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# Existence and uniqueness of a density probability solution for the stationary Doi–Edwards equation

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**Abstract**

We prove the existence, uniqueness and non-negativity of solutions for a nonlinear stationary Doi–Edwards equation. The existence is proved by a perturbation argument. We get the uniqueness and the non-negativity by showing the convergence in time of the solution of the evolutionary Doi–Edwards equation towards any stationary solution.


*Keywords:* Polymeric fluids; Doi–Edwards equation; Stationary equation; Well-posedness

## 1. Introduction

It is well established that the modelling of non-Newtonian and viscoelastic flows bases on molecular theories. In such theories, kinetical concepts are used to obtain a mathematical description of the configuration of polymer chains. One of the most popular theories used to predict the behaviour of the melted polymers is that of Doi and Edwards (see for example [8] and [9]). It makes use of de Gennes reptation concept [10]. In the Doi–Edwards model, chains of polymer are confined within a tube of surrounding chains, and chains cannot move freely. This description of the entanglement phenomenon leads to the concept of a primitive chain (the tube centerline). The primitive chain is not the real chain, and is shorter. Nevertheless, the goal of Doi–Edwards theory is to describe the dynamics of the primitive chain. Basically, short time fluctuations of the polymer chain happen near the primitive chain in a wriggling motion, while fluctuations on larger time scales (say $t \geq T_{equilibrium}$, see [7]) account for the chain ability to move inside the tube (roughly speaking, $T_{equilibrium}$ is the time after which the primitive chain "feels" the constraints imposed by the tube). This is the "snakelike" diffusive motion. Since diffusion concerns the primitive chain, the primitive chain finally disengages from the original tube. This is a major complication in the theory, and for more details the reader is referred

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to [7,8] and [9]. Nevertheless, notice that on the average (say on $\Delta t = T_{equilibrium}$) the primitive chain and the real chain coincide. Finally, for details on the thermodynamics of the model, see for instance [9,17].

From a mathematical point of view, a primitive chain is represented as a curve in $\mathbb{R}^3$. The position on the primitive chain is given by a curvilinear coordinate $s \in [0, 1]$ (from now on, all the primitive chains are supposed to have the same length which is normalised to 1). Moreover, the orientation for any $s$ is given by a unitary vector $u$ tangent to the curve; we then have $u \in S_2$ where $S_2$ is the unit sphere in $\mathbb{R}^3$, that is:

$$S_2 = \{u \in \mathbb{R}^3, \quad \|u\| = 1\}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$. The tangent vector $(s, u)$ is the microscopic variable of the model.

The rheology of such a fluid is obtained with the help of the so-called *configurational probability density* of the molecules, denoted here by $F$. It is a probability density with respect to the variable $u$. Assuming space independence, we have $F = F(t, s, u)$ where $t \geq 0$ is the time variable. In the general case $F = F(t, x, s, u)$ and one should write equation (1.1) below with a convective term, i.e. replace $\partial F/\partial t$ by the material derivative $\partial F/\partial t + v \cdot \nabla_x F$. It would lead to serious complications since, in that case, a complementary equation (conservation law) is required to determine $v$. Here, as usual, $v$ stands for the macroscopic speed of the fluid.

The probability density satisfies the following PDE, known under the name of *Doi–Edwards equation*, and which is of Fokker–Planck–Smoluchowski type:

$$\frac{\partial F}{\partial t} - D\frac{\partial^2 F}{\partial s^2} + \frac{\partial}{\partial u}.(\mathcal{G}F) - \epsilon F\kappa : u \otimes u + \epsilon\frac{\partial}{\partial s}\Big(F\kappa : \lambda(F)\Big) = 0 \text{ on } S_2 \times ]0, 1[ \tag{1.1}$$

The orientation at the chain ends is assumed to be isotropic (see [8]), which gives the boundary condition:

$$F(s = 0) = F(s = 1) = (1/4\pi). \tag{1.2}$$

We also have the initial condition:

$$F(t = 0) = F_0(s, u) \tag{1.3}$$

(see also [9,17] and [5]).

In the equation (1.1) $D > 0$ and $\epsilon \geq 0$ are physical coefficients and $\kappa = \kappa(t) \in \mathcal{M}_3(\mathbb{R})$ is the velocity gradient; in order to comply with the hypothesis that $F$ is independent on the space variable $x$ we assume that $\kappa$ is a function of $t$ only. We also have

$$\mathcal{G} = \kappa.u - (\kappa : u \otimes u)u$$

and

$$\lambda(F)(s) = \int_0^s \int_{S_2} F(s', u)u \otimes u d\mu ds'.$$

The case $\epsilon = 0$ corresponds to the so-called *Independent Alignment Approximation* (IAA) for which explicit solutions of the evolutionary configurational PDE are known (see [8]). In the case $\epsilon > 0$, the two mechanism described by the terms $-\epsilon F\kappa : u \otimes u$ and $\epsilon \frac{\partial}{\partial s}\Big(F\kappa : \lambda(F)\Big)$ compensate, keeping constant the number of segments by unit length:

$$\int_0^1 \int_{S_2} \Big[ -\epsilon F\kappa : u \otimes u + \epsilon\frac{\partial}{\partial s}\Big(F\kappa : \lambda(F)\Big)\Big] d\mu(u)ds = 0 \tag{1.4}$$

In fact, we first obtain the following “non-integrated in $s$ version” of (1.4):

$$\int_{S_2} \Big[ -\epsilon F\kappa : u \otimes u + \epsilon\frac{\partial}{\partial s}\Big(F\kappa : \lambda(F)\Big)\Big] d\mu = \epsilon\,\frac{\partial}{\partial s}\left[\left(\int_{S_2} F(s, u)\, d\mu - 1\right)\kappa : \lambda(F)\right]. \tag{1.5}$$

Integrating in $s$ this inequality, we obtain (1.4) with the help of (1.2). In the present paper, we will make little use of (1.4), but it is likely that a thorough analysis of the stationary problem (i.e. for large $\epsilon$) would appeal to such cancellation property. Note also that this is an "ad hoc" compensation since these two terms arise from two different phenomena. The first one quantifies the creation of new segments, while the second one is due to the extension–retraction mechanism by which the chain keeps constant its curvilinear length.

Existence, uniqueness, and regularity of solutions of (1.1), (1.2), (1.3) are proved in [5], as well as the fact that $F$ is a probability density. For existence results in the case of related – but different – molecular models, see [15,6,13]. As an aside, notice that the Doi–Edwards model should not be mixed up with what is commonly called the *Doi model* (see [16]), this latter being used for dilute polymers. In Doi theory, molecules are considered as rigid dumbbells.

In this paper we focus on the following stationary problem associated with (1.1), (1.2):

$$-\frac{\partial^2 F}{\partial s^2}+\frac{\partial}{\partial u}.(\mathcal{G}F)-\epsilon F\kappa : u\otimes u+\epsilon\frac{\partial}{\partial s}\Big(F\kappa:\lambda(F)\Big)=0 \text{ on } S_2\times\,]0,1[ \tag{1.6}$$

$$F(s=0)=F(s=1)=(1/4\pi) \tag{1.7}$$

In equation (1.6), we set $D=1$, which is not restrictive, and we assume that the tensor $\kappa$ does not depend on $t$. Notice that stationary Fokker–Planck equations with degenerate constitutive functions, but elliptic principal part, are studied for example in [2,3] and [4].

The two points that are addressed in the sequel are the well posedness and the non-negativity of solutions of equations (1.6)–(1.7) (remark that, in contrast with $F\geq 0$, equality $\int_{S_2}F(u)d\mu(u)=1$ can easily be obtained in formal manner by integrating (1.6) on $S_2$ and making use of (1.5) and (1.7)). We will essentially restrict to $|\epsilon|$ small, since global estimates on the sphere $S_2$ do not seem easy to obtain for $|\epsilon|$ large. As a matter of fact, even for $\epsilon=0$, well posedness of the stationary problem may not be obvious due to the lack of ellipticity in the $u$ variable. Moreover, due to the probabilistic features of the equations, the problem has to be well posed in $L^1(S_2)$, with some extra smoothness due for instance to the $FLog(F)$ entropy estimates on the associated time dependent problem (see for instance [6]). But $L^2(S_2)$ estimates are not expected. Anyhow, proceeding as in [8], i.e. writing

$$f(s,u):=F(s,u)-\frac{1}{4\pi}=\sum_{n\in\mathbb{N}^*}f_n(u)\sin(n\pi s)$$

the original problem (1.6)–(1.7) with $\epsilon=0$ is reduced to a set of well posed problems in $L^r(S_2)$ with $r-1\geq 0$ small enough (see Section 3):

$$\frac{\partial}{\partial u}\cdot\Big(\mathcal{G}f_n\Big)+n^2\pi^2 f_n=g_n,\quad n\in\mathbb{N}^* \tag{1.8}$$

Therefore, we proceed as follows. We first introduce our main functional space (Section 2), and prove an existence and uniqueness theorem in the case $\epsilon=0$ (Section 3). The existence result is extended in Section 4 to the case $|\epsilon|$ small, by means of the implicit function theorem. Notice that the proof provides quite strong $L^r(S_2)$ estimates frequency by frequency with $r\geq 1$ close enough to 1, but not for $r=2$, due to the low frequencies. This excludes all the usual Hilbert spaces as suitable working frames. We use instead a subspace of $W^{1,\infty}\big(0,1,L^r(S_2)\big)$, subspace which is not easily characterised in term of the classical functional spaces. As a matter of fact, the restriction $r<2$ on low frequencies also causes some difficulties in the proof of the positivity of $F$.

The above arguments could be used to show uniqueness of solutions of problem (1.6)–(1.7) by duality. Nevertheless, we proceed differently and prove at the same time the uniqueness and the non-negativity of $F$. In order to establish this last result, we show that the solutions of problem (1.6)–(1.7) are the limits when $t\to\infty$ of solutions of the time dependent Doi–Edwards problems (see Section 5 and Section 6). In fact, these evolutionary solutions are already known to be probability densities (see [5]). The proof of the above convergence consists essentially in bounding globally in time nonlinear terms such as $\frac{\partial}{\partial s}\Big(F\kappa:\lambda(F)\Big)$.

## 2. Presentation of the problem and of the main results

Throughout this paper we write $Q=\,]0,1[\,\times S_2$. Making use of the Riemannian metric induced by the canonical inner product . of $\mathbb{R}^3$, we can define the usual surface measure $d\mu$ (or $d\mu(u)$), the gradient $\frac{\partial}{\partial u}$ and the divergence $\frac{\partial}{\partial u}\cdot$ operators on $S_2$ (see [1]). Since $S_2$ is a Riemannian submanifold of $\mathbb{R}^3$, the gradient of a smooth scalar valued function $g:S_2\to\mathbb{R}$ can alternatively be defined as the following projection (see [12]):

$$\frac{\partial}{\partial u} g = \nabla_u \tilde{g} - (\nabla_u \tilde{g} \cdot u) u$$

where $\tilde{g}$ is any smooth extension of $g$ in a neighbourhood of $S_2$ in $\mathbb{R}^3$ and $\nabla_u$ is the usual gradient in $\mathbb{R}^3$. Similarly, for any smooth vector valued vector field of $S_2$, identified with $X \in C^1(S_2, \mathbb{R}^3)$ with $X \cdot u = 0$, the divergence of $X$ can be defined as (see [12]):

$$\frac{\partial}{\partial u} \cdot X = \nabla_u \cdot \tilde{X} - \tilde{X}' u \cdot u$$

where $\tilde{X}$ is any smooth extension of $X$ in a neighbourhood of $S_2$ in $\mathbb{R}^3$. Notation $\tilde{X}'$ stands for the usual Jacobian matrix of $\tilde{X}$. In what follows, we will essentially use Stokes formula:

$$\int_{S_2} X \cdot \frac{\partial g}{\partial u} d\mu = -\int_{S_2} \Big(\frac{\partial}{\partial u} \cdot X\Big) g d\mu \tag{2.1}$$

valid for any smooth functions $X : S_2 \to \mathbb{R}^3$ with $X \cdot u = 0$, and $g : S_2 \to \mathbb{R}$. In particular, for $g = 1$, we get:

$$\int_{S_2} \Big(\frac{\partial}{\partial u} \cdot X\Big) d\mu = 0 \tag{2.2}$$

Formulas (2.1) and (2.2) will be used to neglect or discard terms coming from $\frac{\partial}{\partial u} \cdot (\mathcal{G} f)$.

Using the following change of unknown function $f = F - \frac{1}{4\pi}$ and making use of:

$$\frac{\partial}{\partial u} \cdot \mathcal{G} = -3\kappa : u \otimes u \tag{2.3}$$

$$\kappa : Id_3 = tr(\kappa) = 0 \tag{2.4}$$

problem (1.6)–(1.7) becomes a problem with homogeneous Dirichlet boundary conditions:

$$-\frac{\partial^2 f}{\partial s^2} + \frac{\partial}{\partial u} \cdot (\mathcal{G} f) - \epsilon f \kappa : u \otimes u + \epsilon \frac{\partial}{\partial s}\Big(f \kappa : \lambda(f)\Big) + \frac{\epsilon}{4\pi} \int_{S_2} \kappa : v \otimes v f(s, v) d\mu(v) = \frac{3+\epsilon}{4\pi} \kappa : u \otimes u \text{ on } Q \tag{2.5}$$

$$f(s = 0) = f(s = 1) = 0 \tag{2.6}$$

In the following, we use a Hilbertian basis of eigenvectors of the Laplacian in $]0, 1[$ with Dirichlet boundary conditions. Namely, family $(H_n)_{n \in \mathbb{N}^*}$ is defined by

$$H_n(s) = \sqrt{2} \sin(n\pi s)$$

For any $g \in L^1(Q)$, $n \in \mathbb{N}^*$, we write $g_n(u) = \int_0^1 g(s, u) H_n(s) ds$. For any $r \geq 1$, we define the vector spaces $X_r$ by:

$$X_r = \Big\{ g \in W^{1,\infty}\big(0, 1, L^r(S_2)\big) \text{ such that for any } n \in \mathbb{N}^*, \; \mathcal{G} \cdot \frac{\partial g_n}{\partial u} \in L^r(S_2) \text{ and } \sup_{n \in \mathbb{N}^*} \Big(n^3 \|g_n\|_{L^r(S_2)}\Big) + \sup_{n \in \mathbb{N}^*} \Big(n \Big\|\mathcal{G} \cdot \frac{\partial g_n}{\partial u}\Big\|_{L^r(S_2)}\Big) < \infty \Big\} \tag{2.7}$$

We will see in Section 3 that $X_r$ is a Banach space when endowed with its natural norm $\|.\|_{X_r}$:

$$\|g\|_{X_r} = \sup_{n \in \mathbb{N}^*} \Big(n^3 \|g_n\|_{L^r(S_2)}\Big) + \sup_{n \in \mathbb{N}^*} \Big(n \Big\|\mathcal{G} \cdot \frac{\partial g_n}{\partial u}\Big\|_{L^r(S_2)}\Big) \tag{2.8}$$

Moreover, we shall prove that any $g \in X_r$ satisfies the homogeneous condition (see Remark 3.1 below):

$$g(s = 0) = g(s = 1) = 0$$

Remark also that if $r_2 \geq r_1 \geq 1$ then $X_{r_2}$ is continuously embedded in $X_{r_1}$.

**Remark 2.1.** **a)** Definition of $X_r$ is a simple but useful step in our analysis. Classical spaces such as $W^{1,\infty}(0,1; W^{1,r}(S_2))$ cannot be used in place of $X_r$. In fact, $L^r(S_2)$ estimates are not likely to be obtained directly from (2.5) (with $\epsilon = 0$) since we cannot take advantage of the $s$-elliptic term $-\frac{\partial^2 f}{\partial s^2}$. In contrast, after a Fourier decomposition in $s$ we shall get the needed estimates. Notice that after such partial decomposition, the non-elliptic divergence term $\frac{\partial}{\partial u}\cdot(\mathcal{G}f)$ is still controlled by integration on $S_2$.

**b)** The space $X_r$ is formally obtained by counting the powers of $n$ in the equation obtained from (2.5) by formal expansion in the $(H_n)_{n\in N^*}$ basis. Notice for instance that we write $n^3\|g_n\|_{L^r(S_2)}$ in place of $n^2\|g_n\|_{L^r(S_2)}$ as a corresponding term to $-\partial^2 f/\partial s^2$. This gain of one power in definition of $X_r$ arises from the right-hand side of equation (2.5), which does not depend on the $s$ variable. Indeed, for any $n \in \mathbb{N}^*$:

$$|\int_0^1 (\kappa : u \otimes u)\sin(n\pi s)ds| \le C/n$$

supplying one power of $n$.

Before giving the weak formulation of equations (2.5)–(2.6) in the $X_r$ functional frame, notice that for any $g, h \in X_r$ we have $\lambda(h) \in W^{2,\infty}(0,1)$ and $g \in W^{1,\infty}\big(0,1,L^r(S_2)\big)$ which implies:

$$\text{for any } g,h \in X_r,\ \frac{\partial}{\partial s}\big(g\kappa : \lambda(h)\big) \text{ is well defined and belongs to } L^\infty\big(0,1,L^r(S_2)\big) \tag{2.9}$$

**Definition 2.1.** We say that $f$ is a weak solution of (2.5)–(2.6) if $f$ belongs to $X_1$ and satisfies:

$$\int_Q \Big[\frac{\partial f}{\partial s}\frac{\partial \phi}{\partial s} - f\mathcal{G}\cdot\frac{\partial \phi}{\partial u} - \epsilon f\kappa : u\otimes u\phi + \epsilon\frac{\partial}{\partial s}\big[f\kappa : \lambda(f)\big]\phi + \frac{\epsilon}{4\pi}\int_{S_2}\kappa : v\otimes v f d\mu(v)\phi\Big]dQ$$
$$= \frac{3+\epsilon}{4\pi}\int_Q \kappa : u\otimes u\phi dQ \quad \forall\,\phi \in H_0^1\big(0,1,H^2(S_2)\big) \tag{2.10}$$

The main result of this paper is:

**Theorem 2.1.** *There exist $\epsilon_0 > 0$ such that, for any $\epsilon \in\, ]-\epsilon_0, \epsilon_0[$, there exists a unique weak solution $f_\epsilon$ of equations (2.5)–(2.6). Moreover:*

- *There exists $r > 1$ such that $f_\epsilon \in X_r$ $\forall\,\epsilon \in\, ]-\epsilon_0,\epsilon_0[$ (regularity result)*
- *$f_\epsilon + (1/4\pi)$ is a probability density on $S_2$. That is, for any $s \in\, ]0,1[$, we have*

$$\Big(f_\epsilon + \frac{1}{4\pi}\Big)(s) \ge 0 \text{ a.e. in } u \in S_2 \text{ and } \int_{S_2}\Big(f_\epsilon + \frac{1}{4\pi}\Big)d\mu(u) = 1 \tag{2.11}$$

In the sequel, we often drop the index $\epsilon$ (or $r \ge 1$) in the notations. In particular, from now on, we write $f$ in place of $f_\epsilon$.

The above theorem is proved in two steps. In a first step, the existence of a solution is established via the implicit function theorem. The rest of the theorem is obtained by showing that solution $F$ of problem (1.6)–(1.7) is the limit for $t \to +\infty$ of a family of density probabilities $\big(F(t)\big)_{t\ge 0}$, namely, the solution of an evolutionary Doi–Edwards equation.

## 3. The case $\epsilon = 0$

We give results related to the functional spaces used in this paper. The existence part of Theorem 2.1 for $\epsilon = 0$ will follow from a priori estimates in these spaces.

**Lemma 3.1.** *For any $r \in [1, +\infty[$, $X_r$ is a Banach space, continuously embedded in $W^{1,\infty}\big(0,1,L^r(S_2)\big)$. Moreover, for any $\phi \in X_r$, we have:*

$$\phi(s,u) = \sum_{n=1}^{\infty} \phi_n(u) H_n(s) \tag{3.1}$$

*with absolute convergence in $W^{1,\infty}\big(0,1,L^r(S_2)\big)$.*

**Proof.** It is clear that $\|.\|_{X_r}$ is a seminorm on the vectorial space $X_r$. The fact that $\|.\|_{X_r}$ is a norm will be a straightforward consequence of equality (3.1).

Let $\phi \in X_r$ and $n \in \mathbb{N}^*$. Then:

$$\begin{aligned}\|\phi_n H_n\|_{W^{1,\infty}(0,1,L^r(S_2))} &\le C_1(1+\pi n)\frac{\|\phi\|_{X_r}}{n^3} \\ &\le C_2 \frac{\|\phi\|_{X_r}}{n^2}\end{aligned}$$

It implies that $\sum_{n=1}^{\infty} \phi_n(u) H_n(s)$ is absolutely convergent in $W^{1,\infty}\big(0,1,L^r(S_2)\big)$.

Now, for any $\psi \in L^{r'}(S_2)$, $r^{-1} + (r')^{-1} = 1$ and $N \in \mathbb{N}^*$, we have:

$$\int_0^1 \int_{S_2} \Big[\phi(s,u) - \sum_{n=1}^{\infty} \phi_n(u) H_n(s)\Big] H_N(s)\psi(u)\,ds\,d\mu(u)$$

$$= \int_{S_2} \phi_N(u)\psi(u)\,d\mu(u) - \sum_{n=1}^{\infty} \int_{S_2} \phi_n(u)\psi(u)\,d\mu(u) \int_0^1 H_n(s) H_N(s)\,ds = 0 \tag{3.2}$$

due to the absolute convergence of $\sum_{n=1}^{\infty} \phi_n(u)\psi(u) H_n(s) H_N(s)$ in $L^{\infty}\big(0,1,L^1(S_2)\big)$. This proves (3.1) and the fact that $X_r$ is continuously embedded in $W^{1,\infty}\big(0,1,L^r(S_2)\big)$.

It remains to prove the completeness of the space $\big(X_r, \|.\|_{X_r}\big)$. Let $(\phi^p)_{p\in\mathbb{N}}$ be a Cauchy sequence in $X_r$. Since $X_r$ is continuously embedded in $W^{1,\infty}\big(0,1,L^r(S_2)\big)$, $(\phi^p)_{p\in\mathbb{N}}$ is also a Cauchy sequence in $W^{1,\infty}\big(0,1,L^r(S_2)\big)$. We denote by $\phi$ its limit in $W^{1,\infty}\big(0,1,L^r(S_2)\big)$. For any $n \in \mathbb{N}^*$:

$$\|\phi_n^p - \phi_n\|_{L^r(S_2)} \le \sqrt{2} \int_0^1 \|\phi^p - \phi\|_{L^r(S_2)}(s)\,ds \to 0 \text{ when } p \to +\infty$$

Hence, $\phi_n^p \to \phi_n$ in $L^r(S_2)$ when $p \to +\infty$, uniformly in $n \in \mathbb{N}^*$. We also have that $\Big(\mathcal{G} \cdot \dfrac{\partial \phi_n^p}{\partial u}\Big)_{p\in\mathbb{N}^*}$ is a Cauchy sequence in $L^r(S_2)$, hence convergent in $L^r(S_2)$. By identification, we deduce that $\mathcal{G} \cdot \dfrac{\partial \phi_n}{\partial u}$ belongs to $L^r(S_2)$ and that $\mathcal{G} \cdot \dfrac{\partial \phi_n^p}{\partial u} \to \mathcal{G} \cdot \dfrac{\partial \phi_n}{\partial u}$ in $L^r(S_2)$ for $p \to +\infty$. From the inequalities:

$$\|\phi_n^p - \phi_n^q\|_{L^r(S_2)} \le (1/n^3)\|\phi^p - \phi^q\|_{X_r}$$

and

$$\Big\|\mathcal{G} \cdot \frac{\partial \phi_n^p}{\partial u} - \mathcal{G} \cdot \frac{\partial \phi_n^q}{\partial u}\Big\|_{L^r(S_2)} \le \frac{1}{n}\|\phi^p - \phi^q\|_{X_r}$$

we classically deduce, taking $q \to +\infty$, that $\phi \in X_r$ and $\phi^p \to \phi$ in $X_r$ for $p \to +\infty$. □

**Remark 3.1.** Formula (3.1) implies that $\phi(s=0) = \phi(s=1) = 0$ for any $\phi \in X_r$.

Let us define for any $r \geq 1$ the space:

$$Z_r = \{\phi \in L^r(S_2) \text{ such that } \mathcal{G} \cdot \frac{\partial \phi}{\partial u} \in L^r(S_2)\}$$

which is clearly a Banach space with norm

$$\|\phi\|_{X_r} = \|\phi\|_{L^r(S_2)} + \left\| \mathcal{G} \cdot \frac{\partial \phi}{\partial u} \right\|_{L^r(S_2)}$$

The space $Z_r$ will be used in the existence proof for $\epsilon = 0$. In order to perform estimates in $Z_r$, we first establish a useful formula (Lemma 3.2). Since this formula shall also be used for the evolution Doi–Edwards equation, we add the variable $t$ in the statement. Notice also that Lemma 3.2 cannot be reduced locally to the case $\mathcal{G}_{local\,chart} = Cst$ due to the zeros of $\mathcal{G}$ on $S_2$.

**Lemma 3.2.** *For any $T > 0$, $r \geq 1$ and $\phi \in L^r(]0, T[ \times S_2)$ with $\mathcal{G} \cdot \frac{\partial \phi}{\partial u} \in L^r(]0, T[ \times S_2)$ we have*

$$r|\phi|^{r-1} \operatorname{sgn}(\phi)\mathcal{G} \cdot \frac{\partial \phi}{\partial u} = \mathcal{G} \cdot \frac{\partial}{\partial u}(|\phi|^r) \tag{3.3}$$

**Proof.** Using local charts, this amounts essentially to prove that for any open bounded set $\Omega \subset \mathbb{R}^3$, $A \in C^\infty(\Omega, \mathbb{R}^3)$, $\psi \in L^r(\Omega)$ with $A \cdot \nabla\psi \in L^r(\Omega)$, we have:

$$r|\psi|^{r-1} \operatorname{sgn}(\psi) A \cdot \nabla\psi = A \cdot \nabla(|\psi|^r) \tag{3.4}$$

Let us consider a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $C^\infty(\Omega)$ endowed with the two following properties (see Lemma II.1 of [11]):

$$\psi_n \to \psi \text{ in } L^r(\Omega) \text{ for } n \to +\infty \tag{3.5}$$

$$A \cdot \nabla\psi_n \to A \cdot \nabla\psi \text{ in } L^r(\Omega) \text{ for } n \to +\infty \tag{3.6}$$

and, for any $\delta > 0$, define functions $h_\delta : \mathbb{R} \to \mathbb{R}$ and $j_\delta : \mathbb{R} \to \mathbb{R}$ by $h_\delta(y) = \sqrt{y^2 + \delta}$ and $j_\delta(y) = y/\sqrt{y^2 + \delta}$. We classically have:

$$r\big[h_\delta(\psi_n)\big]^{r-1} j_\delta(\psi_n) A \cdot \nabla\psi_n = A \cdot \nabla\Big[\big(h_\delta(\psi_n)\big)^r\Big] \tag{3.7}$$

($\delta > 0, n \in \mathbb{N}^*$).

We extract a subsequence of $(\psi_n)_{n\in\mathbb{N}^*}$, still denoted by $(\psi_n)_{n\in\mathbb{N}^*}$, such that:

$$\psi_n \to \psi \text{ a.e. for } n \to +\infty \tag{3.8}$$

$$|\psi_n| \leq \psi^* \text{ a.e., with } \psi^* \in L^r(\Omega) \tag{3.9}$$

Fix $\delta > 0$. Our goal is to pass to the limit when $n \to +\infty$ in (3.7). We have:

$$\begin{aligned} |h_\delta(\psi_n) - h_\delta(\psi)| &\leq \frac{|\psi_n - \psi||\psi_n + \psi|}{\sqrt{\psi_n^2 + \delta} + \sqrt{\psi^2 + \delta}} \\ &\leq |\psi_n - \psi| \end{aligned}$$

Hence, from (3.5), we get $\big[h_\delta(\psi_n)\big]^r \to \big[h_\delta(\psi)\big]^r$ for $n \to \infty$. It implies that, for $n \to +\infty$:

$$A \cdot \nabla\Big[h_\delta(\psi_n)^r\Big] \to A \cdot \nabla\Big[h_\delta(\psi)^r\Big] \text{ in } \mathscr{D}'(\Omega) \tag{3.10}$$

Observe that:

$$\begin{aligned} |j_\delta(\psi_n)||h_\delta(\psi_n)|^{r-1} &\leq \big[(\psi^*)^2 + \delta\big]^{(r-1)/2} \\ &\leq \big[\psi^* + \delta\big]^{r-1} \in L^{r'}(\Omega) \end{aligned} \tag{3.11}$$

with $r' \in [1, +\infty]$ such that $r^{-1} + r'^{-1} = 1$. For $r > 1$, using the dominated convergence theorem, we deduce from (3.8) and (3.11) that:

$$j_\delta(\psi_n)h_\delta(\psi_n)^{r-1} \to j_\delta(\psi)h_\delta(\psi)^{r-1} \text{ in } L^{r'}(\Omega) \text{ for } n \to +\infty \tag{3.12}$$

Using (3.6) and (3.12), we conclude that, for $r > 1$:

$$j_\delta(\psi_n)h_\delta(\psi_n)^{r-1}A \cdot \nabla\psi_n \to j_\delta(\psi)h_\delta(\psi)^{r-1}A \cdot \nabla\psi \text{ in } L^1(\Omega) \text{ for } n \to +\infty \tag{3.13}$$

For $r = 1$, we easily obtain:

$$j_\delta(\psi_n)A \cdot \nabla\psi_n \to j_\delta(\psi)A \cdot \nabla\psi \text{ in } L^1(\Omega) \text{ for } n \to +\infty \tag{3.14}$$

We deduce from (3.7), (3.10), (3.13), (3.14), that:

$$r\Big[h_\delta(\psi)\Big]^{r-1} j_\delta(\psi)A \cdot \nabla\psi = A \cdot \nabla\Big[\big(h_\delta(\psi)\big)^r\Big] \tag{3.15}$$

for $r \geq 1$. In order to pass to the limit $\delta \to 0$ in the above equality, notice that:

$$|h_\delta(\psi)| \leq |\psi| + 1 \text{ for } \delta \leq 1 \tag{3.16}$$

For $\delta \to 0$, we have $h_\delta \to |.|$ everywhere. Due to (3.16), $\psi \in L^r(\Omega)$ and the dominated convergence theorem, we conclude that, for any $r \geq 1$:

$$h_\delta(\psi)^r \to |\psi|^r \text{ in } L^1(\Omega) \text{ when } \delta \to 0 \tag{3.17}$$

for any $r \geq 1$. Arguing similarly, we also prove that:

$$j_\delta(\psi)h_\delta(\psi)^{r-1}A.\nabla\psi \to \text{sgn}(\psi)|\psi|^{r-1}A.\nabla\psi \text{ in } L^1(\Omega) \text{ when } \delta \to 0 \tag{3.18}$$

Using (3.17), (3.18) and (3.15), we obtain the result. □

Let us introduce for any $r \geq 1$ the following space of $L^r(S_2)$ sequences:

$$Y_r = \{(a_n)_{n\in\mathbb{N}^*} \text{ such that: } \forall n \in \mathbb{N}^*, a_n \in L^r(S_2) \text{ and } \sup_{n\in\mathbb{N}^*}\Big(n\|a_n\|_{L^r(S_2)}\Big) < \infty\}$$

It is clear that $Y_r$, when endowed with its natural norm:

$$\|(a_n)_{n\in\mathbb{N}^*}\|_{Y_r} = \sup_{n\in\mathbb{N}^*}\Big(n\|a_n\|_{L^r(S_2)}\Big)$$

is a Banach space. Next, we introduce the linear, bounded operator $\mathscr{T}_0 : X_r \to Y_r$ defined for any $g \in X_r$ by $\mathscr{T}_0(g) = (a_n)_{n\in\mathbb{N}^*}$ with:

$$a_n = n^2\pi^2 g_n + \frac{\partial}{\partial u} \cdot (\mathcal{G} g_n)$$

where we recall that

$$g_n = \int_0^1 g(s)H_n(s)ds \tag{3.19}$$

This operator is formally obtained by projecting the left-hand side of equation (2.5) for $\epsilon = 0$ on the Hilbertian basis $(H_n)_{n\in\mathbb{N}^*}$ of $L^2(]0, 1[)$.

In order to study $\mathscr{T}_0$, we first introduce the following unbounded linear operator defined for any $n \in \mathbb{N}^*$ and $r > 1$ by:

$$L_n : L^r(S_2) \to L^r(S_2)$$

where $D(L_n) = Z_r$ and for any $h \in Z_r$:

$$L_n(h) = n^2\pi^2 h + \frac{\partial}{\partial u} \cdot (\mathcal{G} h)$$

It is clear that $L_n$ is closed and densely defined.

Let us consider $r' > 1$ such that

$$\frac{1}{r} + \frac{1}{r'} = 1. \tag{3.20}$$

One can easily prove that the adjoint operator

$$L_n^* : L^{r'}(S_2) \to L^{r'}(S_2)$$

is such that $D(L_n^*) = Z_{r'}$ and for any $\psi \in Z_{r'}$:

$$L_n^*(\psi) = n^2\pi^2\psi - \mathcal{G} \cdot \frac{\partial \psi}{\partial u}$$

**Lemma 3.3.** *There exists $r_0 > 1$ such that $L_n : Z_r \to L^r(S_2)$ is a Banach isomorphism for any $r \in ]0, r_0[$ and $n \in \mathbb{N}^*$. Moreover, there exists $C > 0$ such that*

$$\|L_n^{-1}(\psi)\|_{Z_r} \le C\|\psi\|_{L^r(S_2)} \tag{3.21}$$

$$\|L_n^{-1}(\psi)\|_{L^r(S_2)} \le \frac{C}{n^2}\|\psi\|_{L^r(S_2)} \tag{3.22}$$

*for any $\psi \in L^r(S_2)$, $n \in \mathbb{N}^*$ and $r \in ]0, r_0[$.*

**Proof.** Let $r' > 1$ satisfying (3.20). The surjectivity of $L_n$ is a consequence of the following a priori estimate:

$$\forall \varphi \in Z_{r'},\ \|\varphi\|_{L^{r'}(S_2)} \le C_1\|L_n^*(\varphi)\|_{L^{r'}(S_2)} \tag{3.23}$$

In order to prove (3.23), set $h = L_n^*(\varphi)$. We have:

$$n^2\pi^2\varphi - \mathcal{G} \cdot \frac{\partial \varphi}{\partial u} = h \tag{3.24}$$

We multiply this inequality by $|\varphi|^{r'-1}\operatorname{sgn}(\varphi)$, integrate over $S_2$ and use Lemma 3.2 and we get:

$$n^2\pi^2 \int_{S_2} |\varphi|^{r'} d\mu - \frac{1}{r'} \int_{S_2} \mathcal{G} \cdot \frac{\partial(|\varphi^{r'}|)}{\partial u} d\mu = \int_{S_2} h|\varphi|^{r'-1}\operatorname{sgn}(\varphi) d\mu \tag{3.25}$$

Using the Stokes formula and Hölder inequality we obtain:

$$\int_{S_2} \Big(n^2\pi^2 - \frac{3}{r'}\kappa : u \otimes u\Big)|\varphi|^{r'} d\mu \le \|h\|_{L^{r'}(S_2)}\|\varphi\|^{r'-1}_{L^{r'}(S_2)}$$

Taking $r'$ large enough, that is $r - 1$ small enough, we get (3.23), which proves that $L_n$ is surjective.

We now prove the injectivity of $L_n$. Let us denote $\psi = L_n(g)$, with $g \in Z_r$, $\psi \in L^r(S_2)$. Hence:

$$n^2\pi^2 g + \frac{\partial}{\partial u} \cdot (\mathcal{G}g) = \psi \tag{3.26}$$

Using Lemma 3.2, we get:

$$\begin{aligned} |g|^{r-1}\operatorname{sgn}(g)\frac{\partial}{\partial u} \cdot (\mathcal{G}g) &= |g|^r \frac{\partial}{\partial u} \cdot \mathcal{G} + \frac{1}{r}\mathcal{G} \cdot \frac{\partial}{\partial u}(|g|^r) \\ &= \frac{\partial}{\partial u} \cdot (\mathcal{G}|g|^r) + (\frac{1}{r} - 1)\mathcal{G} \cdot \frac{\partial}{\partial u}(|g|^r) \end{aligned} \tag{3.27}$$

We now multiply (3.26) by $|g|^{r-1}\operatorname{sgn}(g)$ and integrate over $S_2$ to get:

$$\int_{S_2} \Big(n^2\pi^2 - \frac{3(r-1)}{r}\kappa : u \otimes u\Big)|g|^r d\mu \le \|\psi\|_{L^r(S_2)}\|g\|^{r-1}_{L^r(S_2)} \tag{3.28}$$

Taking again $r-1$ small enough we obtain at the same time that $L_n$ is one to one and estimate (3.22). Estimate (3.21) follows from equality

$$\mathcal{G}\cdot\frac{\partial g}{\partial u}=\psi-\big(n^2\pi^2-3\kappa:u\otimes u\big)g$$

(see eqs. (3.26) and (2.3)) and estimate (3.22). □

**Remark 3.2.** In the above proof, the condition $n^2\frac{r}{r-1}$ large enough is required. Hence, we can take $r$ large provided that $n$ is large enough. This limitation on low frequencies forbids to work in a Hilbertian frame ($r=2$).

Since for any $g\in X_r$ we have $\big(\mathscr{T}_0(g)\big)_n=L_n(g_n)$ where $g_n$ is given by (3.19), we easily obtain the following:

**Corollary 3.1.** *There exists $r_0>1$ such that for any $r\in\,]0,r_0[$, $\mathscr{T}_0$ is a Banach isomorphism.*

## 4. Proof of the existence result for $\epsilon$ small

For $|\epsilon|$ small enough, existence of solutions for the problem (2.5)–(2.6) will be a consequence of Corollary 3.1 and the implicit function theorem for an appropriate operator $\mathscr{T}:\mathbb{R}\times X_r\to Y_r$. In order to handle the nonlinearity of such an operator, we prove a preliminary lemma. Notice first that for any $n\in\mathbb{N}^*$, due to Remark 2.9:

$$b_n=\int_0^1\frac{\partial}{\partial s}\big[\phi\kappa:\lambda(\psi)\big](s)H_n(s)ds \tag{4.1}$$

is well defined and belongs to $L^r(S_2)$ for any $\phi,\psi\in X_r$.

**Lemma 4.1.** *For any $r\geq 1$ let $B:X_r\times X_r\to Y_r$ be given by $B(\phi,\psi)=(b_n)_{n\in\mathbb{N}^*}$ where $b_n$ is given by (4.1).*

*The function $B$ is well defined, bilinear and continuous. Moreover, for any $\phi,\psi\in X_r$ and $r\geq 1$ we have:*

$$\|b_n\|_{L^r(S_2)}\leq\frac{C}{n^2}\|\phi\|_{X_r}\|\psi\|_{X_r} \tag{4.2}$$

*where $C>0$ is a constant.*

**Proof.** In order to prove inequality (4.2), we integrate by parts equation (4.1). We get:

$$b_n=-\sqrt{2}n\pi\int_0^1\phi\kappa:\lambda(\psi)\cos(n\pi s)ds \tag{4.3}$$

Hence, we just have to prove that:

$$\|\int_0^1\phi\kappa:\lambda(\psi)e^{-in\pi s}ds\|_{L^r(S_2)}\leq\frac{C}{n^3}\|\phi\|_{X_r}\|\psi\|_{X_r} \tag{4.4}$$

for any $n\in\mathbb{N}^*$, with $C>0$ independent of $n,\phi,\psi$.

Observe that:

$$\begin{aligned}\kappa:\lambda(\psi)(s)&=\int_0^s\int_{S_2}\Big[\kappa:v\otimes v\sum_{q=1}^{+\infty}\psi_q(v)H_q(s')\Big]dvds'\\&=\sqrt{2}\sum_{q=1}^{+\infty}\Big\{\frac{1}{q\pi}\big[1-\cos(q\pi s)\big]\int_{S_2}\psi_q(v)\kappa:v\otimes v\,dv\Big\}\end{aligned} \tag{4.5}$$

We have by definition of $\|.\|_{X_r}$ and Hölder inequality:

$$\Big|\int_{S_2} \psi_q(s)(v)\kappa : v\otimes v dv\Big| \le \frac{C}{q^3}\|\psi\|_{X_r} \tag{4.6}$$

Hence:

$$\sum_{q=1}^{+\infty}\Big\{\frac{1}{q}\Big|\int_{S_2} \psi_q(v)\kappa : v\otimes v dv\Big|\Big\} \le C\|\psi\|_{X_r} \tag{4.7}$$

It follows from (4.5), (4.6) and (4.7) that:

$$\kappa : \lambda(\psi)(s) = \sum_{p\in\mathbb{Z}} \lambda_p e^{ip\pi s} \tag{4.8}$$

with:

- For $p=0$, $|\lambda_0| \le C\|\psi\|_{X_r}$ (4.9)
- For $p\in\mathbb{Z}^*$, $|\lambda_p| \le \frac{C}{p^4}\|\psi\|_{X_r}$ (4.10)

As a consequence, $\sum_{p\in\mathbb{Z}}\lambda_p e^{ip\pi s}$ is absolutely convergent in $L^\infty(0,1)$.

On the other hand, we can write:

$$\begin{aligned}\phi(s) &= \sum_{q=1}^{\infty}\phi_q \sin(q\pi s)\\ &= \sum_{q\in\mathbb{Z}}\tilde{\phi}_q e^{iq\pi s}\end{aligned} \tag{4.11}$$

with $\tilde{\phi}_0 = 0$, $\tilde{\phi}_q = -(i/2)\phi_q$ for $q>0$ and $\tilde{\phi}_q = (i/2)\phi_{-q}$ for $q<0$. Hence, for any $q\in\mathbb{Z}^*$:

$$\begin{aligned}\|\tilde{\phi}_q\|_{L^r(S_2)} &\le \frac{1}{2}\|\phi_q\|_{L^r(S_2)}\\ &\le \frac{1}{2|q|^3}\|\phi\|_{X_r}\end{aligned} \tag{4.12}$$

It follows that $\sum_{q\in\mathbb{Z}}\tilde{\phi}_q e^{iq\pi s}$ is absolutely convergent in $L^\infty(0,1,L^r(S_2))$. Invoking a classical result on the product of absolutely convergent series in Banach spaces, we find that:

$$\phi\kappa : \lambda(\psi) = \sum_{n\in\mathbb{Z}} h_n e^{in\pi s}$$

with absolute convergence in $L^\infty(0,1,L^r(S_2))$. Moreover, since, for any $n\in\mathbb{Z}$ we have:

$$h_n = \sum_{q\in\mathbb{Z}}\lambda_{n-q}\tilde{\phi}_q \tag{4.13}$$

we can write, restricting to $n\in\mathbb{N}^*$ and making use of inequalities (4.9), (4.10), (4.12):

$$\begin{aligned}\|h_n\|_{L^r(S_2)} &\le |\lambda_0|\|\tilde{\phi}_n\|_{L^r(S_2)} + \sum_{|q|\ge(n/2)}|\lambda_{n-q}|\|\tilde{\phi}_q\|_{L^r(S_2)} + \sum_{0<|q|<(n/2)}|\lambda_{n-q}|\|\tilde{\phi}_q\|_{L^r(S_2)}\\ &\le \|\psi\|_{X_r}\|\phi\|_{X_r}\Big[\frac{C}{n^3} + C\Big(\sum_{k\in\mathbb{Z}^*}\frac{1}{k^4}\Big)\Big(\frac{2}{n}\Big)^3 + C\Big(\sum_{q\in\mathbb{Z}^*}\frac{1}{q^3}\Big)\Big(\frac{2}{n}\Big)^4\Big]\\ &\le \frac{C}{n^3}\|\phi\|_{X_r}\|\psi\|_{X_r}\end{aligned}$$

This implies (4.4). Due to equality (4.3), we finally get (4.2). □

Let $r \geq 1$. We introduce the operator:

$$\mathscr{T} : \mathbb{R} \times X_r \to Y_r$$

defined for any $\epsilon \in \mathbb{R}$ and $g \in X_r$ by $\mathscr{T}(\epsilon, g) = (d_n)_{n\in\mathbb{N}^*}$ with:

$$d_n = n^2\pi^2 g_n + \frac{\partial}{\partial u} \cdot \big(\mathcal{G} g_n\big) - \epsilon\kappa : u \otimes u g_n + \epsilon\big(B(g, g)\big)_n + \frac{\epsilon}{4\pi} \int_{S_2} \kappa : v \otimes v g_n(v) dv - \frac{3+\epsilon}{4\pi}\kappa : u \otimes u 1_n \tag{4.14}$$

In this writing, $g_n$ is given by (3.19). Coefficient $1_n$ is such that $1 = \sum_{n\in\mathbb{N}^*} 1_n H_n(s)$, with convergence in $L^2(0, 1)$, that is:

$$1_n = \sqrt{2} \int_{S_2} \sin(n\pi s) ds = \frac{\sqrt{2}}{n\pi}[1 - (-1)^n] \tag{4.15}$$

We can formulate problem (2.5)–(2.6) in term of operator $\mathscr{T}$:

**Lemma 4.2.** *Let $(\epsilon, f) \in \mathbb{R} \times X_r$ with $r \geq 1$. Function $f$ is a weak solution of (2.5)–(2.6) if and only if $\mathscr{T}(\epsilon, f) = 0$.*

**Proof.** Let $f \in X_r$ be a weak solution of (2.5)–(2.6). Taking in (2.10) $\phi(s, u) = \psi(u) \sin(n\pi s)$ with arbitrary $\psi \in H^2(S_2)$ and $n \in \mathbb{N}^*$, we obtain after integration by parts in $s$ that $\mathscr{T}(\epsilon, f) = 0$.

Conversely, let us consider $f \in X_r$ such that $\big(\mathscr{T}(\epsilon, f)\big)_n = 0$ for any $n \in \mathbb{N}^*$. This implies that for any $\phi \in H_0^1\big(0, 1, H^2(S_2)\big)$ and any $m \in \mathbb{N}^*$ we have:

$$\int_{S_2}\int_0^1 \Big[-\frac{\partial^2 f^{(m)}}{\partial s^2}\phi - f^{(m)}\mathcal{G} \cdot \frac{\partial \phi}{\partial u} - \epsilon f^{(m)}\kappa : u \otimes u\phi + \epsilon h^{(m)}\phi + \frac{\epsilon}{4\pi} \int_{S_2} \kappa : v \otimes v f^{(m)}(v, s) d\mu(v)\phi - \frac{3+\epsilon}{4\pi}\kappa : u \otimes u 1^{(m)}\phi\Big] ds d\mu(u) = 0 \tag{4.16}$$

In the above equation, exponent $^{(m)}$ indicates an $L^2$ projection on $span(H_1, \ldots, H_m)$, i.e.:

$$f^{(m)}(s) = \sum_{n=1}^{m} \Big[ \int_0^1 f(\sigma) H_n(\sigma) d\sigma \Big] H_n(s),$$

$$h^{(m)}(s) = \sum_{n=1}^{m} \Big[ \int_0^1 \frac{\partial}{\partial s}\big(f\kappa : \lambda(f)\big)(\sigma) H_n(\sigma) d\sigma \Big] H_n(s),$$

$$1^{(m)}(s) = \sum_{n=1}^{m} 1_n H_n(s).$$

We integrate by parts with respect to the $s$ variable the first term of (4.16). Using convergences $f^{(m)} \to f$ in $W^{1,\infty}\big(0, 1, L^r(S_2)\big)$ (see (3.1)) and $h^{(m)} \to \frac{\partial}{\partial s}\big(f\kappa : \lambda(f)\big)$ in $L^\infty\big(0, 1, L^r(S_2)\big)$ (see (4.2)), we obtain the result. □

We are in position to prove the existence and regularity parts in Theorem 2.1:

**Proof.** We consider $r \in ]1, r_0[$ with $r_0 > 1$ given by Lemma 3.3. It is clear from Lemma 4.1 that $\mathscr{T}$ is a $C^\infty$ function. Remark that, for any $g \in X_r$, $\mathscr{T}(0, g) = \mathscr{T}_0(g) - \alpha$ where $\alpha \in Y_r$ is given by $\alpha_n = \dfrac{3}{4\pi}\kappa : u \otimes u 1_n$. Now, Corollary 3.1 ensures that the hypothesis of the implicit function theorem are satisfied. It provides the existence part as well as the regularity part in Theorem 2.1. □

**Remark 4.1.** In the case $\epsilon = 0$ we have $f_n \in L^r(S_2)$ for $r \in [1, r_0[$. But we also know, from Remark 3.2, that there exists $N \in \mathbb{N}^*$ such that $f_n \in L^2(S_2)$ for $n \geq N$. Hence for $\alpha \in [0, \frac{5}{2}[$ we have:

$$\|\sum_{n=1}^{\infty} n^\alpha f_n(u)H_n(s)\|_{L^2(0,1;L^r(S_2))} \leq \sum_{n=1}^{N-1} n^\alpha \|f_n\|_{L^r(S_2)} + C\|\sum_{n=N}^{\infty} n^\alpha f_n(u)H_n(s)\|_{L^2(0,1;L^2(S_2))}$$
$$\leq C(N,\alpha) + C\left(\sum_{n=N}^{\infty} n^{2\alpha}\|f_n\|^2_{L^2(S_2)}\right)^{1/2} < +\infty$$

since $\|f_n\|_{L^2(S_2)} \leq \frac{c}{n^3}$ for $n \geq N$. It follows that $f \in H^{5/2-\delta}(0, 1; L^r(S_2))$ for any $\delta > 0$ arbitrary small.

In contrast, we are unable to obtain such smoothness for $\epsilon \neq 0$. It comes from the non-linear term $\epsilon\frac{\partial}{\partial s}(Fk : \lambda(F))$ which couples the "bad" low frequencies with high frequencies.

The main issue in the following is the non-negativity of $F$, the other properties could be obtained by rather simple means. For instance the uniqueness can be proved by Holmgren's principle, but we now argue differently.

## 5. Some results on the evolution problem

We prove in the sequel (Sections 5 and 6) that the solution $F = f + (4\pi)^{-1}$ obtained in Theorem 2.1 is the $L^1(Q)$ limit as $t \to +\infty$ of a family $\left((f^e + (4\pi)^{-1})(t)\right)_{t>0}$ of probability densities which is solution of the corresponding evolution problem. In the rest of this paper, we mostly restrict to exponent $r = 1$, in order to get uniqueness in the $L^1(S_2)$ frame. To begin with, consider the following evolution problem associated with equations (2.5)–(2.6):

Find $f^e(t, s, u)$ solution of (5.1), (5.2), (5.3):

$$\frac{\partial f^e}{\partial t} - \frac{\partial^2 f^e}{\partial s^2} + \frac{\partial}{\partial u}\cdot(\mathcal{G}f^e) - \epsilon f^e\kappa : u\otimes u + \epsilon\frac{\partial}{\partial s}\Big(f^e\kappa : \lambda(f)\Big)$$
$$+ \frac{\epsilon}{4\pi}\int_{S_2} \kappa : v\otimes v f^e(s,v)d\mu(v) = \frac{3+\epsilon}{4\pi}\kappa : u\otimes u \text{ on } Q_T \tag{5.1}$$

$$f^e(s=0) = f^e(s=1) = 0 \tag{5.2}$$
$$f^e(t=0) = f_0^e \tag{5.3}$$

with $Q_T = [0, T] \times Q$, $T > 0$. Function $f_0^e : Q \to \mathbb{R}$ is the initial data.

Existence and uniqueness results for problem (5.1), (5.2), (5.3) have been obtained in [5]:

**Theorem 5.1.** *Assume that* $f_0^e \in L^2(Q)$ *and* $\dfrac{\partial f_0^e}{\partial u} \in (L^2(Q))^3$. *Then, there exists a unique variational solution* $f^e \in L^2\big(0, T, H_0^1(Q)\big)$ *with* $\dfrac{\partial f^e}{\partial t} \in L^2\big(0, T, H^{-1}(Q)\big)$ *in the following sense:*

$$-\int_{Q_T} f^e\frac{\partial\phi}{\partial t}dQ_T - \int_Q f_0^e\phi(t=0)dQ + \int_{Q_T}\Big[\frac{\partial f^e}{\partial t}\frac{\partial\phi}{\partial s} + \frac{\partial}{\partial u}\cdot(\mathcal{G}f^e)\phi$$
$$-\epsilon\kappa : u\otimes u f^e\phi + \frac{\epsilon}{4\pi}\int_{S_2}\kappa : v\otimes v f^e dv\phi - \epsilon f^e\kappa : \lambda(f^e)\frac{\partial\phi}{\partial s}\Big]dQ_T = \frac{3+\epsilon}{4\pi}\int_{Q_T}\kappa : u\otimes u\phi\, dQ_T \tag{5.4}$$

*for any* $\phi \in H^1\big(0, T : H_0^1(Q)\big)$ *with* $\phi(t = T) = 0$*. Moreover, if:*

$$f_0^e + \frac{1}{4\pi} \geq 0 \text{ a.e. } (s,u) \in Q \text{ and } \int\limits_{S_2} \Big(f_0^e + \frac{1}{4\pi}\Big) d\mu = 1 \text{ a.e. } s \in ]0,1[$$

*then:*

$$f^e + \frac{1}{4\pi} \geq 0 \text{ a.e. } (t,s,u) \in Q_T \text{ and } \int\limits_{S_2} \Big(f^e + \frac{1}{4\pi}\Big) d\mu = 1 \text{ a.e. } (t,s) \in ]0,T[ \times ]0,1[$$

From now on, we assume that $f_0^\epsilon \in H_0^1(Q)$ with $f_0^e + \dfrac{1}{4\pi} \geq 0$ and $\int_{S_2} \Big(f_0^e + \dfrac{1}{4\pi}\Big) d\mu = 1$. From Theorem 5.1, this implies the very useful estimate (uniform in $t \in \mathbb{R}_+$):

$$\int\limits_{S_2} |f^e| d\mu \leq 2 \text{ a.e. } (t,s) \in \mathbb{R}_+ \times ]0,1[ \tag{5.5}$$

As a consequence, we deduce from (5.5) that:

$$\|\lambda(f^e)\|_{L^\infty(0,T:W^{1,\infty}(0,1))} \leq 2 \tag{5.6}$$

with $C \geq 0$ independent of $T \geq 0$. We also have:

$$\sum_{n=1}^{\infty} n^2 \| f_{0,n}^e \|_{L^1(S_2)}^2 \leq C \sum_{n=1}^{\infty} n^2 \| f_{0,n}^e \|_{L^2(S_2)}^2 < \infty \tag{5.7}$$

where

$$f_{0,n}^e = \int\limits_0^1 f_0^e(s) H_n(s) ds \tag{5.8}$$

for any $n \in \mathbb{N}^*$. For $n \in \mathbb{N}^*$, let us denote $f_n^e = \int_0^1 f^e(s) H_n(s) ds \in L^2\big(0, T, H^1(S_2)\big)$. In (5.4), taking $\phi(t,s,u) = \psi(t,u) H_n(s)$ with $\psi \in H^1\big(0, T, H^1(S_2)\big)$ and $\psi(t = T) = 0$ as a test function, we easily obtain, for any $n \in \mathbb{N}^*$:

$$\frac{\partial f_n^e}{\partial t} + n^2 \pi^2 f_n^e + \frac{\partial}{\partial u} \cdot \big(\mathcal{G} f_n^e\big) - \epsilon \kappa u \otimes u f_n^e + \frac{\epsilon}{4\pi} \int\limits_{S_2} \kappa : v \otimes v f_n^e(v) dv$$
$$- \sqrt{2} \epsilon n \pi \int\limits_0^1 \big(f^e \kappa : \lambda(f^e)\big)(s) \cos(n\pi s) ds = \frac{3+\epsilon}{4\pi} \kappa : u \otimes u 1_n \tag{5.9}$$

All the terms appearing in the above equality belongs to $L^2\big(0, T, L^2(S_2)\big)$.

For the initial data, we have:

$$f_n^e(t = 0) = f_{0,n}^e \tag{5.10}$$

where $f_{0,n}^e$ is given by (5.8). For future reference note that:

$$f^e(t,s,u) = \sum_{n=1}^{\infty} f_n^e(t,u) H_n(s) \tag{5.11}$$

with convergence in $L^2\Big(0, 1, L^2\big(0, T, H^1(S_2)\big)\Big)$. Last:

$$\frac{\partial f_n^e}{\partial s}(s) = \sum_{n=1}^{\infty} n\pi \sqrt{2} f_n^e(s) \cos(n\pi s) \tag{5.12}$$

with convergence in $L^2(Q_T)$.

It is well known that for the heat equation, estimates of two derivatives with respect to the space variables can be obtained in suitable spaces. From that point of view, estimates with respect to the $s$ derivatives in Theorem 5.1 do not seem to be optimal. The following simple estimate will be enough for our purposes:

**Lemma 5.1.** *With the notations, and under the hypothesis of Theorem 5.1, there exist* $\epsilon_0 > 0$ *such that:*

$$\sum_{n=1}^{\infty} \int_0^T n^4 \|f_n^e(t)\|^2_{L^1(S_2)} dt \le C(T) < \infty \tag{5.13}$$

*for any* $T \ge 0$ *and* $\epsilon \in ]0, \epsilon_0[$

**Proof.** Let us denote $g^e = \frac{\partial}{\partial s}\Big[f^e \kappa : \lambda(f^e)\Big]$. Function $g^e$ is an element of $L^2(Q_T)$ due to (5.6). For $n \in \mathbb{N}^*$, we write as usual $g_n^e = \int_0^1 g^e(s) H_n(s) ds$. Function $g_n^e$ belongs to $L^2(]0, T[ \times S_2)$, and by the Hölder and Bessel inequalities:

$$\sum_{n=1}^{\infty} \|g_n^e\|^2_{L^2(0,T;L^1(S_2))} \le C \sum_{n=1}^{\infty} \|g_n^e\|^2_{L^2(]0,T[\times S_2)} \le C\|g^e\|^2_{L^2(Q_T)} \tag{5.14}$$

We multiply (5.9) by $\operatorname{sgn}(f_n^e)$ and integrate on $S_2$. It gives, for $n \in \mathbb{N}^*$ and $\epsilon$ small enough:

$$\frac{d}{dt}\|f_n^e\|_{L^1(S_2)} + n^2\pi^2 \|f_n^e\|_{L^1(S_2)} \le C\epsilon \|f_n^e\|_{L^1(S_2)} + \|g_n^e\|_{L^1(S_2)} + C|1_n| \tag{5.15}$$

In the above inequality, we have used Lemma 3.2 with $r = 1$ and identity $\int_{S_2} \frac{\partial}{\partial u} \cdot \big(\mathcal{G}|f_n^e|\big) d\mu = 0$. Remark that $\|f_n^e\|_{L^1(S_2)}$ belongs to $H^1(0, T)$. Now, we fix $m \in \mathbb{N}^*$, multiply (5.15) by $n^2\|f_n^e\|_{L^1(S_2)}$ and take the sum from $n = 1$ to $m$. Using the fact that:

$$\Big(\|g_n^e\|_{L^1(S_2)} + C|1_n|\Big) n^2 \|f_n^e\|_{L^1(S_2)} \le \frac{\pi^2 n^4}{4}\|f_n^e\|^2_{L^1(S_2)} + 2\|g_n^e\|^2_{L^1(S_2)} + 2C^2|1_n|^2 \tag{5.16}$$

we deduce from (5.15) for $\epsilon$ small enough and using also (4.15) that:

$$\frac{d}{dt}\Big(\sum_{n=1}^{m} n^2 \|f_n^e\|^2_{L^1(S_2)}\Big) + \pi^2 \sum_{n=1}^{m} n^4 \|f_n^e\|^2_{L^1(S_2)} \le C + 4\sum_{n=1}^{\infty} \|g_n^e(t)\|^2_{L^1(S_2)} \tag{5.17}$$

with $C > 0$ independent of $m \in \mathbb{N}^*$. Integrating this inequality with respect to $t$ and appealing to (5.7) and (5.14), we obtain the result. □

## 6. Proof of uniqueness. Function $F$ is a probability density

We denote by $f^d = f^e - f$. As before, function $f$ is a stationary solution constructed in Section 4 and $f^e$ is the solution of the evolutionary Doi–Edwards equation (see Section 5). We now prove that $f^d(t) \to 0$ in a suitable norm when $t \to +\infty$. This will provide at the same time uniqueness of $f$ and the fact that $f + (4\pi)^{-1}$ is a probability density. Notice that long time behaviour of some systems arising in the theory of polymeric fluids (Hookean model and FENE model) are studied for instance in [14].

Since

$$f^e \in H^1\Big(0, 1, L^2\big(]0, T[ \times S_2\big)\Big) \tag{6.1}$$

and

$$f \in W^{1,\infty}\big(0, 1, L^r(S_2)\big) \text{ with } r \in [1, r_0] \tag{6.2}$$

we have:

$$f^d \in H^1\Big(0, 1, L^2\big(0, T, L^r(S_2)\big)\Big) \tag{6.3}$$

We also have:

$$f^d(s) = \sum_{n=1}^{\infty} f_n^d H_n(s) \tag{6.4}$$

with convergence in $H^1\Big(0, 1, L^2\big(0, T, L^r(S_2)\big)\Big)$, with $f_n^d = f_n^e - f_n$ (see (5.11), (5.12), and Lemma 3.1). Remark that from Lemma 5.1 and the fact that $f \in X_1$ we have:

$$\sum_{n=1}^{\infty} \int_0^T n^4 \| f_n^d(t)\|^2_{L^1(S_2)} dt \le C(T) < \infty.$$

Now, from (5.9) and equality $\mathscr{T}(\epsilon, f) = 0$, we obtain:

$$\frac{\partial f_n^d}{\partial t} + n^2\pi^2 f_n^d + \frac{\partial}{\partial u} \cdot \big(\mathcal{G} f_n^d\big) - \epsilon\kappa u \otimes u f_n^d + \frac{\epsilon}{4\pi} \int_{S_2} \kappa : v \otimes v f_n^d(v) dv$$
$$+ \epsilon \int_0^1 \frac{\partial}{\partial s}\big[\kappa : \lambda(f^d) f\big](s) H_n(s) ds + \epsilon \int_0^1 \frac{\partial}{\partial s}\big[\kappa : \lambda(f^e) f^d\big](s) H_n(s) ds = 0 \tag{6.5}$$

for any $n \in \mathbb{N}^*$. Remark that:

$$I_{1n} := \int_0^1 \frac{\partial}{\partial s}\big[\kappa : \lambda(f^e) f^d\big] H_n(s)\, ds \in L^2\big(]0, T[, L^r(S_2)\big) \tag{6.6}$$

and:

$$I_{2n} := \int_0^1 \frac{\partial}{\partial s}\big[\kappa : \lambda(f^d) f\big] H_n(s)\, ds \in L^\infty\big(]0, T[, L^r(S_2)\big) \tag{6.7}$$

due to (6.1), (6.2), (6.3) and (5.6). Notice also that $\frac{\partial}{\partial t}\| f_n^d\|_{L^1(S_2)} \in L^2(0, T)$. We multiply (6.5) by $\operatorname{sgn}(f_n^d)$ and integrate on $S_2$. Using Lemma 3.2 with $r = 1$, we obtain:

$$\frac{\partial}{\partial t}\| f_n^d\|_{L^1(S_2)} + n^2\pi^2 \| f_n^d\|_{L^1(S_2)} \le 2\epsilon \int_{S_2} |\kappa : u \otimes u f_n^d| d\mu + \epsilon \| I_{1n}\|_{L^1(S_2)} + \epsilon \| I_{2n}\|_{L^1(S_2)} \tag{6.8}$$

The goal is now to multiply (6.8) by $n^2\| f_n^d\|_{L^1(S_2)}$ and take the sum from $n = 1$ to $\infty$. We will need some preliminary lemmas.

**Lemma 6.1.**

$$\sum_{n=1}^{\infty} n^2 \| f_n^d\|_{L^1(S_2)} \| I_{1n}\|_{L^1(S_2)} \le C \sum_{n=1}^{\infty} n^4 \| f_n^d\|^2_{L^1(S_2)} \tag{6.9}$$

*with $C > 0$ independent of $t$.*

**Proof.** We have:

$$I_{1n}(t, u) = \sum_{q=1}^{\infty} a_{qn}(t) f_q^d(t, u) \tag{6.10}$$

where $a_{qn}(t) = \int_0^1 \frac{\partial}{\partial s}\big[\kappa : \lambda(f^e)H_q(s)\big]H_n(s)ds$. The right-hand side of (6.10) is convergent in $L^2\big(0, T, L^1(S_2)\big)$ due to the convergence in $H^1\Big(0, 1, L^2\big(0, T, L^1(S_2)\big)\Big)$ of $\sum_{q=1}^{\infty} f_q^d H_q$. From (6.10), we deduce:

$$\|I_{1n}(t)\|_{L^1(S_2)} \le \sum_{q=1}^{\infty} |a_{qn}(t)|\|f_q^d(t)\|_{L^1(S_2)} \tag{6.11}$$

Now, we observe that, due to (5.6):

$$\sum_{n=1}^{\infty} |a_{qn}|^2 = \|\frac{\partial}{\partial s}\Big(\kappa : \lambda(f^e)H_q\Big)\|^2_{L^2(0,1)} \le Cq^2 \tag{6.12}$$

with $C > 0$ independent of $t$. We deduce from (6.11) that

$$\begin{aligned}\sum_{n=1}^{\infty} n^2\|f_n^d(t)\|_{L^1(S_2)}\|I_{1n}\|_{L^1(S_2)} &\le \sum_{n=1}^{\infty}\sum_{q=1}^{\infty} n^2|a_{qn}|\|f_n^d\|_{L^1(S_2)}\|f_q^d\|_{L^1(S_2)} \\ &\le \sum_{n=1}^{\infty}\sum_{q=1}^{\infty} \frac{|a_{qn}|}{q^2}n^2\|f_n^d\|_{L^1(S_2)}q^2\|f_q^d\|_{L^1(S_2)}\end{aligned} \tag{6.13}$$

Using Cauchy– Schwarz inequality, we obtain:

$$\sum_{n=1}^{\infty} n^2\|f_n^d(t)\|_{L^1(S_2)}\|I_{1n}\|_{L^1(S_2)} \le \Big(\sum_{n=1}^{\infty}\sum_{q=1}^{\infty} \frac{|a_{qn}|^2}{q^4}\Big)^{1/2}\Big(\sum_{n=1}^{\infty} n^4\|f_n^d\|^2_{L^1(S_2)}\Big) \tag{6.14}$$

With (6.12), this gives the result. □

Before giving bounds on $\|I_{2n}\|_{L^1(S_2)}$, we establish the following:

**Lemma 6.2.** *There exists $C > 0$ such that for any $N \in \mathbb{N}^*$, we have:*

$$\|\int_0^1 f(s)\cos(N\pi s)ds\|_{L^1(S_2)} \le \frac{C}{N^2}\|f\|_{X_1} \tag{6.15}$$

**Proof.** We have:

$$\begin{aligned}\int_0^1 f(s)\cos(N\pi s)ds &= \sqrt{2}\sum_{p=1}^{\infty} f_p \int_0^1 \sin(p\pi s)\cos(N\pi s)ds \\ &= \sqrt{2}\sum_{p\ne N} \frac{p}{p^2 - N^2}\big[1 + (-1)^{p+N+1}\big]f_p\end{aligned} \tag{6.16}$$

Now, from definition of $\|f\|_{X_r}$ we have:

$$\|\int_0^1 f(s)\cos(N\pi s)ds\|_{L^1(S_2)} \le 2\sqrt{2}\|f\|_{X_1}\sum_{p\ne N} \frac{1}{p^2|p^2 - N^2|} \tag{6.17}$$

Next, remark that:

$$\begin{aligned}&\sum_{p\ne N} \frac{1}{p^2|p^2 - N^2|} \\ &= \sum_{1\le|p-N|<N/2} \frac{1}{p^2}\frac{1}{|p-N||p+N|} + \sum_{1\le p\le(N/2)} \frac{1}{p^2|p^2-N^2|} + \sum_{p\ge(3N/2)} \frac{1}{p^2|p^2-N^2|}\end{aligned}$$

$$\le \Big(\frac{2}{N}\Big)^2\Big(\sum_{N/2<p<3N/2}\frac{1}{N}\Big)+\frac{4}{3N^2}\Big(\sum_{p=1}^{\infty}\frac{1}{p^2}\Big)+\frac{4}{5N^2}\Big(\sum_{p=1}^{\infty}\frac{1}{p^2}\Big)$$
$$\le \frac{C}{N^2}$$

Together with (6.16), (6.17), this ends the proof. □

As a consequence, we have the following estimate on the nonlinear term:

**Lemma 6.3.** *There exists $C>0$ such that for any $q\in\mathbb{N}^*$, $n\in\mathbb{N}^*$ we have:*

$$\Big\|\int_0^1 \frac{\partial}{\partial s}\Big[\int_0^s H_q(\tau)d\tau f(s,u)\Big]H_n(s)ds\Big\|_{L^1(S_2)} \le C\frac{q}{n}\|f\|_{X_1} \tag{6.18}$$

**Proof.** Since $\int_0^s H_q(\tau)d\tau = \frac{\sqrt{2}}{q\pi}\big[1-\cos(q\pi s)\big]$, we obtain, integrating by parts:

$$\int_0^1 \frac{\partial}{\partial s}\Big[\int_0^s H_q(\tau)d\tau f(s,u)\Big]H_n(s)ds = \frac{n}{q}(E_1+E_2+E_3) \tag{6.19}$$

where:

- $E_1 = -2\int_0^1 f(s)\cos(n\pi s)ds$
- $E_2 = \int_0^1 f(s)\cos\big((n+q)\pi s\big)ds$
- $E_3 = \int_0^1 f(s)\cos\big((n-q)\pi s\big)ds$

Using Lemma 6.2, we have:

$$\|E_j\|_{L^1(S_2)} \le \frac{C}{n^2}\|f\|_{X_1}, \ \text{for } j=1,2 \tag{6.20}$$

and also:

$$\bullet\ \|E_3\|_{L^1(S_2)} \le C\|f\|_{X_1}, \ \text{for } q=n \tag{6.21}$$
$$\bullet\ \|E_3\|_{L^1(S_2)} \le \frac{C}{|n-q|^2}\|f\|_{X_1}, \ \text{for } q\ne n$$

Now, notice that:

$$\frac{1}{|n-q|} = \frac{1}{n}\Big|1+\frac{q}{n-q}\Big| \le 2\frac{q}{n} \quad \text{for} \quad q\ne n \tag{6.22}$$

Hence:

$$\|E_3\|_{L^1(S_2)} \le C\frac{q^2}{n^2}\|f\|_{X_1} \quad \forall q,n\in\mathbb{N}^*. \tag{6.23}$$

From (6.19), (6.20), (6.23), we get the result. □

We deduce from Lemma 6.3 the required estimate on $\|I_{2n}\|_{L^r(S_2)}$:

**Lemma 6.4.**

$$\sum_{n=1}^{\infty} n^2\|f_n^d\|_{L^1(S_2)}\|I_{2n}\|_{L^1(S_2)} \le C\sum_{n=1}^{\infty} n^4\|f_n^d\|^2_{L^1(S_2)}$$

*with $C>0$ independent of $t$.*

**Proof.** We can write

$$I_{2n} = \sum_{q=1}^{\infty} \kappa : \tilde{\lambda}(f_q^d) \int_0^1 \frac{\partial}{\partial s}\Big[\int_0^s H_q(\tau)d\tau f(s)\Big] H_n(s)ds$$

where we denote

$$\tilde{\lambda}(f_q^d) = \int_{S_2} f_q^d(t,v) v \otimes v\, dv.$$

Using Lemma 6.3, we deduce:

$$\|I_{2n}\|_{L^1(S_2)} \leq \sum_{q=1}^{\infty} \|f_q^d\|_{L^1(S_2)} \frac{q}{n} \|f\|_{X_1}$$

which gives:

$$\begin{aligned} \sum_{n=1}^{\infty} n^2 \|f_n^d\|_{L^1(S_2)} \|I_{2n}\|_{L^1(S_2)} &\leq \sum_{q=1}^{\infty}\sum_{n=1}^{\infty} nq \|f_n^d\|_{L^1(S_2)} \|f_q^d\|_{L^1(S_2)} \|f\|_{X_1} \\ &= \Big[\sum_{n=1}^{\infty} n\|f_n^d\|_{L^1(S_2)}\Big]^2 \|f\|_{X_1} \end{aligned} \tag{6.24}$$

From Cauchy–Schwarz inequality, we get:

$$\sum_{n=1}^{\infty} n\|f_n^d\|_{L^1(S_2)} \leq \Big[\sum_{n=1}^{\infty} \frac{1}{n^2}\Big]^{1/2} \Big[\sum_{n=1}^{\infty} n^4 \|f_n^d\|^2_{L^1(S_2)}\Big]^{1/2} \tag{6.25}$$

Inequalities (6.24) and (6.25) provides the result. □

We finally give the proof of the last part of Theorem 2.1.

**Proof.** We multiply (6.8) by $n^2\|f_n^d\|_{L^1(S_2)}$ and take the sum from $n = 1$ to $m \in \mathbb{N}^*$. For $|\epsilon|$ small enough, and using Lemma 6.1 and Lemma 6.4, we obtain:

$$\frac{d}{dt}\xi_m(\tau) + \chi_m(\tau) \leq C|\epsilon|\chi(\tau) \tag{6.26}$$

where we denote

$$\xi_m(\tau) = \sum_{n=1}^{m} n^2 \|f_n^d(\tau)\|^2_{L^1(S_2)}$$

$$\chi_m(\tau) = \sum_{n=1}^{m} n^4 \|f_n^d(\tau)\|^2_{L^1(S_2)}$$

$$\xi(\tau) = \sum_{n=1}^{\infty} n^2 \|f_n^d(\tau)\|^2_{L^1(S_2)}$$

$$\chi(\tau) = \sum_{n=1}^{\infty} n^4 \|f_n^d(\tau)\|^2_{L^1(S_2)}$$

Let $t \in [0, T]$. We multiply (6.26) by $e^{\tau/2}$, integrate from $\tau = 0$ to $\tau = t$ and we get:

$$\xi_m(t)e^{t/2} - \frac{1}{2}\int_0^t \xi_m(\tau)e^{\tau/2}d\tau + \int_0^t \chi_m(\tau)e^{\tau/2}d\tau \leq C|\epsilon| \int_0^t \chi(\tau)e^{\tau/2}d\tau + \xi_m(0). \tag{6.27}$$

Now we pass to the limit $m \to +\infty$ in (6.27), which gives for $|\epsilon|$ small enough

$$\xi(t)e^{t/2} - \frac{1}{2}\int_0^t \xi(\tau)e^{\tau/2}d\tau + \frac{1}{2}\int_0^t \chi(\tau)e^{\tau/2}d\tau \le C.$$

Since $\chi(t) \ge \xi(t)$ we deduce $\xi(t) \le Ce^{-t/2}$ which implies that $\xi(t) \to 0$ when $t \to +\infty$. Since by the Cauchy–Schwarz inequality and the definition of function $\xi$ we have:

$$\sum_{n=1}^{\infty} \|f_n^d(t)\|_{L^1(S_2)} \le C\Big[\sum_{n=1}^{\infty}\frac{1}{n^2}\Big]^{1/2}\Big[\sum_{n=1}^{\infty} n^2\|f_n^d(t)\|^2_{L^1(S_2)}\Big]^{1/2} \le C\sqrt{\xi(t)} \tag{6.28}$$

we finally get (see (6.4)):

$$\|f^d(t)\|_{L^\infty(0,1,L^1(S_2))} \to 0 \text{ when } t \to +\infty \tag{6.29}$$

As a consequence,

$$\Big\|\int_{S_2} f^d(t)d\mu\Big\|_{L^\infty(0,1)} \to 0 \text{ when } t \to +\infty$$

Recall that $\int_{S_2} f^e(t)d\mu = 1$. Hence $\int_{S_2} f d\mu = 1$ for almost every $s \in ]0,1[$, and in fact for every $s \in [0,1]$ due to (6.2) and Sobolev embeddings. The uniqueness follows also from (6.29) since for another solution $g$ of (2.5)–(2.6), we have that $\|f-g\|_{L^\infty(0,1,L^1(S_2))} \to 0$ when $t \to +\infty$, hence $f = g$.

It remains to prove the non-negativity of $f + \frac{1}{4\pi}$. To do that, let us consider an arbitrary function $\varphi \in C(S_2)$ with $\varphi \ge 0$ on $S_2$. We have

$$\int_{S_2}\Big(f + \frac{1}{4\pi}\Big)\varphi\, d\mu = \int_{S_2}\Big(f^e + \frac{1}{4\pi}\Big)\varphi\, d\mu - \int_{S_2} f^d\varphi\, d\mu \tag{6.30}$$

Since $f^e + \frac{1}{4\pi} \ge 0$ we obtain with the help of (6.29) that $\int_{S_2}\Big(f + \frac{1}{4\pi}\Big)\varphi\, d\mu \ge 0$ for almost every $s \in ]0,1[$, and in fact for every $s \in [0,1]$. This completes the proof. □

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## References


[1] R. Abraham, J.E. Marsden, Foundation of Mechanics, second edition, Addison–Wesley Publishing Company, Inc., 1987.
[2] L. Chupin, Fokker–Planck equation in bounded domain, Ann. Inst. Fourier 60 (1) (2010) 217–255.
[3] L. Chupin, The FENE model for viscoelastic thin film flows, Methods Appl. Anal. 16 (2) (2009) 217–261.
[4] I.S. Ciuperca, L.I. Palade, The steady state configurational distribution diffusion equation of the standard FENE dumbbell polymer model: existence and uniqueness of solutions for arbitrary velocity gradients, Math. Models Methods Appl. Sci. 19 (2009) 2039–2064.
[5] I.S. Ciuperca, A. Heibig, L.I. Palade, Existence and uniqueness results for the Doi–Edwards polymer melt model: the case of the (full) nonlinear configurational probability density equation, Nonlinearity 25 (4) (2012) 991–1009.
[6] P. Constantin, N. Masmoudi, Global well-posedness for a Smoluchowski equation coupled with Navier–Stokes equations in 2D, Commun. Math. Phys. 278 (2008) 179–191.
[7] M. Doi, S.F. Edwards, Dynamics of concentrated polymer systems, Part 1. Brownian motion in the equilibrium state, J. Chem. Soc. Faraday Trans. II 74 (1978) 789–1801, http://dx.doi.org/10.1039/F29787401789.
[8] M. Doi, S.F. Edwards, Dynamics of concentrated polymer systems, Part 3. The constitutive equation, J. Chem. Soc. Faraday Trans. II 74 (1978) 1818–1832.
[9] M. Doi, S.F. Edwards, The Theory of Polymer Dynamics, Oxford University Press, Oxford, 1989.
[10] P.G. de Gennes, Scaling Concepts in Polymer Physics, Cornell University Press, Ithaca, London, 1979.

[11] R.J. DiPerna, P.L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (1989) 511–547.
[12] A. Henrot, M. Pierre, Variation et optimisation de forme, une analyse géométrique, Springer, Berlin, 2005.
[13] B. Jourdain, T. Lelièvre, C. Le Bris, Existence of solution for a micro–macro model of polymeric fluid: the FENE model, J. Funct. Anal. 209 (2004) 162–193.
[14] B. Jourdain, T. Lelièvre, C. Le Bris, F. Otto, Long time asymptotics of a multiscale model for polymeric fluid flows, Arch. Ration. Mech. Anal. 181 (2006) 97–148, http://dx.doi.org/10.1007/s00205-005-0411-4.
[15] N. Masmoudi, Well-posedness for the FENE dumbbell model of polymeric flows, Commun. Pure Appl. Math. 61 (12) (2008) 1685–1714.
[16] F. Otto, A. Tzavaras, Continuity of velocity gradients in suspensions of rod-like molecules, Commun. Math. Phys. 277 (2008) 729–758.
[17] J.F. Palierne, Rheothermodynamics of the Doi–Edwards reptation model, Phys. Rev. Lett. 93 (2004) 136001-1–136001-4.