varepsilon}\right)-f_{\beta,\varepsilon}\left(\tilde u^-_{k,\beta,\varepsilon}\right), \tag{5.14}$$

where $f_{\beta,\varepsilon}((u^+_{k,\beta,\varepsilon})_1)\ge 0$, $\forall\varepsilon>0$, $\lim_{\varepsilon\to0} f_{\beta,\varepsilon}((\tilde u^+_{k,\beta,\varepsilon})_1)=0$ and $\lim_{\varepsilon\to0} f_{\beta,\varepsilon}(u^-_{k,\beta,\varepsilon})>\int_\Omega|D\tilde u^-_{k,\beta}|^2dx\ge \lim_{\varepsilon\to0} f_{\beta,\varepsilon}(\tilde u^-_{k,\beta,\varepsilon})$.

It follows that $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})>f_{\beta,\varepsilon}(\tilde u_{k,\beta,\varepsilon})$ for $\varepsilon>0$ small enough, which is a contradiction because $\tilde u_{k,\beta,\varepsilon}\in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$ and $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})=\min\{f_{\beta,\varepsilon}(u) : u\in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}\}$. Thus, we can conclude that $u^-_{k,\beta,\varepsilon}\to u^-_{k,\beta}$ in $H^1_0(\Omega)$ as $\varepsilon\to0$ and $\int_\Omega|D(\bar u^+_{k,\beta})_i|^2dx=\beta$, $\forall i\in\{1,\dots,k\}$.

In a similar way we can prove (5.3). Arguing again by contradiction, assume that there exists $\bar u\in E^\beta_{x_{1,\beta},\dots,x_{k,\beta}}$ such that $\|\bar u^-\|_{L^2(\Omega)}=1$, $\int_\Omega|D\bar u^+_i|^2dx=\beta$, $\|\bar u^+_i\|_{L^2(\Omega)}=1$, $\forall i\in\{1,\dots,k\}$, and $\int_\Omega|D\bar u^-|^2dx<\int_\Omega|D\bar u^-_{k,\beta}|^2dx$.

In this case, by slight modifications of the supports of $\bar u^-$ and $\bar u^+_i$ for $i=1,\dots,k$, one can find $\check u_{k,\beta}\in E^\beta_{x_{1,\beta},\dots,x_{k,\beta}}$ such that $\int_\Omega|D(\check u^+_{k,\beta})_i|^2dx<\beta\int_\Omega(\check u^+_{k,\beta})^2_i dx$, $\forall i\in\{1,\dots,k\}$, $\|\check u^-_{k,\beta}\|_{L^2(\Omega)}=1$ and $\int_\Omega|D\check u^-_{k,\beta}|^2dx<\int_\Omega|D\bar u^-_{k,\beta}|^2dx$.

It follows that there exist k positive numbers $\check t_{1,\varepsilon},\dots,\check t_{k,\varepsilon}$ such that $f'_{\beta,\varepsilon}(\check t_{i,\varepsilon}(\check u^+_{k,\beta})_i)[(\check u^+_{k,\beta})_i]=0$, $\forall i\in\{1,\dots,k\}$, and we can consider the function $\check u_{k,\beta,\varepsilon}$ in $M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$ defined in the following way: for $i=1,\dots,k$, $(\check u^+_{k,\beta,\varepsilon})_i(x)=(\check u^+_{k,\beta})_i(x-x_{i,\beta,\varepsilon}+x_{i,\beta})$, $\forall x\in\Omega$, and $\check u^-_{k,\beta,\varepsilon}$ is the nonnegative function in $H^1_0(\Omega)$ such that $\check u^-_{k,\beta,\varepsilon}(x)=0$, $\forall x\in\bigcup_{i=1}^k\operatorname{supp}(u^+_{k,\beta,\varepsilon})_i$, $\|\check u^-_{k,\beta,\varepsilon}\|_{L^2(\Omega)}=1$ and

$$\int_\Omega |D\check{u}^-_{k,\beta,\varepsilon}|^2 dx = \min\Bigg\{\int_\Omega |Du|^2 dx : u \in H^1_0(\Omega),\ u \geq 0 \text{ in } \Omega,$$
$$u(x) = 0,\ \forall x \in \bigcup_{i=1}^k \operatorname{supp}(\check{u}^+_{k,\beta,\varepsilon})_i,\ \|u\|_{L^2(\Omega)} = 1\Bigg\}. \tag{5.15}$$

Then, by direct computation, we obtain

$$f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) - f_{\beta,\varepsilon}(\check{u}_{k,\beta,\varepsilon}) = f_{\beta,\varepsilon}(u^-_{k,\beta,\varepsilon}) - f_{\beta,\varepsilon}(\check{u}^-_{k,\beta,\varepsilon}) + \sum_{i=1}^k f_{\beta,\varepsilon}((u^+_{k,\beta,\varepsilon})_i) - \sum_{i=1}^k f_{\beta,\varepsilon}((\check{u}^+_{k,\beta,\varepsilon})_i), \tag{5.16}$$

where $f_{\beta,\varepsilon}((u^+_{k,\beta,\varepsilon})_i) \geq 0$, $\forall \varepsilon > 0$, $\forall i \in \{1,\dots,k\}$, $\lim_{\varepsilon\to 0} f_{\beta,\varepsilon}((\check{u}^+_{k,\beta,\varepsilon})_i) = 0$, $\forall i \in \{1,\dots,k\}$, and

$$\lim_{\varepsilon\to 0} f_{\beta,\varepsilon}(u^-_{k,\beta,\varepsilon}) = \int_\Omega |D\bar{u}^-_{k,\beta}|^2 dx > \int_\Omega |D\check{u}^-_{k,\beta}|^2 dx \geq \lim_{\varepsilon\to 0} f_{\beta,\varepsilon}(\check{u}^-_{k,\beta,\varepsilon}). \tag{5.17}$$

It follows that $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) > f_{\beta,\varepsilon}(\check{u}_{k,\beta,\varepsilon})$ for $\varepsilon > 0$ small enough; so we have again a contradiction because $\check{u}_{k,\beta,\varepsilon} \in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$ and $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \min\{f_{\beta,\varepsilon}(u) : u \in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}\}$. □

Now, notice that we can consider the function $\varphi_{k,\beta} : \Omega_{k,\beta} \to \mathbb{R}$ such that, for all $(x_1,\dots,x_k) \in \Omega_{k,\beta}$

$$\varphi_{k,\beta}(x_1,\dots,x_k) = \min\Bigg\{\int_\Omega |Du^-|^2 dx : u \in E^\beta_{x_1,\dots,x_k},\ \|u^-\|_{L^2(\Omega)} = 1,$$
$$\int_\Omega |Du^+_i|^2 dx = \beta,\ \|u^+_i\|_{L^2(\Omega)} = 1 \text{ for } i = 1,\dots,k\Bigg\}. \tag{5.18}$$

In fact, this minimum exists as we can infer from Proposition 3.1 and Lemma 5.1 (where we choose $(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}) = (x_1,\dots,x_k)$, $\forall \beta > 0$, $\forall \varepsilon > 0$).

Lemma 5.2. *If in Lemma 5.1 we assume in addition that $\varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon}$, $\forall \varepsilon > 0$, then $\varphi_{k,\beta}(x_{1,\beta},\dots,x_{k,\beta}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta}$.*

Proof. Arguing by contradiction, assume that there exists $(y_{1,\beta},\dots,y_{k,\beta}) \in \Omega_{k,\beta}$ such that $\varphi_{k,\beta}(x_{1,\beta},\dots,x_{k,\beta}) < \varphi_{k,\beta}(y_{1,\beta},\dots,y_{k,\beta})$.

Taking into account Lemma 5.1, we have $\int_\Omega |D\bar{u}^-_{k,\beta}|^2 dx = \varphi_{k,\beta}(x_{1,\beta},\dots,x_{k,\beta})$. Then, slight modifications of the supports of $\bar{u}^-_{k,\beta}$ and $(\bar{u}^+_{k,\beta})_i$, for $i = 1,\dots,k$, allow us to construct a function $v_{k,\beta} \in E^\beta_{x_{1,\beta},\dots,x_{k,\beta}}$ such that $\|v^-_{k,\beta}\|_{L^2(\Omega)} = 1$,

$$\int_\Omega |Dv^-_{k,\beta}|^2 dx < \varphi_{k,\beta}(y_{1,\beta},\dots,y_{k,\beta}) \tag{5.19}$$

and

$$\int_\Omega |D(v^+_{k,\beta})_i|^2 dx < \beta \int_\Omega (v^+_{k,\beta})^2_i dx, \quad \forall i \in \{1,\dots,k\}, \tag{5.20}$$

which implies the existence of k positive numbers $t_{1,\varepsilon},\dots,t_{k,\varepsilon}$ such that $f'_{\beta,\varepsilon}(t_{i,\varepsilon}(v^+_{k,\beta})_i)[(v^+_{k,\beta})_i] = 0$, $\forall i \in \{1,\dots,k\}$. Let us consider the function $v_{k,\beta,\varepsilon}$ in $M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$ such that $(v^+_{k,\beta,\varepsilon})_i(x) = (v^+_{k,\beta})_i(x - x_{i,\beta,\varepsilon} + x_{i,\beta})$, $\forall x \in \Omega$, $\forall i \in \{1,\dots,k\}$, $\forall \varepsilon > 0$, and $v^-_{k,\beta,\varepsilon}$ is the nonnegative function in $H^1_0(\Omega)$ such that $v^-_{k,\beta,\varepsilon}(x) = 0$, $\forall x \in \bigcup_{i=1}^k \operatorname{supp}(v^+_{k,\beta,\varepsilon})_i$, $\|v^-_{k,\beta,\varepsilon}\|_{L^2(\Omega)} = 1$ and

$$\int_\Omega |Dv^-_{k,\beta,\varepsilon}|^2 dx = \min\Bigg\{\int_\Omega |Dv|^2 dx : v \in H^1_0(\Omega),\ v \geq 0 \text{ in } \Omega,$$

$$v(x) = 0,\ \forall x \in \bigcup_{i=1}^k \operatorname{supp}\big(v^+_{k,\beta,\varepsilon}\big)_i,\ \|v\|_{L^2(\Omega)} = 1\Bigg\}. \tag{5.21}$$

Moreover, let us consider a function $w_{k,\beta,\varepsilon}$ in $M^{\beta,\varepsilon}_{y_{1,\beta},\dots,y_{k,\beta}}$ such that $f_{\beta,\varepsilon}(w_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(y_{1,\beta},\dots,y_{k,\beta})$, $\forall \varepsilon > 0$.

Then, since $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon})$ and $v_{k,\beta,\varepsilon} \in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$, we obtain

$$f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) \leq f_{\beta,\varepsilon}(v_{k,\beta,\varepsilon}) = f_{\beta,\varepsilon}\big(v^-_{k,\beta,\varepsilon}\big) + \sum_{i=1}^k f_{\beta,\varepsilon}\big(\big(v^+_{k,\beta,\varepsilon}\big)_i\big), \quad \forall \varepsilon > 0, \tag{5.22}$$

where $\lim_{\varepsilon\to 0} f_{\beta,\varepsilon}((v^+_{k,\beta,\varepsilon})_i) = 0$, $\forall i \in \{1,\dots,k\}$, and $\lim_{\varepsilon\to 0} f_{\beta,\varepsilon}(v^-_{k,\beta,\varepsilon}) = \int_\Omega |Dv^-_{k,\beta}|^2 dx < \varphi_{k,\beta}(y_{1,\beta},\dots,y_{k,\beta})$.

Moreover, we have

$$f_{\beta,\varepsilon}(w_{k,\beta,\varepsilon}) = f_{\beta,\varepsilon}\big(w^-_{k,\beta,\varepsilon}\big) + \sum_{i=1}^k f_{\beta,\varepsilon}\big(\big(w^+_{k,\beta,\varepsilon}\big)_i\big) \tag{5.23}$$

where $f_{\beta,\varepsilon}((w^+_{k,\beta,\varepsilon})_i) \geq 0$, $\forall \varepsilon > 0$, and, by Lemma 5.1, $\lim_{\varepsilon\to 0} f_{\beta,\varepsilon}(w^-_{k,\beta,\varepsilon}) = \varphi_{k,\beta}(y_{1,\beta},\dots,y_{k,\beta})$. It follows that, for $\varepsilon > 0$ small enough, $\varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}) < \varphi_{k,\beta,\varepsilon}(y_{1,\beta},\dots,y_{k,\beta})$ which is a contradiction because $\varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon}$. □

Proposition 5.3. *Let us consider $(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon})$ in $\Omega_{k,\beta}$ and $u_{k,\beta,\varepsilon}$ in $M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$, satisfying the same assumptions as in Proposition 4.7.*

Then, up to a subsequence, $(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}) \to (x_{1,\beta},\dots,x_{k,\beta})$ as $\varepsilon \to 0$ and $u_{k,\beta,\varepsilon}$ converges in $H^1_0(\Omega)$ to a function $u_{k,\beta} \in E^\beta_{x_{1,\beta},\dots,x_{k,\beta}}$, for all $\beta > \tilde\beta_k$ (where $\tilde\beta_k$ is the number obtained in Proposition 4.7). Moreover, for all $\beta > \tilde\beta_k$, $u_{k,\beta}$ solves the equation

$$\Delta u - \alpha_{k,\beta} u^- + \beta u^+ = 0 \quad \text{in } \Omega, \tag{5.24}$$

where $\alpha_{k,\beta} = \int_\Omega |Du^-_{k,\beta}|^2 dx$.

Proof. As we proved in Proposition 4.7, for all $\beta > \tilde\beta_k$ and $\varepsilon \in]0, \frac{1}{2}\frac{U(3\bar r_1)\max_\Omega e_1}{\lim_{|x|\to\infty} U(x) - U(3\bar r_1)}[$, $u_{k,\beta,\varepsilon}$ is a weak solution of the equation

$$\Delta u - \alpha_{k,\beta,\varepsilon} u^- + g_{\beta,\varepsilon}(u) = 0 \quad \text{in } \Omega, \tag{5.25}$$

where $\alpha_{k,\beta,\varepsilon} = \int_\Omega |Du^-_{k,\beta,\varepsilon}|^2 dx$.

Moreover, by Lemma 5.1, $-u^-_{k,\beta,\varepsilon} + \sum_{i=1}^k (u^+_{k,\beta,\varepsilon})_i \|(u^+_{k,\beta,\varepsilon})_i\|^{-1}_{L^2(\Omega)}$ converges in $H^1_0(\Omega)$, as $\varepsilon \to 0$, to a function $\bar u_{k,\beta} \in E^\beta_{x_{1,\beta},\dots,x_{k,\beta}}$. Let us prove that

$$\liminf_{\varepsilon\to 0} \big\|\big(u^+_{k,\beta,\varepsilon}\big)_i\big\|_{L^2(\Omega)} > 0, \quad \forall \beta > \tilde\beta_k,\ \forall i \in \{1,\dots,k\}. \tag{5.26}$$

Arguing by contradiction, assume that, up to a subsequence, $\lim_{\varepsilon\to 0} \|(u^+_{k,\beta,\varepsilon})_i\|_{L^2(\Omega)} = 0$ for suitable $\beta > \tilde\beta_k$ and $i \in \{1,\dots,k\}$.

In this case, we have $(u^+_{k,\beta,\varepsilon})_i \to 0$ in $H^1_0(\Omega)$ as $\varepsilon \to 0$ (because $f'_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})[(u^+_{k,\beta,\varepsilon})_i] = 0$, $\forall \varepsilon > 0$). Therefore, if we let $\varepsilon \to 0$, from (5.25) we obtain

$$\int_{B(x_{i,\beta},r_\beta)} \big[D\bar u^-_{k,\beta} \cdot D\psi - \bar\alpha_{k,\beta} \bar u^-_{k,\beta} \psi\big] dx = 0, \quad \forall \psi \in H^1_0\big(B(x_{i,\beta},r_\beta)\big), \tag{5.27}$$

where $\bar\alpha_{k,\beta} = \int_\Omega |D\bar u^-_{k,\beta}|^2 dx$. Thus, we have a contradiction because $D\bar u_{k,\beta} \not\equiv 0$ on $B(x_{i,\beta},r_\beta) \cap \partial(\operatorname{supp} \bar u^-_{k,\beta})$.

Now, let us prove that

$$\limsup_{\varepsilon\to 0}\left\|\left(u^+_{k,\beta,\varepsilon}\right)_i\right\|_{L^2(\Omega)} < +\infty, \quad \forall\beta \geq \tilde\beta_k,\ \forall i\in\{1,\dots,k\}. \tag{5.28}$$

Arguing again by contradiction, assume that, up to a subsequence, $\lim_{\varepsilon\to 0}\|(u^+_{k,\beta,\varepsilon})_i\|_{L^2(\Omega)} = +\infty$ for suitable $\beta > \tilde\beta_k$ and $i\in\{1,\dots,k\}$. Then, as $\varepsilon\to 0$, from (5.25) we obtain

$$\int_\Omega \left[D\left(\bar u^+_{k,\beta}\right)_i\cdot D\psi - \beta\left(\bar u^+_{k,\beta}\right)_i\psi\right]dx = 0, \quad \forall\psi\in H^1_0\left(B(x_{i,\beta},r_\beta)\right). \tag{5.29}$$

Thus, we still have a contradiction because $D\bar u_{k,\beta}\not\equiv 0$ on $\partial(\mathrm{supp}(\bar u^+_{k,\beta})_i)$.

Therefore, we can say that for all $\beta > \tilde\beta_k$ (up to a subsequence) $u_{k,\beta,\varepsilon}$ converges in $H^1_0(\Omega)$, as $\varepsilon\to 0$, to a function $u_{k,\beta}\in E^\beta_{x_{1,\beta},\dots,x_{k,\beta}}$. Moreover, if we let $\varepsilon\to 0$ in (5.25), we infer that, for all $\beta > \tilde\beta_k$, $u_{k,\beta}$ is a weak solution of the equation

$$\Delta u - \alpha_{k,\beta}u^- + \beta u^+ = 0 \quad \text{in } \Omega \tag{5.30}$$

with $\alpha_{k,\beta} = \int_\Omega |Du^-_{k,\beta}|^2 dx$. So the proof is complete. □

Proposition 5.4. *For all $\beta > \tilde\beta_k$, let $u_{k,\beta}\in E^\beta_{x_{1,\beta},\dots,x_{k,\beta}}$ be the function obtained in Proposition 5.3 and set $\alpha_{k,\beta} = \int_\Omega |Du^-_{k,\beta}|^2 dx$.*

Then, for every positive integer k, $u_{k,\beta}\to -e_1$ in $H^1_0(\Omega)$ as $\beta\to+\infty$,

$$\lim_{\beta\to+\infty}\beta^{\frac{N-2}{2}}(\alpha_{k,\beta}-\lambda_1) = k\,\mathrm{cap}(\bar r_1)\left(\max_\Omega e_1\right)^2, \tag{5.31}$$

$$\lim_{\beta\to+\infty} e_1(x_{i,\beta}) = \max_\Omega e_1, \quad \forall i\in\{1,\dots,k\}, \tag{5.32}$$

and

$$\lim_{\beta\to+\infty}\sqrt\beta|x_{i,\beta}-x_{j,\beta}| = \infty \quad \textit{for } i\neq j. \tag{5.33}$$

Moreover, $u_{k,\beta}(\frac{x}{\sqrt\beta}+x_{i,\beta})\to -[\lim_{|x|\to\infty}U(x)]^{-1}(\max_\Omega e_1)U(x)$, $\forall x\in\mathbb{R}^N$, $\forall i\in\{1,\dots,k\}$, and the convergence is uniform on the compact subsets of $\mathbb{R}^N$.

For the proof, it suffices to argue as in the proof of Proposition 4.1, taking into account Lemmas 5.1 and 5.2.

As a direct consequence of Proposition 5.4 (see (5.31)), we can state the following corollary.

Corollary 5.5. *For all positive integer k and for $\beta > \tilde\beta_k$, let $\alpha_{k,\beta}$ be as in Proposition 5.4. Then, there exists a sequence $(b_k)_k$ such that*

$$b_k \geq \tilde\beta_k, \qquad b_k\leq b_{k+1} \quad \textit{and} \quad \alpha_{k,\beta} < \alpha_{k+1,\beta}, \quad \forall k\in\mathbb{N},\ \forall\beta > b_{k+1}. \tag{5.34}$$

Proposition 5.6. *Let b_k and $\alpha_{k,\beta}$ be as in Corollary 5.5 for every positive integer k and for $\beta > b_k$. Then, $\alpha_{k,\beta}$ depends continuously on β in $]b_k,+\infty[$, $\forall k\in\mathbb{N}$.*

Proof. Taking into account Lemma 5.1, we have $\alpha_{k,\beta} = \int_\Omega |Du^-_{k,\beta}|^2 dx = \varphi_{k,\beta}(x_{1,\beta},\dots,x_{k,\beta})$, $\forall k\in\mathbb{N}$, $\forall\beta > b_k$.

Let us prove that $\lim_{\beta\to\bar\beta}\alpha_{k,\beta} = \alpha_{k,\bar\beta}$, $\forall\bar\beta\in\,]b_k,+\infty[$. First notice that, by lower semicontinuity arguments with respect to the weak $H^1_0(\Omega)$ convergence, we have $\liminf_{\beta\to\bar\beta}\alpha_{k,\beta}\geq\alpha_{k,\bar\beta}$. Then, arguing by contradiction, assume that there exists a sequence $(\beta'_n)_n$ such that $\lim_{n\to\infty}\beta'_n = \bar\beta$ and $\lim_{n\to\infty}\alpha_{k,\beta'_n} > \alpha_{k,\bar\beta}$, namely

$$\lim_{n\to\infty}\int_\Omega \left|Du^-_{k,\beta'_n}\right|^2 dx > \int_\Omega \left|Du^-_{k,\bar\beta}\right|^2 dx. \tag{5.35}$$

Let us set $\bar u_n = -u^-_{k,\beta'_n} + \sum_{i=1}^k (u^+_{k,\beta'_n})_i \|(u^+_{k,\beta'_n})_i\|^{-1}_{L^2(\Omega)}$. Since $\int_\Omega |D(\bar u^+_n)_i|^2 dx = \beta'_n$, $\forall n \in \mathbb{N}$, $\bar u_n$ converges to a function $\bar u \in E^{\bar\beta}_{x_{1,\bar\beta},\dots,x_{k,\bar\beta}}$ in $L^2(\Omega)$, weakly in $H^1_0(\Omega)$ and a.e. in Ω. It follows that $\int_\Omega |D\bar u^+_i|^2 dx \le \bar\beta$ and $\|\bar u^+_i\|_{L^2(\Omega)} = 1$, $\forall i \in \{1,\dots,k\}$. Therefore, if (5.35) holds, one can find a function $\tilde u \in E^{\bar\beta}_{x_{1,\bar\beta},\dots,x_{k,\bar\beta}}$ such that $\int_\Omega |D\tilde u^+_i|^2 dx = \bar\beta$, $\|\tilde u^+_i\|_{L^2(\Omega)} = 1$, $\forall i \in \{1,\dots,k\}$, and

$$\lim_{n\to\infty} \int_\Omega \left|Du^-_{k,\beta'_n}\right|^2 dx > \int_\Omega \left|D\tilde u^-\right|^2 dx. \tag{5.36}$$

Now, let us consider the function $\tilde u_n \in E^{\beta'_n}_{x_{1,\beta'_n},\dots,x_{k,\beta'_n}}$ such that $(\tilde u^+_n)_i(x) = \tilde u^+_i(\sqrt{\beta'_n\bar\beta^{-1}}(x - x_{i,\beta'_n}) + x_{i,\bar\beta})$, $\forall x \in \Omega$, $\forall i \in \{1,\dots,k\}$, $\forall n \in \mathbb{N}$ and $\tilde u^-_n$ is the nonnegative function in $H^1_0(\Omega)$ such that $\tilde u^-_n(x) = 0$, $\forall x \in \bigcup_{i=1}^k \operatorname{supp}(\tilde u^+_n)_i$, $\|\tilde u^-_n\|_{L^2(\Omega)} = 1$ and

$$\int_\Omega \left|D\tilde u^-_n\right|^2 dx = \min\left\{\int_\Omega |Du|^2 dx : u \in H^1_0(\Omega),\ u \ge 0 \text{ in } \Omega,\right.$$
$$\left. u(x) = 0,\ \forall x \in \bigcup_{i=1}^k \operatorname{supp}\left(\tilde u^+_n\right)_i,\ \|u\|_{L^2(\Omega)} = 1\right\}, \quad \forall n \in \mathbb{N}. \tag{5.37}$$

Notice that $\lim_{n\to\infty} \int_\Omega |D\tilde u^-_n|^2 dx = \int_\Omega |D\tilde u^-|^2 dx$; moreover, since $\|(\tilde u^+_n)_i\|^{-2}_{L^2(\Omega)} \int_\Omega |D(\tilde u^+_n)_i|^2 dx = \beta'_n$, we have

$$\alpha_{k,\beta'_n} = \int_\Omega \left|Du^-_{k,\beta'_n}\right|^2 dx = \varphi_{k,\beta'_n}(x_{1,\beta'_n},\dots,x_{k,\beta'_n}) \le \int_\Omega \left|D\tilde u^-_n\right|^2 dx, \quad \forall n \in \mathbb{N}, \tag{5.38}$$

and, as $n \to \infty$,

$$\lim_{n\to\infty} \int_\Omega \left|Du^-_{k,\beta'_n}\right|^2 dx \le \lim_{n\to\infty} \int_\Omega \left|D\tilde u^-_n\right|^2 dx = \int_\Omega \left|D\tilde u^-\right|^2 dx, \tag{5.39}$$

in contradiction with (5.36).

Thus, we can conclude that $\alpha_{k,\beta}$ depends continuously on β in $]b_k, +\infty[$. □

Proof of Theorem 2.1. For every positive integer k, for $\beta > 0$ large enough so that $\Omega_{k,\beta} \ne \emptyset$ and for $\varepsilon > 0$, let us consider a point $(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}) \in \Omega_{k,\beta}$ and a function $u_{k,\beta,\varepsilon} \in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}$ such that $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon}$ (here we apply Propositions 3.1 and 3.2).

As $\varepsilon \to 0$ (up to a subsequence) $(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon})$ tends to a point $(x_{1,\beta},\dots,x_{k,\beta}) \in \Omega_{k,\beta}$ and $u_{k,\beta,\varepsilon}$ converges in $H^1_0(\Omega)$ to a function $u_{k,\beta} \in E^\beta_{x_{1,\beta},\dots,x_{k,\beta}}$ which, for $\beta > 0$ large enough, satisfies the equation $\Delta u - \alpha_{k,\beta}u^- + \beta u^+ = 0$ in Ω with $\alpha_{k,\beta} = f_\beta(u_{k,\beta}) = \int_\Omega |Du^-_{k,\beta}|^2 dx = \varphi_{k,\beta}(x_{1,\beta},\dots,x_{k,\beta}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta} > \lambda_1$ (here we apply Lemmas 5.1 and 5.2 and Proposition 5.3).

Thus $(\alpha_{k,\beta}, \beta)$ belongs to the Fučík spectrum Σ for $\beta > 0$ large enough. Moreover, from Proposition 5.4 we infer that, for every positive integer k, $\alpha_{k,\beta} \to \lambda_1$ as $\beta \to +\infty$ while $u_{k,\beta} \to -e_1$ in $H^1_0(\Omega)$. Corollary 5.5 guarantees the existence of a nondecreasing sequence $(b_k)_k$ of positive numbers such that $\alpha_{k,\beta} < \alpha_{k+1,\beta}$, $\forall \beta > b_{k+1}$. Proposition 5.6 shows that $\alpha_{k,\beta}$ depends continuously on β in $]b_k, +\infty[$.

All the other assertions in Theorem 2.1 follow directly from Proposition 5.4 as one can easily verify. □

Remark 5.7. Assume that the domain Ω satisfies in addition the following condition: there exists an open subset A of Ω such that $\sup_{\partial A} e_1 < \sup_A e_1$. Then, the method used to prove Theorem 2.1 may be easily adapted in order to construct eigenfunctions $u_{k,\beta}$ as in Theorem 2.1, with k bumps localized near k concentration points $x_{1,\beta},\dots,x_{k,\beta}$, with rescaled bumps having the same asymptotic profile (still described by the radial solution U of (2.1)), but with the concentration points that, as $\beta \to +\infty$, approach maximum points of e_1 in A (i.e. $x_{i,\beta} \to x_i$ as $\beta \to +\infty$, with $x_i \in A$ and $e_1(x_i) = \max_A e_1$ for $i = 1,\dots,k$).

Remark 5.8. Notice that (as we show in a paper in preparation) one can also obtain infinitely many curves of the Fučík spectrum Σ, asymptotic to the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ and corresponding to eigenfunctions of different type, with bumps localized near points of the boundary of Ω (while the eigenfunctions $u_{k,\beta}$ given by Theorem 2.1 present k bumps localized near the maximum points of e_1).

In fact, under the same assumptions as in Theorem 2.1, there exists a nondecreasing sequence $(\bar{b}_k)_k$ of positive numbers, having the following properties. For all $\beta > \bar{b}_k$ there exists $\bar{\alpha}_{k,\beta} > \lambda_1$ and $v_{k,\beta} \in H_0^1(\Omega)$, $v_{k,\beta}^+ \not\equiv 0$ and $v_{k,\beta}^- \not\equiv 0$, such that (1.1), with $\alpha = \bar{\alpha}_{k,\beta}$ and $u = v_{k,\beta}$, is satisfied for all $\beta > \bar{b}_k$. Moreover, for every $k \in \mathbb{N}$, $\bar{\alpha}_{k,\beta}$ depends continuously on β, $\bar{\alpha}_{k,\beta} < \bar{\alpha}_{k+1,\beta}$, $\forall \beta > \bar{b}_{k+1}$, and $\bar{\alpha}_{k,\beta} \to \lambda_1$, as $\beta \to +\infty$, while $v_{k,\beta} \to -e_1$ in $H_0^1(\Omega)$. Furthermore, $v_{k,\beta}$ present k bumps that, as $\beta \to +\infty$, concentrate near k points approaching the boundary of Ω; the concentration rate is greater than the approaching rate between two distinct concentration points or between the concentration points and the boundary (so that the k bumps remain quite distinct).

The eigenfunctions $v_{k,\beta}$ have lower energy and they have a different variational nature compared to the eigenfunctions $u_{k,\beta}$. In fact, their bumps present a different asymptotic profile which is not described by the function U, as it happens for the eigenfunctions $u_{k,\beta}$ (see Theorem 2.1). Notice that, since $v_{k,\beta}$ has lower energy than $u_{k,\beta}$, we can also say that, even in the case $k = 1$, Theorem 2.1 does not give the first curve of the Fučík spectrum (see for instance [15]) because, for all $\beta > b_1$, the pair $(\alpha_{1,\beta}, \beta)$ does not belong to the first curve; the eigenfunctions corresponding to pairs (α, β) of the first curve have lower energy and only one bump which, for α or β large enough, is localized near the boundary of Ω (see [31] and [32]).

Remark 5.9. It is interesting to know from where the curves of the Fučík spectrum we obtain come from. They might come from bifurcations of the first curve of the Fučík spectrum, which emanates from the pair (λ_2, λ_2), or they might come from pairs (λ_i, λ_i) of higher eigenvalues, or might be they do not meet the line $\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha = \beta\}$, etc. The fact that the corresponding eigenfunctions present several nodal regions (as the Fučík eigenfunctions related to pairs (α, β) close to pairs (λ_i, λ_i) of higher eigenvalues) seems to suggest that they might be curves emanating from the pairs (λ_i, λ_i). However notice that, for the Fučík eigenfunctions we obtained in this paper, only the positive part presents several nodal regions while the negative part has only one nodal region (on the contrary, it is natural to expect that for the Fučík eigenfunctions corresponding to pairs (α, β) close to pairs (λ_i, λ_i), both positive and negative parts present several nodal regions); on the other hand, also in the case $N > 1$, one can find simple examples of curve in the Fučík spectrum that pass through pairs (λ_i, λ_i) of higher eigenvalues and are asymptotic to lines $\{\lambda\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda\}$ with $\lambda > \lambda_1$. Thus, the problem is widely open and might give rise to interesting results. Most probably, if Ω is a bounded domain of $\mathbb{R}^N$ with $N > 1$, for each pair (λ_i, λ_i) of eigenvalues, the smallest curve of the Fučík spectrum emanating from (λ_i, λ_i), corresponding to lower energy eigenfunctions, is asymptotic to $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ while the other curves passing through (λ_i, λ_i) are asymptotic to lines $\{\lambda\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda\}$ with $\lambda > \lambda_1$.

Remark 5.10. The difference between the case of dimension $N = 1$ and the case $N > 1$ becomes even more evident if in (1.1) we replace the Dirichlet boundary condition by the Neumann condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

In fact, if we denote by $\tilde{\lambda}_1 < \tilde{\lambda}_2 \le \tilde{\lambda}_3 \le \ldots$ and by $\tilde{\Sigma}$, respectively, the eigenvalues of $-\Delta$ and the Fučík spectrum with Neumann boundary conditions, we have $\tilde{\lambda}_1 = 0$ and, if $N = 1$, no curve of $\tilde{\Sigma}$ is asymptotic to the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$. Indeed, if $N = 1$, a direct computation shows that the Fučík spectrum consists of the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$ and of infinitely many curves $C_2, C_3, \ldots$ having the following properties: for every $i \ge 2$, C_i is a smooth, unbounded, decreasing curve, emanating from $(\tilde{\lambda}_i, \tilde{\lambda}_i)$ and asymptotic to the lines $\{\frac{\tilde{\lambda}_i}{4}\} \times \mathbb{R}$ and $\mathbb{R} \times \{\frac{\tilde{\lambda}_i}{4}\}$ (notice that $\frac{\tilde{\lambda}_i}{4}$ is an eigenvalue of $-\Delta$ in $H^1(\Omega)$ if and only if i is an odd positive integer and, in this case, $\frac{\tilde{\lambda}_i}{4} = \lambda_{(i+1)/2}$). Therefore, if $N = 1$, no curve of $\tilde{\Sigma}$ is asymptotic to the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$ and every nontrivial pair (α, β) of $\tilde{\Sigma}$ satisfies $\alpha > \frac{\tilde{\lambda}_2}{4}$ and $\beta > \frac{\tilde{\lambda}_2}{4}$ (with $\tilde{\lambda}_2 > \tilde{\lambda}_1 = 0$).

On the contrary, the situation is quite different in the case $N > 1$. In fact (as we show in a paper in preparation) in this case there exist infinitely many curves contained in $\tilde{\Sigma}$ and asymptotic to the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$; the corresponding eigenfunctions have an arbitrarily large number of bumps which may be localized in the interior of Ω or near prescribed connected components of $\partial\Omega$; both, interior and boundary bumps, present the same asymptotic profile (still described by the function U).

Conflict of interest statement

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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