



Available online at www.sciencedirect.com



Ann. I. H. Poincaré - AN 33 (2016) 169-197



www.elsevier.com/locate/anihpc

# On the planar Schrödinger-Poisson system

Silvia Cingolani<sup>a,\*</sup>, Tobias Weth<sup>b</sup>

<sup>a</sup> Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via Orabona 4, 70125 Bari, Italy <sup>b</sup> Institut für Mathematik, Goethe-Universität Frankfurt, D-60054 Frankfurt am Main, Germany

Received 31 January 2014; received in revised form 17 September 2014; accepted 23 September 2014

Available online 7 October 2014

## Abstract

We develop a variational framework to detect high energy solutions of the planar Schrödinger-Poisson system

 $\begin{cases} -\Delta u + a(x)u + \gamma wu = 0, & \text{in } \mathbb{R}^2\\ \Delta w = u^2 \end{cases}$ 

with a positive function  $a \in L^{\infty}(\mathbb{R}^2)$  and  $\gamma > 0$ . In particular, we deal with the periodic setting where the corresponding functional is invariant under  $\mathbb{Z}^2$ -translations and therefore fails to satisfy a global Palais–Smale condition. The key tool is a surprisingly strong compactness condition for Cerami sequences which is not available for the corresponding problem in higher space dimensions. In the case where the external potential *a* is a positive constant, we also derive, as a special case of a more general result, the existence of nonradial solutions (u, w) such that *u* has arbitrarily many nodal domains. Finally, in the case where *a* is constant, we also show that solutions of the above problem with u > 0 in  $\mathbb{R}^2$  and  $w(x) \to -\infty$  as  $|x| \to \infty$  are radially symmetric up to translation. Our results are also valid for a variant of the above system containing a local nonlinear term in *u* in the first equation. @ 2014 Elsevier Masson SAS. All rights reserved.

#### MSC: 35J50; 35Q40

Keywords: Schrödinger-Poisson system; Logarithmic convolution potential; Standing wave solutions

# 1. Introduction

The present paper is concerned with standing (or solitary) wave solutions of Schrödinger–Poisson systems of the type

$$\begin{cases} i\psi_t - \Delta\psi + E(x)\psi + \gamma w\psi = 0, \\ \Delta w = |\psi|^2 \end{cases} \quad \text{in } \mathbb{R}^d \times \mathbb{R}.$$

$$(1.1)$$

\* Corresponding author.

E-mail addresses: silvia.cingolani@poliba.it (S. Cingolani), weth@math.uni-frankfurt.de (T. Weth).

http://dx.doi.org/10.1016/j.anihpc.2014.09.008

<sup>0294-1449/© 2014</sup> Elsevier Masson SAS. All rights reserved.

Here  $\psi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$  is the (time-dependent) wave function,  $x \mapsto E(x)$  is a real external potential, and  $\gamma \in \mathbb{R}$  is a parameter. The function *w* represents an internal potential for a nonlocal self-interaction of the wave function  $\psi$ . The standing wave ansatz  $\psi(x, t) = e^{-i\lambda t}u(x)$ ,  $\lambda \in \mathbb{R}$  reduces (1.1) to the system

$$\begin{cases} -\Delta u + a(x)u + \gamma \ wu = 0, \\ \Delta w = u^2 \end{cases} \quad \text{for } u : \mathbb{R}^d \to \mathbb{R}$$

$$(1.2)$$

with  $a(x) = E(x) + \lambda$ . The second equation determines  $w : \mathbb{R}^d \to \mathbb{R}$  only up to harmonic functions, but it is natural to choose w as the Newton potential of  $u^2$ , i.e., the convolution of  $u^2$  with the fundamental solution  $\Phi_d$  of the Laplacian. With this formal inversion of the second equation in (1.2), we obtain the integro-differential equation

$$-\Delta u + a(x)u + \gamma \left[ \Phi_d * |u|^2 \right] u = 0 \quad \text{in } \mathbb{R}^d,$$
(1.3)

where  $\Phi_d(x) = \frac{1}{d(2-d)\omega_d} |x|^{2-d}$  in case  $d \ge 3$  and  $\Phi_d(x) = \frac{1}{2\pi} \log |x|$  in case d = 2. Here, as usual,  $\omega_d$  denotes the volume of the unit ball in  $\mathbb{R}^d$ .

In the three dimensional case, (1.3) has been extensively studied. For d = 3,  $a \equiv \lambda > 0$  and  $\gamma > 0$ , it was introduced by Pekar [27] in 1954 for describing the quantum mechanics of a polaron at rest and by Choquard in 1976 for describing an electron trapped in its hole. In [20], Lieb proved the existence of a unique ground state of (1.2) which is positive and spherically symmetric, using a minimization argument. In [22], Lions proved the existence of infinitely many distinct spherically symmetric solutions when a(x) is a nonnegative, radially symmetric potential. In [28] Penrose derived (1.3) in his discussion about the self gravitational collapse of a quantum-mechanical system. Recently, existence and regularity results have also been obtained for  $a(x) \neq \lambda$  and for more general convolution potentials, see [1,13,18,23,25].

In some recent works, local nonlinear terms of the form  $b|u|^{p-2}u$ ,  $b \in \mathbb{R}$ , p > 2 have been added to the right hand side of (1.3). Those nonlinear terms are frequently used in Schrödinger equations to model the interaction among particles [7,9]. A large amount of papers are devoted to the study of existence of solutions of the corresponding Schrödinger–Poisson system (also called Schrödinger–Maxwell system) in the three dimensional case. In particular for  $\gamma < 0$ , b > 0,  $2 , we quote the results [3,4,16,30,6]. Conversely, for <math>\gamma > 0$  and b < 0, the corresponding system has been studied in [8] and it represents a Hartree model for crystals (see also [26]).

The literature is scantier for the planar case d = 2, which is the focus of the present paper. In this case, Masaki [24] proved global well-posedness of the Cauchy problem for (1.1) in a subspace of  $H^1(\mathbb{R}^2)$ . Moreover, Stubbe and Vuffray [12] established existence and uniqueness of positive, spherically symmetric solutions of (1.2) in the case  $a \equiv \lambda$ ,  $d \leq 6$ , using shooting methods for the associated ODE system (see also [11] for the one-dimensional case). Unlike in the case d = 3, variational methods have rarely been used in the planar case in which (1.3) becomes

$$-\Delta u + au + \frac{\gamma}{2\pi} \left( \log(|\cdot|) * |u|^2 \right) u = 0 \quad \text{in } \mathbb{R}^2.$$

$$\tag{1.4}$$

Note that, at least formally, (1.3) has a variational structure related to the energy functional

$$u \mapsto \frac{1}{2} \int_{\mathbb{R}^2} \left( |\nabla u|^2 + a(x)u^2 \right) dx + \frac{\gamma}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Phi_d \left( |x - y|^2 \right) u^2(x) u^2(y) dx dy.$$

In case  $d \ge 3$  and  $a \in L^{\infty}(\mathbb{R}^d)$ , this functional is well-defined and of class  $C^1$  on  $H^1(\mathbb{R}^d)$ , but it is not well defined on  $H^1(\mathbb{R}^2)$  in case d = 2. Considering the case d = 2,  $a \equiv \lambda \in \mathbb{R}$  and  $\gamma > 0$ , Stubbe [32] set up a variational framework for (1.4) within a subspace of  $H^1(\mathbb{R}^2)$ . By using strict rearrangement inequalities, he proved that there exists, for any  $\lambda \ge 0$ , a unique ground state, which is a positive spherically symmetric decreasing function. In addition, he proved that there exists a negative number  $\lambda^*$  such that for any  $\lambda \in (\lambda^*, 0)$  there exist two ground states with different  $L^2$  norm. In the limiting case  $\lambda = \lambda^*$ , there is again a unique ground state.

In the present work we focus on (1.4) in the case  $\gamma > 0$ , and by rescaling we may assume  $\gamma = 2\pi$ . More precisely, we will consider a generalization of (1.4) given by

$$-\Delta u + a(x)u + (\log(|\cdot|) * |u|^2)u = b|u|^{p-2}u \quad \text{in } \mathbb{R}^2$$
(1.5)

with  $b \ge 0$ ,  $p \ge 2$  and  $a \in L^{\infty}(\mathbb{R}^2)$ . We wish to emphasize that all of our results are new even for b = 0, i.e., for Eq. (1.4). As remarked before, the energy functional

$$u \mapsto I(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( |\nabla u|^2 + a(x)u^2 \right) dx + \frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x - y|^2) u^2(x) u^2(y) \, dx \, dy - \frac{b}{p} \int_{\mathbb{R}^2} |u|^p \, dx$$

is not well defined on  $H^1(\mathbb{R}^2)$ . Inspired by Stubbe [32], we will consider the smaller Hilbert space

$$X := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \log(1+|x|) u^2(x) \, dx < \infty \right\}$$

It is not difficult to see that *I* defines a  $C^1$ -functional on *X*. Moreover, critical points  $u \in X$  of *I* are strong solutions of (1.5) in  $W^{2,p}(\mathbb{R}^2)$  for all  $p \ge 1$ , and they are classical solutions in  $C^2(\mathbb{R}^2)$  if *a* is Hölder continuous, see Proposition 2.3 below. So *X* provides a variational framework for (1.5), but some difficulties appear due to the following unpleasant facts. First, the norm of *X* is *not* translation invariant, whereas the functional *I* is invariant under every translation which leaves the external potential *a* invariant. Second, the quadratic part of *I* is not coercive on *X* even if  $\inf_{\mathbb{R}^2} a > 0$ . These difficulties enforce the implementation of new ideas and estimates within the variational framework. On the other hand, somewhat surprisingly, the specific form of *I* also allows to establish much better existence and multiplicity results than those available in the case  $d \ge 3$  or for the simpler equation

$$-\Delta u + a(x)u = |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^2).$$
(1.6)

In fact, our main results suggest that the structure of the solution set of (1.5) is in general much richer than the one of (1.6) or related problems in higher dimensions.

Our first main result is concerned with the periodic setting.

**Theorem 1.1.** Suppose that  $p \ge 4$ ,  $b \ge 0$ , and that  $a : \mathbb{R}^2 \to (0, \infty)$  is continuous and  $\mathbb{Z}^2$ -periodic. Then (1.5) admits a sequence of solution pairs  $\pm u_n \in X$  such that  $I(u_n) \to \infty$  as  $n \to \infty$ . Moreover, the restriction of I to the associated Nehari manifold  $\mathcal{N} := \{u \in X \setminus \{0\} : I'(u)u = 0\}$  attains a global minimum, and every minimizer  $u \in \mathcal{N}$  of  $I|_{\mathcal{N}}$  is a solution of (1.5) which does not change sign and obeys the variational characterization

$$I(u) = \inf_{u \in X} \sup_{t \in \mathbb{R}} I(tu).$$

We remark that Theorem 1.1 yields infinitely many geometrically distinct solutions (i.e., solutions generating different  $\mathbb{Z}^2$ -orbits) and also ground state solutions of (1.5), i.e., energy minimizers within the set of nontrivial solutions. This seems to be the first existence result for (1.5) in the periodic setting even in the special case b = 0. We also note that a result on the existence of solutions with arbitrarily high energy is not available for (1.6) without additional nondegeneracy assumptions (see e.g. [15]).

Our second main result is concerned with a different symmetric setting in the case where *a* is a positive constant. We need to fix some notation. Let *G* be a closed subgroup of the orthogonal group O(2) such that

$$\operatorname{Fix}(G) = \{0\}, \quad \text{where } \operatorname{Fix}(G) := \left\{ x \in \mathbb{R}^2 : Ax = x \text{ for all } A \in G \right\}.$$
(1.7)

Moreover, let  $\tau : G \to \{-1, 1\}$  be a group homomorphism. The pair  $(G, \tau)$  gives rise to a group action of G on X defined by

$$A * u(x) := \tau(A)u(A^{-1}x) \quad \text{for } A \in G, \ u \in X, \text{ and } x \in \mathbb{R}^2.$$

$$(1.8)$$

The following result is concerned with solutions of (1.5) in the invariant subspace

$$X_G := \{ u \in X : A * u = u \text{ for all } A \in G \}.$$

**Theorem 1.2.** Suppose that  $p \ge 4$ ,  $b \ge 0$ , and that a in (1.5) is a positive constant. Let G,  $\tau$  be as above, and suppose that  $\tau \equiv 1$  or that G is finite. Then (1.5) admits a sequence of solution pairs  $\pm u_n \in X_G$  such that  $I(u_n) \to \infty$  as  $n \to \infty$ . Moreover, the restriction of I to the associated Nehari manifold  $\mathcal{N}_G := \{u \in X_G \setminus \{0\} : I'(u)u = 0\}$  attains a global minimum, and every minimizer  $u \in \mathcal{N}_G$  of  $I|_{\mathcal{N}_G}$  is a G-invariant solution of (1.5) which obeys the variational characterization

$$I(u) = \inf_{u \in X_G} \sup_{t \in \mathbb{R}} I(tu),$$

and it does not change sign if  $\tau \equiv 1$ .

Some remarks seem appropriate. The assumptions on G and  $\tau$  ensure that the closed subspace  $X_G \subset X$  is infinite dimensional. If  $\tau$  is nontrivial and  $A \in G$  is given with  $\tau(A) = -1$ , then every  $u \in X_G$  satisfies  $u(A^{-1}x) = -u(x)$ . In particular, u vanishes on the set  $\{x \in \mathbb{R}^2 : Ax = x\}$  and changes sign in  $\mathbb{R}^2$  if  $u \neq 0$ . Let us briefly discuss some examples. In case  $G = O(2), \tau \equiv 1$  the space  $X_G$  consists of radial functions in X. In this case Theorem 1.2 yields the existence of infinitely many radial solutions. Next, let  $G = \{id, A_1, A_2\}$ , where  $A_i$  is the reflection at the coordinate hyperplane  $\{x_i = 0\}$  for i = 1, 2. Moreover, let  $\tau : G \to \{-1, 1\}$  be the homomorphism defined by  $\tau(A_1) = -1$  and  $\tau(A_2) = 1$  (this defines a homomorphism since  $A_1$  and  $A_2$  commute). Then  $u \in X_G$  if and only if

$$u(-x_1, x_2) = -u(x_1, x_2)$$
 and  $u(x_1, -x_2) = u(x_1, x_2)$  for all  $x = (x_1, x_2) \in \mathbb{R}^2$ . (1.9)

Hence Theorem 1.2 yields a sequence of solution pairs  $\pm u_n$ ,  $n \in \mathbb{N}$  of (1.5) with unbounded energy and such that  $u_n$  is odd with respect to the hyperplane  $\{x_1 = 0\}$  and even with respect to the hyperplane  $\{x_2 = 0\}$ . We point out that such a result is not available for (1.6) in the case where a > 0 is constant; in fact, (1.6) does not admit any nontrivial solutions which vanish on a hyperplane, see e.g. [17]. As a third example we may consider, for given  $k \in \mathbb{N}$ , the subgroup G of O(2) of order 2k generated by the (counter-clockwise)  $\frac{\pi}{k}$ -rotation

$$A \in O(2),$$
  $A(x) = (x_1 \cos \pi/k - x_2 \sin \pi/k, x_1 \sin \pi/k + x_2 \cos \pi/k)$  for  $x = (x_1, x_2) \in \mathbb{R}^2.$ 

Let  $\tau : G \to \{-1, 1\}$  be the homomorphism defined by  $\tau(A^j) = (-1)^j$  for  $j \in \mathbb{N}$ . Then Theorem 1.2 applies and yields sign changing solutions which are invariant under the corresponding action defined by (1.8). Note that any such solution has at least 2k (conical) nodal domains in  $\mathbb{R}^2$ . We also point out that the assumption (1.7) is not essential for the existence of the sequence  $(u_n)_n$  in Theorem 1.2, since it can always be achieved by enlargening G suitably. More precisely, if  $F := \operatorname{Fix}(G) \neq \{0\}$ , we may consider the orthogonal splitting  $\mathbb{R}^2 := E \oplus F$  and define  $B \in O(2) \setminus G$  by Bx = -x for  $x \in F$  and Bx = x for  $x \in E$ . Then B is an involution which commutes with every element of G, so we may consider  $\tilde{G} = G \cup \{B\}$  and  $\tilde{\tau} : \tilde{G} \to \{-1, 1\}$  defined by  $\tilde{\tau}|_G = \tau$  and  $\tilde{\tau}(B) = -1$  in place of G and  $\tau$ . Denoting by  $X_{\tilde{G}}$  the invariant subspace of X with respect to the corresponding group action defined by (1.8) with  $\tilde{G}, \tilde{\tau}$  in place of  $G, \tau$ , we then have  $\operatorname{Fix}(\tilde{G}) = 0$  and  $X_{\tilde{G}} \subset X_G$ .

In view of the fact that, by the results above, (1.5) has a large solution set in general, it is a natural aim to classify different types of solutions via their geometric properties. The first step within this rather ambitious program is the study of the shape and possible uniqueness of positive solutions (1.5). Related to this, we have the following result.

**Theorem 1.3.** Suppose that  $p \ge 2$ , and that a is a positive constant in (1.5). Then every positive solution  $u \in X$  of (1.5) is radially symmetric up to translation and strictly decreasing in the distance from the symmetry center. Moreover, if b = 0, then u is unique up to translation.

We note that Ma and Zhao [23] proved the corresponding statement for (1.3) in the case d = 3,  $a \equiv 1$  and  $\gamma = 1$ . Their approach relies on a reformulation of (1.3) as an integral equation and a moving plane argument. The results of [23] apply to a general class of integral equations, but they do not apply to (1.5) since the logarithmic integral kernel is sign-changing. Our proof of Theorem 1.3 relies on a more direct and simpler variant of the moving plane method for the corresponding system, see Section 6 below. We also point out that the uniqueness claim in Theorem 1.3 is a simple consequence of the uniqueness result for positive, radial solutions in [12].

The paper is organized as follows. In Section 2 we set up the variational framework and establish useful preliminary estimates. We also address the notion and regularity of weak solutions, see Proposition 2.3 below. Section 3 contains the key compactness condition, Proposition 3.1 which is fundamental for our existence results. Roughly speaking, Proposition 3.1 asserts that the functional I satisfies the Cerami condition at arbitrary energy levels' up to translations' if a is periodic and positive. In Section 4 we then prove Theorem 1.1. The proof is based on deformation arguments which differ considerably from previous works. In particular, we need to implement new estimates on the Krasnoselski genus of neighborhoods of the *noncompact* set of critical points of I at a given energy level. Moreover, special care is needed at some points since the norm of X is not translation invariant. Section 5 is devoted to the proof of

Theorem 1.2. The key observation here is the fact that, in the *G*-invariant setting described above, the restriction of *I* to the invariant space  $X_G$  satisfies the Cerami condition. Finally, in Section 6 we will prove a slightly more general variant of Theorem 1.3, and we will deduce Theorem 1.3 in the end of this section.

# 2. Preliminaries

In the following, we assume that  $a \in L^{\infty}(\mathbb{R}^2)$  satisfies  $\inf_{\mathbb{R}} a > 0$ . Then we may endow  $H^1(\mathbb{R}^2)$  with the scalar product

$$\langle u, v \rangle = \int_{\mathbb{R}^2} \left( \nabla u \cdot \nabla v + a(x)uv \right) dx, \quad \text{for } u, v \in H^1(\mathbb{R}^2).$$

The corresponding norm given by

$$\|u\|^{2} := \int_{\mathbb{R}^{2}} \left( |\nabla u|^{2} + a(x)u^{2} \right) dx, \quad \text{for } u \in H^{1}(\mathbb{R}^{2})$$

is equivalent to the standard  $H^1(\mathbb{R}^2)$ -norm. We define the symmetric bilinear forms

$$(u, v) \mapsto B_{1}(u, v) = \iint_{\mathbb{R}^{2} \mathbb{R}^{2}} \log(1 + |x - y|)u(x)v(y) \, dx \, dy,$$
  

$$(u, v) \mapsto B_{2}(u, v) = \iint_{\mathbb{R}^{2} \mathbb{R}^{2}} \log\left(1 + \frac{1}{|x - y|}\right)u(x)v(y) \, dx \, dy,$$
  

$$(u, v) \mapsto B_{0}(u, v) = B_{1}(u, v) - B_{2}(u, v) = \iint_{\mathbb{R}^{2} \mathbb{R}^{2}} \log(|x - y|)u(x)v(y) \, dx \, dy.$$

Here the definition is restricted, in each case, to measurable functions  $u, v : \mathbb{R}^2 \to \mathbb{R}$  such that the corresponding double integral is well defined in Lebesgue sense. Note that, since  $0 \le \log(1 + r) \le r$  for r > 0, we have by the Hardy–Littlewood–Sobolev inequality [21]

$$\left|B_{2}(u,v)\right| \leq \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \frac{1}{|x-y|} \left|u(x)v(y)\right| dx dy \leq C_{0} |u|_{\frac{4}{3}} |v|_{\frac{4}{3}} \quad \text{for } u, v \in L^{\frac{4}{3}}(\mathbb{R}^{2})$$

$$(2.1)$$

with a constant  $C_0 > 0$ . Here and in the following,  $|u|_p$  stands for the  $L^p$ -norm of a function  $u \in L^p(\mathbb{R}^2)$ ,  $1 \le p \le \infty$ . We now define the functionals

$$V_{1}: H^{1}(\mathbb{R}^{2}) \to [0, \infty], \qquad V_{1}(u) = B_{1}(u^{2}, u^{2}) = \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \log(1 + |x - y|) u^{2}(x) u^{2}(y) \, dx \, dy,$$
  

$$V_{2}: L^{\frac{8}{3}}(\mathbb{R}^{2}) \to [0, \infty), \qquad V_{2}(u) = B_{2}(u^{2}, u^{2}) = \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \log\left(1 + \frac{1}{|x - y|}\right) u^{2}(x) u^{2}(y) \, dx \, dy,$$
  

$$V_{0}: H^{1}(\mathbb{R}^{2}) \to \mathbb{R} \cup \{\infty\}, \qquad V_{0}(u) = B_{0}(u^{2}, u^{2}) = \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \log(|x - y|) u^{2}(x) u^{2}(y) \, dx \, dy.$$

Note that, as a consequence of (2.1), we have

$$|V_2(u)| \le C_0 |u|_{\frac{8}{3}}^4$$
 for all  $u \in L^{\frac{8}{3}}(\mathbb{R}^2)$ , (2.2)

so  $V_2$  only takes finite values on  $L^{\frac{8}{3}}(\mathbb{R}^2) \subset H^1(\mathbb{R}^2)$ . Next we define, for any measurable function  $u : \mathbb{R}^2 \to \mathbb{R}$ ,

$$|u|_*^2 := \int_{\mathbb{R}^2} \log(1+|x|) u^2(x) \, dx \in [0,\infty].$$

We note that, since

$$\log(1+|x-y|) \le \log(1+|x|+|y|) \le \log(1+|x|) + \log(1+|y|) \quad \text{for } x, y \in \mathbb{R}^2,$$
(2.3)

we have the estimate

$$B_{1}(uv, wz) \leq \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \left[ \log(1+|x|) + \log(1+|y|) \right] |u(x)v(x)| |w(y)z(y)| dxdy$$
  
$$\leq |u|_{*}|v|_{*}|w|_{2}|z|_{2} + |u|_{2}|v|_{2}|w|_{*}|z|_{*} \quad \text{for } u, v, w, z \in L^{2}(\mathbb{R}^{2})$$
(2.4)

with the conventions  $\infty \cdot 0 = 0$  and  $\infty \cdot s = \infty$  for s > 0. The following lemma will also be useful in the sequel.

**Lemma 2.1.** Let  $(u_n)_n$  be a sequence in  $L^2(\mathbb{R}^2)$  such that  $u_n \to u \in L^2(\mathbb{R}^2) \setminus \{0\}$  pointwise a.e. on  $\mathbb{R}^2$ . Moreover, let  $(v_n)_n$  be a bounded sequence in  $L^2(\mathbb{R}^2)$  such that

$$\sup_{n\in\mathbb{N}}B_1\left(u_n^2, v_n^2\right) < \infty.$$
(2.5)

Then there exists  $n_0 \in \mathbb{N}$  and C > 0 such that  $|v_n|_* < C$  for  $n \ge n_0$ .

If, moreover,

$$B_1(u_n^2, v_n^2) \to 0 \quad and \quad |v_n|_2 \to 0 \quad as \ n \to \infty,$$
(2.6)

then

$$|v_n|_* \to 0 \quad as \ n \to \infty, \ n \ge n_0. \tag{2.7}$$

**Proof.** By assumption and Egorov's Theorem, there exists  $n_0 \in \mathbb{N}$ ,  $R, \delta > 0$  and a measurable subset  $A \subset B_R(0)$  such that |A| > 0 and  $u_n^2(x) \ge \delta$  for every  $n \ge n_0$ . Since

$$1 + |x - y| \ge 1 + \frac{|y|}{2} \ge \sqrt{1 + |y|} \quad \text{for every } x \in B_R(0), \ y \in \mathbb{R}^2 \setminus B_{2R}(0).$$

we may then estimate

$$\begin{split} B_1(u_n^2, v_n^2) &\geq \int\limits_{\mathbb{R}^2 \setminus B_{2R}(0)} \int\limits_A \log(1 + |x - y|) u_n^2(x) v_n^2(y) \, dx \, dy \\ &\geq \frac{\delta |A|}{2} \int\limits_{\mathbb{R}^2 \setminus B_{2R}(0)} \log(1 + |y|) v_n^2(y) \, dy = \frac{\delta |A|}{2} \left( |v_n|_*^2 - \int\limits_{B_{2R}(0)} \log(1 + |y|) v_n^2(y) \, dy \right) \\ &\geq \frac{\delta |A|}{2} \left( |v_n|_*^2 - \log(1 + 2R) |v_n|_2^2 \right). \end{split}$$

Since  $|v_n|_2^2$  and  $B_1(u_n^2, v_n^2)$  remain bounded in *n* by assumption, it follows that  $|v_n|_*$  also remains bounded in *n*, as claimed. If moreover (2.6) holds, then the estimate above yields (2.7).  $\Box$ 

In the following, we fix  $b \ge 0$ ,  $p \ge 4$  and consider the functional

$$I: H^1(\mathbb{R}^2) \to \mathbb{R} \cup \{\infty\}, \qquad I(u) = \frac{1}{2} \|u\|^2 + \frac{1}{4} V_0(u) - \frac{b}{p} \|u\|_p^p.$$

We also define the Hilbert space

 $X := \left\{ u \in H^1(\mathbb{R}^2) : |u|_*^2 < \infty \right\} \text{ with norm given by } u \mapsto \|u\|_X^2 := \|u\|^2 + |u|_*^2.$ 

Note that, by (2.4), the restriction of *I* to *X* (also denoted by *I* in the following) only takes finite values in  $\mathbb{R}$ . We have the following properties.

# Lemma 2.2.

- (i) The space X is compactly embedded in  $L^{s}(\mathbb{R}^{2})$  for all  $s \in [2, \infty)$ .
- (ii) The functionals  $V_0$ ,  $V_1$ ,  $V_2$  and I are of class  $C^1$  on X.

*Moreover*,  $V'_i(u)v = 4B_i(u^2, uv)$  for  $u, v \in X$  and i = 0, 1, 2.

- (iii)  $V_2$  is continuous (in fact continuously differentiable) on  $L^{\frac{8}{3}}(\mathbb{R}^2)$ .
- (iv)  $V_1$  is weakly lower semicontinuous on  $H^1(\mathbb{R}^2)$ .
- (v) I is weakly lower semicontinuous on X.
- (vi) I is lower semicontinuous on  $H^1(\mathbb{R}^2)$ .

**Proof.** (i) follows from Rellich's Theorem (see Theorem XIII.65 in [29]).

We prove (ii) and (iii). Let  $u_n$  be a sequence in X converging to some  $u \in X$ . It follows that  $u_n$  is bounded and

$$\begin{aligned} \left| V_{1}(u_{n}) - V_{1}(u) \right| &= \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |x - y|) u_{n}^{2}(x) (u_{n}^{2}(y) - u^{2}(y)) dx dy \\ &+ \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |x - y|) (u_{n}^{2}(x) - u^{2}(x)) u^{2}(y) dx dy \\ &\leq \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |x|) u_{n}^{2}(x) |u_{n}(y) - u(y)| |u_{n}(y) + u(y)| \\ &+ \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |y|) u_{n}^{2}(x) |u_{n}(y) - u(y)| |u_{n}(y) + u(y)| \\ &+ \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |y|) |u_{n}(x) - u(x)| |u_{n}(x) + u(x)| u^{2}(y) dx dy \\ &+ \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |y|) |u_{n}(x) - u(x)| |u_{n}(x) + u(x)| u^{2}(y) dx dy \\ &\leq |u_{n}|_{*}^{2} |u_{n} - u|_{2} |u_{n} + u|_{2} + |u_{n}|_{2}^{2} |u_{n} - u|_{2} |u_{n} + u|_{*} \\ &+ |u|_{2}^{2} |u_{n} - u|_{*} |u_{n} + u|_{*} + |u|_{*}^{2} |u_{n} - u|_{2} |u_{n} + u|_{2} \\ &\leq C ||u_{n} - u||_{X} \end{aligned}$$

for a suitable positive constant C > 0. Hence we derive that  $V_1(u_n)$  tends to  $V_1(u)$ , as  $n \to \infty$ . For any  $v \in X$  the Gateaux derivative of  $V_1$  at  $u \in X$  is given by

$$V_1'(u)v = 4 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log(1 + |x - y|) u^2(x) u(y) v(y) dx dy.$$

Since

$$\begin{aligned} \left| V_{1}'(u)v \right| &\leq 4 \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1+|x|)u^{2}(x)u(y)v(y) + 4 \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1+|y|)u^{2}(x)u(y)v(y) \\ &\leq |u|_{*}^{2}|u|_{2}|v|_{2} + |u|_{2}^{2}|u|_{*}|v|_{*} \leq K \|u\|_{X}^{3} \|v\|_{X} \end{aligned}$$

$$(2.9)$$

we deduce that  $V'_1(u) \in X^*$  and  $||V'_1(u)|| \le K ||u||_X^3$  for any  $u \in X$ . Now we prove that  $V'(u_n)$  tends to V'(u) in  $X^*$  if  $u_n \to u$  in X.

Suppose that  $u_n$  tends to u in X. It follows that  $u_n$  is bounded and for any  $v \in X$  we have

$$\frac{|V_{1}^{\prime}(u_{n})v - V_{1}^{\prime}(u)v|}{4} = \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |x - y|) (u_{n}^{2}(x)u_{n}(y) - u^{2}(x)u(y))v(y)dxdy$$

$$= \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |x - y|) (u_{n}^{2}(x) - u^{2}(x))u_{n}(y)v(y)dxdy$$

$$+ \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |x - y|) (u_{n}(y) - u(y))u^{2}(x)v(y)dxdy$$

$$\leq \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |x|) |u_{n}(x) - u(x)| |u_{n}(x) + u(x)| |u_{n}(y)| |v(y)|dxdy$$

$$+ \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |y|) |u_{n}(x) - u(x)| |u_{n}(x) + u(x)| |u_{n}(y)| |v(y)|dxdy$$

$$+ \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |y|) |u_{n}(y) - u(y)|u^{2}(x)|v(y)|dxdy$$

$$+ \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log(1 + |y|) |u_{n}(y) - u(y)|u^{2}(x)|v(y)|dxdy$$

$$\leq |u_{n} - u|_{*}|u_{n} + u|_{*}|u_{n}|_{2}|v|_{2} + |u_{n} - u|_{2}|u_{n} + u|_{2}|u_{n}|_{*}|v|_{*}$$

$$+ |u_{n} - u|_{2}|v|_{2}|u|_{*}^{2} + |u_{n} - u|_{*}|v|_{*}|u|_{2}^{2}$$

$$\leq K ||u_{n} - u|_{X}||v||_{X}$$

$$(2.10)$$

for a suitable some K > 0, independent of v. Hence  $||V'_1(u_n) - V'_1(u)|| \le K ||u_n - u||_X$  and thus  $V_1$  is continuously differentiable on X. Moreover  $V'_1(u)v = 4B_1(u^2, uv)$  holds. Let  $u_n$  be a sequence on  $L^{8/3}(\mathbb{R}^2)$  converging to some  $u \in L^{8/3}(\mathbb{R}^2)$  in  $L^{8/3}(\mathbb{R}^2)$ . Now we evaluate

$$\begin{aligned} \left| V_{2}(u_{n}) - V_{2}(u) \right| &\leq \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log \left( 1 + \frac{1}{|x - y|} \right) u_{n}^{2}(x) \left| u_{n}^{2}(y) - u^{2}(y) \right| dx dy \\ &+ \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log \left( 1 + \frac{1}{|x - y|} \right) \left| u_{n}^{2}(x) - u^{2}(x) \right| u^{2}(y) dx dy \\ &\leq \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \frac{1}{|x - y|} u_{n}^{2}(x) \left| u_{n}(y) - u(y) \right| \left| u_{n}(y) + u(y) \right| \\ &+ \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \frac{1}{|x - y|} \left| u_{n}(x) - u(x) \right| \left| u_{n}(x) + u(x) \right| u^{2}(y) dx dy \\ &\leq C \left| u_{n} - u \right|_{8/3} \left| u_{n} + u \right|_{8/3} \left| u_{n} \right|_{8/3}^{2} + \left| u_{n} - u \right|_{8/3} \left| u_{n} + u \right|_{8/3} \left| u \right|_{8/3}^{2} \\ &\leq C \left| u_{n} - u \right|_{8/3} \left[ \left( \left| u_{n} \right|_{8/3} + \left| u \right|_{8/3} \right) \left| u_{n} \right|_{8/3}^{2} + \left( \left| u \right|_{8/3} + \left| u \right|_{8/3} \right) \left| u \right|_{8/3}^{2} \right] \\ &\leq C \left| u_{n} - u \right|_{8/3} \left[ \left( \left| u_{n} \right|_{8/3} + \left| u \right|_{8/3} \right) \left| u_{n} \right|_{8/3}^{2} + \left| u \right|_{8/3}^{2} \right] \end{aligned}$$

$$(2.11)$$

From (2.11) we derive that if  $u_n$  tends to u in  $L^{8/3}$  as  $n \to \infty$ , then  $V_2(u_n)$  tends to  $V_2(u)$  as  $n \to \infty$ . Hence  $V_2$  is continuous on  $L^{\frac{8}{3}}(\mathbb{R}^2)$ . For any  $v \in L^{8/3}(\mathbb{R}^2)$  the Gateaux derivative of  $V_2$  at  $u \in L^{8/3}(\mathbb{R}^2)$  along v is given by

$$V_2'(u)v = 4\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log\left(1 + \frac{1}{|x - y|}\right) u^2(x)u(y)v(y)dxdy$$

Since

$$\begin{aligned} \left| V_{2}'(u)v \right| &\leq 4 \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \frac{1}{|x-y|} u^{2}(x) \left| u(y) \right| \left| v(y) \right| \\ &\leq 4C_{0} |u^{2}|_{4/3} |uv|_{4/3} \leq 4C_{0} |u|_{8/3}^{3} |v|_{8/3} \end{aligned}$$

$$(2.12)$$

we deduce  $||V'_2(u)|| \le K |u|^3_{8/3}$  for any  $u \in L^{8/3}(\mathbb{R}^2)$ .

Now we prove that  $V_2$  is continuously differentiable on  $L^{8/3}(\mathbb{R}^2)$ . Suppose that  $u_n$  tends to u in  $L^{8/3}(\mathbb{R}^2)$ . It follows that  $u_n$  is bounded and for any  $v \in L^{8/3}(\mathbb{R}^2)$ 

$$\frac{|V_{2}'(u_{n})v - V_{2}'(u)v|}{4} = \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log\left(1 + \frac{1}{|x - y|}\right) \left(u_{n}^{2}(x)u_{n}(y) - u^{2}(x)u(y)\right)v(y)dxdy$$

$$= \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \log\left(1 + \frac{1}{|x - y|}\right) \left(u_{n}^{2}(x) - u^{2}(x)\right)u_{n}(y)v(y)dxdy$$

$$+ \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \frac{1}{|x - y|} |u_{n}(y) - u(y)|u^{2}(x)|v(y)|dxdy$$

$$\leq \iint_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \frac{1}{|x - y|} |u_{n}(x) - u(x)| |u_{n}(x) + u(x)| |u_{n}(y)| |v(y)|dxdy$$

$$\leq C|u_{n} - u|_{8/3}|v|_{8/3}|u|_{8/3}^{2} + |u_{n} - u|_{8/3}|u_{n} + u|_{8/3}|u_{n}|_{8/3}|v|_{8/3}$$

$$= C|u_{n} - u|_{8/3}|v|_{8/3}(|u|_{8/3}^{2} + |u_{n} + u|_{8/3}|u_{n}|_{8/3})$$

$$\leq C_{1}|u_{n} - u|_{8/3}|v|_{8/3}$$

$$(2.13)$$

for some suitable some positive constants  $C, C_1$ , independent of v. Hence  $||V'_2(u_n) - V'_2(u)|| \le C_1 |u_n - u|_{8/3}$  and thus  $V_2$  is continuous differentiable on  $L^{8/3}(\mathbb{R}^2)$ .

Furthermore we deduce that  $V_2$  is  $C^1$  on X and  $V'_2(u)v = 4B_2(u^2, uv)$  holds.

It follows that  $V_0 = V_1 - V_2$  is  $C^1$  on X and  $V'_0(u)v = 4B_0(u^2, uv)$  holds. We also infer that I is  $C^1$  on X.

We prove (iv). Let  $u_n$  be a sequence in  $H^1(\mathbb{R}^2)$  and  $u \in H^1(\mathbb{R}^2)$ . Suppose that  $u_n$  weakly converges to u in  $H^1(\mathbb{R}^2)$ . For R > 0 fixed,  $u_n$  strongly converges to u in  $L^2(B_R)$  and

$$\lim_{n \to \infty} \iint_{B_R(0) \times B_R(0)} \log(1 + |x - y|) u_n^2(x) u_n^2(y) dx dy = \iint_{B_R(0) \times B_R(0)} \log(1 + |x - y|) u^2(x) u^2(y) dx dy.$$

It follows that

$$\liminf_{n \to \infty} V_1(u_n) \ge \iint_{B_R(0) \times B_R(0)} \log(1 + |x - y|) u^2(x) u^2(y) dx dy.$$
(2.14)

By Monotone Convergence theorem, we derive

$$\lim_{R \to \infty} \iint_{B_R(0) \times B_R(0)} \log(1 + |x - y|) u^2(x) u^2(y) dx dy = V_1(u)$$

and then

 $\liminf_{n\to\infty} V_1(u_n) \ge V_1(u).$ 

We prove (v). Now let  $u_n$  be a sequence in X and  $u \in X$ . Assume that  $u_n$  weakly converges to u in X. It follows that  $u_n$  strongly converges to u in  $L^{8/3}(\mathbb{R}^2)$  and thus from (ii) we have  $\lim_{n\to\infty} V_2(u_n) = V_2(u)$ .

Moreover  $u_n$  weakly converges to u in  $H^1(\mathbb{R}^2)$  and by (iv), we have  $\liminf_{n\to\infty} V_1(u_n) \ge V_1(u)$ . Taking also into account (i), we conclude that

$$\liminf_{n\to\infty} I(u_n) \ge I(u).$$

Finally, (vi) follows from (iv) and the continuity of the functional

$$u \mapsto I(u) - \frac{1}{4}V_1(u) = \frac{1}{2}||u||^2 - \frac{1}{4}V_2(u) - \frac{b}{p}|u|_p^p$$

with respect to the  $H^1$ -norm.  $\Box$ 

As a consequence of 2.2(ii), we find that

$$I'(u)v = \langle u, v \rangle + B_0(u^2, uv) - b \int_{\mathbb{R}^2} |u|^{p-2} uv \, dx \quad \text{for } u, v \in X,$$
(2.15)

so  $u \in X$  is a critical point of I if and only if it is a weak solution of (1.5) in the sense that the RHS of (2.15) vanishes for every  $v \in X$ . Moreover, we have the following regularity result.

**Proposition 2.3.** Let  $u \in X$  be a weak solution of (1.5), i.e., a critical point of I. Then  $u \in W^{2,p}(\mathbb{R}^2)$  for every p > 1. and u is a strong solution of (1.5). Moreover, we have:

- (i)  $|u(x)|e^{\alpha|x|} \to 0$  as  $|x| \to \infty$  for every  $\alpha > 0$ . (ii) The function  $w : \mathbb{R}^2 \to \mathbb{R}$ ,  $w(x) = \int_{\mathbb{R}^2} \log |x y| u^2(y) \, dy$  is of class  $C^3$  on  $\mathbb{R}^2$  and satisfies

 $\Delta w = u^2 \quad in \mathbb{R}^2, \qquad w(x) - |u|_2^2 \log |x| \to 0 \quad as \ |x| \to \infty.$ 

Furthermore, if a is Hölder continuous, then  $u \in C^2(\mathbb{R}^2)$  is a classical solution of (1.5).

**Proof.** We first show that the function w defined in (ii) is locally bounded and has the asserted asymptotics. If |x| < 1, we have

$$|w(x)| \le \int_{B_2(x)} |\log |x - y| |u^2(y) \, dy + \int_{\mathbb{R}^2 \setminus B_2(x)} \log |x - y| u^2(y) \, dy,$$
(2.16)

Since  $1 \le |x - y| \le 1 + |y|$  for  $y \in \mathbb{R}^2 \setminus B_2(x)$ , we find that

$$\int_{\mathbb{R}^2 \setminus B_2(x)} \log |x - y| u^2(y) \, dy \le \|u\|_X^2.$$

Moreover, by Young's inequality and Sobolev embeddings, we have

$$\int_{B_2(x)} \left| \log |x - y| \left| u^2(y) \, dy \le \left( \int_{B_2(0)} \left| \log |y| \right|^2 dy \right)^{1/2} \|u\|_{L^4(B_2(x))}^2 \le c_0 \|u\|_X^2, \tag{2.17}$$

where  $c_0 > 0$  is a constant. Hence (2.16) implies that  $w \in L^{\infty}(B_1(0))$ . Next, we consider  $x \in \mathbb{R}^2$  with  $|x| \ge 1$ , and we note that

$$w(x) - |u|_2^2 \log |x| = \int_{\mathbb{R}^2} h(x, y) u^2(y) \, dy \quad \text{with } h(x, y) = \log |x - y| - \log |x| = \log \frac{|x - y|}{|x|}.$$

Note that  $h(x, y) \to 0$  as  $|x| \to \infty$  for every  $y \in \mathbb{R}^2$ . Moreover,

$$\log \frac{1}{2} \le h(x, y) \mathbb{1}_{\{|y-x| \ge \frac{1}{2}\}}(y) \le \log(1+|y|) \text{ for all } x, y \in \mathbb{R}^2 \text{ with } |x| \ge 1.$$

Since the functions  $[\log \frac{1}{2}]u^2$  and  $\log(1 + |\cdot|)u^2$  are in  $L^1(\mathbb{R}^2)$ , Lebesgue's Theorem implies that

$$\int_{|y-x| \ge \frac{1}{2}} h(x, y)u^2(y) \, dy \to 0 \quad \text{as } |x| \to \infty.$$
(2.18)

Moreover, since  $u \in X$ , we have

$$0 \le \log |x| \int_{|y-x| \le \frac{1}{2}} u^2(y) \, dy \le \int_{|y| \ge \frac{|x|}{2}} \log \left( 2 \left[ 1 + |y| \right] \right) u^2(y) \, dy \to 0 \quad \text{as } |x| \to \infty,$$

and, similarly as in (2.17),

$$\int_{|y-x| \le \frac{1}{2}} \left| \log |x-y| \right| u^2(y) \, dy \le c_1 \|u\|_{L^4(B_{\frac{1}{2}}(x))}^2 \to 0 \quad \text{as } |x| \to \infty$$

with a constant  $c_1 > 0$ . Combining these estimates with (2.18), we conclude that

$$\int_{\mathbb{R}^2} h(x, y) u^2(y) \, dy \to 0 \quad \text{as } |x| \to \infty.$$

and thus the asymptotics in (ii) are proved. By Agmon's Theorem (see [2]), it follows that *u* satisfies (i), and elliptic regularity theory then yields that  $u \in W^{2,p}(\mathbb{R}^2)$  for every  $p \in [1, \infty)$ , and that *u* is a strong solution of (1.5). Moreover,  $u \in C_{loc}^{1,\beta}(\mathbb{R}^2)$  for all  $\beta \in (0, 1)$  by Sobolev embeddings. Hence elliptic regularity implies that  $w \in C_{loc}^{3,\beta}(\mathbb{R}^2)$  satisfies  $\Delta w = u^2$ .

Finally, if *a* is Hölder continuous, then *u* satisfies an equation of the form  $-\Delta u = f$  with (locally) Hölder continuous  $f : \mathbb{R}^2 \to \mathbb{R}$ , so  $u \in C^2(\mathbb{R}^2)$  by elliptic regularity.  $\Box$ 

We add some observations on the functional geometry of I.

**Lemma 2.4.** There exists  $\alpha > 0$  such that

$$\inf\{I(u) : u \in X : ||u|| = \beta\} > 0 \quad for \ 0 < \beta \le \alpha$$
(2.19)

and

$$\inf\{I'(u)u : u \in X : \|u\| = \beta\} > 0 \quad \text{for } 0 < \beta \le \alpha.$$
(2.20)

**Proof.** By (2.2) and Sobolev embeddings we have

$$I(u) \ge \frac{\|u\|^2}{2} - \frac{V_2(u)}{4} - \frac{b}{p} |u|_p^p \ge \frac{\|u\|^2}{2} - \frac{C_0}{4} |u|_{\frac{8}{3}}^4 - \frac{b}{p} |u|_p^p$$
  
$$\ge \frac{1}{2} (\|u\|^2 - C_1 \|u\|^4 - C_2 \|u\|^p) = \frac{\|u\|^2}{2} (1 - C_1 \|u\|^2 - C_2 \|u\|^{p-2})$$

for  $u \in X$  with constants  $C_1, C_2 > 0$ . This readily implies that (2.19) holds for  $\alpha > 0$  sufficiently small. Since

$$I'(u)(u) = ||u||^2 + V_0(u) - b|u|_p^p \ge ||u||^2 - V_2(u) - b|u|_p^p$$

for  $u \in X$ , a similar estimate shows that (2.20) holds for  $\alpha > 0$  sufficiently small.  $\Box$ 

**Lemma 2.5.** Let  $u \in X \setminus \{0\}$ . Then the function  $\varphi_u : \mathbb{R} \to \mathbb{R}$ ,  $\varphi_u(t) = I(tu)$  is even and has the following properties.

(i) *If* 

$$V_0(u) - b|u|_4^4 < 0 \quad \text{in case } p = 4 \quad \text{and} \quad V_0(u) < 0 \quad \text{or} \quad b > 0 \quad \text{in case } p > 4,$$
(2.21)

then there exists a unique  $t_u \in (0, \infty)$  such that  $\varphi'_u > 0$  on  $(0, t_u)$  and  $\varphi'_u < 0$  on  $(t_u, \infty)$ . Moreover,  $\varphi_u(t) \to -\infty$  as  $t \to \infty$ .

(ii) If (2.21) does not hold, then  $\varphi'_u > 0$  on  $(0, \infty)$ , and  $\varphi_u(t) \to \infty$  as  $t \to \infty$ .

Proof. The assertions follow easily from the fact that

$$\frac{\varphi'_u(t)}{t} = \|u\|^2 + t^2 V_0(u) - bt^{p-2} |u|_p^p \quad \text{for } t > 0. \qquad \Box$$

We also need the following continuity property of the bilinear form  $B_1$ .

**Lemma 2.6.** Let  $(u_n)_n, (v_n)_n, (w_n)_n$  be bounded sequences in X such that  $u_n \to u$  weakly in X. Then, for every  $z \in X$ , we have  $B_1(v_nw_n, z(u_n - u)) \to 0$  as  $n \to \infty$ .

**Proof.** We put  $c_1 := \sup_{n \in \mathbb{N}} |v_n|_2 |w_n|_2$  and  $c_2 := \sup_{n \in \mathbb{N}} |v_n|_* |w_n|_*$ . Then  $c_1, c_2 < \infty$  by assumption. Using (2.3) again, we estimate

$$\begin{aligned} \left| B_1 \left( v_n w_n, z(u_n - u) \right) \right| &\leq |v_n|_2 |w_n|_2 \int_{\mathbb{R}^2} \log(1 + |x|) |z(x)| |u_n(x) - u(x)| \, dx + |v_n|_* |w_n|_* |z|_2 |u_n - u|_2 \\ &\leq c_1 \int_{\mathbb{R}^2} \log(1 + |x|) |z(x)| |u_n(x) - u(x)| \, dx + o(1) \end{aligned}$$

$$(2.22)$$

as  $n \to \infty$ , since  $u_n \to u$  strongly in  $L^2(\mathbb{R}^2)$  by Lemma 2.2(i). For fixed R > 0, we have

$$\int_{\mathbb{R}^2} \log(1+|x|) |z(x)| |u_n(x) - u(x)| dx = f_n(R) + g_n(R)$$

with

$$f_n(R) := \int_{B_R(0)} \log(1+|x|) |z(x)| |u_n(x) - u(x)| dx \le \log(1+R)|z|_2|u_n - u|_2 \to 0$$
(2.23)

as  $n \to \infty$  and

$$g_{n}(R) := \int_{\mathbb{R}^{2} \setminus B_{R}(0)} \log(1+|x|) |z(x)| |u_{n}(x) - u(x)| dx$$
  
$$\leq \left(\int_{\mathbb{R}^{2} \setminus B_{R}(0)} \log(1+|x|) |z(x)|^{2} dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{2}} \log(1+|x|) |u_{n} - u|^{2} dx\right)^{\frac{1}{2}} \leq h(R),$$
(2.24)

where

$$h(R) := \left(\int_{\mathbb{R}^2 \setminus B_R(0)} \log(1+|x|) |z(x)|^2 dx\right)^{\frac{1}{2}} \sup_{n \in \mathbb{N}} (|u_n|_* + |u|_*) \to 0 \quad \text{as } R \to \infty.$$
(2.25)

Combining (2.22), (2.23) and (2.24), we find that

 $\limsup_{n \to \infty} \left| B_1(v_n w_n, z(u_n - u)) \right| \le c_1 h(R) \quad \text{for every } R > 0$ 

and therefore  $|B_1(v_n w_n, z(u_n - u))| \to 0$  as  $n \to \infty$  by (2.25).  $\Box$ 

# 3. A compactness condition

In the following, we assume that

$$a: \mathbb{R}^2 \to (0, \infty)$$
 is continuous and  $\mathbb{Z}^2$ -periodic. (3.1)

Then the functional *I* is invariant under translations with respect to  $\mathbb{Z}^2$ . In the following, we fix some notation for the action of translation. For a function  $u : \mathbb{R}^2 \to \mathbb{R}$  and  $x \in \mathbb{R}^2$ , we define

 $x * u : \mathbb{R}^2 \to \mathbb{R}, \qquad [x * u](y) = u(y - x) \text{ for } y \in \mathbb{R}^2.$ 

The main result of this section is the following.

**Proposition 3.1.** Let  $(u_n)_n$  be a sequence in X such that

$$I(u_n) \to d > 0 \quad and \quad \left\| I'(u_n) \right\|_{X'} \left( 1 + \|u_n\|_X \right) \to 0 \quad as \ n \to \infty.$$

$$(3.2)$$

Then, after passing to a subsequence, there exist points  $x_n \in \mathbb{Z}^2$ ,  $n \in \mathbb{N}$  such that

 $x_n * u_n \rightarrow u$  strongly in X as  $n \rightarrow \infty$ 

for some nonzero critical point  $u \in X$  of I.

The remainder of this section will be occupied with the proof of Proposition 3.1. So we assume from now on that we are given a sequence  $(u_n)_n \subset X$  satisfying (3.2).

**Lemma 3.2.** If  $(t_n)_n$  is a bounded sequence in  $[0, \infty)$ , then

$$I(t_n u_n) \le I(u_n) + o(1) \quad as \ n \to \infty.$$

$$(3.3)$$

*Moreover, if*  $t_n \rightarrow 0$  *as*  $n \rightarrow \infty$ *, then* 

$$\liminf_{n \to \infty} I(t_n u_n) \ge 0. \tag{3.4}$$

Proof. We have

$$I(t_n u_n) - I(u_n) = \frac{t_n^2 - 1}{2} \|u_n\|^2 + \frac{t_n^4 - 1}{4} V_0(u_n) - b \frac{t_n^p - 1}{p} |u_n|_p^p$$

and

$$V_0(u_n) = I'(u_n)(u_n) + b|u_n|_p^p - ||u_n||^2 = o(1) + b|u_n|_p^p - ||u_n||^2 \quad \text{as } n \to \infty,$$
(3.5)

so that

$$I(t_n u_n) - I(u_n) = -\left(\frac{t_n^4 - 1}{4} - \frac{t_n^2 - 1}{2}\right) \|u_n\|^2 - b\left(\frac{t_n^p - 1}{p} - \frac{t_n^4 - 1}{4}\right) \|u_n\|_p^p + o(1)$$
  
=  $-\frac{(t_n^2 - 1)^2}{4} \|u_n\|^2 - \frac{b}{p}\left(t_n^p - \frac{p}{4}t_n^4 + \frac{p}{4} - 1\right) \|u_n\|_p^p + o(1)$   
 $\leq o(1) \quad \text{as } n \to \infty,$ 

where the last step follows from the fact that the map  $t \mapsto t^p - \frac{p}{4}t^4 + \frac{p}{4} - 1$  is nonnegative on  $[0, \infty)$  since  $p \ge 4$  (with global minimum 0 attained at t = 1). This shows (3.3). If moreover  $t_n \to 0$  as  $n \to \infty$ , then we have, using (3.5) again

$$I(t_n u_n) = \left(\frac{t_n^2}{2} - \frac{t_n^4}{4}\right) ||u_n||^2 + b\left(\frac{t_n^4}{4} - \frac{t_n^p}{p}\right) |u_n|^p + o(1) \ge o(1)$$

as  $n \to \infty$ . Hence (3.4) holds.  $\Box$ 

**Lemma 3.3.** The sequence  $(u_n)_n$  is bounded in  $H^1(\mathbb{R}^2)$ .

Proof. Suppose by contradiction that, after passing to a subsequence,

$$\|u_n\| \to \infty \quad \text{as } n \to \infty. \tag{3.6}$$

Put  $v_n := \frac{u_n}{\|u_n\|}$  for  $n \in \mathbb{N}$ , so that  $\|v_n\| = 1$  for all *n*. We claim that

$$\inf_{n\in\mathbb{N}}\sup_{x\in\mathbb{Z}^2}\int_{B_2(x)}v_n^2(y)\,dy>0.$$
(3.7)

Suppose by contradiction that this is false. Then, after passing to a subsequence,  $v_n \to 0$  in  $L^s(\mathbb{R}^2)$  for every s > 2 by Lions' Lemma, see e.g. [33], and thus, by Lemma 2.2(iii),

$$V_2(tv_n) \to 0$$
 and  $|tu_n|_p \to 0$  as  $n \to \infty$  for every  $t \ge 0$ .

Hence

$$I(tv_n) = \frac{t^2}{2} \|v_n\|^2 + \frac{t^4}{4} V_1(tv_n) + o(1) \ge \frac{t^2}{2} + o(1) \quad \text{for every } t \ge 0$$

as  $n \to \infty$ , whereas, on the other hand,

$$I(tv_n) = I\left(\frac{t}{\|u_n\|}u_n\right) \le I(u_n) + o(1) = d + o(1) \quad \text{as } n \to \infty \text{ for every } t \ge 0$$
(3.8)

by Lemma 3.2. This is a contradiction for  $t > \sqrt{2d}$ , so we conclude that (3.7) holds. Thus there exists a sequence of points  $x_n \in \mathbb{Z}^2$ ,  $n \in \mathbb{N}$  such that, after passing to a subsequence, the sequence of the functions

 $w_n := x_n * v_n \in X, \quad n \in \mathbb{N}$ 

converges weakly in  $H^1(\mathbb{R}^2)$  to some nonzero function  $w \in H^1(\mathbb{R}^2)$ . Passing to a subsequence again if necessary, we may also assume that  $w_n \to w$  pointwise a.e. on  $\mathbb{R}^2$ . Since, by (3.8), we have  $I(w_n) = I(v_n) \le d + o(1)$  as  $n \to \infty$ , we find that

$$V_1(w_n) = I(w_n) - \frac{1}{2} + V_2(w_n) + \frac{b}{p} |w_n|_p^p \le C \quad \text{for all } n \in \mathbb{N}$$
(3.9)

with a constant C > 0. We now distinguish two cases:

*Case 1: b* > 0 *and p* > 4. In this case we estimate, using (3.9), that

$$I(tv_n) = I(tw_n) = \frac{t^2}{2} + \frac{t^4}{4} (V_1(w_n) - V_2(w_n)) - b\frac{t^p}{p} |w_n|_p^p$$
$$\leq \frac{t^2}{2} + \frac{t^4}{4}C - b\frac{t^p}{p} |w|_p^p + o(1),$$

so that, since b > 0 and p > 4, there exists  $n_0 \in \mathbb{N}$  and  $t_0 > 0$  such that

$$I\left(\frac{t_0}{\|u_n\|}u_n\right) = I(t_0v_n) \le -1 \quad \text{for } n \ge n_0.$$
(3.10)

Since  $\frac{t_0}{\|u_n\|} \to 0$  as  $n \to \infty$ , (3.10) contradicts (3.4).

Case 2: b = 0 or p = 4. In this case we have

$$I(tv_n) = I(tw_n) = \frac{t^2}{2} + \frac{t^4}{4}\rho_n \quad \text{for } t \ge 0$$
(3.11)

with  $\rho_n := V_0(w_n) - \frac{b}{p} |w_n|_p^p$  for  $n \in \mathbb{N}$ . We claim that

$$\rho := \limsup_{n \to \infty} \rho_n < 0. \tag{3.12}$$

Indeed, if, by contradiction,  $\liminf_{n\to\infty} \rho_n \ge 0$  after passing to a subsequence, then Lemma 3.2 and (3.11), applied with  $t_1 = \sqrt{4d}$  imply that

$$d + o(1) = I(u_n) \ge I(t_1v_n) + o(1) \ge \frac{t_1^2}{2} + o(1) = 2d + o(1)$$
 as  $n \to \infty$ ,

a contradiction. Hence (3.12) holds, and thus (3.11) implies that there exists  $n_0 \in \mathbb{N}$  and  $t_0 > 0$  such that (3.10) also holds in this case. Again, we arrive at a contradiction to (3.4).

Since in both cases we reached a contradiction, we conclude that the sequence  $(u_n)_n$  is bounded in  $H^1(\mathbb{R}^2)$ , as claimed.  $\Box$ 

Proof of Proposition 3.1 (completed). We first show that

$$\liminf_{n \in \mathbb{N}} \sup_{x \in \mathbb{Z}^2} \int_{B_2(x)} u_n^2(y) \, dy > 0.$$
(3.13)

Suppose by contradiction that (3.13) is false. Since the sequence  $(u_n)_n$  is bounded in  $H^1(\mathbb{R}^2)$ , it then follows from Lions' Lemma (see e.g. [33]) that, after passing to a subsequence,  $u_n \to 0$  in  $L^s(\mathbb{R}^2)$  for every s > 2, and thus, by (2.2) and (3.2),

$$||u_n||^2 + V_1(u_n) = I'(u_n)u_n + V_2(u_n) + b|u_n|_p^p \to 0 \text{ as } n \to \infty,$$

so that  $||u_n||^2 \to 0$ ,  $V_1(u_n) \to 0$  and therefore

$$I(u_n) = \frac{\|u_n\|^2}{2} + \frac{1}{4} (V_1(u_n) - V_2(u_n)) - \frac{b}{p} |u_n|_p^p \to 0 \quad \text{as } n \to \infty,$$

contradicting the assumption that  $I(u_n) \to d \neq 0$ . Hence (3.13) holds. Thus there exists a sequence of points  $x_n \in \mathbb{Z}^2$ ,  $n \in \mathbb{N}$  such that, after passing to a subsequence, the sequence of the functions

$$\tilde{u}_n := x_n * u_n \in X, \quad n \in \mathbb{N},$$

converges weakly in  $H^1(\mathbb{R}^2)$  to some function  $u \in H^1(\mathbb{R}^2) \setminus \{0\}$ . We may also assume that  $\tilde{u}_n \to u$  pointwise almost everywhere in  $\mathbb{R}^2$ . Moreover, we have, invoking (3.5) again, that

$$B_1(\tilde{u}_n^2, \tilde{u}_n^2) = V_1(\tilde{u}_n) = V_1(u_n) = o(1) + V_2(u_n) + b|u_n|_p^p - ||u_n||^2,$$

and the RHS of this inequality remains bounded in *n*. Consequently, Lemma 2.1 implies that  $|\tilde{u}_n|_*$  remains bounded in *n*, so that the sequence  $(\tilde{u}_n)_n$  is bounded in *X*. We may thus assume, passing to a subsequence again if necessary, that  $\tilde{u}_n \rightarrow u$  weakly in *X*, so that  $u \in X$ . It then follows by Lemma 2.2(i) that  $\tilde{u}_n \rightarrow u$  strongly in  $L^s(\mathbb{R}^2)$  for  $s \in [2, \infty)$ . Next we claim that

$$I'(\tilde{u}_n)(\tilde{u}_n-u) \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Indeed, we have

$$\left|I'(\tilde{u}_n)(\tilde{u}_n-u)\right| = \left|I'(u_n)\left(u_n-(-x_n)*u\right)\right| \le \left\|I'(u_n)\right\|_{X'}\left(\|u_n\|_X + \left\|(-x_n)*u\right\|_X\right)$$
(3.15)

for every n. Moreover, it is easy to see that

$$|u_n|_* = \int_{\mathbb{R}^2} \log(1 + |x - x_n|) \tilde{u}_n^2 \, dx \ge c_1 \log(1 + |x_n|) \quad \text{for all } n$$

with a constant  $c_1 > 0$  and

$$|(-x_n) * u|_* = \int_{\mathbb{R}^2} \log(1 + |x - x_n|) u^2 dx \le c_2 \log(1 + |x_n|)$$
 for all  $n$ 

with a constant  $c_2 > 0$ . Since moreover  $||u_n||$  and  $||(-x_n) * u||$  are bounded in *n*, there exists a constant  $c_3 > 0$  such that

 $\|(-x_n) * u\|_X \le c_3 \|u_n\|_X$  for all n.

Hence (3.15) implies that

$$\left| I'(\tilde{u}_n)(\tilde{u}_n - u) \right| \le (1 + c_3) \left\| I'(u_n) \right\|_{X'} \|u_n\|_X \to 0 \quad \text{as } n \to \infty,$$

as claimed in (3.14). Using (3.14), we find that

$$o(1) = I'(\tilde{u}_n)(\tilde{u}_n - u)$$
  
=  $o(1) + \|\tilde{u}_n\|^2 - \|u\|^2 + \frac{1}{4}V'(\tilde{u}_n)(\tilde{u}_n - u) - b \int_{\mathbb{R}^2} |\tilde{u}_n|^{p-2}\tilde{u}_n(\tilde{u}_n - u) dx$   
=  $o(1) + \|\tilde{u}_n\|^2 - \|u\|^2 + \frac{1}{4} [V'_1(\tilde{u}_n)(\tilde{u}_n - u) - V'_2(\tilde{u}_n)(\tilde{u}_n - u)],$ 

where

$$\left|\frac{1}{4}V_{2}'(\tilde{u}_{n})(\tilde{u}_{n}-u)\right| = \left|B_{2}(\tilde{u}_{n}^{2},\tilde{u}_{n}(\tilde{u}_{n}-u))\right| \le |\tilde{u}_{n}|_{\frac{8}{3}}^{3}|\tilde{u}_{n}-u|_{\frac{8}{3}} \to 0$$

as  $n \to \infty$  and

$$\frac{1}{4}V_1'(\tilde{u}_n)(\tilde{u}_n-u) = B_1(\tilde{u}_n^2, \tilde{u}_n(\tilde{u}_n-u)) = B_1(\tilde{u}_n^2, (\tilde{u}_n-u)^2) + B_1(\tilde{u}_n^2, u(\tilde{u}_n-u))$$

with

$$B_1(\tilde{u}_n^2, u(\tilde{u}_n - u)) \to 0 \quad \text{as } n \to \infty$$

by Lemma 2.6. Combining these estimates, we obtain that

$$o(1) = \|\tilde{u}_n\|^2 - \|u\|^2 + B_1(\tilde{u}_n^2, (\tilde{u}_n - u)^2) + o(1) \ge \|\tilde{u}_n\|^2 - \|u\|^2 + o(1) \ge o(1) \quad \text{as } n \to \infty,$$

which implies that  $\|\tilde{u}_n\| \to \|u\|$  and  $B_1(\tilde{u}_n^2, (\tilde{u}_n - u)^2) \to 0$  as  $n \to \infty$ . Hence  $\|\tilde{u}_n - \tilde{u}\| \to 0$  and, by Lemma 2.1,  $\|\tilde{u}_n - \tilde{u}\|_* \to 0$  as  $n \to \infty$ . We thus conclude that  $\|\tilde{u}_n - u\|_X \to 0$  as  $n \to \infty$ , as claimed.

We still need to show that I'(u) = 0. Let  $v \in X$ . Then we have

 $\|(-x_n) * v\|_X \le c_4 \|u_n\|_X$  for all *n* with a constant  $c_4 > 0$ 

by the same argument which leads to (3.16). Moreover,  $I'(u)v = \lim_{n \to \infty} I'(\tilde{u}_n)v$ , where

$$|I'(\tilde{u}_n)v| = |I'(u_n)(-x_n) * v| \le ||I'(u_n)||_X ||(-x_n) * v||_X \le c_4 ||I'(u_n)||_X ||u_n||_X \to 0$$

as  $n \to \infty$ . Hence I'(u)v = 0, and the proof is finished.  $\Box$ 

# 4. The periodic setting

In this section, we will complete the proof of Theorem 1.1. As in the last section, we assume that  $a : \mathbb{R}^2 \to (0, \infty)$  is continuous and  $\mathbb{Z}^2$ -periodic. For c > 0 we put

$$K_c := \{ u \in X : I'(u) = 0, I(u) = c \}.$$

Moreover, for  $\rho > 0$  we also consider the set

$$A_{c,\rho} := \left\{ u \in X : \|u - v\| \le \rho \text{ for some } v \in K_c \right\}.$$

$$(4.1)$$

We note that  $A_{c,\rho}$  is closed in X and in  $H^1(\mathbb{R}^2)$ . Indeed, the closedness in  $H^1(\mathbb{R}^2)$  follows easily from Proposition 3.1. We also note that  $A_{c,\rho}$  is symmetric with respect to the reflection  $u \mapsto -u$  and invariant under  $\mathbb{Z}^2$ -translations for every  $c \in \mathbb{R}$ ,  $\rho > 0$ . In the following,  $\gamma(A) \in \mathbb{N} \cup \{0, \infty\}$  denotes the Krasnoselski genus of a closed and symmetric subset  $A \subset X$  (defined with respect to continuity in  $\|\cdot\|_X$ ), see e.g. [31, Chapter II.5] for the definition of  $\gamma$ . The following finiteness property is a key step in the proof of Theorem 1.1.

(3.16)

**Proposition 4.1.** Let c > 0. Then there exists  $\rho_0 = \rho_0(c) > 0$  such that  $\gamma(A_{c,\rho}) < \infty$  for  $\rho \in (0, \rho_0)$ .

**Proof.** We fix a continuous map

$$\beta: L^2(\mathbb{R}^2) \setminus \{0\} \to \mathbb{R}^2$$

which is equivariant under  $\mathbb{Z}^2$ -translations, i.e.,

$$\beta(x * u) = x + \beta(u) \text{ for } x \in \mathbb{R}^2 \text{ and } u \in L^2(\mathbb{R}^2) \setminus \{0\}.$$

We also require that  $\beta(-u) = \beta(u)$  for every  $u \in L^2(\mathbb{R}^2) \setminus \{0\}$ . A map with these properties has been constructed independently in [10] and [5], and in [5] it is called *generalized barycenter map*. We also put

 $\tilde{K}_c := \{ u \in K_c : \beta(u) \in [-4, 4]^2 \}.$ 

We first show that

$$K_c$$
 is compact in X. (4.2)

To prove this, let  $u_n \in \tilde{K}_c$ ,  $n \in \mathbb{N}$  be arbitrary. By Proposition 3.1, there exists a sequence  $(x_n)_n \subset \mathbb{R}^2$  such that, after passing to a subsequence,

 $w_n := x_n * u_n \to u \in K_c$  with respect to  $\|\cdot\|_X$  as  $n \to \infty$ .

Consequently,

 $x_n + \beta(u_n) = \beta(w_n) \to \beta(u) \text{ as } n \to \infty.$ 

Since  $\beta(u_n) \in [-4, 4]^2$  for all *n*, this implies that the sequence  $(x_n)_n$  is bounded in  $\mathbb{R}^2$ , so that  $x_n \to x \in \mathbb{R}^2$  after passing to a subsequence. Consequently,

 $u_n \to u_0 := (-x) * u$  with respect to  $\|\cdot\|_X$  as  $n \to \infty$ .

Indeed, although  $\|\cdot\|_X$  is not invariant under  $\mathbb{Z}^2$ -translations, we still have

$$\|u_n - (-x) * u\|_X \le \|u_n - (-x_n) * u\|_X + \|(-x_n) * u - (-x) * u\|_X \le C \|x_n * u_n - u\|_X + o(1) = o(1)$$

as  $n \to \infty$  with a constant C > 0, since the sequence  $(x_n)_n \subset \mathbb{R}^2$  is bounded. Moreover,

$$\beta(u_0) = \lim_{n \to \infty} \beta(u_n) \in [-4, 4]^2,$$

so that  $u_0 \in \tilde{K}_c$ . Hence (4.2) is proved.

By (4.2) and a standard result (see e.g. [31, Chapter II.5]), we have  $k := \gamma(\tilde{K}_c) < \infty$ , hence there exists an odd and continuous map  $g_0 : \tilde{K}_c \to \mathbb{R}^k \setminus \{0\}$ . A priori,  $g_0$  is only continuous with respect to  $\|\cdot\|_X$ , but it easily follows from the compactness of  $\tilde{K}_c$  in X that  $g_0$  is also continuous with respect to  $\|\cdot\|$ . By Tietze's extension theorem, we may extend  $g_0$  to an odd and continuous map  $g : H^1(\mathbb{R}^2) \to \mathbb{R}^k$  (continuous with respect to  $\|\cdot\|$ ). Since  $\tilde{K}_c$  is also compact in  $H^1(\mathbb{R}^2)$ , there exists  $\rho > 0$  such that

$$g(u) \neq 0$$
 for every  $u \in U_{\rho}$ , (4.3)

where

 $U_{\rho} := \left\{ u \in H^1(\mathbb{R}^2) : \|u - v\| \le \rho \quad \text{for some } v \in \tilde{K}_c \right\}.$ 

In the following, we let  $[\cdot]$  denote the Gauss bracket, i.e.

 $[\cdot] : \mathbb{R} \to \mathbb{Z}, \qquad [s] := \max\{n \in \mathbb{Z} : n \le s\}.$ 

We also consider  $[\cdot]$  as a map from  $\mathbb{R}^2 \to \mathbb{Z}^2$ , defining

 $[(x, y)] = ([x], [y]) \in \mathbb{Z}^2$  for  $(x, y) \in \mathbb{R}^2$ .

Let

$$L_1 := \left\{ u \in L^2(\mathbb{R}^2) \setminus \{0\} : \left| \beta(u) - \left[ \beta(u) \right] \right|_{\infty} \le \frac{1}{2} \right\}$$

where  $|x|_{\infty} := \max\{|x_1|, |x_2|\}$  for  $x \in \mathbb{R}^2$ . Then  $L_1$  is invariant under  $\mathbb{Z}^2$ -translations, i.e.

$$q * L_1 = L_1$$
 for every  $q \in \mathbb{Z}^2$ .

Moreover, it is easy to see that the map

$$h_1: L_1 \to L_1, \qquad h_1(u) = \left(-\left[\beta(u)\right]\right) * u$$

is an isometry with respect to  $\|\cdot\|$  which is odd and invariant under  $\mathbb{Z}^2$ -translations, i.e.

$$h_1(q * u) = h_1(u) = -h_1(-u)$$
 for every  $u \in L_1, q \in \mathbb{Z}^2$ .

We also put

$$a_2 := \left(\frac{1}{2}, 0\right), \qquad a_3 := \left(0, \frac{1}{2}\right), \qquad a_4 := \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R}^2$$

and define the sets

$$L_i = a_i * L_1 \subset L^2(\mathbb{R}^2) \setminus \{0\}$$
 for  $i = 2, 3, 4$ .

Then, by construction,

$$L^2(\mathbb{R}^2)\setminus\{0\}\subset \bigcup_{i=1}^4 L_i.$$

We define the maps  $h_i : L_i \to L_i, i = 2, 3, 4$  by

$$h_i(u) = a_i * [h_1((-a_i) * u)] \text{ for } u \in L_i$$

These maps are also isometries with respect to  $\|\cdot\|$ , and they are odd and invariant under translations. Moreover, for every  $i \in 1, ..., 4$  and  $u \in L_i$  we have  $\beta(h_i(u)) \in [0, 1]^2$ . We now consider the sets

$$A_i := h_i^{-1}(U_\rho) \subset L_i \text{ for } i = 1, \dots, 4.$$

Then the sets  $A_i$  are closed in  $H^1(\mathbb{R}^2)$ , symmetric and invariant under  $\mathbb{Z}^2$ -translations for i = 1, ..., 4. Let  $A := \bigcup_{i=1}^4 A_i$ . Using (4.3) and standard arguments in the context of the Krasnoselski genus, one may construct an odd map

$$\hat{g}: A \to \mathbb{R}^{4k} \setminus \{0\}$$

which is continuous with respect to  $\|\cdot\|$ . It remains to prove that

 $A_{c,\rho} \subset A$  for  $\rho > 0$  sufficiently small.

Once this is established, the restriction of the map  $\hat{g}$  to  $A_{c,\rho}$  is odd and continuous with respect to  $\|\cdot\|_X$ , and thus it follows that  $\gamma(A_{c,\rho}) \leq 4k$ .

To prove (4.4), we argue by contradiction, assuming that  $A_{c,\rho} \not\subset A$  for every  $\rho > 0$ . Then there exists

$$u_n \in A_{c,\frac{1}{n}} \setminus A \quad \text{for every } n \in \mathbb{N}.$$
 (4.5)

Moreover, there exist  $v_n \in K_c$  such that  $||u_n - v_n|| \le \frac{1}{n}$  for every *n*. Without loss, we may assume that  $u_n \in L_1$  for every  $n \in \mathbb{N}$ . Moreover, by the translation invariance of  $L_1$ ,  $K_c$  and statement (4.5), we may assume that  $\beta(v_n) \in [0, 1]^2$  for every  $n \in \mathbb{N}$ , so in particular  $v_n \in \tilde{K}_c$  for every *n*. By compactness of  $\tilde{K}_c$ , we may pass to a subsequence such that  $||v_n - v|| \to 0$  as  $n \to \infty$  for some  $v \in \tilde{K}_c$  with  $\beta(v) \in [0, 1]^2$ . Since also  $||u_n - v|| \to 0$  as  $n \to \infty$ , it follows that  $\beta(u_n) \in [-2, 2]^2$  for *n* sufficiently large. Consequently,

 $w_n := (-[\beta(u_n)]) * v_n \in \tilde{K}_c$  for *n* sufficiently large,

186

(4.4)

whereas  $||h_1(u_n) - w_n|| = ||u_n - v_n|| \to 0$  as  $n \to \infty$ . Hence  $h_1(u_n) \in U_\rho$  for *n* sufficiently large, which implies that  $u_n \in A_1 \subset A$  for *n* sufficiently large. This contradicts (4.5). Hence (4.4) follows, and the proof is finished.  $\Box$ 

Next we recall the notion of relative genus.

**Definition 4.2.** Let  $D \subset Y$  be closed symmetric subsets of X. We define the genus of Y relative to D, denoted  $\gamma_D(Y)$ , as the smallest number k such that Y can be covered by closed and symmetric subsets U, V with the following properties:

(i)  $D \subset U$ , and there exists an odd continuous map  $\chi : U \to D$  such that  $\chi(u) = u$  for every  $u \in D$ . (ii)  $\gamma(V) \le k$ .

If no such covering exists we set  $\gamma_D(Y) := \infty$ .

The following properties are easily verified (see e.g. [14, Proposition 3.4] for similar statements); we include the proof for the convenience of the reader.

**Lemma 4.3.** Let D, Y and Z be closed symmetric subsets of X with  $D \subset Y$ . Then we have:

- (i) (Subadditivity)  $\gamma_D(Y \cup Z) \le \gamma_D(Y) + \gamma(Z)$ .
- (ii) If  $D \subset Z$ , and if there exists an odd continuous map  $\varphi : Y \to Z$  with  $\varphi(u) = u$  for every  $u \in D$ , then  $\gamma_D(Y) \leq \gamma_D(Z)$ .

Note in particular that (ii) implies the monotonicity property  $\gamma_D(Y) \leq \gamma_D(Z)$  if  $D \subset Y \subset Z$ .

**Proof.** (i) By definition of  $\gamma_D(Y)$ , there exist closed and symmetric sets  $U, V \subset X$  with  $D \subset U, Y \subset U \cup V, \gamma(V) = \gamma_D(Y)$  and an odd continuous map  $\chi : U \to D$  with  $\chi(u) = u$  for every  $u \in D$ . Since  $V \cup Z$  is also closed and symmetric and  $Y \cup Z \subset U \cup (V \cup Z)$ , we conclude that  $\gamma_D(Y \cup Z) \leq \gamma(V \cup Z) \leq \gamma(V) + \gamma(Z) = \gamma_D(Y) + \gamma(Z)$  by the subadditivity of  $\gamma$  (see [31, Chapter II.5]).

(ii) By definition of  $\gamma_D(Z)$ , there exist closed and symmetric sets  $U, V \subset X$  with  $D \subset U, Z \subset U \cup V, \gamma(V) = \gamma_D(Z)$  and an odd continuous map  $\chi : U \to D$  with  $\chi(u) = u$  for every  $u \in D$ . We then put  $\tilde{U} = \varphi^{-1}(U \cap Z)$  and  $\tilde{V} = \varphi^{-1}(V \cap Z)$ , so that  $Y \subset \tilde{U} \cup \tilde{V}$ . Since the map  $\tilde{\chi} = \chi \circ \varphi : \tilde{U} \to D$  is odd and continuous and satisfies  $\tilde{\chi}(u) = u$  for every  $u \in D$ , it follows from the definition of  $\gamma$  that  $\gamma_D(Y) \leq \gamma(\tilde{V}) \leq \gamma(V) = \gamma_D(Z)$ .  $\Box$ 

In the following, we put

 $I^c := \{ u \in X : I(u) \le c \} \text{ for } c \in \mathbb{R} \text{ and } D := I^0.$ 

Moreover we define

$$c_k := \inf \{ c \ge 0 : \gamma_D(I^c) \ge k \} \quad \text{for } k \in \mathbb{N}.$$

**Lemma 4.4.** As in *Theorem 1.1*, let  $\mathcal{N} := \{u \in X \setminus \{0\} : I'(u)u = 0\}$ . Then we have

$$c_1 = \inf_{\mathcal{N}} I = \inf_{u \in X \setminus \{0\}} \sup_{t \in \mathbb{R}} I(tu) > 0, \tag{4.6}$$

**Proof.** The second equality in (4.6) is an easy consequence of Lemma 2.5, since  $u \in \mathcal{N}$  if and only if  $\varphi'_u(1) = 0$ . The last inequality is an immediate consequence of Lemma 2.4.

Next we show that  $c_1 \ge \inf_{\mathcal{N}} I$ . We argue by contradiction, assuming that  $c_1 < \inf_{\mathcal{N}} I$ , and we fix  $c \in (c_1, \inf_{\mathcal{N}} I)$ . We also define  $\mathcal{N}^{\pm} := \{u \in X : \pm I'(u)u > 0\}$  and the function

 $\tau: \mathcal{N}^- \to [1, \infty), \qquad \tau(u) := \inf \{ t \ge 1 : I(tu) \le 0 \}.$ 

It follows from Lemma 2.5(i) that  $\tau$  is a continuous (and even) function. Next we define the map

$$\chi: I^c \to X, \qquad \chi(u) = \begin{cases} 0 & u = 0, \\ 0 & \text{if } u \in \mathcal{N}^+, \\ \tau(u)u & \text{if } u \in \mathcal{N}^-. \end{cases}$$

Since  $c < \inf_{\mathcal{N}} I$ ,  $\chi$  is well-defined, and it is odd and continuous by (2.20) and the properties of  $\tau$ . Since we also have  $\chi|_D = \operatorname{id}|_D$ , we conclude that  $\gamma_D(I^c) = 0$ . Since  $c > c_1$ , this contradicts the definition of  $c_1$ .

It remains to prove that  $c_1 \leq \inf_{u \in X \setminus \{0\}} \sup_{t \in \mathbb{R}} I(tu)$ . For this let  $u \in X \setminus \{0\}$ , such that  $\sup_W I < \infty$  for  $W := \mathbb{R}u$ . It then follows from Lemma 2.5 that there exists R > 0 such that  $\{v \in W : ||v|| \geq R\} \subset D$ . We need to show that  $c_1 \leq \sup_W I$ , which follows immediately one we have shown that  $\gamma_D(W \cup D) \geq 1$ . Arguing by contradiction, we suppose that  $\gamma_D(W \cup D) = 0$ , which implies that there exists an odd continuous map  $\chi : W \cup D \to D$  such that  $\chi(v) = v$  for every  $v \in D$ . Since  $\chi(0) = 0$  and  $\chi(v) = v$  for  $v \in W$  with  $||v|| \geq R$ , there exists  $v_\alpha \in W$  with  $||\chi(v_\alpha)|| = \alpha$ , where  $\alpha > 0$  is given in Lemma 2.4. Hence  $\chi(v_\alpha) \notin D$ , a contradiction.  $\Box$ 

**Lemma 4.5.** We have  $c_k < \infty$  for  $k \in \mathbb{N}$ .

**Proof.** Let *W* be a *k*-dimensional subspace of functions in *X* with support contained in  $B_{\frac{1}{4}}(0) \subset \mathbb{R}^2$ . Then for any two points *x*, *y* contained in the support of function  $u \in W$  we have |x - y| < 1, and this implies that  $V_0(u) < 0$  for every  $u \in W \setminus \{0\}$ . As a consequence, it is easy to deduce from Lemma 2.5 that

$$\{u \in W : ||u|| \ge R\} \subset D \quad \text{for some } R > 0, \tag{4.7}$$

and that  $c_W := \sup_{u \in W} I(u) < \infty$ . We claim that

$$\gamma_D(I^{c_W}) \ge k$$
 and therefore  $c_k \le c_W$ . (4.8)

Indeed, suppose the opposite; then there exist closed and symmetric subsets  $U, V \subset X$  with  $I^{cw} \subset U \cup V$  and such that

(i)  $D \subset U$ , and there exists an odd continuous map  $\chi : U \to D$  such that  $\chi(u) = u$  for every  $u \in D$ ; (ii)  $\gamma(V) \le k - 1$ .

By Tietze's Theorem we may extend  $\chi$  to an odd and continuous map  $X \to X$  (still denoted by  $\chi$ ). We define

$$\mathcal{O} := \left\{ u \in W : \left\| \chi(u) \right\| < \alpha \right\},\$$

where  $\alpha$  is given by Lemma 2.4. By (4.7) we see that  $\mathcal{O}$  is a bounded and symmetric neighborhood of 0 in W. Hence  $\gamma(\partial_W \mathcal{O}) = k$  by the Borsuk–Ulam Theorem, where  $\partial_W \mathcal{O}$  denotes the relative boundary of  $\mathcal{O}$  in W. We note that  $\partial_W \mathcal{O} \cap U = \emptyset$ , since  $\|\chi(u)\| = \alpha$  for every  $u \in \partial_W \mathcal{O}$  and  $\chi(u) \in D$  for every  $u \in U$ . Consequently,  $\partial_W \mathcal{O} \subset V$  and therefore

$$k = \gamma(\partial_W \mathcal{O}) \le \gamma(V) = k - 1,$$

a contradiction. This shows (4.8), as required.  $\Box$ 

We need the following deformation lemma.

**Lemma 4.6.** Let c > 0. Then there exists  $\rho_1 = \rho_1(c) > 0$  such that the following assertions hold for  $\rho \in (0, \rho_1)$ .

(i)  $A_{c,\rho} \cap D = \emptyset$ .

(ii) There exists  $\varepsilon = \varepsilon(c, \rho) > 0$  and an odd and continuous map  $\varphi : I^{c+\varepsilon} \setminus A_{c,\rho} \to I^{c-\varepsilon}$  such that  $\varphi|_D = id_D$ .

**Proof.** We first show that there exists  $\rho_1 = \rho_1(c) > 0$  such that (i) holds for  $\rho \in (0, \rho_1)$ . Suppose by contradiction that this is false; then there exists numbers  $\rho_n \to 0$  and  $u_n \in A_{c,\rho_n} \cap D$  for every  $n \in \mathbb{N}$ . There also exist corresponding

elements  $v_n \in K_c$  such that  $||u_n - v_n|| \le \rho_n$ . Moreover, by Proposition 3.1 there exists  $x_n \in \mathbb{Z}^2$ ,  $n \in \mathbb{N}$  such that  $x_n * v_n \to v \in K_c$  with respect to  $|| \cdot ||_X$ . By  $\mathbb{Z}^2$ -translation invariance, we have  $x_n * u_n \in A_{c,\rho_n} \cap D$  and

$$\|x_n * u_n - v\| \le \|x_n * (u_n - v_n)\| + \|x_n * v_n - v\| \le \|u_n - v_n\| + \|x_n * v_n - v\| \to 0 \quad \text{as } n \to \infty$$

Since  $x_n * u_n \in D$  for every *n*, we then infer from Lemma 2.2(vi) that

$$c = I(v) \le \liminf_{n \to \infty} I(x_n * u_n) \le 0$$

contrary to our assumption. Hence there exists  $\rho_1 = \rho_1(c) > 0$  such that (i) holds for  $\rho \in (0, \rho_1)$ , as claimed.

Next, we fix  $\rho \in (0, \rho_1)$  and we consider

$$S := X \setminus A_{c,\rho}$$
 and  $\tilde{S}_{\delta} := \{ u \in X : ||u - v|| \le \delta \text{ for some } v \in S \}$  for  $\delta > 0$ .

We claim that for  $\delta > 0$  sufficiently small we have

$$\|I'(u)\|_{X'}(1+\|u\|_X) \ge 8\delta \quad \text{for all } u \in \tilde{S}_{2\delta} \text{ with } c-2\delta^2 \le I(u) \le c+2\delta^2.$$
(4.9)

Suppose by contradiction that this is false; then there exist sequences of numbers  $\delta_n > 0$  and functions  $u_n \in \tilde{S}_{2\delta_n}$  such that  $\delta_n \to 0$  as  $n \to \infty$ , whereas

$$||I'(u_n)||_{X'}(1+||u_n||_X) < 8\delta_n \text{ and } c-2\delta_n^2 \le I(u_n) \le c+2\delta_n^2 \text{ for all } n \in \mathbb{N}$$

By Proposition 3.1, there exists, after passing to a subsequence, points  $x_n \in \mathbb{Z}^2$ ,  $n \in \mathbb{N}$  such that

 $x_n * u_n \to u$  strongly in *X* as  $n \to \infty$ 

for some  $u \in K_c$ . Moreover, there exists  $v_n \in S$  such that

 $||v_n - u_n|| \le 2\delta_n$  for every  $n \in \mathbb{N}$ .

It then follows that

$$||x_n * v_n - u|| \le ||x_n * v_n - x_n * u_n|| + ||x_n * u_n - u|| \le ||v_n - u_n|| + ||x_n * u_n - u|| \to 0 \quad \text{as } n \to \infty.$$

Since, by  $\mathbb{Z}^2$ -translation invariance,  $x_n * v_n \in \tilde{S}_{\delta_n}$  for every *n*, this implies that *u* is contained in the closure of *S* with respect to  $\|\cdot\|$ , so that

 $||u - v|| \ge \rho$  for every  $v \in K_c$ .

This contradicts the fact that *u* itself is contained in  $K_c$ . We conclude that (4.9) holds for  $\rho > 0$  sufficiently small. Since  $\|\cdot\| \le \|\cdot\|_X$ , we may now fix  $\delta > 0$  such that

$$\varepsilon := \delta^2 < \frac{c}{2} \tag{4.10}$$

and such that statement (4.9) holds with  $\tilde{S}_{2\delta}$  replaced by the subset

 $S_{2\delta} := \{ u \in X : \|u - v\|_X \le \delta \text{ for some } v \in S \}.$ 

By [19, Lemma 2.6], there exists a continuous function  $\eta$  :  $[0, 1] \times X \to X$  such that

(i)  $\eta(t, u) = u$  if t = 0 or  $u \notin I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$ ;

(ii) 
$$\eta(1, I^{c+\varepsilon} \cap S) \subset I^{c-\varepsilon}$$

(iii)  $t \mapsto I(\eta(t, u))$  is nonincreasing for all  $u \in X$ .

An inspection of the proof of [19, Lemma 2.6] also shows that  $\eta$  can be constructed such that  $\eta(t, -u) = -\eta(t, u)$  for  $t \in [0, 1], u \in X$  since the underlying functional I is even in u. Since  $D \cap I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) = \emptyset$  by (4.10), the map

$$\varphi: I^{c+\varepsilon} \setminus A_{c,\rho} \to I^{c-\varepsilon}, \qquad \varphi(u) = \eta(1,u)$$

has the desired properties.  $\Box$ 

**Corollary 4.7.** *The values*  $c_k$  *are critical values of the functional I.* 

**Proof.** Suppose by contradiction that  $K_{c_k} = \emptyset$  for some *k*. Then also  $A_{c_k,\rho} = \emptyset$  for every  $\rho > 0$ , and thus Lemma 4.6 yields  $\varepsilon > 0$  and an odd and continuous map  $\eta : I^{c_k + \varepsilon} \to I^{c_k - \varepsilon}$  such that  $\eta|_D = id_D$ . This contradicts the definition of  $c_k$ .  $\Box$ 

**Proposition 4.8.**  $c_k \to \infty$  as  $k \to \infty$ .

**Proof.** Suppose by contradiction that

 $c_k \to c < \infty$  as  $k \to \infty$ .

By Proposition 4.1 and Lemma 4.6, there exists  $\rho, \varepsilon > 0$  such that  $\gamma(A_{c,\rho}) < \infty$ ,  $A_{c,\rho} \cap D = \emptyset$  and such that there exists an odd and continuous map  $\varphi: I^{c+\varepsilon} \setminus A_{c,\rho} \to I^{c-\varepsilon}$  such that  $\varphi|_D = id_D$ . Consequently,

 $\gamma_D (I^{c+\varepsilon} \setminus A_{c,\rho}) \le \gamma_D (I^{c-\varepsilon}) < \infty$ 

by definition of c. Moreover, by the subadditivity property,

$$\gamma_D(I^{c+\varepsilon}) \leq \gamma_D(I^{c+\varepsilon} \setminus A_{c,\rho}) + \gamma(A_{c,\rho}) < \infty$$

which contradicts the fact that  $c + \varepsilon > c_k$  for all k. The proof is finished.  $\Box$ 

**Proof of Theorem 1.1 (completed).** By Lemma 4.5, Corollary 4.7 and Proposition 4.8, there exists a sequence of critical points  $\pm u_k \in X$  of  $I, k \in \mathbb{N}$  such that  $I(u_k) = c_k \to \infty$ . Moreover, by Lemma 4.4, we have that

$$c_1 = \inf_{\mathcal{N}} I = \inf_{u \in X \setminus \{0\}} \sup_{t \in \mathbb{R}} I(tu) > 0,$$

and by Corollary 4.7 this value is attained by a critical point of I (which is contained in  $\mathcal{N}$ ).

Next, let  $u \in \mathcal{N}$  be an arbitrary minimizer of  $I|_{\mathcal{N}}$ . We show that u is a critical point of I. Suppose by contradiction that I'(u)v < 0 for some  $v \in X$ . Since I' is continuous, there exists  $\delta$ ,  $\rho > 0$  such that

$$I'(t(u+sv))v < 0 \quad \text{for } t \in [1-\delta, 1+\delta], \ s \in [-\rho, \rho].$$
(4.11)

Moreover, since  $I'((1 - \delta)u)u > 0 > I'((1 + \delta)u)u$ , there exists  $\hat{s} \in (0, \rho)$  with

$$I'((1-\delta)(u+\hat{s}v))(u+\hat{s}v) > 0 > I'((1+\delta)(u+\hat{s}v))(u+\hat{s}v),$$

which implies that  $\hat{t}(u + \hat{s}v) \in \mathcal{N}$  for some  $\hat{t} \in (1 - \delta, 1 + \delta)$ . By Lemma 2.5 and (4.11), we find that

$$I(\hat{t}(u+\hat{s}v)) - I(u) \le I(\hat{t}(u+\hat{s}v)) - I(\hat{t}u) = -\int_{0}^{\hat{s}} I'(\hat{t}(u+sv))v \, ds < 0,$$

contradicting the fact that u is a minimizer of  $I|_{\mathcal{N}}$ . Hence u is a critical point of I, as claimed.

Finally, if  $u \in \mathcal{N}$  is a minimizer of  $I|_{\mathcal{N}}$ , it is easy to see that  $|u| \in \mathcal{N}$  is a minimizer of  $I|_{\mathcal{N}}$  as well, so it is a critical point of I by the considerations above. By Proposition 2.3,  $|u| \in W^{2,p}(\mathbb{R}^2)$  for every  $p \ge 1$ , and  $-\Delta |u| + q|u| = 0$  on  $\mathbb{R}^2$  with some function  $q \in L^{\infty}_{loc}(\mathbb{R}^2)$ . The strong maximum principle and the fact that  $u \ne 0$  therefore imply that |u| > 0 on  $\mathbb{R}^2$ , which shows u does not change sign.  $\Box$ 

# 5. The G-invariant setting

This section is devoted to the proof of Theorem 1.2. Since the proof is similar but easier than the proof of Theorem 1.1, we skip some of the details. We thus assume from now on that

*a* is constant on  $\mathbb{R}^2$ .

Moreover, we let  $G \subset O(2)$  and  $\tau : G \to \{-1, 1\}$  satisfy the assumptions of Theorem 1.2. Our aim is to detect critical points of the restriction of the function *I* to the invariant subspace

 $X_G := \{ u \in X : A * u = u \text{ for all } A \in G \},\$ 

where \* denotes the group action defined in (1.8). By the principle of symmetric criticality, any critical point of the restriction of *I* to  $X_G$  (which we will denote by *I* as well in the following) is a critical point of *I*. We have the following compactness condition which improves Proposition 3.1 in the symmetric setting.

**Proposition 5.1.** Suppose that G,  $\tau$  satisfy the assumptions of Theorem 1.2. Then the restriction of I to  $X_G$  satisfies the Cerami condition at every energy level d > 0. More precisely, if  $(u_n)_n$  is a sequence in  $X_G$  satisfying

$$I(u_n) \to d > 0$$
 and  $\|I'(u_n)\|_{X'_G} (1 + \|u_n\|_X) \to 0$  as  $n \to \infty$ ,

then there exists a nonzero critical point  $u \in X_G$  of I such that, after passing to a subsequence,

 $u_n \rightarrow u$  in X as  $n \rightarrow \infty$ .

**Proof.** The invariance of *I* under the action *G* implies that I'(v)w = 0 for all  $v \in X_G$ ,  $w \in X_G^{\perp}$ , where  $X_G^{\perp}$  denotes the orthogonal complement of  $X_G$  in *X*. Consequently, we have  $||I'(v)||_{X'_G} = ||I'(v)||_{X'}$  for all  $v \in X_G$ . Hence the sequence  $(u_n)_n$  satisfies the assumption of Proposition 3.1, and thus we may pass to a subsequence such that

$$x_n * u_n \to u_0$$
 strongly in X as  $n \to \infty$  (5.2)

for some nonzero critical point  $u_0 \in X$  of I and a suitable sequence  $(x_n)_n \subset \mathbb{R}^2$ . We show that the sequence  $(x_n)_n$  is bounded. Suppose by contradiction that  $|x_n| \to \infty$  after passing to a subsequence. We may assume that  $\frac{x_n}{|x_n|} \to z_0$  as  $n \to \infty$ . By assumption (1.7), there exists  $A \in G$  with  $Az_0 \neq z_0$ . Put  $y_n := -x_n$  for  $n \in \mathbb{N}$ . Then we have, since  $A * u_n = u_n$  for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} \left| A * (y_n * u_0) - y_n * u_0 \right|_2 &\leq \left| A * (y_n * u_0 - u_n) \right|_2 + |u_n - y_n * u_0|_2 = 2|y_n * u_0 - u_n|_2 \\ &= 2|u_0 - x_n * u_n|_2 \to 0 \quad \text{as } n \to \infty. \end{aligned}$$
(5.3)

However,

$$|A * (y_n * u_0) - y_n * u_0|_2^2 - 2|u_0|_2^2 = -2\tau(A) \int_{\mathbb{R}^2} u_0 (A^{-1}x - y_n) u_0(x - y_n) dx$$
  
=  $-2\tau(A) \int_{\mathbb{R}^2} u_0 (A^{-1}(z - z_n)) u_0(z) dz$   
=  $-2 \int_{\mathbb{R}^2} (z_n * v) u_0 dz$  (5.4)

with  $v := A * u_0$  and  $z_n := Ay_n - y_n$ . Moreover, since  $Az_0 \neq z_0$ , we find that

 $|z_n| = |x_n - Ax_n| = |x_n| |Az_0 - z_0 + o(1)| \to \infty \text{ as } n \to \infty,$ 

and therefore

$$\int_{\mathbb{R}^2} (z_n * v) u_0 \, dx \to 0 \quad \text{as } n \to \infty.$$

Consequently, (5.4) implies that

$$|A * (y_n * u_0) - y_n * u_0|_2 \rightarrow 2|u_0|_2 \neq 0$$

contrary to (5.3). The contradiction shows that  $(x_n)_n$  is bounded, so that  $x_n \to x_0$  as  $n \to \infty$  after passing to a subsequence. Hence (5.2) implies that  $u_n \to u := (-x_0) * u_0$  in X, where u is also a nonzero critical point of I. Moreover,  $u \in X_G$  since  $X_G$  is closed and  $u_n \in X_G$  for every  $n \in \mathbb{N}$ .  $\Box$ 

As a consequence of Proposition 5.1, the sets

$$K_c := \{ u \in X_G : I'(u) = 0, I(u) = c \} \quad c > 0$$

are compact and therefore satisfy  $\gamma(K_c) < \infty$ , where, as before, the Krasnoselski genus  $\gamma$  is defined with respect to continuity in the norm  $\|\cdot\|_X$ . Setting

$$I^c := \{ u \in X_G : I(u) \le c \} \quad \text{for } c \in \mathbb{R} \quad \text{and} \quad D := I^0,$$

we have the following deformation lemma.

**Lemma 5.2.** Let c > 0. Then for every neighborhood U of  $K_c$  in  $X_G$  with  $U \cap D = \emptyset$  there exists  $\varepsilon > 0$  and an odd and continuous map  $\varphi : I^{c+\varepsilon} \setminus U \to I^{c-\varepsilon}$  such that  $\varphi|_D = id_D$ .

#### Proof. Let

$$S := X_G \setminus U \quad \text{and} \quad S_\delta := \{ u \in X_G : \|u - v\|_{X_G} \le \delta \text{ for some } v \in S \} \quad \text{for } \delta > 0.$$

It is standard to deduce from Proposition 5.1 that there exists  $\delta \in (0, \sqrt{c/2})$  such that

$$\|I'(u)\|_{X'_G} (1 + \|u\|_{X_G}) \ge 8\delta \quad \text{for all } u \in S_{2\delta} \text{ with } c - 2\delta^2 \le I(u) \le c + 2\delta^2.$$
 (5.5)

Let  $\varepsilon := \delta^2$ . By [19, Lemma 2.6], there exists a continuous function  $\eta : [0, 1] \times X_G \to X_G$  such that

- (i)  $\eta(t, u) = u$  if t = 0 or  $u \notin I^{-1}([c 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$ ;
- (ii)  $\eta(1, I^{c+\varepsilon} \cap S) \subset I^{c-\varepsilon};$
- (iii)  $t \mapsto I(\eta(t, u))$  is nonincreasing for all  $u \in X_G$ .

Moreover,  $\eta$  can be constructed such that  $\eta(t, -u) = -\eta(t, u)$  for  $t \in [0, 1]$ ,  $u \in X_G$  since the underlying functional I is even in u. Since  $D \cap I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) = \emptyset$ , the map

 $\varphi: I^{c+\varepsilon} \setminus U \to I^{c-\varepsilon}, \qquad \varphi(u) = \eta(1, u)$ 

has the asserted properties.  $\Box$ 

For closed symmetric subsets  $D \subset Y \subset X_G$  we now define the relative genus  $\gamma_D(Y)$  as in Definition 4.2, noting that Lemma 4.3 still holds. We then define the nondecreasing sequence of values

$$c_k := \inf\{c \ge 0 : \gamma_D(I^c) \ge k\}, \quad k \in \mathbb{N}.$$

Precisely as in the proof of Lemma 4.4, we see that

$$c_1 := \inf_{\mathcal{N}_G} I = \inf_{u \in X_G \setminus \{0\}} \sup_{t \in \mathbb{R}} I(tu) > 0,$$
(5.6)

where  $\mathcal{N}_G := \{u \in X_G \setminus \{0\} : I'(u)u = 0\}$ . Moreover we have

$$c_k < \infty \quad \text{for } k \in \mathbb{N}.$$
 (5.7)

The proof of (5.7) is similar to the one of Lemma 4.5. The only difference is that we have to choose, for given  $k \in \mathbb{N}$ , a *k*-dimensional subspace *W* of functions in  $X_G$  with support contained in  $B_{\frac{1}{4}}(0) \subset \mathbb{R}^2$ . In the case where  $\tau \equiv 1$ , we may simply choose a space *W* consisting of radial  $C^1$ -functions supported in  $B_{\frac{1}{4}}(0)$ . In the case where  $\tau \not\equiv 1$ , *G* is assumed to be finite. This implies that there exists  $x_0 \in \mathbb{R}^2 \setminus \{0\}$  with  $Ax_0 \neq x_0$  for all  $A \in G \setminus \{id\}$ , since the set

$$\{x \in \mathbb{R}^2 : Ax = x \text{ for some } A \in G, A \neq \text{id}\}$$

is a finite union of subspaces of  $\mathbb{R}^2$  of dimension less than or equal to one. As a consequence,

$$Ax_0 \neq Bx_0 \quad \text{for all } A, B \in G \text{ with } \tau(A) = 1, \ \tau(B) = -1.$$

$$(5.8)$$

Without loss, we may assume that  $|x_0| < \frac{1}{4}$ . It then follows from (5.8) that there exists an open neighborhood  $U \subset B_{\frac{1}{4}}(0)$  of  $x_0$  such that

$$A(U) \cap B(U) = \emptyset \quad \text{for all } A, B \in G \text{ with } \tau(A) = 1, \ \tau(B) = -1.$$
(5.9)

We may therefore pick a k-dimensional subspace  $\tilde{W}$  of  $C^1$ -functions supported in U and define

$$W := \bigg\{ \sum_{A \in \tau^{-1}(1)} \psi \circ A^{-1} - \sum_{B \in \tau^{-1}(-1)} \psi \circ B^{-1} : \psi \in \tilde{W} \bigg\}.$$

By (5.9), W is a k-dimensional subspace of  $X_G$  of functions supported in  $B_{\frac{1}{4}}(0)$ . We may thus proceed as in the proof of Lemma 4.5 to obtain (5.7).

With the help of Lemma 5.2, we now deduce (similarly as in Corollary 4.7) that

the values  $c_k, k \in \mathbb{N}$ , are critical values of the functional  $I \in C^1(X_G, \mathbb{R})$ . (5.10)

We also observe that

$$c_k \to \infty \quad \text{as } k \to \infty.$$
 (5.11)

This follows as in the proof of Proposition 4.8, with  $A_{c,\rho}$  replaced by an arbitrary neighborhood U of  $K_c$  with  $\gamma(U) < \infty$  and  $U \cap D = \emptyset$ . The existence of such a neighborhood follows from the compactness of  $K_c$  (see e.g. [31, Chapter II.5]).

By combining the statements (5.6), (5.7), (5.10) and (5.11), we may now finish the proof of Theorem 1.2 by the same arguments as in the proof of Theorem 1.1 (see the end of Section 4). The only difference is given by the fact that the implication

$$u \in X_G \implies |u| \in X_G$$

is only true if  $\tau \equiv 1$ , and thus the last assertion in Theorem 1.2 requires this extra assumption.

## 6. Symmetry and uniqueness of positive solutions

This section is concerned with the proof of Theorem 1.3. More precisely, we will prove a symmetry result in a slightly more general setting and deduce Theorem 1.3 afterwards. Let  $f : \mathbb{R} \to \mathbb{R}$  be locally Lipschitz with f(0) = 0. We study classical solutions (u, w) of the nonlinear system

$$\begin{cases} -\Delta u = wu + f(u) & \text{in } \mathbb{R}^2, \\ -\Delta w = 2\pi u^2 & \text{in } \mathbb{R}^2, \end{cases}$$
(6.1)

subject to the conditions

$$u \in L^{\infty}(\mathbb{R}^2)$$
 and  $w(x) \to -\infty$  as  $|x| \to \infty$ . (6.2)

By Agmon's Theorem (see [2]), (6.1) and (6.2) imply that

$$u(x) = o(e^{-\alpha |x|}) \quad \text{as } |x| \to \infty \text{ for every } \alpha > 0.$$
(6.3)

Moreover, since every semibounded harmonic function  $\mathbb{R}^2 \to \mathbb{R}$  is constant, we have

$$w(x) = c - \int_{\mathbb{R}^2} \log|x - y|u^2(y) \, dy \quad \text{for } x \in \mathbb{R}^2 \text{ with a constant } c \in \mathbb{R}.$$
(6.4)

The main result of this section is the following.

**Theorem 6.1.** Every classical solution (u, w) of (6.1), (6.2) with u > 0 in  $\mathbb{R}^2$  is radially symmetric up to translation and strictly decreasing in the distance from the symmetry center. Moreover, if f(u) = -au with a constant a > 0, then (u, w) is unique up to translation.

The proof will be given by a variant of the moving plane method. For  $\lambda \in \mathbb{R}$ , we put

$$H_{\lambda} := \{ x \in \mathbb{R}^2 : x_1 > \lambda \}, \qquad T_{\lambda} = \partial H_{\lambda} = \{ x \in \mathbb{R}^2 : x_1 = \lambda \}.$$

Moreover, we let  $\mathbb{R}^2 \to \mathbb{R}^2$ ,  $x \mapsto x^{\lambda}$  denote the reflection of x at  $T_{\lambda}$ . From now on, we consider a fixed solution of (6.1), (6.2) with u > 0, and we set

$$u^{\lambda}(x) = u(x^{\lambda}), \qquad w^{\lambda}(x) = w(x^{\lambda}) \quad \text{for } x \in \mathbb{R}^2, \ \lambda \in \mathbb{R}.$$

On  $H_{\lambda}$  we define the difference functions

$$u_{\lambda} = u^{\lambda} - u, \qquad w_{\lambda} = w^{\lambda} - w$$

which satisfy the system of equations

$$\begin{cases} -\Delta u_{\lambda} + u_{\lambda} = w_{\lambda}u^{\lambda} + (w + h_{\lambda})u_{\lambda}, \\ -\Delta w_{\lambda} = 2\pi \left[ \left( u^{\lambda} \right)^{2} - u^{2} \right] = 2\pi \left( u^{\lambda} + u \right)u_{\lambda} \end{cases}$$
 in  $\mathcal{H}_{\lambda}$  (6.5)

with

$$h_{\lambda}(x) := \begin{cases} \frac{f(u^{\lambda}(x)) - f(u(x))}{u(x)} + 1, & \text{if } u(x) \neq 0, \\ 1, & \text{if } u(x) = 0. \end{cases}$$

Since f is Lipschitz continuous on  $[0, ||u||_{L^{\infty}(\mathbb{R}^2)}]$ , there exists a constant C = C(u) > 0 such that

$$\|h_{\lambda}\|_{L^{\infty}(H_{\lambda})} \le C \quad \text{for every } \lambda \in \mathbb{R}.$$
(6.6)

It follows from (6.4) that

$$w_{\lambda}(x) = \int_{H_{\lambda}} \log \frac{|x - y^{\lambda}|}{|x - y|} (u^{\lambda}(y) + u(y)) u_{\lambda}(y) dy \quad \text{for } x \in H_{\lambda}.$$
(6.7)

Since  $\log \frac{|x-y^{\lambda}|}{|x-y|} > 0$  for every  $x, y \in H_{\lambda}$ , we have the implication

$$u_{\lambda} \ge 0 \quad \text{in } H_{\lambda} \implies w_{\lambda} \ge 0 \quad \text{in } H_{\lambda} \tag{6.8}$$

for every  $\lambda \in \mathbb{R}$ . In the following, we let  $v^- := \min\{v, 0\}$  denote the negative part of a function defined on a subset of  $\mathbb{R}^2$ . Note that  $v^-$  is a nonpositive function with this convention. We need the following estimate:

**Lemma 6.2.** There exists a constant  $\kappa > 0$  such that

$$\left\| w_{\lambda}^{-} \right\|_{L^{2}(H_{\lambda})} \leq \kappa c_{\lambda} \left\| u_{\lambda}^{-} \right\|_{L^{2}(H_{\lambda})} \quad for \ every \ \lambda \in \mathbb{R},$$

where

$$c_{\lambda} = \left(\int_{M_{\lambda}} (y_1 - \lambda)^2 u^2(y) \, dy\right)^{\frac{1}{2}} \quad and \quad M_{\lambda} := \left\{ x \in H_{\lambda} : u_{\lambda}(x) < 0 \right\}.$$

**Proof.** We note that

$$0 \le \log \frac{|x - y^{\lambda}|}{|x - y|} \le \log \left( 1 + \frac{|y - y^{\lambda}|}{|x - y|} \right) \le \frac{|y - y^{\lambda}|}{|x - y|} = \frac{2(y_1 - \lambda)}{|x - y|} \quad \text{for } x, y \in H_{\lambda}.$$

Since also  $u_{\lambda}(y) < 0$  implies that  $0 \le u^{\lambda}(y) \le u(y)$ , we may use the integral representation (6.7) to conclude that

$$w_{\lambda}^{-}(x) \ge \int_{M_{\lambda}} \frac{2(y_1 - \lambda)}{|x - y|} \left( u^{\lambda}(y) + u(y) \right) u_{\lambda}^{-}(y) \, dy \ge 4 \int_{M_{\lambda}} \frac{(y_1 - \lambda)}{|x - y|} u(y) u_{\lambda}^{-}(y) \, dy$$

for  $x \in H_{\lambda}$ , so that, by the Hardy–Littlewood–Sobolev inequality,

$$\left\|w_{\lambda}^{-}\right\|_{L^{2}(H_{\lambda})} \leq \kappa c_{\lambda} \left\|u_{\lambda}^{-}\right\|_{L^{2}(H_{\lambda})}$$

with a constant  $\kappa > 0$  independent of  $\lambda$ , as claimed.  $\Box$ 

194

**Lemma 6.3.** There exists  $\overline{\lambda} > 0$  such that  $u_{\lambda} \ge 0$  in  $H_{\lambda}$  for  $\lambda \ge \overline{\lambda}$ .

**Proof.** By (6.2) and (6.6), we may choose  $\lambda_1 > 0$  such that  $w + h_{\lambda} \le 0$  in  $H_{\lambda}$  for  $\lambda \ge \lambda_1$ . Multiplying the first equation in (6.5) by  $u_{\lambda}^-$  and integrating, we may estimate with the help of Lemma 6.2

$$\begin{aligned} \left\| u_{\lambda}^{-} \right\|_{L^{2}(H_{\lambda})}^{2} &\leq \left\| u_{\lambda}^{-} \right\|_{H^{1}(H_{\lambda})}^{2} = \int_{H_{\lambda}} \left[ w_{\lambda} u^{\lambda} u_{\lambda}^{-} + (w + h_{\lambda}) \left( u_{\lambda}^{-} \right)^{2} \right] dx \leq \int_{H_{\lambda}} w_{\lambda}^{-} u^{\lambda} u_{\lambda}^{-} dx \\ &\leq \left\| w_{\lambda}^{-} \right\|_{L^{2}(H_{\lambda})} \left\| u^{\lambda} \right\|_{L^{\infty}(H_{\lambda})} \left\| u_{\lambda}^{-} \right\|_{L^{2}(H_{\lambda})} \leq \kappa c_{\lambda} \left\| u \right\|_{L^{\infty}(\mathbb{R}^{2})} \left\| u_{\lambda}^{-} \right\|_{L^{2}(H_{\lambda})}^{2}. \end{aligned}$$

Since  $c_{\lambda} \to 0$  as  $\lambda \to \infty$  by (6.3), there exists  $\overline{\lambda} \ge \lambda_1$  such that

$$\kappa c_{\lambda} \|u\|_{L^{\infty}(\mathbb{R}^2)} < 1 \quad \text{for } \lambda \geq \lambda,$$

so that  $u_{\lambda}^{-} \equiv 0$  on  $H_{\lambda}$  for  $\lambda \geq \overline{\lambda}$ , as claimed.  $\Box$ 

**Lemma 6.4.** If  $\lambda \in \mathbb{R}$  is such that  $u_{\lambda} \ge 0$  in  $H_{\lambda}$ , then also  $w_{\lambda} \ge 0$  on  $H_{\lambda}$ . Moreover, either  $u_{\lambda} \equiv 0 \equiv w_{\lambda}$  or

$$u_{\lambda} > 0, \qquad w_{\lambda} > 0 \quad on \ H_{\lambda}$$

$$(6.9)$$

and

$$\frac{\partial u}{\partial x_1} < 0, \qquad \frac{\partial w}{\partial x_1} < 0 \quad on \ T_{\lambda}.$$
 (6.10)

**Proof.** We already noted in (6.8) that  $u_{\lambda} \ge 0$  in  $H_{\lambda}$  implies  $w_{\lambda} \ge 0$  in  $H_{\lambda}$ . Moreover, if  $u_{\lambda} \ne 0$ , then  $w_{\lambda}$  is strictly positive in  $H_{\lambda}$  and

$$\frac{\partial w_{\lambda}}{\partial x_1} = -2\frac{\partial w}{\partial x_1} > 0 \quad \text{on } T_{\lambda}$$

by the Hopf lemma. Conversely, if  $w_{\lambda} \neq 0$ , then also  $u_{\lambda} \neq 0$  by (6.7), and  $u_{\lambda}$  satisfies

$$-\Delta u_{\lambda} + (1 - w - h_{\lambda})u_{\lambda} \ge u^{\lambda}w_{\lambda} \ge 0 \quad \text{in } H_{\lambda}$$

Hence  $u_{\lambda} > 0$  in  $H_{\lambda}$  by the maximum principle, and

$$\frac{\partial u_{\lambda}}{\partial x_1} = -2\frac{\partial u}{\partial x_1} > 0 \quad \text{on } T_{\lambda}$$

by the Hopf lemma.  $\Box$ 

**Lemma 6.5.** Let  $\lambda \in \mathbb{R}$ . If  $u_{\lambda} > 0$  in  $H_{\lambda}$ , then there exists  $\varepsilon > 0$  such that  $u_{\mu} \ge 0$  in  $H_{\mu}$  for  $\mu \in (\lambda - \varepsilon, \lambda)$ .

**Proof.** Let  $B_R := B_R(0)$  for R > 0. By (6.2), (6.3) and (6.6), we may fix R > 1 large enough such that

$$w + h_{\mu} \le 0$$
 in  $H_{\mu} \setminus B_R$  for every  $\mu \in \mathbb{R}$  (6.11)

and

$$\left(\int_{\mathbb{R}^2 \setminus B_R} (y_1 - \mu)^2 u^2(y) \, dy\right)^{\frac{1}{2}} < \frac{1}{\kappa \|u\|_{L^\infty(\mathbb{R}^2)}} \quad \text{for every } \mu \in [\lambda - 1, \lambda],\tag{6.12}$$

where  $\kappa$  is as in Lemma 6.2. Moreover, by (6.9), (6.10) and the continuity of u,  $\frac{\partial u}{\partial x_1}$ , there exists  $0 < \varepsilon < 1$  such that

$$u_{\mu} > 0 \quad \text{in } H_{\mu} \cap B_R \text{ for } \mu \in (\lambda - \varepsilon, \lambda].$$
(6.13)

Arguing similarly as in the proof of Lemma 6.2 with  $\mu \in (\lambda - \varepsilon, \lambda]$  in place of  $\lambda$ , we may multiply the first equation in (6.5) – with  $\mu$  in place of  $\lambda$  – with  $u_{\mu}^{-}$  and integrate, obtaining the estimate

S. Cingolani, T. Weth / Ann. I. H. Poincaré - AN 33 (2016) 169-197

$$\|u_{\mu}^{-}\|_{L^{2}(H_{\mu})}^{2} \leq \|u_{\mu}^{-}\|_{H^{1}(H_{\mu})}^{2} = \int_{H_{\mu}\setminus B_{R}} \left[w_{\mu}u^{\mu}u_{\mu}^{-} + (w+h_{\mu})(u_{\mu}^{-})^{2}\right]dx \leq \int_{H_{\mu}\setminus B_{R}} w_{\mu}^{-}u^{\mu}u_{\mu}^{-}dx \\ \leq \|w_{\mu}^{-}\|_{L^{2}(H_{\mu})}\|u^{\mu}\|_{L^{\infty}(H_{\mu})}\|u_{\mu}^{-}\|_{L^{2}(H_{\mu})} \leq \kappa c_{\mu}\|u\|_{L^{\infty}(\mathbb{R}^{2})}\|u_{\mu}^{-}\|_{L^{2}(H_{\mu})}^{2}.$$

$$(6.14)$$

Here we used (6.13) in the second step, (6.11) in the third step and Lemma 6.2 in the last step. We note that (6.12) and (6.13) also imply that

$$c_{\mu} \leq \left(\int_{\mathbb{R}^2 \setminus B_R} (y_1 - \mu)^2 u^2(y) \, dy\right)^{\frac{1}{2}} < \frac{1}{\kappa \|u\|_{L^{\infty}(\mathbb{R}^2)}} \quad \text{for } \mu \in (\lambda - \varepsilon, \lambda].$$

Consequently, (6.14) implies that  $u_{\mu}^{-} \equiv 0$  for  $\mu \in (\lambda - \varepsilon, \lambda]$ .  $\Box$ 

## Proof of Theorem 6.1 (completed). Put

 $\lambda_1 := \inf\{\lambda \in \mathbb{R} : u_\lambda \ge 0 \text{ in } H_\lambda\}$ 

By Lemma 6.3 we have  $\lambda_1 < \infty$ , whereas the positivity of u and (6.3) imply that  $\lambda_1 > -\infty$ . Moreover, as a consequence of Lemmas 6.4 and 6.5, we have that  $u_{\lambda_1} \equiv 0$  and  $w_{\lambda_1} \equiv 0$ . Repeating the same argument with  $x_1$  replaced by the second coordinate direction  $x_2$ , we also find  $\lambda_2 \in \mathbb{R}$  such that u and w are symmetric with respect to the hyperplane  $\{x \in \mathbb{R}^2 : x_2 = \lambda_2\}$ . Now consider  $\lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2$  and the translated functions

$$\tilde{u}, \tilde{w} : \mathbb{R}^2 \to \mathbb{R}, \qquad \tilde{u}(x) = u(x - \lambda), \qquad \tilde{w}(x) = w(x - \lambda).$$

These functions also solve (6.1), (6.2) and satisfy

$$\tilde{u}(x) = \tilde{u}(-x), \qquad \tilde{w}(x) = \tilde{w}(-x) \quad \text{for } x \in \mathbb{R}^2.$$

It is then easy to see that every symmetry hyperplane of  $\tilde{u}$  and  $\tilde{w}$  must contain the origin. Consequently, repeating the moving plane procedure in an arbitrary direction in place of the  $x_1$ -coordinate direction, we obtain that  $\tilde{u}$  and  $\tilde{w}$  are symmetric with respect to any hyperplane containing zero, hence radially symmetric. The uniqueness of positive, radial solutions in the case where f(u) = -au with a > 0 is proved in [12, Theorem 1.1.].

**Proof of Theorem 1.3 (completed).** Let  $u \in X$  be a solution of (1.5) with constant *a*. By Proposition 2.3, *u* and the function  $w : \mathbb{R}^2 \to \mathbb{R}$  defined by (6.4) satisfy condition (6.2), and (u, w) is a classical solution of (6.1) with the Lipschitz continuous function  $f(u) = b|u|^{p-2}u - au$  (here we need the assumption  $p \ge 2$ ). Consequently, the claim follows from Theorem 6.1.  $\Box$ 

#### **Conflict of interest statement**

No conflict of interest is declared.

## Acknowledgements

This work was started during a visit of T.W. at the Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari. He would like to thank this institution for the kind hospitality. The research of the authors was supported by GNAMPA project "Metodi variazionali e problemi ellittici non lineari" and MIUR project "Variational and topological methods in the study of nonlinear phenomena" (PRIN 2009). S.C. would like to thank Professor Louis Jeanjean for stimulating discussions about the results in [32].

# References

196

<sup>[1]</sup> N. Ackermann, On a periodic Schrödinger equation with nonlocal superlinear part, Math. Z. 248 (2004) 248-443.

<sup>[2]</sup> S. Agmon, Lectures on Exponential Decay of Solutions of Second-order Elliptic Equations: Bounds on Eigenfunctions of N-Body Schrödinger Operators, Math. Notes, vol. 29, Princeton University Press, Princeton, NJ, 1982, p. 118.

- [3] A. Ambrosetti, D. Ruiz, Multiple bound state for the Schödinger–Poisson problem, Commun. Contemp. Math. 10 (3) (2008) 391–404.
- [4] A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear Schrödinger–Maxwell equations, J. Math. Anal. Appl. 345 (2008) 90–108.
- [5] T. Bartsch, T. Weth, Three nodal solutions of singularly perturbed elliptic equations on domains without topology, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 22 (3) (2005) 259–281.
- [6] J. Bellazzini, L. Jeanjean, T. Luo, Existence and instability of standing waves with prescribed norm for a class of Schrödinger–Poisson equations, Proc. Lond. Math. Soc. (2013) 303–339.
- [7] V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger–Maxwell equations, Topol. Methods Nonlinear Anal. 11 (1998) 283–293.
- [8] I. Catto, C. Le Bris, P.-L. Lions, On some periodic Hartree-type models for crystals, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 19 (2002) 143–190.
- [9] I. Catto, O. Sanchez, J. Soler, Nondispersive dynamics stemming from  $X^{\alpha}$  corrections to the Schrödinger–Poisson systems.
- [10] G. Cerami, D. Passaseo, The effect of concentrating potentials in some singularly perturbed problems, Calc. Var. Partial Differ. Equ. 17 (3) (2003) 257–281.
- [11] P. Choquard, J. Stubbe, The one dimensional Schrödinger-Newton equations, Lett. Math. Phys. 81 (2007) 177-184.
- [12] P. Choquard, J. Stubbe, M. Vuffray, Stationary solutions of the Schrödinger–Newton model—an ODE approach, Differ. Integral Equ. 21 (7–8) (2008) 665–679.
- [13] S. Cingolani, M. Clapp, S. Secchi, Multiple solutions to a magnetic nonlinear Choquard equation, Z. Angew. Math. Phys. 63 (2012) 233-248.
- [14] M. Clapp, D. Puppe, Critical point theory with symmetries, J. Reine Angew. Math. 418 (1991) 1-29.
- [15] V. Coti-Zelati, P.H. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on  $\mathbb{R}^n$ , Commun. Pure Appl. Math. 45 (1992) 1217–1269.
- [16] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrodinger–Maxwell equations, Proc. R. Soc. Edinb. A 134 (5) (2004) 893–906.
- [17] M.J. Esteban, P.-L. Lions, Existence and nonexistence results for semilinear elliptic problems in unbounded domains, Proc. R. Soc. Edinb. A 93 (1–2) (1982/1983) 1–14.
- [18] J. Frölich, E. Lenzmann, Mean-field limit of quantum Bose gases and nonlinear Hartree equation, Equ. Dériv. Partielles, Ecole Polytech. Palaiseau 134 (XIX) (2003).
- [19] G. Li, C. Wang, The existence of a nontrivial solution to a nonlinear elliptic problem of linking type without the Ambrosetti–Rabinowitz condition, Ann. Acad. Sci. Fenn., Math. 36 (2) (2011) 461–480.
- [20] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Stud. Appl. Math. 57 (1977) 93–105.
- [21] E.H. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities, Ann. Math. 118 (1983) 349–374.
- [22] P.-L. Lions, Solutions of Hartree–Fock equations for Coulomb systems, Commun. Math. Phys. 109 (1984) 33–97.
- [23] L. Ma, L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, Arch. Ration. Mech. Anal. 195 (2010) 455–467.
- [24] S. Masaki, Energy solution to a Schrödinger–Poisson system in the two-dimensional whole space, SIAM J. Math. Anal. 43 (6) (2011) 2719–2731.
- [25] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, J. Funct. Anal. 265 (2) (2013) 153–184.
- [26] D. Mugnai, The Schrödinger–Poisson System with Positive Potential, Commun. Partial Differ. Equ. 36 (2013) 1009–1117.
- [27] S.I. Pekar, Untersuchungen über die Elektronentheorie der Kristalle, Akademie-Verlag, Berlin, 1954.
- [28] R. Penrose, On gravity's role in quantum state reduction, Gen. Relativ. Gravit. 28 (5) (1996) 581-600.
- [29] M. Reed, S. Barry, Methods Modern Math. Phys. 4 (1978).
- [30] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, J. Funct. Anal. 137 (2006) 655-674.
- [31] M. Struwe, Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer-Verlag, Berlin, Heidelberg, 1990.
- [32] J. Stubbe, Bound states of two-dimensional Schrödinger-Newton equations, arXiv:0807.4059v1, 2008.
- [33] M. Willem, Minimax Theorems, vol. 24, 1996.