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A constructive approach to positive solutions of $\Delta_p u + f(u, \nabla u) \le 0$ on Riemannian manifolds [☆]

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Abstract

Grigor'yan–Sun in [6] (with $p = 2$) and Sun in [10] (with $p > 1$) proved that if

$$\sup_{r\gg 1} \mathrm{vol}(B(x_0, r)) r^{\frac{p\sigma}{p-\sigma-1}} (\ln r)^{\frac{p-1}{p-\sigma-1}} < \infty$$

then the only non-negative weak solution of $\Delta_p u + u^\sigma \le 0$ on a complete Riemannian manifold is identically 0; moreover, the powers of r and $\ln r$ are sharp. In this note, we present a constructive approach to the sharpness, which is flexible enough to treat the sharpness for $\Delta_p u + f(u, \nabla u) \le 0$. Our construction is based on a perturbation of the fundamental solution to the p-Laplace equation, and we believe that the ideas introduced here are applicable to other nonlinear differential inequalities on manifolds.

MSC: 35J70; 58J05

Keywords: Non-negative solution; Volume growth consideration; Complete Riemannian manifold

1. Introduction

This article stems from an essential understanding of Grigor'yan–Sun's work [6] (with $p = 2$) and Sun's follow-up [10] (with $p > 1$) on how the volume condition

$$\sup_{r\gg 1} \mu(B(x_0, r)) r^{-Q} (\ln r)^{-q} < \infty \tag{1}$$

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(on a complete Riemannian manifold (M, g) with volume element $d\mu$ and x_0-centered geodesic ball $B(x_0, r)$ of radius $r > 0$) ensures weakly

$$\Delta_p u + u^\sigma \le 0, \tag{2}$$

i.e.,

$$-\int_M |\nabla u|^{p-2}(\nabla u, \nabla\psi)\, d\mu + \int_M u^\sigma \psi\, d\mu \le 0 \quad \forall \quad 0 \le \psi \in W_c^{1,p}(M),$$

where $(\cdot,\cdot)$ is the inner product in the tangent bundle T_xM given by the Riemannian metric g, and $W_c^{1,p}(M)$ is the p-Sobolev space of all $W_{loc}^{1,p}(M) = \{f \in L_{loc}^p(M) : |\nabla f| \in L_{loc}^p(M)\}$ functions with compact support and

$$\sigma > p - 1 > 0 \quad \& \quad \Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

1.1. *Grigor'yan–Sun's result for $\Delta u + u^\sigma \le 0$*

A great deal of attention (see e.g. [4,5,2,3] and their references) has been attracted over the last two decades to the non-negative weak solutions of (2) under $p = 2$. In particular, Grigor'yan–Sun established the following result.

Theorem 1.1. *(See [6].) Assume that the inequality (1) with*

$$Q = \frac{2\sigma}{\sigma - 1} \quad \& \quad q = \frac{1}{\sigma - 1} \tag{3}$$

is valid. Then any non-negative weak solution of (2) with $p = 2$ is identically equal to 0. Furthermore, the parameter pair (3) is sharp in the sense that if

$$\text{either} \quad Q > \frac{2\sigma}{\sigma - 1} \quad \text{or} \quad Q = \frac{2\sigma}{\sigma - 1} \quad \& \quad q > \frac{1}{\sigma - 1}$$

then there exists a manifold obeying (1) where (2) with $p = 2$ has a positive solution.

1.2. *Sun's work on $\Delta_p u + u^\sigma \le 0$*

In addition to extending Theorem 1.1 to a semi-linear case [9], Sun in [10] extended Theorem 1.1 from the special case $p = 2$ to the general case $p > 1$ (whose setting on the Euclidean space is also of independent interest; see e.g. [1,7,8]).

Theorem 1.2. *(See [10].) Assume that the inequality (1) with*

$$Q = \frac{p\sigma}{\sigma - p + 1} \quad \& \quad q = \frac{p - 1}{\sigma - p + 1} \tag{4}$$

is valid. Then any non-negative weak solution of (2) is identically equal to 0. Furthermore, the parameter pair (4) is sharp in the sense that if

$$\text{either} \quad Q > \frac{p\sigma}{\sigma - p + 1} \quad \text{or} \quad Q = \frac{p\sigma}{\sigma - p + 1} \quad \& \quad q > \frac{p - 1}{\sigma - p + 1}$$

then there exists a manifold obeying (1) where (2) has a positive solution.

Quite remarkably, the sharpness arguments provided in [6,9,10] are based on an existence theory of some ODEs with polynomial non-linearity u^σ and the eigenvalue theory of Laplace equations. But, there is no evidence showing that their arguments can be used to find out an explicit positive solution even locally, such as near the origin or infinity. In fact, there were somewhat unexpected difficulties in utilizing their methods to handle other relevant inequalities; see for example [11].

1.3. Our approach to $\Delta_p + f(u, \nabla u) \le 0$

In contrast to the foregoing approach, in Section 2 we will construct some explicit solutions of (2), which not only give the sharpness of Theorems 1.1–1.2 automatically, but also are applicable to the following inequality

$$\Delta_p u + f(u, \nabla u) \le 0, \tag{5}$$

equivalently,

$$-\int_M |\nabla u|^{p-2}(\nabla u, \nabla\psi)\, d\mu + \int_M f(u, \nabla u)\psi\, d\mu \le 0 \quad \forall \quad 0 \le \psi \in W_c^{1,p}(M),$$

with more general non-linearity (cf. [12]) including

$$f(u, \nabla u) = \begin{cases} u^{\sigma_1}|\nabla u|^{\sigma_2}; \\ e^u - 1; \\ \ln(1+u); \end{cases}$$

see Theorems 3.1–3.2–3.3 in Section 3. Here, it is perhaps appropriate to mention that with general non-linearity $f(u, \nabla u)$ the sharpness argument in [6,10] seems unsuitable to be conducted.

Notation. In what follows, $\mathsf{U} \sim \mathsf{V}$ means that there is a constant $c > 0$ such that $c^{-1}\mathsf{V} \le \mathsf{U} \le c\mathsf{V}$.

2. Construction

In this section, the manifold M is taken to be $(\mathbb{R}^n, g)$ with the Riemannian metric:

$$g = dr^2 + \psi^2(r)d\theta^2,$$

where (r, θ) are the polar coordinates in $\mathbb{R}^n$, and $r \mapsto \psi(r)$ is a smooth, positive, increasing function on the half line $(0, \infty)$:

$$\psi(r) = \begin{cases} r & \text{for small enough } r; \\ \left(r^{Q-1}\ln^q r\right)^{\frac{1}{n-1}} & \text{for large enough } r. \end{cases} \tag{6}$$

In view of the desired construction, the geodesic ball $B_r = B(0, r)$ on M coincides with the Euclidean ball $\{x : |x| \le r\}$. Letting $S(r) = |\partial B_r|$ be the surface area of B_r in M and ω_n be the surface area of the unit ball in $\mathbb{R}^n$, we use (6) to achieve

$$S(r) = \omega_n \psi^{n-1}(r).$$

The Riemannian volume of the ball B_r is

$$\mu(B_r) = \int_0^r S(\tau)\, d\tau.$$

Since the solution to be constructed is radial, we can write $u = u(r)$, thereby discovering that under the radial assumption, the inequality (5) reduces to

$$\left(S|u'|^{p-2}u'\right)' + Sf(u, u') \le 0, \tag{7}$$

where $S = S(\cdot)$ is the surface area of B_r. Note that

$$S(r) = \omega_n r^{Q-1}\ln^q(r) \quad \forall \ r \gg 1 \tag{8}$$

with

$$Q \ge \frac{\sigma p}{\sigma - p + 1} \quad \& \quad q \ge \frac{p-1}{\sigma - p + 1}.$$

So, it is easy to see that the fundamental solution to

$$\left(S|u'|^{p-2}u'\right)'=0$$

is

$$W(r)=\int_r^\infty (S(\tau))^{-\frac{1}{p-1}}\,d\tau \quad \forall\ r>0. \tag{9}$$

The main observation here is that the nonlinear term $S(r)W^\sigma(r)$ processes a stronger decay once σ, Q or q is suitably large. Thus it is hopeful to construct a positive solution to (7) when $r\gg 1$ by a perturbation of the fundamental solution (9).

A positive solution of (7) with $f(u,u')=u^\sigma$ will be constructed according to the following three steps.

2.1. Step 1: solution near infinity

In what follows, we construct a positive solution to (7) with $f(u,u')=u^\sigma$ near ∞ provided that

$$\text{either}\quad Q>\frac{\sigma p}{\sigma-p+1}\quad \text{or}\quad Q=\frac{\sigma p}{\sigma-p+1}\quad \&\quad q>\frac{p-1}{\sigma-p+1}$$

occurs. To this end, we perturb (9) (as the fundamental solution of the p-Laplace equation) in the following way. Let

$$Y_{\epsilon_1,\epsilon_2}(r)=\int_r^\infty \left(\frac{r^{-\epsilon_1}S(\tau)}{1+\ln^{\epsilon_2}(\tau)}\right)^{-\frac{1}{p-1}}d\tau \quad \forall \quad r\gg 1, \tag{10}$$

where $S=S(\cdot)$ is in (8), and $0<\epsilon_1\ \&\ \epsilon_2<1$ will be determined later. Then, it is easy to check that $Y_{\epsilon_1,\epsilon_2}$ is well-defined for sufficiently small ϵ_1 and ϵ_2.

On the one hand, we have the following asymptotic property

$$Y_{\epsilon_1,\epsilon_2}(r)\sim r^{1+\frac{\epsilon_1}{p-1}}\ln^{\frac{\epsilon_2}{p-1}}(r)S(r)^{-\frac{1}{p-1}} \quad \forall \quad r\gg 1 \tag{11}$$

which implies

$$S(r)Y^\sigma_{\epsilon_1,\epsilon_2}(r)\sim S(r)\left(r^{1+\frac{\epsilon_1}{p-1}}\ln^{\frac{\epsilon_2}{p-1}}(r)S(r)^{-\frac{1}{p-1}}\right)^\sigma \quad \forall\ r\gg 1.$$

Then, by the formula of $S(r)$ in (8) we obtain

$$S(r)Y^\sigma_{\epsilon_1,\epsilon_2}(r)\sim r^{\frac{\sigma p}{p-1}(1-\frac{\sigma-p+1}{\sigma p}Q)-1+\frac{\sigma\epsilon_1}{p-1}}\left(\ln r\right)^{-q\frac{\sigma-p+1}{p-1}+\frac{\sigma\epsilon_2}{p-1}} \quad \forall \quad r\gg 1. \tag{12}$$

On the other hand, by the construction (10) we have

$$Y'_{\epsilon_1,\epsilon_2}=-(r^{-\epsilon_1}S)^{-\frac{1}{p-1}}(1+\ln^{\epsilon_2}(r))^{\frac{1}{p-1}},$$

and thus

$$(S|Y'_{\epsilon_1,\epsilon_2}|^{p-2}Y'_{\epsilon_1,\epsilon_2})'=-(r^{\epsilon_1}(1+\ln^{\epsilon_2}(r)))'. \tag{13}$$

In what follows, we will carefully choose the parameters ϵ_1 and ϵ_2, such that $Y_{\epsilon_1,\epsilon_2}$ is a solution to the inequality (7) with $f(u,u')=u^\sigma$. Differently speaking, if the last-mentioned conditions on the parameter pair (Q,q) are satisfied, then there exist ϵ_1 and ϵ_2 such that (12) is overwhelmed by (13) as r goes to infinity.

Condition (i): $Q>\frac{\sigma p}{\sigma-p+1}$. Taking

$$u=Y_{\epsilon_1,\epsilon_2}\ \text{ with }\ \epsilon_1=p\left(\frac{\sigma-p+1}{\sigma p}Q-1\right)>0=\epsilon_2,$$

and using (12) we achieve

$$S(r)u(r)^\sigma\sim r^{-1}(\ln r)^{-\frac{q(\sigma-p+1)}{p-1}} \quad \forall \quad r\gg 1.$$

By (13), we have

$$(S|u'|^{p-2}u')' = -(2r^{\epsilon_1})' = -2\epsilon_1 r^{-1+\epsilon_1} \ll -r^{-1}(\ln r)^{-\frac{q(\sigma-p+1)}{p-1}} \quad \forall \quad r \gg 1.$$

Thus (7) is satisfied when r is large enough.

Condition (ii): $Q = \frac{\sigma p}{\sigma-p+1}$ & $q > \frac{p-1}{\sigma-p+1}$. Under this assumption, we employ $u = Y_{\epsilon_1,\epsilon_2}$ with

$$\epsilon_1 = 0; \quad \frac{\sigma\epsilon_2}{p-1} = \frac{q(\sigma-p+1)}{p-1} - 1,$$

thereby getting via (12)

$$S(r)u(r)^\sigma \sim \frac{1}{r\ln r} \quad \forall \quad r \gg 1.$$

On the other hand, by (13) we have

$$(S|u'|^{p-1}u')' = -\big(1+\ln^{\epsilon_2} r\big)' = -\frac{\epsilon_2}{r\ln^{1-\epsilon_2} r} \ll -\frac{1}{r\ln r} \quad \forall \quad r \gg 1,$$

whence seeing that (7) is satisfied for large r.

In conclusion, we have established the following assertion.

Lemma 2.1. *Assume that*

$$\text{either} \quad Q > \frac{\sigma p}{\sigma-p+1} \quad \text{or} \quad Q = \frac{\sigma p}{\sigma-p+1} \quad \& \quad q > \frac{p-1}{\sigma-p+1}$$

is true. Then there exists a positive solution Y of (7) on (r_0,∞) provided that r_0 is sufficiently large. Furthermore, this Y may have the following property:

$$Y > 0 > Y'. \tag{14}$$

The key-point here is that we give explicitly this positive solution, rather than just the existence of a solution being offered in [6,10,9], whose arguments depend on the particular structure of the inequality, i.e., the polynomial non-linearity. Thus it is difficult to carry out their arguments over other nonlinear models – at this point – the explicit solution constructed in Lemma 2.1 is more flexible.

2.2. *Step 2: solution near origin*

In the meantime, we are required to construct a positive solution to (7) near the origin. In what follows, we always denote by Y the solution constructed in Lemma 2.1. Consider (7) with a parameter on the second term, that is:

$$\big(S|u'|^{p-2}u'\big)' + \lambda_\rho Sf(u,u') \le 0. \tag{15}$$

Let

$$u_\rho(r) := c_\rho \int_r^\rho \big(1-e^{-x/\rho}\big)^{\frac{1}{p-1}} dx \quad \forall \quad r \in [0,\rho) \tag{16}$$

where c_ρ is a normalized constant to make $u_\rho(0) = 1$. Without loss of generality, we may assume

$$\sup_{|x|,|y|\le 2} f(x,y) = C.$$

Clearly, we have

$$0 \le u_\rho, |u'_\rho| \le 1 \quad \forall \quad \rho \gg 1,$$

and thus

$$\lambda_\rho S f(u_\rho, u'_\rho) \le C\lambda_\rho S. \tag{17}$$

Furthermore, we get

$$\begin{aligned}\big(S|u'_\rho|^{p-2}u'_\rho\big)' &= -c_\rho^{p-1}\big(S(1-e^{-r/\rho})\big)' \\ &= -\frac{c_\rho^{p-1}}{\rho} S e^{-r/\rho} - c_\rho^{p-1} S'(1-e^{-r/\rho}) \\ &\le -\frac{c_\rho^{p-1}}{\rho} S e^{-r/\rho}.\end{aligned} \tag{18}$$

Here we have used the non-decreasing property of $S(r)$. If

$$\lambda_\rho = \frac{c_\rho^{p-1}}{Ce\rho},$$

then form (17) and (18) it follows that u_ρ in (16) satisfies the inequality (15) on $[0, \rho)$. By the construction, we see

$$\lim_{\rho\to\infty} c_\rho = \lim_{\rho\to\infty}\left(\int_0^\rho \big(1-e^{-x/\rho}\big)^{\frac{1}{p-1}} dx\right)^{-1} = 0,$$

whence

$$\lim_{\rho\to\infty} u'_\rho(r) = -\lim_{\rho\to\infty} c_\rho (1-e^{-r/\rho})^{\frac{1}{p-1}} = 0.$$

Moreover, an application of the L'Hospital rule derives that for any given $r > 0$,

$$\lim_{\rho\to\infty} u_\rho(r) = \lim_{\rho\to\infty}\frac{u_\rho(r)}{u_\rho(0)} = \lim_{\rho\to\infty}\frac{\int_r^\rho \big(1-e^{-x/\rho}\big)^{\frac{1}{p-1}} dx}{\int_0^\rho \big(1-e^{-x/\rho}\big)^{\frac{1}{p-1}} dx} = 1.$$

In conclusion, we have demonstrated the following fact.

Lemma 2.2. *Assume that $S \in C^1_{loc}([0,\infty))$ is positive and non-decreasing. Then, for a fixed $\rho \gg 1$ there is a small number λ_ρ, such that the inequality (15) has a positive solution u on $[0, \rho)$. Furthermore, such a solution u enjoys the following property:*

$$\lim_{\rho\to\infty} u'_\rho(r) = 0 = \lim_{\rho\to\infty}(u_\rho(r) - 1) \quad \forall \quad r \in (0,\infty). \tag{19}$$

2.3. Step 3: solution from infinity to origin

It remains to glue the above two solutions Y and u_ρ together, whence producing a constructive solution to (7). In doing so, we decide to borrow the gluing technique in [6]. Fix $R_0 > r_0$ and then choose $\rho > R_0$ to be such that

$$\frac{u'_\rho}{u_\rho}(R_0) > \frac{Y'}{Y}(R_0).$$

This choice is possible since

$$\frac{Y'}{Y}(R_0) < 0 \quad \big(\text{from (14)}\big) \quad \& \quad \lim_{\rho\to\infty}\frac{u'_\rho}{u_\rho}(R_0) = 0 \quad \big(\text{from (19)}\big).$$

Then it follows that

$$\left(\frac{Y}{u_\rho}\right)'(R_0) = \frac{Y'u_\rho - Yu'_\rho}{u_\rho^2}(R_0) < 0 \quad \& \quad \lim_{r\to\rho^+}\frac{Y(r)}{u_\rho(r)} = \infty,$$

and consequently, $\frac{Y}{u_\rho}$ has a local minimum m at some point $\eta \in (R_0, \rho)$. This in turns derives

$$\left(\frac{Y}{u_\rho}\right)'(\eta) = 0,$$

whence

$$Y(\eta) = mu_\rho(\eta) \quad \& \quad Y'(\eta) = mu'_\rho(\eta). \tag{20}$$

Upon defining a new function:

$$v(r) = \begin{cases} mu_\rho(r) & \text{for } r \in [0, \eta); \\ Y(r) & \text{for } r \in [\eta, \infty), \end{cases} \tag{21}$$

we use (20) to get that v belongs to $C^1(M)$ (the class of continuously differential functions on M) and even $W^{1,p}_{loc}(M)$. In view of the construction, we have

$$\begin{cases} \Delta_p u + \frac{\lambda_\rho}{m^{\sigma-p+1}} u^\sigma \le 0 & \text{in } B(x_0, \eta); \\ \Delta_p u + u^\sigma \le 0 & \text{in } M \backslash B(x_0, \eta), \end{cases} \tag{22}$$

thereby gaining that v enjoys

$$\Delta_p v + \omega v^\sigma \le 0 \quad \text{with} \quad \omega = \min\{1, \lambda_\rho / m^{\sigma-p+1}\}.$$

In fact, we have established the following assertion.

Lemma 2.3. *The function $u = \omega^{\frac{1}{\sigma-p+1}} v$ is a positive solution to (5) on $M = \left(\mathbb{R}^n, dr^2 + \psi^2(r)\, d\theta^2\right)$.*

3. Application

Three lemmas discovered as above will be applied to treat some Δ_p-based inequalities with more general nonlinearity. Here it is worth mentioning that all inequalities discussed below will be understood weakly.

3.1. Application 1

Firstly, let us consider the inequality

$$\Delta_p u + u^{\sigma_1} |\nabla u|^{\sigma_2} \le 0. \tag{23}$$

Inequality (23) arises naturally from many contexts, for example, the uniqueness of bounded nonnegative p-superharmonic function: $\Delta_p u \le 0$ on a geodesically complete non-compact connected Riemannian manifold M. If

$$v = \ln(1 + u),$$

then v is non-negative and satisfies the following inequality

$$e^{(p-1)v}\left(\Delta_p v + (p-1)|\nabla v|^p\right) \le 0$$

which is equivalent to

$$\Delta_p v + (p-1)|\nabla v|^p \le 0.$$

The factor $p - 1$ could be removed by changing v to cv. Following [11] we introduce the parameter pair:

$$Q = \frac{p\sigma_1 + \sigma_2}{\sigma_1 + \sigma_2 - p + 1} \quad \& \quad q = \frac{p-1}{\sigma_1 + \sigma_2 - p + 1}. \tag{24}$$

Theorem 3.1. *Assume that the inequality (1) for (24) is valid. Then any non-negative weak solution to (23) is identically equal to* 0 [11]. *Furthermore, the parameter pair (24) is sharp in the sense that if*

$$\text{either}\quad Q > \frac{p\sigma_1+\sigma_2}{\sigma_1+\sigma_2-p+1} \quad \text{or} \quad Q = \frac{p\sigma_1+\sigma_2}{\sigma_1+\sigma_2-p+1} \quad \& \quad q > \frac{p-1}{\sigma_1+\sigma_2-p+1}$$

then there exists a manifold obeying (1) where (23) has a positive solution.

Proof. The first part of Theorem 3.1 has been verified in [11]. But [11] contains only a partial result on the sharpness of the parameter pair (24). Here, with the aid of the explicit solutions presented in Lemmas 2.1–2.2–2.3, we are able to completely settle this issue.

As a matter of fact, under the radial assumption, the inequality (23) becomes to

$$\big(S|u'|^{p-2}u'\big)' + Su^{\sigma_1}|u'|^{\sigma_2} \le 0, \tag{25}$$

where $S = S(\cdot)$ is the surface area of B_r. In view of (11) we have

$$SY^{\sigma_1}_{\epsilon_1,\epsilon_2}|Y'_{\epsilon_1,\epsilon_2}|^{\sigma_1} \sim r^{\mu_1}\ln^{\mu_2} r \quad \forall\ r \gg 1,$$

where

$$\begin{cases} \mu_1 = \frac{p\sigma_1+\sigma_2}{p-1}\Big(1 - Q\frac{\sigma_1+\sigma_2-p+1}{p\sigma_1+\sigma_2}\Big) + \frac{\epsilon_1(\sigma_1+\sigma_2)}{p-1} - 1; \\ \mu_2 = -q\frac{\sigma_1+\sigma_2-p+1}{p-1} + \frac{\epsilon_2(\sigma_1+\sigma_2)}{p-1}. \end{cases}$$

We distinguish two cases as seen below.

Case 1: $Q > \frac{p\sigma_1+\sigma_2}{\sigma_1+\sigma_2-p+1}$. Under this situation, we can choose $\epsilon_1 > 0$ such that $\mu_1 = -1$. From choosing $\epsilon_2 = 0$ and using (13) (for Lemma 2.1) it follows that $u = Y_{\epsilon_1,0}$ satisfies

$$\begin{cases} Su^{\sigma_1}|u'|^{\sigma_2} \sim r^{-1}\ln^{\mu_2} r \\ (S|u'|^{p-2}u')' = -2\epsilon_1 r^{-1+\epsilon_1} \end{cases} \quad \forall \quad r \gg 1,$$

which infers that u is a solution to (25) for sufficiently large r.

Case 2: $Q = \frac{p\sigma_1+\sigma_2}{\sigma_1+\sigma_2-p+1}$ & $q > \frac{p-1}{\sigma_1+\sigma_2-p+1}$. Under this situation, we can choose $\epsilon_2 > 0$ such that $\mu_2 = -1$. From letting $\epsilon_1 = 0$ it follows that $u = Y_{0,\epsilon_2}$ satisfies

$$\begin{cases} Su^{\sigma_1}|u'|^{\sigma_2} \sim r^{-1}\ln^{-1} r \\ (S|u'|^{p-2}u')' = -(1+\ln^{\epsilon_2} r)' = -\frac{\epsilon_2}{r\ln^{1-\epsilon_2} r} \end{cases} \quad \forall \quad r \gg 1,$$

which again infers that u is a solution to (25) for sufficiently large r.

On the other hand, since

$$0 \le u_\rho^{\sigma_1}, |u'_\rho|^{\sigma_2} \le 1,$$

we can similarly verify that the function u_ρ constructed in (16) for Lemma 2.2 is a solution to (25).

Finally, an application of Lemma 2.3 derives that the desired positive solution to (25) is obtained by gluing u and u_ρ together. □

Remark 1. Our method can be used to effectively handle more general models such as

$$\operatorname{div}\big(A(x)|\nabla u|^{p-2}\nabla u\big) + V(x)u^{\sigma_1}|\nabla u|^{\sigma_2} \le 0,$$

where $A(x)$ and $V(x)$ are two potential functions with polynomial growth or decay.

3.2. Application 2

Next, we deal with the following inequality

$$\Delta_p u + e^{u^\sigma} - 1 \le 0. \tag{26}$$

The non-linearity $e^u - 1$ naturally appears in [13]. Here we address the existence of a non-negative solution to the inequality (26), thereby discovering the following sharp result.

Theorem 3.2. *Assume that the inequality (1) holds for (4). Then any non-negative weak solution to (26) is identically equal to* 0*. Furthermore, the parameter pair (4) is sharp in the sense that if*

$$\text{either} \quad Q > \frac{p\sigma}{\sigma - p + 1} \quad \text{or} \quad Q = \frac{p\sigma}{\sigma - p + 1} \quad \& \quad q > \frac{p-1}{\sigma - p + 1}$$

then there exists a manifold obeying (1) where (26) has a positive solution.

Proof. The non-linearity $f(u, \nabla u) = e^{u^\sigma} - 1$ is stronger than u^σ in the sense that for all non-negative $\psi \in L^\sigma_{loc}(M)$ with compact support one has

$$\int_M (e^{u^\sigma} - 1)\psi\, d\mu \ge \int_M u^\sigma \psi\, d\mu$$

Thus the non-existence of a non-negative solution can be readily obtained. To be more precise, if u is a non-negative solution to (26), then u solves $\Delta_p u + u^\sigma \le 0$ in the weak sense as well, and hence $u \equiv 0$ in view of Theorem 1.2 and the conclusion follows.

The sharpness of parameter pair (4) follows directly from the above construction since the previously-constructed solution u is bounded with

$$\lim_{r\to\infty} u(r) = 0,$$

which implies

$$f(u, u') \sim u^\sigma \quad \& \quad f(u(r), u'(r)) \sim u^\sigma(r) \quad \forall\ r \gg 1.$$

To be more precise, according to $0 \le u_\rho \le 1$ and (15), there exists a solution u to

$$\Delta_p u_\rho + \lambda_\rho e^{-1}(e^{u_\rho^\sigma} - 1) \le 0.$$

Then, via Lemma 2.3 we can merge u_ρ and Y (cf. (21) & (22)) to produce a solution u of

$$\Delta_p u + \omega e^{-1}(e^{u^\sigma} - 1) \le 0 \quad \text{with} \quad \omega = \min\{1, \lambda_\rho / m^{\sigma - p + 1}\}. \tag{27}$$

It is easy to deduce that $\|u\|_{L^\infty(M)} \le m$. Upon selecting $\theta \in (0, 1)$ such that

$$\theta^{\sigma - p + 1} e^{(m\theta)^\sigma} \le \omega e^{-1},$$

setting $v = \theta u$, and using (27) we obtain

$$\begin{aligned}
\Delta_p v + e^{v^\sigma} - 1 &= \theta^{p-1}\Delta_p u + e^{\theta^\sigma u^\sigma} - 1 \\
&\le \theta^{p-1}\Delta_p u + e^{(\theta m)^\sigma}\theta^\sigma u^\sigma \\
&= \theta^{p-1}\big(\Delta_p u + e^{(\theta m)^\sigma}\theta^{\sigma - p + 1} u^\sigma\big) \\
&\le \theta^{p-1}\big(\Delta_p u + \omega e^{-1} u^\sigma\big) \\
&\le \theta^{p-1}\big(\Delta_p u + \omega e^{-1}(e^{u^\sigma} - 1)\big) \\
&\le 0,
\end{aligned}$$

thereby reaching the desired result. □

Remark 2. Interestingly and importantly, (26) with $\sigma = 1$ may be regarded as the limit ($\sigma \to \infty$) case or the strongest variant of (2).

3.3. Application 3

Finally, we take the following inequality into account

$$\Delta_p u + \ln(1+u^\sigma) \le 0. \tag{28}$$

Note that $\ln(1+u)$ is the inverse function of $e^u - 1$. So (28) may be viewed as the inverse form of (26) (cf. [12]). Upon touching the existence of a non-negative solution of the inequality (28), we obtain the following result.

Theorem 3.3. *Assume that the inequality (1) holds for (4). Then any bounded non-negative weak solution to (28) is identically equal to* 0. *Furthermore, the parameter pair (4) is sharp in the sense that if*

$$\textit{either} \quad Q > \frac{p\sigma}{\sigma - p + 1} \quad \textit{or} \quad Q = \frac{p\sigma}{\sigma - p + 1} \quad \& \quad q > \frac{p-1}{\sigma - p + 1}$$

then there exists a manifold obeying (1) where (28) has a positive solution.

Proof. It suffices to show that if there is a non-negative solution to (28) satisfying

$$\|u\|_{L^\infty(M)} \le N \quad \text{for any given} \quad N > 0,$$

then $u \equiv 0$. To this end, let us first take a look at the inequality

$$\Delta_p u + (eN^\sigma)^{-1} u^\sigma \le 0. \tag{29}$$

It is easy to see that if u solves (28) with $\|u\|_{L^\infty(M)} \le N$, then it also solves (29). Since

$$(eN^\sigma)^{-1} u^\sigma \le \ln(1+u^\sigma),$$

without loss of generality we may assume that u is a non-trivial non-negative solution to (29). Then

$$v = (eN^\sigma)^{-\frac{1}{\sigma-p+1}} u$$

solves (2), which leads to $u \equiv 0$ via the first part of Theorem 1.2.

The sharpness of the pair (Q, q) follows from Lemmas 2.1–2.2–2.3 since we have constructed a positive solution to (5) with $f(u, \nabla u) = u^\sigma$, which solves (28) as well thanks to

$$\ln(1+u^\sigma) \le e^\sigma. \qquad \square$$

Remark 3. Unfortunately, the boundedness assumption on u is needed in our argument for Theorem 3.3. Because (29) with $\sigma = 1$ may be treated as the weakest variant of (2), it is natural to conjecture that all solutions to (28) are bounded; see also [12] for a kind of treatment over $\mathbb{R}^n$.

Conflict of interest statement

We declare no conflict of interest.

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