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Ann. I. H. Poincaré – AN 32 (2015) 1189–1197

ANNALES DE L'INSTITUT HENRI POINCARÉ
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Centro-affine normal flows on curves: Harnack estimates and ancient solutions

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Received 23 February 2014; accepted 9 July 2014

Available online 24 July 2014

Abstract

We prove that the only compact, origin-symmetric, strictly convex ancient solutions of the planar p centro-affine normal flows are contracting origin-centered ellipses.

MSC: primary 53C44, 53A04, 52A10; secondary 53A15

Keywords: Centro-affine normal flow; Affine differential geometry; Affine support function; Ancient solutions

1. Introduction

The setting of this paper is the two-dimensional Euclidean space, $\mathbb{R}^2$. A compact convex subset of $\mathbb{R}^2$ with non-empty interior is called a *convex body*. The set of smooth, strictly convex bodies in $\mathbb{R}^2$ is denoted by $\mathcal{K}$. Write $\mathcal{K}_0$ for the set of smooth, strictly convex bodies whose interiors contain the origin of the plane.

Let K be a smooth, strictly convex body $\mathbb{R}^2$ and let $X_K : \partial K \to \mathbb{R}^2$ be a smooth embedding of ∂K, the boundary of K. Write $\mathbb{S}^1$ for the unit circle and write $\nu : \partial K \to \mathbb{S}^1$ for the Gauss map of ∂K. That is, at each point $x \in \partial K$, $\nu(x)$ is the unit outwards normal at x. The support function of $K \in \mathcal{K}_0$ as a function on the unit circle is defined by $s(z) := \langle X(\nu^{-1}(z)), z\rangle$, for each $z \in \mathbb{S}^1$. We denote the curvature of ∂K by κ which as a function on ∂K is related to the support function by

$$\frac{1}{\kappa(\nu^{-1}(z))} := \mathfrak{r}(z) = \frac{\partial^2}{\partial\theta^2}s(z) + s(z).$$

Here and afterwards, we identify $z = (\cos\theta, \sin\theta)$ with θ. The function $\mathfrak{r}$ is called the radius of curvature. The affine support function of K is defined by $\sigma : \partial K \to \mathbb{R}$ and $\sigma(x) := s(\nu(x))\mathfrak{r}^{1/3}(\nu(x))$. The affine support function is invariant under the group of special linear transformations, $SL(2)$, and it plays a basic role in our argument.

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http://dx.doi.org/10.1016/j.anihpc.2014.07.001
0294-1449/

Let $K \in \mathcal{K}_0$. A family of convex bodies $\{K_t\}_t \subset \mathcal{K}_0$ given by the smooth map $X : \partial K \times [0, T) \to \mathbb{R}^2$ is said to be a solution to the p centro-affine normal flow, in short p-flow, with the initial data X_K, if the following evolution equation is satisfied:

$$\partial_t X(x,t) = -\left(\frac{\kappa(x,t)}{\langle X(x,t), \nu(x,t)\rangle^3}\right)^{\frac{p}{p+2}-\frac{1}{3}} \kappa^{\frac{1}{3}}(x,t)\nu(x,t), \qquad X(\cdot,0) = X_K, \tag{1.1}$$

for a fixed $1 < p < \infty$. In this equation, $0 < T < \infty$ is the maximal time that the solution exists, and $\nu(x,t)$ is the unit normal to the curve $X(\partial K, t) = \partial K_t$ at $X(x,t)$. This family of flows for $p > 1$ was defined by Stancu [12]. The case $p = 1$ is the well-known affine normal flow whose asymptotic behavior was investigated by Sapiro and Tannenbaum [11], and by Andrews in a more general setting [2,4]: Any convex solution to the affine normal flow, after appropriate rescaling converges to an ellipse in the $\mathcal{C}^\infty$ norm. For $p > 1$, similar result was obtained with smooth, origin-symmetric, strictly convex initial data by the author and Stancu [7,8]. Moreover, ancient solutions of the affine normal flow have been also classified: the only compact, convex ancient solutions of the affine normal flow are contracting ellipsoids. This result in $\mathbb{R}^n$, for $n \geq 3$, was proved by Loftin and Tsui [9] and in dimension two by S. Chen [5], and also by the author with a different method. We recall that a solution of flow is called an ancient solution if it exists on $(-\infty, T)$. Here we classify compact, origin-symmetric, strictly convex ancient solutions of the planar p centro-affine normal flows:

Theorem. *The only compact, origin-symmetric, strictly convex ancient solutions of the p-flows are contracting origin-centered ellipses.*

Throughout this paper, we consider origin-symmetric solutions.

2. Harnack estimate

In this section, we follow [1] to obtain the Harnack estimates for p-flows.

Proposition. *Under the flow (1.1) we have* $\partial_t(s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}t^{\frac{p}{2p+2}}) \geq 0$.

Proof. For simplicity we set $\alpha = -\frac{p}{p+2}$. To prove the proposition, using the parabolic maximum principle we prove that the quantity defined by

$$\mathcal{R} := t\mathcal{P} - \frac{\alpha}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha} \tag{2.1}$$

remains negative as long as the flow exists. Here $\mathcal{P}$ is defined as follows

$$\mathcal{P} := \partial_t\left(-s^{1+3\alpha}\mathfrak{r}^{\alpha}\right).$$

Lemma 2.1. *(See [7].)*

- $\partial_t s = -s^{1+3\alpha}\mathfrak{r}^{\alpha}$,
- $\partial_t \mathfrak{r} = -[(s^{1+3\alpha}\mathfrak{r}^{\alpha})_{\theta\theta} + s^{1+3\alpha}\mathfrak{r}^{\alpha}]$.

Using the evolution equations of s and $\mathfrak{r}$ we find

$$\begin{aligned}\mathcal{P} &= (1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha} + \alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta} + s^{1+3\alpha}\mathfrak{r}^{\alpha}\right] \\ &:= (1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha} + \alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{Q}.\end{aligned} \tag{2.2}$$

Lemma 2.2. *We have the following evolution equation for $\mathcal{P}$ as long as the flow exists:*

$$\partial_t\mathcal{P} = -\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}[\mathcal{P}_{\theta\theta}+\mathcal{P}] + \left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha} + \left[-3(3\alpha+1) + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right]s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P} - \frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}}.$$

Proof. We repeatedly use the evolution equation of s and $\mathfrak{r}$ given in Lemma 2.1.

$$\begin{aligned}
&\partial_t\mathcal{P}\\
&= -(1+3\alpha)(1+6\alpha)s^{1+9\alpha}\mathfrak{r}^{3\alpha} - 2\alpha(1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha-1}\big[\big(s^{1+3\alpha}\mathfrak{r}^{\alpha}\big)_{\theta\theta} + s^{1+3\alpha}\mathfrak{r}^{\alpha}\big]\\
&\quad - \alpha(1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha-1}\big[\big(s^{1+3\alpha}\mathfrak{r}^{\alpha}\big)_{\theta\theta} + s^{1+3\alpha}\mathfrak{r}^{\alpha}\big]\\
&\quad - \alpha(\alpha-1)s^{1+3\alpha}\mathfrak{r}^{\alpha-2}\big[\big(s^{1+3\alpha}\mathfrak{r}^{\alpha}\big)_{\theta\theta} + s^{1+3\alpha}\mathfrak{r}^{\alpha}\big]^2 - \alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}[\mathcal{P}_{\theta\theta}+\mathcal{P}]\\
&= -(1+3\alpha)(1+6\alpha)s^{1+9\alpha}\mathfrak{r}^{3\alpha} - 3\alpha(1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha-1}\mathcal{Q}\\
&\quad - \alpha(\alpha-1)s^{1+3\alpha}\mathfrak{r}^{\alpha-2}\mathcal{Q}^2 - \alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}[\mathcal{P}_{\theta\theta}+\mathcal{P}].
\end{aligned}$$

By the definition of $\mathcal{Q}$, (2.2), we have

$$\mathcal{Q}^2 = \frac{\mathcal{P}^2}{\alpha^2 s^{2+6\alpha}\mathfrak{r}^{2\alpha-2}} - \frac{2(3\alpha+1)}{\alpha^2}\frac{\mathcal{P}\mathfrak{r}^2}{s} + \frac{(3\alpha+1)^2}{\alpha^2}s^{6\alpha}\mathfrak{r}^{2\alpha+2}$$

and

$$\mathcal{Q} = \frac{\mathcal{P} - (1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha}}{\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}}.$$

Substituting these expressions into the evolution equation of $\mathcal{P}$ we find that

$$\partial_t\mathcal{P} = -\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}[\mathcal{P}_{\theta\theta}+\mathcal{P}] + \left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha} - 3(3\alpha+1)s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P} - \frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P}.$$

This completes the proof of Lemma 2.2. □

We now proceed to find the evolution equation of $\mathcal{R}$ which is defined by (2.1). First notice that

$$-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta} = -t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P}_{\theta\theta} + \frac{\alpha^2}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\big(s^{1+3\alpha}\mathfrak{r}^{\alpha}\big)_{\theta\theta}.$$

Therefore, by Lemma 2.2 and identity (2.2) we get

$$\begin{aligned}
&\partial_t\mathcal{R}\\
&= -t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}[\mathcal{P}_{\theta\theta}+\mathcal{P}] + t\left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha}\\
&\quad + t\left[-3(3\alpha+1) + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right]s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P} - t\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \mathcal{P} + \frac{\alpha}{\alpha-1}\mathcal{P}\\
&\quad - \alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta} + t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P}_{\theta\theta} - \frac{\alpha^2}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\big(s^{1+3\alpha}\mathfrak{r}^{\alpha}\big)_{\theta\theta}\\
&\quad + \frac{\alpha^2}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\big(s^{1+3\alpha}\mathfrak{r}^{\alpha}\big) - \frac{\alpha^2}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\big(s^{1+3\alpha}\mathfrak{r}^{\alpha}\big)\\
&\quad + \frac{\alpha(3\alpha+1)}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\big(s^{1+6\alpha}\mathfrak{r}^{2\alpha}\big) - \frac{\alpha(3\alpha+1)}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\big(s^{1+6\alpha}\mathfrak{r}^{2\alpha}\big)
\end{aligned}$$

$$
\begin{aligned}
&= -\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta} + t\left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha} \\
&\quad + t\left[-3(3\alpha+1) + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right]s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P} - t\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \mathcal{P} \\
&\quad + \frac{\alpha}{\alpha-1}\mathcal{P} - \frac{\alpha}{\alpha-1}\mathcal{P} - t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P} + \frac{\alpha^2}{\alpha-1}s^{2+6\alpha}\mathfrak{r}^{2\alpha-1} + \frac{\alpha(3\alpha+1)}{\alpha-1}s^{2+9\alpha}\mathfrak{r}^{3\alpha-1} \\
&= -\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta} + t\left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha} \\
&\quad + t\left[-3(3\alpha+1) + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right]s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P} - t\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \mathcal{P} \\
&\quad - t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P} + \frac{\alpha^2}{\alpha-1}s^{2+6\alpha}\mathfrak{r}^{2\alpha-1} + \frac{\alpha(3\alpha+1)}{\alpha-1}s^{2+9\alpha}\mathfrak{r}^{3\alpha-1}.
\end{aligned}
$$

In the last expression, using the definition of $\mathcal{R}$, identity (2.1), we replace $t\mathcal{P}$ by $\mathcal{R} + \frac{\alpha}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha}$. Therefore, at the point where the maximum of $\mathcal{R}$ is achieved we obtain

$$
\begin{aligned}
&\partial_t\mathcal{R} \\
&\le \mathcal{R}\left[-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1} - \frac{\alpha-1}{\alpha}\frac{\mathcal{P}}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \left[\frac{2(\alpha-1)(3\alpha+1)}{\alpha} - 3(3\alpha+1)\right]s^{3\alpha}\mathfrak{r}^{\alpha}\right] \\
&\quad + \frac{\alpha}{\alpha-1}\left[\frac{2(\alpha-1)(3\alpha+1)}{\alpha} - 3(3\alpha+1)\right]s^{2+6\alpha}\mathfrak{r}^{2\alpha} + \frac{\alpha(3\alpha+1)}{\alpha-1}s^{2+9\alpha}\mathfrak{r}^{3\alpha-1} \\
&\quad + t\left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha} \\
&\le \mathcal{R}\left[-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1} - \frac{\alpha-1}{\alpha}\frac{\mathcal{P}}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \left[\frac{2(\alpha-1)(3\alpha+1)}{\alpha} - 3(3\alpha+1)\right]s^{3\alpha}\mathfrak{r}^{\alpha}\right].
\end{aligned}
$$

To get the last inequality, we used the fact that the terms on the second and third line are negative for $p \ge 1$. Hence, by the parabolic maximum principle and the fact that at the time zero we have $\mathcal{R} \le 0$, we conclude $\mathcal{R} = t\mathcal{P} - \frac{\alpha}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha} \le 0$. Negativity of $\mathcal{R}$ is equivalent to $\partial_t \ln(s^{1+3\alpha}\mathfrak{r}^{\alpha}) \ge \frac{\alpha}{1-\alpha}\frac{1}{t}$ for $t > 0$. From this we infer that $\partial_t(s^{1+3\alpha}\mathfrak{r}^{\alpha}t^{\frac{\alpha}{\alpha-1}}) \ge 0$ for $t > 0$. □

Proposition 2.3. *Ancient solutions of the flow (1.1) satisfy* $\partial_t(s(\frac{1}{\mathfrak{r}s^3})^{\frac{p}{p+2}}) \ge 0$.

Proof. By the Harnack estimate every solution of the flow (1.1) satisfies

$$
\partial_t\left(s\left(\frac{1}{\mathfrak{r}s^3}\right)^{\frac{p}{p+2}}\right) + \frac{p}{2t(p+1)}\left(s\left(\frac{1}{\mathfrak{r}s^3}\right)^{\frac{p}{p+2}}\right) \ge 0. \tag{2.3}
$$

We let the flow starts from a fixed time $t_0 < 0$. So the inequality (2.3) becomes

$$
\partial_t\left(s\left(\frac{1}{\mathfrak{r}s^3}\right)^{\frac{p}{p+2}}\right) + \frac{p}{2(t-t_0)(p+1)}\left(s\left(\frac{1}{\mathfrak{r}s^3}\right)^{\frac{p}{p+2}}\right) \ge 0.
$$

Now letting t_0 goes to $-\infty$ proves the claim. □

Corollary 2.4. *Every ancient solution of the flow (1.1) satisfies* $\partial_t(s\mathfrak{r}^{\frac{1}{3}}) \le 0$.

Proof. The $s(\cdot, t)$ is decreasing on the time interval $(-\infty, 0]$. The claim now follows from the previous proposition. □

3. Affine differential setting

We will recall several definitions from affine differential geometry. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedded strictly convex curve with the curve parameter θ. Define $\mathfrak{g}(\theta) := [\gamma_\theta, \gamma_{\theta\theta}]^{1/3}$, where for two vectors u, v in $\mathbb{R}^2$, $[u, v]$ denotes the determinant of the matrix with rows u and v. The affine arc-length is defined as

$$\mathfrak{s}(\theta) := \int_0^\theta \mathfrak{g}(\alpha) d\alpha.$$

Furthermore, the affine normal vector $\mathfrak{n}$ is given by $\mathfrak{n} := \gamma_{\mathfrak{s}\mathfrak{s}}$. In the affine coordinate $\mathfrak{s}$, there hold $[\gamma_\mathfrak{s}, \gamma_{\mathfrak{s}\mathfrak{s}}] = 1$, $\sigma = [\gamma, \gamma_\mathfrak{s}]$, and $\sigma_{\mathfrak{s}\mathfrak{s}} + \sigma\mu = 1$, where $\mu = [\gamma_{\mathfrak{s}\mathfrak{s}}, \gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}}]$ is the affine curvature.

We can express the area of $K \in \mathcal{K}$, denoted by $A(K)$, in terms of affine invariant quantities:

$$A(K) = \frac{1}{2} \int_{\partial K} \sigma d\mathfrak{s}.$$

The p-affine perimeter of $K \in \mathcal{K}_0$ (for $p = 1$ the assumption $K \in \mathcal{K}_0$ is not necessary and we may take $K \in \mathcal{K}$), denoted by $\Omega_p(K)$, is defined as

$$\Omega_p(K) := \int_{\partial K} \sigma^{1-\frac{3p}{p+2}} d\mathfrak{s},$$

[10]. We call the quantity $\Omega_p^{2+p}(K)/A^{2-p}(K)$, the p-affine isoperimetric ratio and mention that it is invariant under $GL(2)$. Moreover, for $p > 1$ the p-affine isoperimetric inequality states that if K has its centroid at the origin, then

$$\frac{\Omega_p^{2+p}(K)}{A^{2-p}(K)} \le 2^{p+2}\pi^{2p} \tag{3.1}$$

and equality cases are obtained only for origin-centered ellipses. In the final section, we will use the 2-affine isoperimetric inequality.

Let $K \in \mathcal{K}_0$. The polar body of K, denoted by K^*, is a convex body in $\mathcal{K}_0$ defined by

$$K^* = \left\{ y \in \mathbb{R}^2 \mid \langle x, y \rangle \le 1, \ \forall x \in K \right\}.$$

The area of K^*, denoted by $A^* = A(K^*)$, can be represented in terms of affine invariant quantities:

$$A^* = \frac{1}{2} \int_{\partial K} \frac{1}{\sigma^2} d\mathfrak{s} = \frac{1}{2} \int_{\mathbb{S}^1} \frac{1}{s^2} d\theta.$$

Let $K \in \mathcal{K}_0$. We consider a family of convex bodies $\{K_t\}_t \subset \mathcal{K}$, given by the smooth embeddings $X : \partial K \times [0, T) \to \mathbb{R}^2$, which are evolving according to (1.1). Then up to a time-dependant diffeomorphism, $\{K_t\}_t$ evolves according to

$$\frac{\partial}{\partial t} X := \sigma^{1-\frac{3p}{p+2}} \mathfrak{n}, \qquad X(\cdot, 0) = X_K(\cdot). \tag{3.2}$$

Therefore, classification of compact, origin-symmetric ancient solutions to (1.1) is equivalent to the classification of compact, origin-symmetric ancient solutions to (3.2). In what follows our reference flow is the evolution equation (3.2).

Notice that as a family of convex bodies evolve according to the evolution equation (3.2), in the Gauss parametrization their support functions and radii of curvature evolve according to Lemma 2.1. Assume Q and $\bar{Q}$ are two smooth functions $Q : \partial K \times [0, T) \to \mathbb{R}$, $\bar{Q} : \mathbb{S}^1 \times [0, T) \to \mathbb{R}$ that are related by $Q(x, t) = \bar{Q}(\nu(x, t), t)$. It can be easily verified that

$$\partial_t \bar{Q} = \partial_t Q - Q_\mathfrak{s} \big(\sigma^{1-\frac{3p}{p+2}}\big)_\mathfrak{s}.$$

In particular, for ancient solutions of (3.2), in views of Corollary 2.4, $Q = \sigma$ must satisfy $0 \ge \partial_t \sigma - \sigma_\mathfrak{s} (\sigma^{1-\frac{3p}{p+2}})_\mathfrak{s}$. The preceding argument proves the next proposition.

Proposition 3.1. *Every ancient solution satisfies* $\partial_t\sigma \le -(\frac{3p}{p+2}-1)\sigma_{\mathfrak{s}}^2\sigma^{-\frac{3p}{p+2}}$.

The next two lemmas were proved in [7].

Lemma 3.2. *(See [7, Lemma 3.1].) The following evolution equations hold:*

(1) $\frac{\partial}{\partial t}\sigma = \sigma^{1-\frac{3p}{p+2}}(-\frac{4}{3}+(\frac{p}{p+2}+1)(1-\frac{3p}{p+2})\frac{\sigma_{\mathfrak{s}}^2}{\sigma}+\frac{p}{p+2}\sigma_{\mathfrak{ss}})$,
(2) $\frac{d}{dt}A = -\Omega_p$.

Lemma 3.3. *(See [7, Section 6].) The following evolution equation for Ω_l holds for every $l \ge 2$ and $p \ge 1$:*

$$\frac{d}{dt}\Omega_l(t) = \frac{2(l-2)}{l+2}\int_{\gamma_t}\sigma^{1-\frac{3p}{p+2}-\frac{3l}{l+2}}d\mathfrak{s} + \frac{18pl}{(l+2)^2(p+2)}\int_{\gamma_t}\sigma^{-\frac{3p}{p+2}-\frac{3l}{l+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}, \tag{3.3}$$

where $\gamma_t := \partial K_t$ is the boundary of K_t.

Lemma 3.4. *(See [12].) The area product, $A(t)A^*(t)$, and the p-affine isoperimetric ratio are both non-decreasing along (3.2).*

Write respectively $\max_{\gamma_t}\sigma$ and $\min_{\gamma_t}\sigma$ for σ_M and σ_m.

Lemma 3.5. *There is a constant $0 < c < \infty$ such that $\frac{\sigma_M}{\sigma_m} \le c$ on $(-\infty, 0]$.*

Proof. By Corollary 3.1 and part (1) of Lemma 3.2 we have

$$\begin{aligned}-\left(\frac{3p}{p+2}-1\right)\frac{\sigma_{\mathfrak{s}}^2}{\sigma^3} &\ge \frac{\partial_t\sigma}{\sigma^{3-\frac{3p}{p+2}}}\\ &= -\frac{4}{3\sigma^2}+\left(\frac{p}{p+2}+1\right)\left(1-\frac{3p}{p+2}\right)\frac{\sigma_{\mathfrak{s}}^2}{\sigma^3}+\frac{p}{p+2}\frac{\sigma_{\mathfrak{ss}}}{\sigma^2}.\end{aligned} \tag{3.4}$$

Integrating the inequality (3.4) against $d\mathfrak{s}$ we obtain

$$\begin{aligned}\frac{4}{3}\int_{\gamma_t}\frac{1}{\sigma^2}d\mathfrak{s} &\ge \frac{p}{p+2}\left(2-\frac{3p}{p+2}\right)\int_{\gamma_t}\frac{\sigma_{\mathfrak{s}}^2}{\sigma^3}d\mathfrak{s}\\ &= \frac{p}{p+2}\left(3-\frac{3p}{p+2}\right)\int_{\gamma_t}\frac{(\ln\sigma)_{\mathfrak{s}}^2}{\sigma}d\mathfrak{s}\\ &\ge \frac{p}{p+2}\left(3-\frac{3p}{p+2}\right)\frac{(\int_{\gamma_t}|(\ln\sigma)_{\mathfrak{s}}|d\mathfrak{s})^2}{\int_{\gamma_t}\sigma\, d\mathfrak{s}}.\end{aligned} \tag{3.5}$$

Set $d_p = \frac{p}{p+2}(3-\frac{3p}{p+2})$. Applying the Hölder inequality to the left-hand side and the right-hand side of inequality (3.5) yields

$$\left(\int_{\gamma_t}|(\ln\sigma)_{\mathfrak{s}}|d\mathfrak{s}\right)^2 \le d_p' A^*(t)A(t),$$

for a new positive constant d_p'. Here we used the identities $\int_{\gamma_t}\frac{1}{\sigma^2}d\mathfrak{s} = 2A^*(t)$ and $\int_{\gamma_t}\sigma\, d\mathfrak{s} = 2A(t)$. Now by Lemma 3.4 we have $A(t)A^*(t) \le A(0)A^*(0)$. This implies that

$$\left(\ln\frac{\sigma_M}{\sigma_m}\right)^2 \le d_p'',$$

for a new positive constant d''_p. Therefore, on $(-\infty, 0]$ we find that

$$\frac{\sigma_M}{\sigma_m} \leq c \tag{3.6}$$

for some positive constant c. $\square$

Let $\{K_t\}_t$ be a solution of (3.2). Then the family of convex bodies, $\{\tilde{K}_t\}_t$, defined by

$$\tilde{K}_t := \sqrt{\frac{\pi}{A(K_t)}} K_t$$

is called a normalized solution to the p-flow, equivalently a solution that the area is fixed and is equal to π.

Furnish every quantity associated with the normalized solution with an over-tilde. For example, the support function, curvature, and the affine support function of $\tilde{K}$ are denoted by $\tilde{s}$, $\tilde{\kappa}$, and $\tilde{\sigma}$, respectively.

Lemma 3.6. *There is a constant* $0 < c < \infty$ *such that on the time interval* $(-\infty, 0]$ *we have*

$$\frac{\tilde{\sigma}_M}{\tilde{\sigma}_m} \leq c. \tag{3.7}$$

Proof. The estimate (3.6) is scaling invariant, so the same estimate holds for the normalized solution. $\square$

Lemma 3.7. $\Omega_2(t)$ *is non-decreasing along the* p*-flow. Moreover, we have*

$$\frac{d}{dt}\Omega_2(t) \geq \frac{9p}{4(p+2)} \int_{\gamma_t} \sigma^{-\frac{3p}{p+2}-\frac{3}{2}} \sigma_{\mathfrak{s}}^2 d\mathfrak{s}.$$

Proof. Use the evolution equation (3.3) for $l = 2$. $\square$

Corollary 3.8. *There exists a constant* $0 < b_p < \infty$ *such that*

$$\frac{1}{\Omega_2^4(t)} < b_p$$

on $(-\infty, 0]$.

Proof. Notice that $\Omega_2(t) = (\int_{\partial\gamma_t} \sigma^{-\frac{1}{2}} d\mathfrak{s})$ is a $GL(2)$ invariant quantity. Therefore, we need only to prove the claim after applying appropriate $SL(2)$ transformations to the normalized solution of the flow. By the estimate (3.7) and the facts that $\Omega_2(\tilde{K}_t)$ is non-decreasing and $A(\tilde{K}_t) = \pi$ we have

$$\frac{c^{\frac{3}{2}}}{2}\tilde{\sigma}_m^{\frac{3}{2}}(t)\tilde{\Omega}_2(0) \geq \frac{1}{2}\tilde{\sigma}_M^{\frac{3}{2}}(t)\tilde{\Omega}_2(0) \geq \frac{1}{2}\tilde{\sigma}_M^{\frac{3}{2}}(t)\tilde{\Omega}_2(t) \geq \tilde{A}(t) = \pi.$$

So we get $(\tilde{s}\tilde{\mathfrak{r}}^{1/3})(t) \geq a > 0$ on $(-\infty, 0]$, for an a independent of t. Moreover, as the affine support function is invariant under $SL(2)$ we may further assume, after applying a length minimizing special linear transformation at each time, that $\tilde{s}(t) < a' < \infty$, for an a' independent of t. Therefore

$$\frac{\tilde{\Omega}_1^3(t)}{\tilde{A}(t)} = \frac{(\int_{\mathbb{S}^1} \tilde{\mathfrak{r}}^{2/3} d\theta)^3}{\pi} > a'' > 0, \tag{3.8}$$

for an a'' independent of t. Now the claim follows from the Hölder inequality:

$$\left(\int_{\gamma_t} \sigma^{-\frac{1}{2}} d\mathfrak{s}\right)\Omega_1^{\frac{1}{2}}(t)A^{\frac{1}{2}}(t) \geq \int_{\gamma_t} \sigma^{-\frac{1}{2}} d\mathfrak{s} \int_{\gamma_t} \sigma^{\frac{1}{2}} d\mathfrak{s} \geq \Omega_1^2(t),$$

so

$$\tilde{\Omega}_2(t) = \Omega_2(t) \geq \left(\frac{\Omega_1^3(t)}{A(t)}\right)^{\frac{1}{2}} = \left(\frac{\tilde{\Omega}_1^3(t)}{\tilde{A}(t)}\right)^{\frac{1}{2}}. \qquad \square$$

Corollary 3.9. *As K_t evolve by (3.2), then the following limit holds:*

$$\liminf_{t\to-\infty}\left(\frac{A(t)}{\Omega_p(t)\Omega_2^5(t)}\right)\int\limits_{\gamma_t}\left(\sigma^{\frac{1}{4}-\frac{3p}{2(p+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s} = 0. \tag{3.9}$$

Proof. Suppose on the contrary that there exists an $\varepsilon > 0$ small enough, such that

$$\left(\frac{A(t)}{\Omega_p(t)\Omega_2^5(t)}\right)\int\limits_{\gamma_t}\left(\sigma^{\frac{1}{4}-\frac{3p}{2(p+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s} \geq \varepsilon\frac{\left(\frac{1}{4}-\frac{3p}{2(p+2)}\right)^2}{\frac{9p}{p+2}}$$

on $(-\infty, -N]$ for N large enough. Then $\frac{d}{dt}\frac{1}{\Omega_2^4(t)} \leq \varepsilon\frac{d}{dt}\ln(A(t))$. So by integrating this last inequality against dt and by Corollary 3.8 we get

$$\begin{aligned} 0 < \frac{1}{\Omega_2^4(-N)} &\leq \frac{1}{\Omega_2^4(t)} + \varepsilon\ln\big(A(-N)\big) - \varepsilon\ln\big(A(t)\big) \\ &< b_p + \varepsilon\ln\big(A(-N)\big) - \varepsilon\ln\big(A(t)\big). \end{aligned}$$

Letting $t \to -\infty$ we reach to a contradiction: $\lim_{t\to-\infty} A(t) = +\infty$, that is, the right-hand side becomes negative for large values of t. $\square$

Corollary 3.10. *For a sequence of times $\{t_k\}$ as t_k converge to $-\infty$ we have*

$$\lim_{t_k\to-\infty}\tilde{\sigma}(t_k) = 1.$$

Proof. Notice that the quantity $\left(\frac{A(t)}{\Omega_p(t)\Omega_2^5(t)}\right)\int_{\gamma_t}\left(\sigma^{\frac{1}{4}-\frac{3p}{2(p+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}$ is scaling invariant and $\frac{\tilde{A}(t)}{\tilde{\Omega}_p(t)\tilde{\Omega}_2^5(t)}$ is bounded from below (by Lemmas 3.4 and 3.7, $\tilde{\Omega}_p(t) \leq \tilde{\Omega}_p(0)$ and $\tilde{\Omega}_2(t) \leq \tilde{\Omega}_2(0)$). Thus Corollary 3.9 implies that there exists a sequence of times $\{t_k\}_{k\in\mathbb{N}}$, such that $\lim_{k\to\infty} t_k = -\infty$ and

$$\lim_{t_k\to-\infty}\int\limits_{\tilde{\gamma}_{t_k}}\left(\tilde{\sigma}^{\frac{1}{4}-\frac{3p}{2(p+2)}}\right)_{\tilde{\mathfrak{s}}}^2 d\tilde{\mathfrak{s}} = 0.$$

On the other hand, by the Hölder inequality

$$\frac{\left(\tilde{\sigma}_M^{\frac{1}{4}-\frac{3p}{2(p+2)}}(t_k) - \tilde{\sigma}_m^{\frac{1}{4}-\frac{3p}{2(p+2)}}(t_k)\right)^2}{\tilde{\Omega}_1(t_k)} \leq \int\limits_{\tilde{\gamma}_{t_k}}\left(\tilde{\sigma}^{\frac{1}{4}-\frac{3p}{2(p+2)}}\right)_{\tilde{\mathfrak{s}}}^2 d\tilde{\mathfrak{s}}.$$

Moreover, $\tilde{\Omega}_1(t)$ is bounded from above: Indeed $\left(\frac{\tilde{\Omega}_1^3(t)}{\tilde{A}(t)}\right)^{\frac{1}{2}} \leq \tilde{\Omega}_2(t) \leq \tilde{\Omega}_2(0)$. Therefore, we find that

$$\lim_{t_k\to-\infty}\left(\tilde{\sigma}_M^{\frac{1}{4}-\frac{3p}{2(p+2)}}(t_k) - \tilde{\sigma}_m^{\frac{1}{4}-\frac{3p}{2(p+2)}}(t_k)\right)^2 = 0.$$

Since $\tilde{\sigma}_m \leq 1$ and $\tilde{\sigma}_M \geq 1$ (see [3, Lemma 10]) the claim follows. $\square$

4. Proof of the main theorem

Proof. For each time $t \in (-\infty, T)$, let $T_t \in SL(2)$ be a special linear transformation that the maximal ellipse contained in $T_t \tilde{K}_t$ is a disk. So by John's ellipsoid lemma we have

$$\frac{1}{\sqrt{2}} \leq s_{T_t \tilde{K}_t} \leq \sqrt{2}.$$

Then by the Blaschke selection theorem, there is a subsequence of times, denoted again by $\{t_k\}$, such that $\{T_{t_k} \tilde{K}_{t_k}\}$ converges in the Hausdorff distance to an origin-symmetric convex body $\tilde{K}_{-\infty}$, as $t_k \to -\infty$. By Corollary 3.10, and by the weak convergence of the Monge–Ampère measures, the support function of $\tilde{K}_{-\infty}$ is the generalized solution of the following Monge–Ampère equation on $\mathbb{S}^1$:

$$s^3(s_{\theta\theta} + s) = 1$$

Therefore, by Lemma 8.1 of Petty [6], $\tilde{K}_{-\infty}$ is an origin-centered ellipse. This in turn implies that $\lim_{t\to-\infty} \tilde{\Omega}_2(t_k) = 2\pi$. On the other hand, by the 2-affine isoperimetric inequality, (3.1), and by Lemma 3.7, for $t \in (\infty, 0]$ we have

$$2\pi \geq \tilde{\Omega}_2(t) \geq \lim_{t_k\to-\infty} \tilde{\Omega}_2(t_k) = 2\pi.$$

Thus $\frac{d}{dt}\tilde{\Omega}_2(t) \equiv 0$ on $(-\infty, 0]$. Hence, in view of Lemma 3.7, K_t is an origin-centered ellipse for every time $t \in (-\infty, T)$. □

Conflict of interest statement

The author declares that there are no conflicts of interest.

References

[1] B. Andrews, Harnack inequalities for evolving hypersurfaces, Math. Z. 217 (1994) 179–197.
[2] B. Andrews, Contraction of convex hypersurfaces by their affine normal, J. Differ. Geom. 43 (1996) 207–230.
[3] B. Andrews, Affine curve-lengthening flow, J. Reine Angew. Math. 506 (1999) 43–83.
[4] B. Andrews, Motion of hypersurfaces by Gauss curvature, Pac. J. Math. 195 (1) (2000) 1–34.
[5] S. Chen, Classifying convex compact ancient solutions to the affine curve shortening flow, J. Geom. Anal. (2013), http://dx.doi.org/10.1007/s12220-013-9456-z.
[6] C.M. Petty, Affine isoperimetric problems, in: Discrete Geometry and Convexity, Ann. N.Y. Acad. Sci. 440 (1985) 113–127, http://dx.doi.org/10.1111/j.1749-6632.1985.tb14545.x.
[7] M.N. Ivaki, Centro-affine curvature flows on centrally symmetric convex curves, Trans. Am. Math. Soc. (2014), in press, arXiv:1205.6456v2.
[8] M.N. Ivaki, A. Stancu, Volume preserving centro-affine normal flows, Commun. Anal. Geom. 21 (2013) 671–685.
[9] J. Loftin, M.P. Tsui, Ancient solutions of the affine normal flow, J. Differ. Geom. 78 (2008) 113–162.
[10] E. Lutwak, The Brunn–Minkowski–Fiery theory II: affine and geominimal surface areas, Adv. Math. 118 (1996) 244–294.
[11] G. Sapiro, A. Tannenbaum, On affine plane curve evolution, J. Funct. Anal. 119 (1994) 79–120.
[12] A. Stancu, Centro-affine invariants for smooth convex bodies, Int. Math. Res. Not. 2012 (10) (2012) 2289–2320.