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On Lipschitz solutions for some forward–backward parabolic equations

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Abstract

We investigate the existence and properties of Lipschitz solutions for some forward–backward parabolic equations in all dimensions. Our main approach to existence is motivated by reformulating such equations into partial differential inclusions and relies on a Baire's category method. In this way, the existence of infinitely many Lipschitz solutions to certain initial-boundary value problem of those equations is guaranteed under a pivotal density condition. Under this framework, we study two important cases of forward–backward anisotropic diffusion in which the density condition can be realized and therefore the existence results follow together with micro-oscillatory behavior of solutions. The first case is a generalization of the Perona–Malik model in image processing and the other that of Höllig's model related to the Clausius–Duhem inequality in the second law of thermodynamics.

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Keywords: Forward–backward parabolic equations; Partial differential inclusions; Convex integration; Baire's category method; Infinitely many Lipschitz solutions

1. Introduction

The evolution process of many quantities in applications can be modeled by a diffusion partial differential equation of the form

$$u_t = \operatorname{div}(A(Du)) \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $T > 0$ is any fixed number, and $u = u(x,t)$ is the density of the quantity at position x and time t, with $Du = (u_{x_1}, \cdots, u_{x_n})$ and u_t denoting its spatial gradient and rate of change, respectively. The vector function $A : \mathbb{R}^n \to \mathbb{R}^n$ here represents the *diffusion flux* of the evolution process. The usual heat equation

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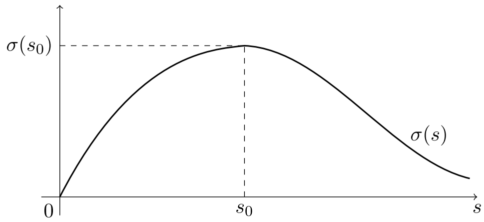

Fig. 1. **Case I:** Perona–Malik type profiles $\sigma(s)$. Note that $\sigma(s)$ is allowed to be non-differentiable at s_0.

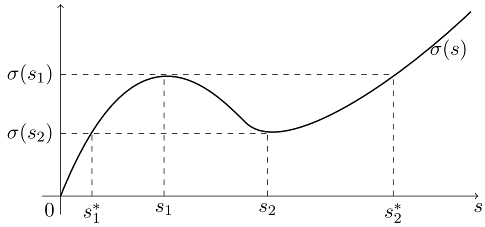

Fig. 2. **Case II:** Höllig type profiles $\sigma(s)$. Note that $\sigma(s)$ is allowed to be non-differentiable on $[s_1, s_2]$ and that $\sigma(s_2)$ can be zero.

corresponds to the case of *isotropic* diffusion given by the Fourier law: $A(p) = kp$ ($p \in \mathbb{R}^n$), where $k > 0$ is the diffusion constant.

For standard diffusion equations, the flux $A(p)$ is assumed to be *monotone*; namely,

$$(A(p) - A(q)) \cdot (p - q) \geq 0 \quad (p, q \in \mathbb{R}^n).$$

In this case, equation (1.1) is parabolic and can be studied by the standard methods of parabolic equations and monotone operators [5,26,27]. In particular, when $A(p)$ is given by a smooth convex function $W(p)$ through $A(p) = D_p W(p)$ ($p \in \mathbb{R}^n$), (1.1) can be viewed and thus studied as a certain gradient flow generated by the energy functional

$$I(u) = \int_\Omega W(Du(x))\, dx.$$

However, for certain applications of the evolution process to some important physical problems, the underlying diffusion fluxes $A(p)$ may not be monotone, yielding non-parabolic equations (1.1). In this paper, we study the diffusion equation (1.1) for some non-monotone diffusion fluxes $A(p)$. We focus on the initial-boundary value problem

$$\begin{cases} u_t = \operatorname{div}(A(Du)) & \text{in } \Omega_T, \\ A(Du) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T), \\ u = u_0 & \text{on } \Omega \times \{t = 0\}, \end{cases} \tag{1.2}$$

where $\Omega_T = \Omega \times (0, T)$, $\mathbf{n}$ is the outer unit normal on $\partial\Omega$, $u_0 = u_0(x)$ is a given initial datum, and the flux $A(p)$ is of the form

$$A(p) = f(|p|^2)p \quad (p \in \mathbb{R}^n), \tag{1.3}$$

given by a function $f : [0, \infty) \to \mathbb{R}$ with *profile* $\sigma(s) = sf(s^2)$ having one of the graphs in Figs. 1 and 2. (Precise structural assumptions on $\sigma(s) = sf(s^2)$ will be given in Section 3.)

The two cases in Figs. 1 and 2 correspond to the applications in image processing proposed by Perona and Malik [33] and in phase transitions of thermodynamics studied by Höllig [19]. For these diffusions, we have

$$\sigma'(\sqrt{s}) = f(s) + 2sf'(s) < 0 \quad \text{for some range of } s > 0.$$

In these cases, the diffusion is *anisotropic* since the diffusion matrix $(A^i_{p_j})$, where

$$A^i_{p_j}(p) = f(|p|^2)\delta_{ij} + 2f'(|p|^2)p_i p_j \quad (i, j = 1, 2, \cdots, n),$$

has the eigenvalues $f(|p|^2)$ (of multiplicity $n-1$) and $f(|p|^2)+2|p|^2 f'(|p|^2)$; hence the diffusion coefficients could also be negative, and problem (1.2) becomes *forward–backward parabolic*. Moreover, setting

$$W(p)=\int_0^{|p|}\sigma(r)\,dr, \quad I(u)=\int_\Omega W(Du)\,dx,$$

the initial-boundary value problem (1.2) becomes a L^2-gradient flow of the energy functional $I(u)$; however, $I(u)$ is *non-convex*. Consequently, neither the standard methods of parabolic equations and monotone operators nor the non-linear semigroup theory can be applied to study (1.2).

Forward–backward parabolic equations have found many important applications in the mathematical modeling of some physical problems, but mathematically, due to the backward parabolicity, the initial-boundary value problem (1.2) of such equations is highly ill-posed, and in many cases even the notion and existence of reasonable solutions remain largely unsettled; see, e.g., [22].

In [33], the original Perona–Malik model (1.2) in image processing used the profile

$$\sigma(s)=se^{-s^2/2s_0^2} \quad \text{or} \quad \sigma(s)=\frac{s}{1+s^2/s_0^2} \tag{1.4}$$

for denoising and edge enhancement of a computer vision; in this case, we call (1.1) the *Perona–Malik equation*. In this model, $u(x,t)$ represents an improved version of the initial gray level $u_0(x)$ of a noisy picture. The anisotropic diffusion $\operatorname{div}(A(Du))$ is forward parabolic in the *subcritical* region where $|Du|<s_0$ and backward parabolic in the *supercritical* region where $|Du|>s_0$. The expectation of the model is that disturbances with small gradient in the subcritical region will be smoothed out by the forward parabolic diffusion, while sharp edges corresponding to large gradient in the supercritical region will be enhanced by the backward parabolic equation. Such expected phenomenology has been implemented and observed in some numerical experiments [13,33], showing the stability and effectiveness of the model. Mathematically, there have been extensive works on the Perona–Malik type equations with profiles $\sigma(s)$ as in Fig. 1; however, most of these works have focused on the analysis of numerical or approximate solutions by different methods. For example, in dimension $n=1$, the singular perturbation and its Γ-limit related to the Perona–Malik type equations were investigated in [2,3]. In [18], a mild regularization of the Perona–Malik equation with a viscous term was used to extract a unique approximate Young measure solution in any dimension. The works [7,35] studied Young measure solutions for the Perona–Malik equation in dimensions $n=1$ and $n=2$. Also the works [13,14] focused on the numerical schemes for the Perona–Malik model. Recently, classical solutions for the Perona–Malik equation were studied in [15,16], where the existence of solutions was proved for certain initial data u_0 that can be *transcritical* in the sense that the sets $\{|Du_0(x)|>s_0\}$ and $\{|Du_0(x)|<s_0\}$ are both non-empty (s_0 is the number in Fig. 1); however, their initial data cannot be arbitrarily prescribed.

The one-dimensional forward–backward parabolic problem (1.2) with a piecewise linear profile $\sigma(s)$ was studied in Höllig [19] and Höllig and Nohel [20], motivated by the Clausius–Duhem inequality in the second law of thermodynamics in continuum mechanics (see, e.g., [11,36]). For such a special profile, it was proved that there exist infinitely many L^2-weak solutions to (1.2) in dimension $n=1$. The piecewise linearity of $\sigma(s)$ in $n=1$ was much relaxed later to include a more general class of profiles $\sigma(s)$ (as in Fig. 2) in the work of Zhang [40], using an entirely different approach from Höllig's.

The question concerning the existence of *exact* weak solutions to problem (1.2) of the Perona–Malik and Höllig types had remained open until Zhang [39,40] first established that the *one-dimensional* problem (1.2) of each type has infinitely many Lipschitz solutions for any suitably given smooth initial data u_0. Zhang's pivotal idea was to reformulate equation (1.1) in $n=1$ into a 2×2 non-homogeneous partial differential inclusion and then to prove the existence by using a modified method of convex integration, following the ideas of [17,25,30]. The study of general partial differential inclusions has stemmed from the successful understanding of homogeneous differential inclusions of the form $Du(x)\in K$, first encountered in the study of crystal microstructures by Ball and James [1], Chipot and Kinderlehrer [8] and Müller and Šverák [29], and in the study of implicit partial differential equations by Dacorogna and Marcellini [10]. Recently, the methods of differential inclusions have been successfully applied to other important problems; see, e.g., [9,12,28,30,31,34,37,38].

In general, a function $u\in W^{1,\infty}(\Omega_T)$ is called a *Lipschitz solution* to problem (1.2) provided that equality

$$\int_\Omega (u(x,s)\zeta(x,s) - u_0(x)\zeta(x,0))dx = \int_0^s \int_\Omega (u\zeta_t - A(Du)\cdot D\zeta)dxdt \tag{1.5}$$

holds for each $\zeta \in C^\infty(\bar{\Omega}_T)$ and each $s \in [0,T]$. Let $\zeta \equiv 1$; then it is immediate from the definition that any Lipschitz solution u to (1.2) conserves the total mass over time:

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx \quad \forall t \in [0,T].$$

In [23], we extended Zhang's method of partial differential inclusions to the Perona–Malik type equations in all dimensions n for balls $\Omega = B_R(0) := \{x \in \mathbb{R}^n \mid |x| < R\}$ and non-constant radially symmetric smooth initial data u_0. In this case, the n-dimensional equation for radial solutions can still be reformulated into a 2×2 non-homogeneous partial differential inclusion, and the existence of infinitely many radial Lipschitz solutions to (1.2) is established. However, for general domains and initial data, the n-dimensional problem (1.2) can only be recast as a $(1+n) \times (n+1)$ non-homogeneous partial differential inclusion that has some uncontrollable gradient components, making the construction of Lipschitz solutions for this inclusion impractical.

In our recent work [24], we overcame this difficulty by developing a suitably modified density method, still motivated by the method of differential inclusions but based on a Baire's category argument. In this work, we have proved that for all bounded convex domains $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^{2+\alpha}$ and initial data $u_0 \in C^{2+\alpha}(\bar{\Omega})$ with $Du_0 \cdot \mathbf{n} = 0$ on $\partial\Omega$, there exist infinitely many Lipschitz solutions to (1.2) for the *exact* Perona–Malik diffusion flux $A(p) = \frac{p}{1+|p|^2}$ ($p \in \mathbb{R}^n$). However, the proof heavily relies on the explicit formula of the rank-one convex hull of a certain matrix set defined by this special flux; such an explicit formula for the general Perona–Malik type flux $A(p)$ (with profile $\sigma(s)$ as in Fig. 1) is unattainable. As a bypass of this obstacle and the main novelty of the current paper, we investigate the first order *partial* rank-one convex hull of a suitable matrix set and obtain its implicit and geometric structure, which is crucial for our construction of the relevant subsolutions (see Section 5).

The main purpose of this paper is to adapt the method of [24] into problem (1.2) in all dimensions for general forward–backward parabolic equations of the Perona–Malik and Höllig types and thus to generalize and sharpen the results of [19,39,40,24]. To state our main theorems, we make the following assumptions on the domain Ω and initial datum u_0:

$$\begin{cases} \Omega \subset \mathbb{R}^n \text{ is a bounded domain with } \partial\Omega \text{ of } C^{2+\alpha}, \\ u_0 \in C^{2+\alpha}(\bar{\Omega}) \text{ is non-constant with } Du_0 \cdot \mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{1.6}$$

where $\alpha \in (0,1)$ is a given number. Without specifying the precise conditions on the profiles $\sigma(s)$ for **Cases I** and **II**, we first state our main existence theorems as below.

Theorem 1.1. *Assume condition (1.6) is fulfilled. Let $A(p)$ be given by (1.3) with the profile $\sigma(s)$. Then problem (1.2) has infinitely many Lipschitz solutions in the following two cases.*

Case I: *$\sigma(s)$ is of the* Perona–Malik type *as in Fig. 1, and in addition Ω is convex.*

Case II: *$\sigma(s)$ is of the* Höllig type *as in Fig. 2, and in addition u_0 satisfies $|Du_0(x_0)| \in (s_1^*, s_2^*)$ at some $x_0 \in \Omega$.*

The precise assumptions on the profiles $\sigma(s)$ and detailed statements of the theorems for both cases will be given later (see Theorems 3.1 and 3.5) along with some discussions about a certain implication of our result on the Perona–Malik model in image processing and the breakdown of uniqueness for the Höllig type equations.

In certain cases, among infinitely many Lipschitz solutions, problem (1.2) admits a unique classical solution; we have the following result (see also [21]).

Theorem 1.2. *Assume condition (1.6) is fulfilled and Ω is convex. Let $A(p)$ be given by (1.3) with the profile $\sigma(s)$. Assume $\|Du_0\|_{L^\infty(\Omega)} < s_0$ if $\sigma(s)$ is of the* Perona–Malik type *or $\|Du_0\|_{L^\infty(\Omega)} < s_1$ if $\sigma(s)$ is of the* Höllig type*. Then problem (1.2) has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T)$ satisfying $\|Du\|_{L^\infty(\Omega_T)} = \|Du_0\|_{L^\infty(\Omega)}$.*

This theorem will be also separated into **Cases I** and **II** in Section 3 in order to be compared with our detailed main results, Theorems 3.1 and 3.5.

Finally, in comparison to the case of radial solutions constructed in our recent paper [23] for the Perona–Malik type problem, we have the following *coexistence* result; although the same result should hold for the equations of the *Höllig type*, we do not pursue it in this paper.

Theorem 1.3. *Assume that $\sigma(s)$ is of the* Perona–Malik type*, that $\Omega = B_R(0)$ is an open ball in $\mathbb{R}^n$, and that u_0 is* radial *and satisfies condition (1.6). Then there are infinitely many radial and non-radial Lipschitz solutions to (1.2).*

The rest of the paper is organized as follows. Section 2 reviews the setup of problem (1.2) as a non-homogeneous partial differential inclusion and the general existence theorem under a key density hypothesis, formulated in the paper [24]; in this section, some general properties of Lipschitz solutions to (1.2) are also provided. In section 3, precise structural conditions on the profiles $\sigma(s)$ of the Perona–Malik and Höllig types are specified, and the detailed statements for the main result, Theorem 1.1, are introduced as Theorems 3.1 and 3.5, followed by some remarks and related results. Throughout the paper, we use the boldface letters for **Cases I** and **II** for clear distinction of the profiles $\sigma(s)$ of the Perona–Malik and Höllig types, respectively, and we follow a parallel exposition to deal with both cases simultaneously; but for better readability, we strongly recommend that the readers follow each case one at a time. Section 4 prepares some useful results that may be of independent interest; for the proof of these results we refer the reader to the paper [24]. As the core analysis of the paper, the geometry of related matrix sets is scrutinized in Section 5, leading to the relaxation result on a homogeneous differential inclusion, Theorem 5.8. Section 6 is devoted to the construction of suitable boundary functions and admissible sets for **Cases I** and **II** in the contexts of Theorems 3.1 and 3.5, respectively. Lastly, in Section 7, the pivotal density hypothesis in each case is realized and the proof of Theorems 3.1, 3.5 and the coexistence result, Theorem 1.3, is given.

2. Existence by a general density approach

Here and below, we assume without loss of generality that the initial datum $u_0 \in W^{1,\infty}(\Omega)$ in problem (1.2) satisfies

$$\int_\Omega u_0(x)\,dx = 0, \tag{2.1}$$

since otherwise we may solve (1.2) with initial datum $\tilde{u}_0 = u_0 - \bar{u}_0$, where $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0\,dx$.

2.1. *Motivation: non-homogeneous partial differential inclusion*

To discuss the approach setup in [24], let us reformulate problem (1.2) into a non-homogeneous partial differential inclusion.

First, assume that we have a function $\Phi = (u^*, v^*) \in W^{1,\infty}(\Omega_T; \mathbb{R}^{1+n})$ satisfying

$$\begin{cases} u^*(x,0) = u_0(x), & x \in \Omega, \\ \operatorname{div} v^*(x,t) = u^*(x,t), & \text{a.e. } (x,t) \in \Omega_T, \\ v^*(\cdot,t)\cdot \mathbf{n}|_{\partial\Omega} = 0, & t \in [0,T], \end{cases} \tag{2.2}$$

which will be called a *boundary function* for the initial datum u_0.

Let $A \in C(\mathbb{R}^n; \mathbb{R}^n)$ be the diffusion flux. For each $l \in \mathbb{R}$, let $K(l)$ be a subset of the matrix space $\mathbb{M}^{(1+n)\times(n+1)}$ defined by

$$K(l) = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \,\middle|\, p \in \mathbb{R}^n,\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = l \right\}. \tag{2.3}$$

Suppose that a function $w = (u, v) \in W^{1,\infty}(\Omega_T; \mathbb{R}^{1+n})$ solves the Dirichlet problem of non-homogeneous partial differential inclusion

$$\begin{cases} \nabla w(x,t) \in K(u(x,t)), & \text{a.e. } (x,t) \in \Omega_T, \\ w(x,t) = \Phi(x,t), & (x,t) \in \partial\Omega_T, \end{cases} \tag{2.4}$$

where ∇w is the space–time Jacobian matrix of w:

$$\nabla w = \begin{pmatrix} Du & u_t \\ Dv & v_t \end{pmatrix}.$$

Then it can be easily shown that u is a Lipschitz solution to (1.2); see [24, Section 3.1] or Proposition 2.1 below.

Although solving problem (2.4) is sufficient to obtain a Lipschitz solution to (1.2), all known general existence results [10,32] are not applicable to solve (2.4). If $n=1$, (2.4) implies $v_x = u$ and thus $v \in W^{1,\infty}$ if $u \in W^{1,\infty}$; so Zhang [39,40] was able to solve (2.4) for the Perona–Malik and Höllig types in dimension $n=1$ by using a modified convex integration method. However, for $n \geq 2$, since it is impossible to bound $\|Dv\|_{L^\infty(\Omega_T)}$ in terms of $\operatorname{div} v$, the function v may not be in $W^{1,\infty}(\Omega_T;\mathbb{R}^n)$ even when $u \in W^{1,\infty}(\Omega_T)$; therefore, solving (2.4) in $W^{1,\infty}(\Omega_T;\mathbb{R}^{1+n})$ is impractical in dimensions $n \geq 2$. To overcome this difficulty, we make the following key observation; for proof see [24, Lemma 3.1].

Proposition 2.1. *Suppose $u \in W^{1,\infty}(\Omega_T)$ is such that $u(x,0)=u_0(x)$, there exists a vector function $v \in W^{1,2}((0,T);L^2(\Omega;\mathbb{R}^n))$ with weak time-derivative v_t satisfying*

$$v_t = A(Du) \quad \text{a.e. in } \Omega_T, \tag{2.5}$$

and for each $\zeta \in C^\infty(\bar{\Omega}_T)$ and each $t \in [0,T]$,

$$\int_\Omega v(x,t)\cdot D\zeta(x,t)\,dx = -\int_\Omega u(x,t)\zeta(x,t)\,dx. \tag{2.6}$$

Then u is a Lipschitz solution to (1.2).

2.2. Admissible set and the density approach

Let $\Phi = (u^*, v^*)$ be any boundary function for u_0 defined by (2.2) above. Denote by $W^{1,\infty}_{u^*}(\Omega_T)$, $W^{1,\infty}_{v^*}(\Omega_T;\mathbb{R}^n)$ the usual *Dirichlet classes* with boundary traces u^*, v^*, respectively.

We say that $\mathcal{U} \subset W^{1,\infty}_{u^*}(\Omega_T)$ is an *admissible set* provided that it is non-empty and bounded in $W^{1,\infty}_{u^*}(\Omega_T)$ and that for each $u \in \mathcal{U}$, there exists a vector function $v \in W^{1,\infty}_{v^*}(\Omega_T;\mathbb{R}^n)$ satisfying

$$\operatorname{div} v = u \text{ a.e. in } \Omega_T, \quad \|v_t\|_{L^\infty(\Omega_T)} \leq R,$$

where $R>0$ is any fixed number. If $\mathcal{U}$ is an admissible set, for each $\epsilon>0$, let $\mathcal{U}_\epsilon$ be the set of all $u \in \mathcal{U}$ such that there exists a function $v \in W^{1,\infty}_{v^*}(\Omega_T;\mathbb{R}^n)$ satisfying

$$\operatorname{div} v = u \text{ a.e. in } \Omega_T, \quad \|v_t\|_{L^\infty(\Omega_T)} \leq R, \quad \text{and} \quad \int_{\Omega_T} |v_t(x,t) - A(Du(x,t))|\,dxdt \leq \epsilon|\Omega_T|.$$

The following general existence theorem relies on a pivotal density hypothesis of $\mathcal{U}_\epsilon$ in $\mathcal{U}$. Although the proof of this theorem appears in [24, Theorem 3.3], the proof itself will be used in the proofs of Theorems 3.1 and 3.5 in Subsection 7.2, and so we include it here.

Theorem 2.2. *Let $\mathcal{U} \subset W^{1,\infty}_{u^*}(\Omega_T)$ be an admissible set satisfying the* density property:

$$\mathcal{U}_\epsilon \text{ is dense in } \mathcal{U} \text{ under the } L^\infty\text{-norm for each } \epsilon > 0. \tag{2.7}$$

Then, given any $\varphi \in \mathcal{U}$, for each $\delta>0$, there exists a Lipschitz solution $u \in W^{1,\infty}_{u^*}(\Omega_T)$ to (1.2) satisfying $\|u-\varphi\|_{L^\infty(\Omega_T)} < \delta$. Furthermore, if $\mathcal{U}$ contains a function which is not a Lipschitz solution to (1.2), then (1.2) itself admits infinitely many Lipschitz solutions.

Proof. For clarity, we divide the proof into several steps.

1. Let $\mathcal{X}$ be the closure of $\mathcal{U}$ in the metric space $L^\infty(\Omega_T)$. Then $(\mathcal{X}, L^\infty)$ is a non-empty complete metric space. By assumption, each $\mathcal{U}_\epsilon$ is dense in $\mathcal{X}$. Moreover, since $\mathcal{U}$ is bounded in $W^{1,\infty}_{u^*}(\Omega_T)$, we have $\mathcal{X} \subset W^{1,\infty}_{u^*}(\Omega_T)$.

2. Let $\mathcal{Y} = L^1(\Omega_T;\mathbb{R}^n)$. For $h>0$, define $T_h\colon \mathcal{X} \to \mathcal{Y}$ as follows. Given any $u \in \mathcal{X}$, write $u = u^* + w$ with $w \in W^{1,\infty}_0(\Omega_T)$ and define

$$T_h(u) = Du^* + D(\rho_h * w),$$

where $\rho_h(z) = h^{-N}\rho(z/h)$, with $z = (x, t)$ and $N = n + 1$, is the standard h-mollifier in $\mathbb{R}^N$, and $\rho_h * w$ is the usual convolution in $\mathbb{R}^N$ with w extended to be zero outside $\bar{\Omega}_T$. Then, for each $h > 0$, the map $T_h\colon (\mathcal{X}, L^\infty) \to (\mathcal{Y}, L^1)$ is continuous, and for each $u \in \mathcal{X}$,

$$\lim_{h\to 0^+} \|T_h(u) - Du\|_{L^1(\Omega_T)} = \lim_{h\to 0^+} \|\rho_h * Dw - Dw\|_{L^1(\Omega_T)} = 0.$$

Therefore, the spatial gradient operator $D\colon \mathcal{X} \to \mathcal{Y}$ is the pointwise limit of a sequence of continuous maps $T_h\colon \mathcal{X} \to \mathcal{Y}$; hence $D\colon \mathcal{X} \to \mathcal{Y}$ is a *Baire-one map*. By Baire's category theorem (e.g., [6, Theorem 10.13]), there exists a *residual set* $\mathcal{G} \subset \mathcal{X}$ such that the operator D is continuous at each point of $\mathcal{G}$. Since $\mathcal{X} \setminus \mathcal{G}$ is of the *first category*, the set $\mathcal{G}$ is *dense* in $\mathcal{X}$. Therefore, given any $\varphi \in \mathcal{X}$, for each $\delta > 0$, there exists a function $u \in \mathcal{G}$ such that $\|u - \varphi\|_{L^\infty(\Omega_T)} < \delta$.

3. We now prove that each $u \in \mathcal{G}$ is a Lipschitz solution to (1.2). Let $u \in \mathcal{G}$ be given. By the density of $\mathcal{U}_\epsilon$ in $(\mathcal{X}, L^\infty)$ for each $\epsilon > 0$, for every $j \in \mathbb{N}$, there exists a function $u_j \in \mathcal{U}_{1/j}$ such that $\|u_j - u\|_{L^\infty(\Omega_T)} < 1/j$. Since the operator $D\colon (\mathcal{X}, L^\infty) \to (\mathcal{Y}, L^1)$ is continuous at u, we have $Du_j \to Du$ in $L^1(\Omega_T; \mathbb{R}^n)$. Furthermore, from (2.2) and the definition of $\mathcal{U}_{1/j}$, there exists a function $v_j \in W^{1,\infty}_{v^*}(\Omega_T; \mathbb{R}^n)$ such that for each $\zeta \in C^\infty(\bar{\Omega}_T)$ and each $t \in [0, T]$,

$$\begin{aligned} &\int_\Omega v_j(x,t)\cdot D\zeta(x,t)\,dx = -\int_\Omega u_j(x,t)\zeta(x,t)\,dx, \\ &\|(v_j)_t\|_{L^\infty(\Omega_T)} \le R, \quad \int_{\Omega_T} |(v_j)_t - A(Du_j)|\,dxdt \le \frac{1}{j}|\Omega_T|. \end{aligned} \tag{2.8}$$

Since $v_j(x, 0) = v^*(x, 0) \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ and $\|(v_j)_t\|_{L^\infty(\Omega_T)} \le R$, it follows that both sequences $\{v_j\}$ and $\{(v_j)_t\}$ are bounded in $L^2(\Omega_T; \mathbb{R}^n) \approx L^2((0, T); L^2(\Omega; \mathbb{R}^n))$. So we may assume

$$v_j \rightharpoonup v \text{ and } (v_j)_t \rightharpoonup v_t \text{ in } L^2((0,T); L^2(\Omega;\mathbb{R}^n))$$

for some $v \in W^{1,2}((0, T); L^2(\Omega; \mathbb{R}^n))$, where $\rightharpoonup$ denotes the weak convergence. Upon taking the limit as $j \to \infty$ in (2.8), since $v \in C([0, T]; L^2(\Omega; \mathbb{R}^n))$ and $A \in C(\mathbb{R}^n; \mathbb{R}^n)$, we obtain

$$\begin{aligned} &\int_\Omega v(x,t)\cdot D\zeta(x,t)\,dx = -\int_\Omega u(x,t)\zeta(x,t)\,dx \quad (t\in[0,T]), \\ &v_t(x,t) = A(Du(x,t)) \quad a.e.\ (x,t)\in\Omega_T. \end{aligned}$$

Consequently, by Proposition 2.1, u is a Lipschitz solution to (1.2).

4. Finally, assume $\mathcal{U}$ contains a function which is not a Lipschitz solution to (1.2); hence $\mathcal{G} \ne \mathcal{U}$. Then $\mathcal{G}$ cannot be a finite set, since otherwise the L^∞-closure $\mathcal{X} = \overline{\mathcal{G}} = \overline{\mathcal{U}}$ would be a finite set, making $\mathcal{U} = \mathcal{G}$. Therefore, in this case, (1.2) admits infinitely many Lipschitz solutions. The proof is complete. □

The following result implies that the density approach streamlined in Theorem 2.2 can be useful only when problem (1.2) is *non-parabolic*, that is, when $A(p)$ is non-monotone.

Proposition 2.3. *If $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is monotone, then (1.2) can have at most one Lipschitz solution.*

Proof. Let $u, \tilde{u} \in W^{1,\infty}(\Omega_T)$ be any two Lipschitz solutions to (1.2). Use the identity (1.5) for both u and $\tilde{u}$, subtract the two equations, take $\zeta = u - \tilde{u}$ (by approximations), and then apply the monotonicity of $A(p)$:

$$(A(Du) - A(D\tilde{u}))\cdot(Du - D\tilde{u}) \ge 0$$

to obtain

$$\int_\Omega (u(x,s)-\tilde{u}(x,s))^2dx \le \int_0^s\int_\Omega (u-\tilde{u})(u-\tilde{u})_t\,dxdt = \frac{1}{2}\int_\Omega (u(x,s)-\tilde{u}(x,s))^2dx \quad (0\le s\le T).$$

Thus $u \equiv \tilde{u}$ in Ω_T. □

2.3. Maximum principle for Lipschitz solutions

For fluxes $A(p)$ satisfying the *positivity* condition below, we prove the following maximum principle for all Lipschitz solutions of (1.2) (see also Kawohl and Kutev [21]). Clearly, the positivity condition is satisfied by the fluxes $A(p)$ given by (1.3) with the profiles $\sigma(s)$ of the Perona–Malik and Höllig types illustrated in Figs. 1 and 2. Note that the positivity condition is consistent with the Clausius–Duhem inequality in the second law of thermodynamics; see, e.g., [11, p. 79] and [36, p. 116].

Proposition 2.4. *Let* $u_0 \in W^{1,\infty}(\Omega)$ *and* $A\colon \mathbb{R}^n \to \mathbb{R}^n$ *satisfy the* positivity condition:

$$A(p)\cdot p \geq 0 \ \ (p \in \mathbb{R}^n).$$

Then any Lipschitz solution u to (1.2) *satisfies*

$$\min_{\bar\Omega} u_0 \leq u \leq \max_{\bar\Omega} u_0 \ \ \text{in } \Omega_T. \tag{2.9}$$

Proof. Let $u \in W^{1,\infty}(\Omega_T)$ be any Lipschitz solution to (1.2). By (1.5), for all $\zeta \in C^\infty(\bar\Omega_T)$,

$$\int_{\Omega_T} u_t(x,t)\zeta(x,t)dxdt = -\int_{\Omega_T} A(Du)\cdot D\zeta\, dxdt;$$

hence by approximation, this equality holds for all $\zeta \in W^{1,\infty}(\Omega_T)$. Taking $\zeta(x,t) = \phi(x,t)\psi(t)$ with arbitrary $\phi \in W^{1,\infty}(\Omega_T)$ and $\psi \in W^{1,\infty}(0,T)$, we deduce that

$$\int_{\Omega} u_t(x,t)\phi(x,t)\,dx = -\int_{\Omega} A(Du(x,t))\cdot D\phi(x,t)\,dx$$

for a.e. $t \in (0,T)$ and all $\phi \in W^{1,\infty}(\Omega_T)$. Now taking $\phi = u^{2k-1}$ with $k = 1, 2, \cdots$, we have for a.e. $t \in (0,T)$,

$$\begin{aligned}\frac{d}{dt}\Big(\int_{\Omega} u^{2k}dx\Big) &= 2k\int_{\Omega} u_t\phi\,dx = -2k\int_{\Omega} A(Du)\cdot D\phi dx\\ &= -2k(2k-1)\int_{\Omega} u^{2k-2}A(Du)\cdot Dudx \leq 0.\end{aligned}$$

From this we deduce that L^{2k}-norm of $u(\cdot,t)$ is non-increasing on $t \in [0,T]$; in particular,

$$\|u(\cdot,t)\|_{L^{2k}(\Omega)} \leq \|u_0\|_{L^{2k}(\Omega)} \quad \forall\, t \in [0,T],\ \ k = 1, 2, \cdots.$$

Letting $k \to \infty$, we obtain $\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$; hence

$$\|u\|_{L^\infty(\Omega_T)} = \|u_0\|_{L^\infty(\Omega)}. \tag{2.10}$$

Now let $m_1 = \min_{\bar\Omega} u_0$ and $m_2 = \max_{\bar\Omega} u_0$. We show $m_1 \leq u(x,t) \leq m_2$ for all $(x,t) \in \Omega_T$ to complete the proof. We proceed with three cases.

(a): $m_2 > 0$ and $|m_1| \leq m_2$. In this case, $\|u_0\|_{L^\infty(\Omega)} = m_2$; so by (2.10)

$$u(x,t) \leq \|u\|_{L^\infty(\Omega_T)} = \|u_0\|_{L^\infty(\Omega)} = m_2.$$

To obtain the lower bound, let $\tilde u_0 = -u_0 + m_2 + m_1$ and $\tilde u = -u + m_2 + m_1$. Then $\tilde u$ is a Lipschitz solution to (1.2) with new flux function $\tilde A(p) = -A(-p)$ and initial data $\tilde u_0$. Since $m_1 \leq \tilde u_0(x) \leq m_2$, as above, we have $\tilde u(x,t) \leq m_2$; hence $u(x,t) \geq m_1$ for all $(x,t) \in \Omega_T$.

(b): $m_2 > 0$ and $m_1 < -m_2$. Let $\tilde u_0 = -u_0$ and $\tilde u = -u$. Then $\tilde u$ is a Lipschitz solution to (1.2) with new flux function $\tilde A(p) = -A(-p)$ and initial data $\tilde u_0$. Since $-m_2 \leq \tilde u_0(x) \leq -m_1$ for all $x \in \Omega$ and $-m_1 > 0$, $|-m_2| = m_2 \leq -m_1$, it follows from Case (a) that $-m_2 \leq \tilde u(x,t) \leq -m_1$ and hence $m_1 \leq u(x,t) \leq m_2$ for all $(x,t) \in \Omega_T$.

(c): $m_2 \leq 0$. In this case $m_1 \leq 0$. If $m_1 = 0$ then $m_2 = 0$ and hence $u_0 \equiv 0$; so, by (2.10), $u \equiv 0$. Now assume $m_1 < 0$. Let again as in Case (b) $\tilde u_0 = -u_0$ and $\tilde u = -u$. Since $-m_2 \leq \tilde u_0(x) \leq -m_1$ for all $x \in \Omega$ and $-m_1 > 0$,

$|-m_2| = -m_2 \leq -m_1$, it follows again from Case (a) that $-m_2 \leq \tilde{u}(x,t) \leq -m_1$ and hence $m_1 \leq u(x,t) \leq m_2$ for all $(x,t) \in \Omega_T$. □

The rest of the paper is devoted to the construction of suitable boundary functions $\Phi = (u^*, v^*)$ and admissible sets $\mathcal{U} \subset W^{1,\infty}_{u^*}(\Omega_T)$ fulfilling the density property (2.7) for **Cases I** and **II**.

3. Structural conditions and detailed statements of main theorems

In this section, we assume the domain Ω and initial datum u_0 satisfy (1.6). We consider the diffusion fluxes $A(p)$ given by (1.3) and present the detailed statements of our main theorems by specifying the structural conditions on the profiles $\sigma(s) = sf(s^2)$ illustrated in Figs. 1 and 2.

3.1. Case I: Perona–Malik type equations

In this case, we assume the following structural condition on the profile $\sigma(s)$.

Hypothesis (PM): (See Figs. 1 and 3.)

(i) There exists a number $s_0 > 0$ such that

$$f \in C^0([0,\infty)) \cap C^3([0,s_0^2)) \cap C^1(s_0^2,\infty).$$

(ii) $\sigma'(s) > 0 \ \forall s \in [0,s_0)$, $\sigma'(s) < 0 \ \forall s \in (s_0,\infty)$, and

$$\lim_{s\to\infty} \sigma(s) = 0.$$

In this case, for each $r \in (0,\sigma(s_0))$, let $s_-(r) \in (0,s_0)$ and $s_+(r) \in (s_0,\infty)$ denote the unique numbers with $r = \sigma(s_\pm(r))$. Then by (ii),

$$\lim_{r\to 0^+} s_-(r) = 0, \qquad \lim_{r\to 0^+} s_+(r) = \infty. \tag{3.1}$$

Note that both profiles in (1.4) for the Perona–Malik model [33] satisfy Hypothesis (PM).

The following is the first main result of this paper in detail.

Theorem 3.1. *Let Ω be convex and $M_0 = \|Du_0\|_{L^\infty(\Omega)}$. Then for each $r \in (0,\sigma(M_0))$, there exists a number $l = l_r \in (0,r)$ such that for all $\tilde{r} \in (l,r)$ and all but at most countably many $\bar{r} \in (0,\tilde{r})$, there exist two disjoint open sets Ω_T^1, $\Omega_T^2 \subset \Omega_T$ with $|\Omega_T^1 \cup \Omega_T^2| = |\Omega_T|$ and infinitely many Lipschitz solutions u to (1.2) satisfying*

$$\begin{aligned} & u \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T^1), \quad u_t = \operatorname{div}(A(Du)) \ \text{pointwise in } \Omega_T^1, \\ & |Du(x,t)| < s_-(\bar{r}) \ \forall (x,t) \in \Omega_T^1, \quad \Omega_0^{\bar{r}} \subset \partial\Omega_T^1, \\ & |S| + |L| = |\Omega_T^2|, \ |L| > 0, \end{aligned}$$

where

$$\begin{aligned} \Omega_0^{\bar{r}} &= \{(x,0) \,|\, x \in \Omega, \ |Du_0(x)| < s_-(\bar{r})\}, \\ S &= \{(x,t) \in \Omega_T^2 \,|\, s_-(\bar{r}) \leq |Du(x,t)| \leq s_-(r)\}, \end{aligned}$$

and

$$L = \{(x,t) \in \Omega_T^2 \,|\, s_+(r) \leq |Du(x,t)| \leq s_+(\tilde{r})\}.$$

Remark 3.2. As we will see later, once the numbers $\tilde{r}$, $\bar{r}$ are chosen as above, the corresponding Lipschitz solutions in the theorem are all identically equal to a single function u^* in Ω_T^1, which is the classical solution to some modified uniformly parabolic Neumann problem. Here the function u^* depends on $\tilde{r}$ but not on $\bar{r}$, whereas Ω_T^1, Ω_T^2 do on $\bar{r}$:

$$\Omega_T^1 = \{(x,t) \in \Omega_T \mid |Du^*(x,t)| < s_-(\bar{r})\},$$
$$\Omega_T^2 = \{(x,t) \in \Omega_T \mid |Du^*(x,t)| > s_-(\bar{r})\}.$$

The choice of $\bar{r}$ is made to guarantee that the interface $\Omega_T \setminus (\Omega_T^1 \cup \Omega_T^2)$ has $(n+1)$-dimensional measure zero; this will be crucial for the proof of the theorem later. Choosing $\bar{r} = \tilde{r}$ may not be safe in this regard, since we have not enough information on the function u^* to be sure that the interface measure $|\{|Du^*| = s_-(\tilde{r})\}| = 0$. This forces us to sacrifice the benefit of the choice $\bar{r} = \tilde{r}$ that would separate the space–time domain Ω_T into two disjoint parts where the Lipschitz solutions are $C^{2+\alpha,1+\alpha/2}$ in one but nowhere C^1 in the other.

Remark 3.3. By (3.1), if $0 < r \ll \sigma(M_0)$, the corresponding Lipschitz solutions u have *large* and *small* gradient regimes L and $\Omega_T^1 \cup S$ in Ω_T up to measure zero, representing the almost constant and sharp edge parts of u in Ω_T, respectively. Although there is a fine mixture of the disjoint regimes $L,\ S \subset \Omega_T^2$ due to a micro-structured ramping with alternate gradients of finite size, such properties together with (2.9) for solutions u are *somehow* reflected in numerical simulations (see, e.g., [33, Fig. 13]). On the other hand, it has been observed in [2,3] that as the limits of solutions to a class of regularized equations, infinitely many different evolutions may arise under the same initial datum u_0. Our non-uniqueness result seems to reflect this pathological behavior of forward–backward problem (1.2).

We restate Theorem 1.2 for **Case I**. The proof of this result appears in Subsection 4.2.

Theorem 3.4. *Let Ω and M_0 be as in Theorem 3.1. If $M_0 < s_0$, then (1.2) has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T)$ satisfying $\|Du\|_{L^\infty(\Omega_T)} = \|Du_0\|_{L^\infty(\Omega)}$.*

For the given initial datum u_0 with $M_0 < s_0$, by Theorem 3.1, there are infinitely many Lipschitz solutions to problem (1.2). On the other hand, by the theorem here, we have a *special* Lipschitz solution u to (1.2), which is also classical. It may be interesting to observe that none of the Lipschitz solutions from Theorem 3.1 coincide with this special solution u.

3.2. Case II: Höllig type equations

In this case, we impose the following structural condition on the profile $\sigma(s)$.
Hypothesis (H): (See Figs. 2 and 4.)

(i) There exist two numbers $s_2 > s_1 > 0$ such that

$$f \in C^0([0,\infty)) \cap C^{1+\alpha}([0, s_1^2) \cup (s_2^2, \infty)).$$

(ii) $\sigma'(s) > 0\ \forall s \in [0, s_1) \cup (s_2, \infty)$, $\sigma(s_1) > \sigma(s_2) \geq 0$, $\lambda \leq \sigma'(s) \leq \Lambda\ \ \forall s \geq 2s_2$, where $\Lambda \geq \lambda > 0$ are constants.
(iii) Let $s_1^* \in [0, s_1)$ and $s_2^* \in (s_2, \infty)$ denote the unique numbers with $\sigma(s_1^*) = \sigma(s_2)$ and $\sigma(s_2^*) = \sigma(s_1)$, respectively.

In addition to Hypothesis (H), if we suppose $\sigma(s) \geq 0$ for all $s \in (s_1, s_2)$, then $A(p) \cdot p \geq 0$ for all $p \in \mathbb{R}^n$, and so (2.9) is satisfied by any Lipschitz solution u to (1.2); but we do not explicitly assume such positivity for **Case II** unless otherwise stated.

With Hypothesis (H), for each $r \in (\sigma(s_2), \sigma(s_1))$, let $s_-(r) \in (s_1^*, s_1)$ and $s_+(r) \in (s_2, s_2^*)$ denote the unique numbers with $r = \sigma(s_\pm(r))$.

The following theorem is the second main result of this paper that generalizes those of [19,40] to any dimension $n \geq 1$.

Theorem 3.5. *Let $M_0 = \|Du_0\|_{L^\infty(\Omega)}$, $M_0' = \min\{M_0, s_1\}$, and $|Du_0(x_0)| \in (s_1^*, s_2^*)$ for some $x_0 \in \Omega$. Then for each $r \in (\sigma(s_2), \sigma(M_0'))$, there exists a number $l = l_r \in (\sigma(s_2), r)$ such that for all $\tilde{r} \in (l, r)$ and all but countably many $\bar{r}_1 \in (\sigma(s_2), \tilde{r})$, $\bar{r}_3 \in (r, \sigma(s_1))$, there exist three disjoint open sets $\Omega_T^1,\ \Omega_T^2,\ \Omega_T^3 \subset \Omega_T$ with $|\Omega_T^1 \cup \Omega_T^2 \cup \Omega_T^3| = |\Omega_T|$ and infinitely many Lipschitz solutions u to (1.2) satisfying*

$$
\begin{aligned}
&u \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T^1 \cup \bar{\Omega}_T^3), \quad u_t = \operatorname{div}(A(Du)) \ \textit{pointwise in}\ \Omega_T^1 \cup \Omega_T^3,\\
&|Du(x,t)| < s_-(\bar{r}_1)\ \forall (x,t) \in \Omega_T^1,\ \Omega_0^{\bar{r}_1} \subset \partial\Omega_T^1,\\
&|Du(x,t)| > s_+(\bar{r}_3)\ \forall (x,t) \in \Omega_T^3,\ \Omega_0^{\bar{r}_3} \subset \partial\Omega_T^3,\\
&|S| + |L| = |\Omega_T^2|,\ \ |S| > 0,\ \ |L| > 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega_0^{\bar{r}_1} &= \{(x,0)\,|\, x \in \Omega,\ |Du_0(x)| < s_-(\bar{r}_1)\},\\
\Omega_0^{\bar{r}_3} &= \{(x,0)\,|\, x \in \Omega,\ |Du_0(x)| > s_+(\bar{r}_3)\},\\
S &= \{(x,t) \in \Omega_T^2 \,|\, s_-(\bar{r}_1) \le |Du(x,t)| \le s_-(r)\},
\end{aligned}
$$

and

$$
L = \{(x,t) \in \Omega_T^2 \,|\, s_+(\tilde{r}) \le |Du(x,t)| \le s_+(\bar{r}_3)\}.
$$

In regard to this theorem, an explanation similar to Remark 3.2 can be made; but we omit this. As a byproduct, we also have the following simple existence result whose proof appears after that of Theorem 3.5.

Corollary 3.6. *For any initial datum $u_0 \in C^{2+\alpha}(\bar{\Omega})$ with $Du_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$, (1.2) has at least one Lipschitz solution.*

We now restate Theorem 1.2 for **Case II**. As in **Case I**, the proof of this result is in Subsection 4.2.

Theorem 3.7. *Let Ω be convex and $M_0 = \|Du_0\|_{L^\infty(\Omega)}$. Assume further that $f \in C^3([0, s_1^2) \cup (s_2^2, \infty))$. If $M_0 < s_1$, then (1.2) has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T)$ satisfying $\|Du\|_{L^\infty(\Omega_T)} = \|Du_0\|_{L^\infty(\Omega)}$.*

Unlike Theorem 3.5 and its corollary, we require the convexity of the domain Ω in this theorem. If $s_1^* < M_0 < s_1$, Theorems 3.5 and 3.7 apply to give infinitely many Lipschitz solutions and one *special* Lipschitz (classical) solution to problem (1.2) under the same initial datum u_0, respectively. However, a subtlety arises when $M_0 \le s_1^*$: Is there a Lipschitz solution to (1.2) other than the classical one in this case? Let us discuss more on this in the remark below.

Remark 3.8 *(Breakdown of uniqueness).* In this remark, let $\sigma(s)$ satisfy

$$
\sigma(s) > 0 \quad \forall s \in (s_1, s_2] \tag{3.2}
$$

in addition to Hypothesis (H) (as in Fig. 2), and let f be as in Theorem 3.7. Then by Corollary 3.6, for any $u_0 \in C^{2+\alpha}(\bar{\Omega})$ with $Du_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$, problem (1.2) has at least one Lipschitz solution. In particular, if $M_0 = 0$ with $M_0 := \|Du_0\|_{L^\infty(\Omega)}$, that is, u_0 is constant in Ω, then by Proposition 2.4, the constant function $u \equiv u_0$ in Ω_T is a unique solution to (1.2). So a natural question is to ask if there is a number $\tilde{M}_0 > 0$ such that (1.2) has a unique Lipschitz solution for all initial data $u_0 \in C^{2+\alpha}(\bar{\Omega})$ with $Du_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $M_0 \le \tilde{M}_0$. To address this question, define

$$
M_0^* = \sup\left\{ \tilde{M}_0 \ge 0 \,\middle|\, \begin{array}{l} \forall u_0 \in C^{2+\alpha}(\bar{\Omega}) \text{ with } Du_0 \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ and} \\ M_0 \le \tilde{M}_0,\ \text{(1.2) has a unique Lipschitz solution} \end{array} \right\},
$$

which we may call as the *critical threshold of uniqueness* (CTU) for (1.2). The CTU M_0^* may depend not only on the profile $\sigma(s)$ but also on the dimension n, final time T, and domain Ω of problem (1.2). By Theorem 3.5, we have an estimate:

$$
0 \le M_0^* \le s_1^*,
$$

which is valid for all dimensions $n \ge 1$, final times $T > 0$, and domains Ω satisfying (1.6). Our question now boils down to the point of asking if $M_0^* > 0$.

When $n = 1$, the answer is affirmative; in this case, the CTU M_0^* is positive and depends only on the profile $\sigma(s)$. This fact can be derived by using an *a priori* estimate on the spatial derivatives of Lipschitz solutions. From [20], when $n = 1$, any Lipschitz solution u to (1.2) satisfies

$$\|u_x\|_{L^\infty(\Omega_T)} \le \frac{\|\sigma(|u_0'|)\|_{L^\infty(\Omega)}}{c}, \tag{3.3}$$

where $c := \inf_{s\in(0,\infty)} \frac{\sigma(s)}{s} > 0$, by Hypothesis (H) and (3.2). Note

$$\frac{\sigma(s_1^*)}{s_1} > \frac{\sigma(s_2)}{s_2} \ge c;$$

that is, $\sigma(s_1^*) > cs_1$. As $\sigma(0) = 0$ and $\sigma(s)$ is strictly increasing on $s \in [0, s_1^*]$, there is a unique number $M_0^l \in (0, s_1^*)$ with $\sigma(M_0^l) = cs_1$. Suppose now that $\|u_0'\|_{L^\infty(\Omega)} < M_0^l$. Then by Theorem 3.7, there exists a classical solution $\tilde u \in C^{2+\alpha,1+\alpha/2}(\bar\Omega_T)$ to (1.2) with $\|\tilde u_x\|_{L^\infty(\Omega_T)} = \|u_0'\|_{L^\infty(\Omega)} < M_0^l$. On the other hand, if u is any Lipschitz solution to (1.2), we have from (3.3) that $\|u_x\|_{L^\infty(\Omega_T)} < s_1$. Modifying the profile $\sigma(s)$ to the right of the threshold $\max\{\|u_0'\|_{L^\infty(\Omega)}, \|u_x\|_{L^\infty(\Omega_T)}\} < s_1$, both $\tilde u$ and u become Lipschitz solutions to some uniformly parabolic problem of *type* (1.2) with monotone flux under the same initial datum u_0. By Proposition 2.3, we thus have $\tilde u \equiv u$ in Ω_T; hence, the Lipschitz solution $\tilde u$ to (1.2) is unique. Therefore, we have an improved estimate for the CTU M_0^* in $n = 1$:

$$0 < M_0^l \le M_0^* \le s_1^*.$$

As a toy example for $n = 1$, we consider the general piecewise linear profile $\sigma(s)$ of the Höllig type. Let s_1, s_2, k_1, k_2 and k_3 be positive numbers with $s_2 > s_1$ and $-k_2(s_2 - s_1) + k_1 s_1 > 0$. Let us take

$$\sigma(s) = \begin{cases} k_1 s, & 0 \le s \le s_1, \\ -k_2(s - s_1) + k_1 s_1, & s_1 \le s \le s_2, \\ k_3(s - s_2) - k_2(s_2 - s_1) + k_1 s_1, & s \ge s_2; \end{cases}$$

then, if $-k_2(s_2 - s_1) + k_1 s_1 < k_3 s_2$, we have

$$0 < -s_1\frac{k_2}{k_1} + \frac{s_1^2}{s_2}\Big(1 + \frac{k_2}{k_1}\Big) = M_0^l \le M_0^* \le s_1^* = -s_2\frac{k_2}{k_1} + s_1\Big(1 + \frac{k_2}{k_1}\Big),$$

or if $-k_2(s_2 - s_1) + k_1 s_1 \ge k_3 s_2$, it follows that

$$0 < s_1\frac{k_3}{k_1} = M_0^l \le M_0^* \le s_1^* = -s_2\frac{k_2}{k_1} + s_1\Big(1 + \frac{k_2}{k_1}\Big).$$

Unfortunately, estimate (3.3) from [20] is not directly generalized to dimensions $n \ge 2$. However, we still expect that the CTU $M_0^* > 0$ even in dimensions $n \ge 2$ for all $T > 0$ as long as Ω is convex. Since this study is beyond the scope of this paper, we leave it to the interested readers.

4. Some useful results

This section prepares some essential ingredients for the proofs of existence theorems, Theorems 3.1 and 3.5. The results in this section have already appeared in [24, Section 2]; we include them here for the convenience of the reader.

4.1. Uniformly parabolic equations

We refer to the standard references (e.g., [26,27]) for some notations concerning functions and domains of class $C^{k+\alpha}$ with an integer $k \ge 0$.

Assume $\tilde f \in C^{1+\alpha}([0,\infty))$ is a function satisfying

$$\theta \le \tilde f(s) + 2s\tilde f'(s) \le \Theta \quad \forall s \ge 0, \tag{4.1}$$

where $\Theta \ge \theta > 0$ are constants. This condition is equivalent to $\theta \le (s\tilde f(s^2))' \le \Theta$ for all $s \in \mathbb{R}$; hence, $\theta \le \tilde f(s) \le \Theta$ for all $s \ge 0$. Let

$$\tilde A(p) = \tilde f(|p|^2)p \quad (p \in \mathbb{R}^n).$$

Then we have

$$\tilde A^i_{p_j}(p) = \tilde f(|p|^2)\delta_{ij} + 2\tilde f'(|p|^2)p_i p_j \quad (i, j = 1, 2, \cdots, n;\ p \in \mathbb{R}^n)$$

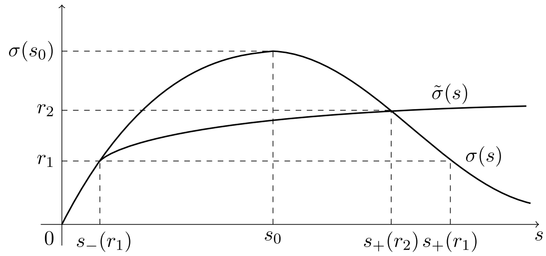

Fig. 3. **Case I:** Perona–Malik type profile $\sigma(s)$ and modified function $\tilde{\sigma}(s)$.

and hence the *uniform ellipticity condition*:

$$\theta|q|^2 \le \sum_{i,j=1}^{n} \tilde{A}^i_{p_j}(p)q_i q_j \le \Theta|q|^2 \quad \forall\, p,\ q \in \mathbb{R}^n. \tag{4.2}$$

Theorem 4.1. *Assume (1.6). Then the initial-Neumann boundary value problem*

$$\begin{cases} u_t = \operatorname{div}(\tilde{A}(Du)) & \text{in } \Omega_T, \\ \partial u/\partial \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega \end{cases} \tag{4.3}$$

has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T)$. Moreover, if $\tilde{f} \in C^3([0,\infty))$ and Ω is convex, then the gradient maximum principle *holds:*

$$\|Du\|_{L^\infty(\Omega_T)} = \|Du_0\|_{L^\infty(\Omega)}. \tag{4.4}$$

Proof. See [24, Theorem 2.1]. □

4.2. Modification of the profile functions

The following elementary results can be proved in a similar way as in [7,39]; we omit the proofs.

Lemma 4.2 *(**Case I:** Perona–Malik type; see Fig. 3). Assume* Hypothesis (PM). *For every $0 < r_1 < r_2 < \sigma(s_0)$, there exists a function $\tilde{\sigma} \in C^3([0,\infty))$ such that*

$$\begin{cases} \tilde{\sigma}(s) = \sigma(s), & 0 \le s \le s_-(r_1), \\ \tilde{\sigma}(s) < \sigma(s), & s_-(r_1) < s < s_+(r_2), \\ \theta \le \tilde{\sigma}'(s) \le \Theta, & 0 \le s < \infty \end{cases} \tag{4.5}$$

for some constants $\Theta \ge \theta > 0$. With such a function $\tilde{\sigma}$, define $\tilde{f}(s) = \tilde{\sigma}(\sqrt{s})/\sqrt{s}$ $(s > 0)$ and $\tilde{f}(0) = f(0)$; then $\tilde{f} \in C^3([0,\infty))$ fulfills condition (4.1).

Lemma 4.3 *(**Case II:** Höllig type; see Fig. 4). Assume* Hypothesis (H). *For every $\sigma(s_2) < r_1 < r_2 < \sigma(s_1)$, there exists a function $\tilde{\sigma} \in C^{1+\alpha}([0,\infty))$ such that*

$$\begin{cases} \tilde{\sigma}(s) = \sigma(s), & 0 \le s \le s_-(r_1), \\ \tilde{\sigma}(s) < \sigma(s), & s_-(r_1) < s \le s_-(r_2), \\ \tilde{\sigma}(s) > \sigma(s), & s_+(r_1) \le s < s_+(r_2), \\ \tilde{\sigma}(s) = \sigma(s), & s_+(r_2) \le s < \infty, \\ \theta \le \tilde{\sigma}'(s) \le \Theta, & 0 \le s < \infty \end{cases} \tag{4.6}$$

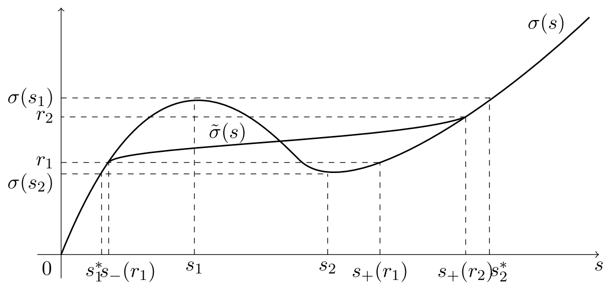

Fig. 4. **Case II:** Höllig type profile $\sigma(s)$ and modified function $\tilde{\sigma}(s)$.

for some constants $\Theta \geq \theta > 0$. *With such a function* $\tilde{\sigma}$, *define* $\tilde{f}(s) = \tilde{\sigma}(\sqrt{s})/\sqrt{s}$ $(s > 0)$ *and* $\tilde{f}(0) = f(0)$; *then* $\tilde{f} \in C^{1+\alpha}([0, \infty))$ *fulfills condition* (4.1). *Moreover, if* $f \in C^3([0, s_1^2) \cup (s_2^2, \infty))$ *in addition, then* $\tilde{\sigma}$, $\tilde{f}$ *can be chosen to be also in* $C^3([0, \infty))$.

4.3. *Proof of Theorems 3.4 and 3.7*

Let $M_0 = \|Du_0\|_{L^\infty(\Omega)}$. In **Case I** (Perona–Malik type), we have $0 < M_0 < s_0$; so we can select $0 < r_1 < r_2 < \sigma(s_0)$ such that $s_-(r_1) = M_0$. In **Case II** (Höllig type), we have $0 < M_0 < s_1$; so we can select $\sigma(s_2) < r_1 < r_2 < \sigma(s_1)$ such that $s_-(r_1) \geq M_0$. Use such a choice of r_1, r_2 in Lemma 4.2 (**Case I**), Lemma 4.3 (**Case II**) to obtain a C^3 function $\tilde{f}$ as stated in the lemma. Let $\tilde{A}(p) = \tilde{f}(|p|^2)p$. For this $\tilde{A}(p)$, problem (4.3) is uniformly parabolic; hence, by Theorem 4.1, (4.3) has a unique solution $u \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega}_T)$ satisfying

$$\|Du\|_{L^\infty(\Omega_T)} = \|Du_0\|_{L^\infty(\Omega)}.$$

Since $\tilde{A}(p) = A(p)$ on $|p| \leq s_-(r_1)$ and $s_-(r_1) \geq M_0$, it follows that $\tilde{A}(Du(x,t)) = A(Du(x,t))$ in Ω_T; this proves u is a classical solution to problem (1.2). On the other hand, we easily see that any classical solution to (1.2) satisfying $\|Du\|_{L^\infty(\Omega_T)} = M_0$ is also a classical solution to (4.3) and hence must be unique. This completes the proofs of Theorems 3.4 and 3.7.

4.4. *Right inverse of the divergence operator*

The following result has been proved in [24, Theorem 2.3], following an argument of Bourgain and Brézis [4, Lemma 4].

Theorem 4.4. *For each* $u \in W_0^{1,\infty}(Q \times I)$ *satisfying* $\int_Q u(x,t)\,dx = 0$ *for all* $t \in I$, *there exists a function* $v = \mathcal{R}u \in W_0^{1,\infty}(Q \times I; \mathbb{R}^n)$ *such that* $\operatorname{div} v = u$ *a.e. in* $Q \times I$ *and*

$$\|v_t\|_{L^\infty(Q\times I)} \leq C_n\,(|J_1| + \cdots + |J_n|)\|u_t\|_{L^\infty(Q\times I)}, \tag{4.7}$$

where $Q = J_1 \times \cdots \times J_n$ *and* C_n *is a constant depending only* n. *Moreover, if* $u \in C^1(\overline{Q \times I})$ *then* $v = \mathcal{R}u \in C^1(\overline{Q \times I}; \mathbb{R}^n)$.

5. Geometry of the relevant matrix sets

Let $A(p)$ be given by (1.3). We assume Hypothesis (PM) or (H) unless one is chosen specifically. Recall the definition (2.3) with $l = 0$:

$$K_0 = K(0) = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \,\middle|\, p \in \mathbb{R}^n,\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0 \right\}. \tag{5.1}$$

Under Hypothesis (PM) or (H), certain structures of the set K_0 turn out to be still quite useful, especially when it comes to the relaxation of homogeneous partial differential inclusion $\nabla\omega(z) \in K_0$ with $z = (x,t)$ and $\omega = (\varphi, \psi)$. We investigate these structures and establish such relaxation results throughout this section.

5.1. *Geometry of the matrix set K_0*

We study some subsets of K_0, depending on the different types of profiles.

Case I: Hypothesis (PM)

(See Fig. 3.) In this case, we assume the following. Fix any two numbers $0 < r_1 < r_2 < \sigma(s_0)$, and let $F_0 = F_{r_1,r_2}(0)$ be the subset of K_0 defined by

$$F_0 = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \;\middle|\; \begin{array}{l} p \in \mathbb{R}^n,\ |p| \in (s_-(r_1), s_-(r_2)) \cup (s_+(r_2), s_+(r_1)), \\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0 \end{array} \right\}.$$

We decompose the set F_0 into two disjoint subsets as follows:

$$F_- = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \;\middle|\; \begin{array}{l} p \in \mathbb{R}^n,\ |p| \in (s_-(r_1), s_-(r_2)), \\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0 \end{array} \right\},$$

$$F_+ = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \;\middle|\; \begin{array}{l} p \in \mathbb{R}^n,\ |p| \in (s_+(r_2), s_+(r_1)), \\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0 \end{array} \right\}.$$

Case II: Hypothesis (H)

(See Fig. 4.) In this case, we assume the following. Fix any two numbers $\sigma(s_2) < r_1 < r_2 < \sigma(s_1)$, and let $F_0 = F_{r_1,r_2}(0)$ be the subset of K_0 given by

$$F_0 = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \;\middle|\; \begin{array}{l} p \in \mathbb{R}^n,\ |p| \in (s_-(r_1), s_-(r_2)) \cup (s_+(r_1), s_+(r_2)), \\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0 \end{array} \right\}.$$

The set F_0 is also decomposed into two disjoint subsets as follows:

$$F_- = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \;\middle|\; \begin{array}{l} p \in \mathbb{R}^n,\ |p| \in (s_-(r_1), s_-(r_2)), \\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0 \end{array} \right\},$$

$$F_+ = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \;\middle|\; \begin{array}{l} p \in \mathbb{R}^n,\ |p| \in (s_+(r_1), s_+(r_2)), \\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0 \end{array} \right\}.$$

In order to study the homogeneous differential inclusion $\nabla\omega(z) \in K_0$, we first scrutinize the rank-one structure of the set $F_0 \subset K_0$. We introduce the following notation.

Definition 5.1. For a given set $E \subset \mathbb{M}^{(1+n)\times(n+1)}$, $L(E)$ is defined to be the set of all matrices $\xi \in \mathbb{M}^{(1+n)\times(n+1)}$ that are not in E but are representable by $\xi = \lambda\xi_1 + (1-\lambda)\xi_2$ for some $\lambda \in (0,1)$ and $\xi_1, \xi_2 \in E$ with $\operatorname{rank}(\xi_1 - \xi_2) = 1$, or equivalently,

$$L(E) = \{\xi \notin E \mid \xi + t_\pm \eta \in E \text{ for some } t_- < 0 < t_+ \text{ and } \operatorname{rank}\eta = 1\}.$$

For the matrix set F_0, we define

$$R(F_0) = \bigcup_{\xi_\pm \in F_\pm,\ \operatorname{rank}(\xi_+ - \xi_-) = 1} (\xi_-, \xi_+),$$

where (ξ_-, ξ_+) is the open line segment in $\mathbb{M}^{(1+n)\times(n+1)}$ joining $\xi_\pm$. Then from careful analyses, one can actually deduce

$$L(F_0) = R(F_0) \cup L(F_+) \quad \text{in } \textbf{Case I} \tag{5.2}$$

and

$$L(F_0) = R(F_0) \quad \text{in } \textbf{Case II}. \tag{5.3}$$

In (5.2), due to the backward nature of $\sigma(s)$ on $(s_+(r_2), s_+(r_1))$ for **Case I**, the set $L(F_+)$ turns out to be non-empty. On the other hand, as only forward parts of σ are involved in F_0 for **Case II**, no such set appears in (5.3). Regardless of this discrepancy, we can only stick to the analysis of the set $R(F_0)$ for both cases towards the existence results, Theorems 3.1 and 3.5.

We perform the step-by-step analysis of the set $R(F_0)$ for both cases simultaneously.

5.1.1. Alternate expression for $R(F_0)$

We investigate more specific criteria for matrices in $R(F_0)$.

Lemma 5.2. *Let $\xi \in \mathbb{M}^{(1+n)\times(n+1)}$. Then $\xi \in R(F_0)$ if and only if there exist numbers $t_- < 0 < t_+$ and vectors $q, \gamma \in \mathbb{R}^n$ with $|q| = 1$, $\gamma \cdot q = 0$ such that for each $b \in \mathbb{R} \setminus \{0\}$, if $\eta = \begin{pmatrix} q & b \\ \frac{1}{b} q \otimes \gamma & \gamma \end{pmatrix}$, then $\xi + t_\pm \eta \in F_\pm$.*

Proof. Assume $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0)$. By definition, $\xi + t_\pm \tilde{\eta} \in F_\pm$, where $t_- < 0 < t_+$ and $\tilde{\eta}$ is a rank-one matrix given by

$$\tilde{\eta} = \begin{pmatrix} a \\ \alpha \end{pmatrix} \otimes (q, \tilde{b}) = \begin{pmatrix} aq & a\tilde{b} \\ \alpha \otimes q & \tilde{b}\alpha \end{pmatrix}, \quad a^2 + |\alpha|^2 \neq 0, \quad \tilde{b}^2 + |q|^2 \neq 0,$$

for some $a, \tilde{b} \in \mathbb{R}$ and $\alpha, q \in \mathbb{R}^n$; here $\alpha \otimes q$ denotes the rank-one or zero matrix $(\alpha_i q_j)$ in $\mathbb{M}^{n\times n}$. Condition $\xi + t_\pm \tilde{\eta} \in F_\pm$ with $t_- < 0 < t_+$ is equivalent to the following: For **Case I**,

$$\begin{aligned} &\operatorname{tr} B = 0, \quad \alpha \cdot q = 0, \quad A(p + t_\pm aq) = \beta + t_\pm \tilde{b}\alpha, \\ &|p + t_+ aq| \in (s_+(r_2), s_+(r_1)), \quad |p + t_- aq| \in (s_-(r_1), s_-(r_2)). \end{aligned} \tag{5.4}$$

For **Case II**,

$$\begin{aligned} &\operatorname{tr} B = 0, \quad \alpha \cdot q = 0, \quad A(p + t_\pm aq) = \beta + t_\pm \tilde{b}\alpha, \\ &|p + t_+ aq| \in (s_+(r_1), s_+(r_2)), \quad |p + t_- aq| \in (s_-(r_1), s_-(r_2)). \end{aligned} \tag{5.5}$$

Therefore, $aq \neq 0$. Upon rescaling $\tilde{\eta}$ and $t_\pm$, we can assume $a = 1$ and $|q| = 1$; namely,

$$\tilde{\eta} = \begin{pmatrix} q & \tilde{b} \\ \alpha \otimes q & \tilde{b}\alpha \end{pmatrix}, \quad |q| = 1, \quad \alpha \cdot q = 0.$$

We now let $\gamma = \tilde{b}\alpha$. Let $b \in \mathbb{R} \setminus \{0\}$ and

$$\eta = \begin{pmatrix} q & b \\ \frac{1}{b}\gamma \otimes q & \gamma \end{pmatrix}.$$

From (5.4) or (5.5), it follows that $\xi + t_\pm \eta \in F_\pm$.

The converse directly follows from the definition of $R(F_0)$. □

5.1.2. Diagonal components of matrices in $R(F_0)$

The following gives a description for the diagonal components of matrices in $R(F_0)$.

Lemma 5.3.

$$R(F_0) = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \,\middle|\, c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0,\ (p, \beta) \in \mathcal{S} \right\} \tag{5.6}$$

for some set $\mathcal{S} = \mathcal{S}_{r_1,r_2} \subset \mathbb{R}^{n+n}$.

Proof. Let $(c, B), (c', B') \in \mathbb{R} \times \mathbb{M}^{n\times n}$ be such that $\operatorname{tr} B = \operatorname{tr} B' = 0$, and define

$$\begin{aligned} \mathcal{S}_{(c,B)} &= \left\{ (p, \beta) \in \mathbb{R}^{n+n} \,\middle|\, \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0) \right\}, \\ \mathcal{S}_{(c',B')} &= \left\{ (p, \beta) \in \mathbb{R}^{n+n} \,\middle|\, \begin{pmatrix} p & c' \\ B' & \beta \end{pmatrix} \in R(F_0) \right\}. \end{aligned}$$

It is sufficient to show that $\mathcal{S}_{(c,B)} = \mathcal{S}_{(c',B')} =: \mathcal{S}$. Let $(p,\beta) \in \mathcal{S}_{(c,B)}$, that is, $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0)$. Then $\xi_\pm := \xi + t_\pm \eta \in F_\pm$ for some $t_- < 0 < t_+$ and $\operatorname{rank} \eta = 1$. Observe that $\xi = \lambda \xi_+ + (1-\lambda)\xi_-$ with $\lambda = \frac{-t_-}{t_+ - t_-} \in (0,1)$ and that

$$\xi' := \begin{pmatrix} p & c' \\ B' & \beta \end{pmatrix} = \xi + \begin{pmatrix} 0 & \tilde{c} \\ \tilde{B} & 0 \end{pmatrix} = \lambda \tilde{\xi}_+ + (1-\lambda)\tilde{\xi}_-$$

where $\tilde{c} = c' - c$, $\tilde{B} = B' - B$, and $\tilde{\xi}_\pm = \xi_\pm + \begin{pmatrix} 0 & \tilde{c} \\ \tilde{B} & 0 \end{pmatrix}$. Since $\xi_\pm \in F_\pm$ and $\operatorname{tr} \tilde{B} = 0$, we have $\tilde{\xi}_\pm \in F_\pm$, and so $\xi' \in R(F_0)$. This implies $(p,\beta) \in \mathcal{S}_{(c',B')}$; hence $\mathcal{S}_{(c,B)} \subset \mathcal{S}_{(c',B')}$. Likewise, $\mathcal{S}_{(c',B')} \subset \mathcal{S}_{(c,B)}$; that is, $\mathcal{S}_{(c,B)} = \mathcal{S}_{(c',B')}$. □

5.1.3. Selection of approximate collinear rank-one connections for $R(F_0)$

We begin with a 2-dimensional description for the rank-one connections of diagonal components of matrices in $R(F_0)$ in a general form. The following lemma deals with **Cases I** and **II** in a parallel manner.

Lemma 5.4. *For all positive numbers a, b, c with $b > a$, there exists a continuous function*

$$h_1(a,b,c;\cdot,\cdot,\cdot) : I^1_{a,b,c} = [0,a) \times [0,\infty) \times [0,c) \to [0,\infty) \quad (\textbf{Case I})$$
$$h_2(a,b,c;\cdot,\cdot,\cdot) : I^2_{a,b,c} = [0,a) \times [0,b-a) \times [0,c) \to [0,\infty) \quad (\textbf{Case II})$$

with $h_i(a,b,c;0,0,0) = 0$ satisfying the following:

Let δ_1, δ_2 and η be any positive numbers with

$$0 < a - \delta_1 < a < b < b + \delta_2, \quad 0 < c - \eta < c, \quad (\textbf{Case I})$$
$$0 < a - \delta_1 < a < b - \delta_2 < b, \quad 0 < c - \eta < c, \quad (\textbf{Case II})$$

*and let $R_1 \in [a-\delta_1, a]$, $R_2 \in [b, b+\delta_2]$ (**Case I**), $R_2 \in [b-\delta_2, b]$ (**Case II**), and $\tilde{R}_1, \tilde{R}_2 \in [c-\eta, c]$. Suppose $\theta \in [-\pi/2, \pi/2]$ and*

$$\Big(\tilde{R}_1\big(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta)\big) - \tilde{R}_2\big(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta)\big)\Big)$$
$$\cdot\Big(R_1\big(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta)\big) - R_2\big(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta)\big)\Big) = 0.$$

Then $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\tilde{R}_1 \geq \tilde{R}_2$, and

$$\max\Big\{ \big|(0,a) - R_1\big(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta)\big)\big|, \big|(0,b) - R_2\big(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta)\big)\big|$$
$$\big|(0,c) - \tilde{R}_1\big(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta)\big)\big|, \big|(0,c) - \tilde{R}_2\big(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta)\big)\big|\Big\}$$
$$\leq h_1(a,b,c;\delta_1,\delta_2,\eta), \quad (\textbf{Case I})$$
$$\max\Big\{ \big|(0,a) - R_1\big(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta)\big)\big|, \big|(0,b) - R_2\big(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta)\big)\big|$$
$$\big|(0,c) - \tilde{R}_1\big(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta)\big)\big|, \big|(0,c) - \tilde{R}_2\big(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta)\big)\big|\Big\}$$
$$\leq h_2(a,b,c;\delta_1,\delta_2,\eta). \quad (\textbf{Case II})$$

Proof. By assumption,

$$\begin{aligned} 0 &= (\tilde{R}_1(-\sin\theta, \cos\theta) - \tilde{R}_2(\sin\theta, \cos\theta)) \cdot (R_1(-\sin\theta, \cos\theta) - R_2(\sin\theta, \cos\theta)) \\ &= (-(\tilde{R}_1 + \tilde{R}_2)\sin\theta, (\tilde{R}_1 - \tilde{R}_2)\cos\theta) \cdot (-(R_1+R_2)\sin\theta, (R_1 - R_2)\cos\theta) \\ &= (\tilde{R}_1 + \tilde{R}_2)(R_1+R_2)\sin^2\theta + (\tilde{R}_1 - \tilde{R}_2)(R_1 - R_2)\cos^2\theta, \end{aligned}$$

that is,

$$(R_2 - R_1)(\tilde{R}_1 - \tilde{R}_2)\cos^2\theta = (R_1 + R_2)(\tilde{R}_1 + \tilde{R}_2)\sin^2\theta;$$

hence, $\theta \neq \pm\frac{\pi}{2}$, $\tilde{R}_1 \geq \tilde{R}_2$, and

$$\theta = \pm \tan^{-1}\left(\sqrt{\frac{(R_2 - R_1)(\tilde{R}_1 - \tilde{R}_2)}{(R_1 + R_2)(\tilde{R}_1 + \tilde{R}_2)}}\right).$$

So

$$|\theta| \leq \tan^{-1}\left(\sqrt{\frac{(b-a+\delta_1+\delta_2)\eta}{2(a+b-\delta_1)(c-\eta)}}\right) =: g_1(a,b,c;\delta_1,\delta_2,\eta), \quad \textbf{(Case I)}$$
$$|\theta| \leq \tan^{-1}\left(\sqrt{\frac{(b-a+\delta_1)\eta}{2(a+b-\delta_1-\delta_2)(c-\eta)}}\right) =: g_2(a,b,c;\delta_1,\delta_2,\eta). \quad \textbf{(Case II)}$$

Note that the function $g_i(a,b,c;\cdot,\cdot,\cdot) : I^i_{a,b,c} \to [0,\pi/2)$ is well-defined and continuous and that $g_i(a,b,c;\delta_1,\delta_2,\eta) = 0$ for all $(\delta_1,\delta_2,\eta) \in I^i_{a,b,c}$ with $\eta = 0$.

Observe now that

$$\begin{aligned}
|(0,a) - & R_1(\cos(\tfrac{\pi}{2}+\theta), \sin(\tfrac{\pi}{2}+\theta))| \\
&\leq \max\{|(0,a) - a(-\sin\theta,\cos\theta)|, |(0,a)-(a-\delta_1)(-\sin\theta,\cos\theta)|\} \\
&= \max\left\{\sqrt{a^2\sin^2\theta + a^2(1-\cos\theta)^2}, \sqrt{(a-\delta_1)^2\sin^2\theta + (a-(a-\delta_1)\cos\theta)^2}\right\} \\
&= \max\left\{\sqrt{2}a\sqrt{1-\cos\theta}, \sqrt{(a-\delta_1)^2 + a^2 - 2a(a-\delta_1)\cos\theta}\right\} \\
&\leq \max\Big\{\sqrt{2}a\sqrt{1-\cos(g_i(a,b,c;\delta_1,\delta_2,\eta))}, \\
&\qquad \sqrt{(a-\delta_1)^2 + a^2 - 2a(a-\delta_1)\cos(g_i(a,b,c;\delta_1,\delta_2,\eta))}\Big\} \\
&=: h_{i,1}(a,b,c;\delta_1,\delta_2,\eta), \quad \mathbf{(i=1,\ 2)} \\
|(0,b) - & R_2(\cos(\tfrac{\pi}{2}-\theta), \sin(\tfrac{\pi}{2}-\theta))| \\
&\leq \max\{|(0,b) - b(\sin\theta,\cos\theta)|, |(0,b)-(b+\delta_2)(\sin\theta,\cos\theta)|\} \\
&= \max\left\{\sqrt{b^2\sin^2\theta + b^2(1-\cos\theta)^2}, \sqrt{(b+\delta_2)^2\sin^2\theta + (b-(b+\delta_2)\cos\theta)^2}\right\} \\
&= \max\left\{\sqrt{2}b\sqrt{1-\cos\theta}, \sqrt{(b+\delta_2)^2 + b^2 - 2b(b+\delta_2)\cos\theta}\right\} \\
&\leq \max\Big\{\sqrt{2}b\sqrt{1-\cos(g_1(a,b,c;\delta_1,\delta_2,\eta))}, \\
&\qquad \sqrt{(b+\delta_2)^2 + b^2 - 2b(b+\delta_2)\cos(g_1(a,b,c;\delta_1,\delta_2,\eta))}\Big\} \\
&=: h_{1,2}(a,b,c;\delta_1,\delta_2,\eta), \quad \textbf{(Case I)} \\
|(0,b) - & R_2(\cos(\tfrac{\pi}{2}-\theta), \sin(\tfrac{\pi}{2}-\theta))| \\
&\leq \max\Big\{\sqrt{2}b\sqrt{1-\cos(g_2(a,b,c;\delta_1,\delta_2,\eta))}, \\
&\qquad \sqrt{(b-\delta_2)^2 + b^2 - 2b(b-\delta_2)\cos(g_2(a,b,c;\delta_1,\delta_2,\eta))}\Big\} \\
&=: h_{2,2}(a,b,c;\delta_1,\delta_2,\eta), \quad \textbf{(Case II)} \\
|(0,c) - & \tilde{R}_1(\cos(\tfrac{\pi}{2}+\theta), \sin(\tfrac{\pi}{2}+\theta))| \\
&\leq \max\Big\{\sqrt{2}c\sqrt{1-\cos(g_i(a,b,c;\delta_1,\delta_2,\eta))},
\end{aligned}$$

$$\sqrt{(c-\eta)^2+c^2-2c(c-\eta)\cos(g_i(a,b,c;\delta_1,\delta_2,\eta))}\Big\}$$
$$=: h_{i,3}(a,b,c;\delta_1,\delta_2,\eta), \quad (\mathbf{i=1,\ 2})$$
$$|(0,c)-\tilde R_2(\cos(\frac{\pi}{2}-\theta),\sin(\frac{\pi}{2}-\theta))|\le h_{i,3}(a,b,c;\delta_1,\delta_2,\eta). \quad (\mathbf{i=1,\ 2})$$

Define $h_i(a,b,c;\delta_1,\delta_2,\eta)=\max_{1\le j\le 3} h_{i,j}(a,b,c;\delta_1,\delta_2,\eta)$ for $i=1,2$ corresponding to **Cases I** and **II**, respectively. Then it is easy to see that the function $h_i(a,b,c;\cdot,\cdot,\cdot): I^i_{a,b,c}\to[0,\infty)$ is well-defined and satisfies the desired properties. □

We now apply the previous lemma to choose *approximate* collinear rank-one connections for the diagonal components of matrices in $R(F_0)$.

Theorem 5.5. *Let $p_\pm\in\mathbb{R}^n$ satisfy*

$$s_-(r_1)<|p_-|<s_-(r_2)<s_+(r_2)<|p_+|<s_+(r_1), \quad (\textbf{Case I})$$
$$s_-(r_1)<|p_-|<s_-(r_2)<s_+(r_1)<|p_+|<s_+(r_2), \quad (\textbf{Case II})$$

and $(A(p_+)-A(p_-))\cdot(p_+-p_-)=0$. Then there exists a vector $\zeta^0\in\mathbb{S}^{n-1}$ such that, with $p^0_\pm=s_\pm(r_2)\zeta^0$, $A(p^0_\pm)=r_2\zeta^0$, we have

$$\max\{\,|p^0_- - p_-|,|p^0_+ - p_+|,|A(p^0_-)-A(p_-)|,|A(p^0_+)-A(p_+)|\}$$
$$\le h_1(s_-(r_2),s_+(r_2),r_2;s_-(r_2)-s_-(r_1),s_+(r_1)-s_+(r_2),r_2-r_1), \quad (\textbf{Case I})$$
$$\max\{\,|p^0_- - p_-|,|p^0_+ - p_+|,|A(p^0_-)-A(p_-)|,|A(p^0_+)-A(p_+)|\}$$
$$\le h_2(s_-(r_2),s_+(r_2),r_2;s_-(r_2)-s_-(r_1),s_+(r_2)-s_+(r_1),r_2-r_1), \quad (\textbf{Case II})$$

where h_1, h_2 are the functions in Lemma 5.4.

Proof. Let Σ_2 denote the 2-dimensional linear subspace of $\mathbb{R}^n$ spanned by the two vectors $p_\pm$. (In the case that $p_\pm$ are collinear, we choose Σ_2 to be any 2-dimensional space containing $p_\pm$.) Set

$$\zeta^0=\frac{\frac{p_+}{|p_+|}+\frac{p_-}{|p_-|}}{\left|\frac{p_+}{|p_+|}+\frac{p_-}{|p_-|}\right|}\in\mathbb{S}^{n-1}\cap\Sigma_2.$$

Since the vectors $p_\pm$, $A(p_\pm)$ and ζ^0 all lie in Σ_2, we can recast the problem into the setting of the previous lemma via one of the two linear isomorphisms of Σ_2 onto $\mathbb{R}^2$ with correspondence $\zeta^0\leftrightarrow(0,1)\in\mathbb{R}^2$. Then the results follow with the following choices in applying Lemma 5.4: $a=s_-(r_2)$, $b=s_+(r_2)$, $c=r_2$, $\delta_1=s_-(r_2)-s_-(r_1)$, $\delta_2=s_+(r_1)-s_+(r_2)$ (**Case I**), $\delta_2=s_+(r_2)-s_+(r_1)$ (**Case II**), $\eta=r_2-r_1$, $R_1=|p_-|$, $R_2=|p_+|$, $\tilde R_1=\sigma(|p_-|)$, $\tilde R_2=\sigma(|p_+|)$, and $\theta\in[0,\pi/2]$ is the half of the angle between p_+ and p_-. □

5.1.4. *Final characterization of $R(F_0)$*

We are now ready to establish the result concerning the essential structures of $R(F_0)$. For this purpose, it suffices to stick only to the diagonal components of matrices in $R(F_0)$.

Theorem 5.6. *Let $0<r_2<\sigma(s_0)$ (**Case I**), $\sigma(s_2)<r_2<\sigma(s_1)$ (**Case II**). Then there exists a number $l_2=l_{r_2}\in(0,r_2)$ (**Case I**), a number $l_2=l_{r_2}\in(\sigma(s_2),r_2)$ (**Case II**) such that for any $l_2<r_1<r_2$, the set $\mathcal{S}=\mathcal{S}_{r_1,r_2}\subset\mathbb{R}^{n+n}$ in (5.6) satisfies the following:*

(i) $\sup_{(p,\beta)\in\mathcal{S}}|p|\le s_+(r_1)$ (**Case I**), $\sup_{(p,\beta)\in\mathcal{S}}|p|\le s_+(r_2)$ (**Case II**) *and* $\sup_{(p,\beta)\in\mathcal{S}}|\beta|\le r_2$*; hence $\mathcal{S}$ is bounded.*
(ii) *$\mathcal{S}$ is open.*
(iii) *For each $(p_0,\beta_0)\in\mathcal{S}$, there exist an open set $\mathcal{V}\subset\subset\mathcal{S}$ containing (p_0,β_0) and C^1 functions $q:\bar{\mathcal{V}}\to\mathbb{S}^{n-1}$, $\gamma:\bar{\mathcal{V}}\to\mathbb{R}^n$, $t_\pm:\bar{\mathcal{V}}\to\mathbb{R}$ with $\gamma\cdot q=0$ and $t_-<0<t_+$ on $\bar{\mathcal{V}}$ such that for every $\xi=\begin{pmatrix} p & c\\ B & \beta\end{pmatrix}\in R(F_0)=R(F_{r_1,r_2}(0))$ with $(p,\beta)\in\bar{\mathcal{V}}$, we have*

$$\xi + t_\pm \eta \in F_\pm,$$

where $t_\pm = t_\pm(p,\beta)$, $\eta = \begin{pmatrix} q(p,\beta) & b \\ \frac{1}{b}\gamma(p,\beta)\otimes q(p,\beta) & \gamma(p,\beta) \end{pmatrix}$, *and* $b \neq 0$ *is arbitrary.*

Proof. Fix any $0 < r_2 < \sigma(s_0)$ (**Case I**), $\sigma(s_2) < r_2 < \sigma(s_1)$ (**Case II**). For the moment, we let r_1 be any number in $(0, r_2)$ (**Case I**), in $(\sigma(s_2), r_2)$ (**Case II**) and prove (i). Then we choose later a lower bound $l_2 = l_{r_2} \in (0, r_2)$ (**Case I**), $l_2 = l_{r_2} \in (\sigma(s_2), r_2)$ (**Case II**) of r_1 for the validity of (ii) and (iii) above.

We divide the proof into several steps.

1. To show that (i) holds, choose any $(p,\beta) \in \mathcal{S}$. By Lemma 5.3, $\xi := \begin{pmatrix} p & 0 \\ O & \beta \end{pmatrix} \in R(F_0) = R(F_{r_1,r_2}(0))$, where O is the $n \times n$ zero matrix. By the definition of $R(F_0)$, there exist two matrices $\xi_\pm = \begin{pmatrix} p_\pm & c_\pm \\ B_\pm & \sigma(|p_\pm|)\frac{p_\pm}{|p_\pm|} \end{pmatrix} \in F_\pm$ and a number $0 < \lambda < 1$ such that $\xi = \lambda\xi_+ + (1-\lambda)\xi_-$. So

$$|p| = |\lambda p_+ + (1-\lambda)p_-| \le s_+(r_1), \quad \textbf{(Case I)}$$
$$|p| = |\lambda p_+ + (1-\lambda)p_-| \le s_+(r_2), \quad \textbf{(Case II)}$$
$$|\beta| = \left|\lambda\sigma(|p_+|)\frac{p_+}{|p_+|} + (1-\lambda)\sigma(|p_-|)\frac{p_-}{|p_-|}\right| \le r_2;$$

hence, $\sup_{(p,\beta)\in\mathcal{S}}|p| \le s_+(r_1)$ (**Case I**), $\sup_{(p,\beta)\in\mathcal{S}}|p| \le s_+(r_2)$ (**Case II**), $\sup_{(p,\beta)\in\mathcal{S}}|\beta| \le r_2$, and $\mathcal{S}$ is bounded. So (i) is proved.

2. We now turn to the remaining assertions that the set $\mathcal{S} = \mathcal{S}_{r_1,r_2}$ fulfills (ii) and (iii) for all $r_1 < r_2$ sufficiently close to r_2. In this step, we still assume r_1 is any fixed number in $(0, r_2)$ (**Case I**), in $(\sigma(s_2), r_2)$ (**Case II**).

Let $(p_0,\beta_0) \in \mathcal{S}$. Since $\xi_0 := \begin{pmatrix} p_0 & 0 \\ O & \beta_0 \end{pmatrix} \in R(F_0)$, it follows from Lemma 5.2 that there exist numbers $s_0 < 0 < t_0$ and vectors $q_0, \gamma_0 \in \mathbb{R}^n$ with $|q_0| = 1$, $\gamma_0 \cdot q_0 = 0$ such that $\xi_0 + s_0\eta_0 \in F_-$ and $\xi_0 + t_0\eta_0 \in F_+$, where $\eta_0 = \begin{pmatrix} q_0 & b \\ \frac{1}{b}q_0\otimes\gamma_0 & \gamma_0 \end{pmatrix}$ and $b \neq 0$ is any fixed number. Let $q_0' = t_0 q_0 \neq 0$, $\gamma_0' = t_0\gamma_0$, and $s_0' = s_0/t_0 < 0$; then

$$\begin{cases} \gamma_0' \cdot q_0' = 0, \quad s_-(r_1) < |p_0 + s_0'q_0'| < s_-(r_2), \\ s_+(r_2) < |p_0 + q_0'| < s_+(r_1), \quad \textbf{(Case I)} \\ s_+(r_1) < |p_0 + q_0'| < s_+(r_2), \quad \textbf{(Case II)} \\ \sigma(|p_0 + s_0'q_0'|)\frac{p_0+s_0'q_0'}{|p_0+s_0'q_0'|} = \beta_0 + s_0'\gamma_0', \\ \sigma(|p_0+q_0'|)\frac{p_0+q_0'}{|p_0+q_0'|} = \beta_0 + \gamma_0'. \end{cases} \tag{5.7}$$

Observe also

$$\begin{aligned} t_0 - s_0 &\ge |(p_0 + t_0q_0)| - |(p_0 + s_0q_0)| > s_+(r_2) - s_-(r_2), \quad \textbf{(Case I)} \\ t_0 - s_0 &\ge |(p_0 + t_0q_0)| - |(p_0 + s_0q_0)| > s_+(r_1) - s_-(r_2). \quad \textbf{(Case II)} \end{aligned} \tag{5.8}$$

Next, consider the function F defined by

$$F(\gamma', q', s'; p, \beta) = \begin{pmatrix} \sigma(|p+s'q'|)\frac{p+s'q'}{|p+s'q'|} - \beta - s'\gamma' \\ \sigma(|p+q'|)\frac{p+q'}{|p+q'|} - \beta - \gamma' \\ \gamma' \cdot q' \end{pmatrix} \in \mathbb{R}^{n+n+1} \tag{5.9}$$

for all $\gamma', q', p, \beta \in \mathbb{R}^n$ and $s' \in \mathbb{R}$ with $s_-(r_1) < |p + s'q'| < s_-(r_2)$, $s_+(r_2) < |p+q'| < s_+(r_1)$ (**Case I**), $s_+(r_1) < |p+q'| < s_+(r_2)$ (**Case II**). Then F is C^1 in the described open subset of $\mathbb{R}^{n+n+1+n+n}$, and the observation (5.7) yields that

$$F(\gamma_0', q_0', s_0'; p_0, \beta_0) = 0.$$

Suppose for the moment that the Jacobian matrix $D_{(\gamma',q',s')}F$ is invertible at the point $(\gamma_0', q_0', s_0'; p_0, \beta_0)$; then the Implicit Function Theorem implies the following: There exist a bounded domain $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_{(p_0,\beta_0)} \subset \mathbb{R}^{n+n}$ containing (p_0, β_0) and C^1 functions $\tilde{q}$, $\tilde{\gamma} \in \mathbb{R}^n$, $\tilde{s} \in \mathbb{R}$ of $(p, \beta) \in \tilde{\mathcal{V}}$ such that

$$\tilde{\gamma}(p_0, \beta_0) = \gamma_0', \;\; \tilde{q}(p_0, \beta_0) = q_0', \;\; \tilde{s}(p_0, \beta_0) = s_0'$$

and that

$$\begin{aligned}
&\tilde{s}(p, \beta) < 0, \;\; s_-(r_1) < |p + \tilde{s}(p, \beta)\tilde{q}(p, \beta)| < s_-(r_2),\\
&s_+(r_2) < |p + \tilde{q}(p, \beta)| < s_+(r_1), \quad \textbf{(Case I)}\\
&s_+(r_1) < |p + \tilde{q}(p, \beta)| < s_+(r_2), \quad \textbf{(Case II)}\\
&F(\tilde{\gamma}(p, \beta), \tilde{q}(p, \beta), \tilde{s}(p, \beta); p, \beta) = 0 \quad \forall (p, \beta) \in \tilde{\mathcal{V}}.
\end{aligned}$$

Define functions

$$\gamma = \frac{\tilde{\gamma}}{|\tilde{q}|}, \;\; q = \frac{\tilde{q}}{|\tilde{q}|}, \;\; t_- = \tilde{s}|\tilde{q}|, \;\; t_+ = |\tilde{q}| \quad \text{in } \tilde{\mathcal{V}};$$

then

$$\begin{aligned}
&s_-(r_1) < |p + t_- q| < s_-(r_2),\\
&s_+(r_2) < |p + t_+ q| < s_+(r_1), \quad \textbf{(Case I)}\\
&s_+(r_1) < |p + t_+ q| < s_+(r_2), \quad \textbf{(Case II)}\\
&\sigma(|p + t_\pm q|)\frac{p + t_\pm q}{|p + t_\pm q|} = \beta + t_\pm \gamma, \;\; |q| = 1, \;\; \gamma \cdot q = 0, \;\; t_- < 0 < t_+,
\end{aligned}$$

where $(p, \beta) \in \tilde{\mathcal{V}}$, $\gamma = \gamma(p, \beta)$, $q = q(p, \beta)$, and $t_\pm = t_\pm(p, \beta)$.

Let $(p, \beta) \in \tilde{\mathcal{V}}$, $B \in \mathbb{M}^{n\times n}$, $\operatorname{tr} B = 0$, $b, c \in \mathbb{R}$, $b \neq 0$, $q = q(p, \beta)$, $\gamma = \gamma(p, \beta)$, $t_\pm = t_\pm(p, \beta)$, $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix}$, and $\eta = \begin{pmatrix} q & b \\ \frac{1}{b}\gamma \otimes q & \gamma \end{pmatrix}$. Then $\xi_\pm := \xi + t_\pm \eta \in F_\pm$. By the definition of $R(F_0)$, $\xi \in (\xi_-, \xi_+) \subset R(F_0)$. By Lemma 5.3, we thus have $(p, \beta) \in \mathcal{S}$; hence $\tilde{\mathcal{V}} \subset \mathcal{S}$. This proves that $\mathcal{S}$ is open. Choosing any open set $\mathcal{V} \subset\subset \tilde{\mathcal{V}}$ with $(p_0, \beta_0) \in \mathcal{V}$, the assertion (iii) will hold.

3. In this step, we continue Step 2 to deduce an equivalent condition for the invertibility of the Jacobian matrix $D_{(\gamma',q',s')}F$ at $(\gamma_0', q_0', s_0'; p_0, \beta_0)$. By direct computation,

$$D_{(\gamma',q',s')}F = \begin{pmatrix} -s' I_n & M_{s'} & \omega_{s'}^- \\ -I_n & M_1 & 0 \\ q' & \gamma' & 0 \end{pmatrix} \in \mathbb{M}^{(n+n+1)\times(n+n+1)},$$

where I_n is the $n \times n$ identity matrix,

$$\begin{aligned}
M_{s'} &= s'\left(\sigma'(|p + s'q'|) - \frac{\sigma(|p + s'q'|)}{|p + s'q'|}\right)\frac{p + s'q'}{|p + s'q'|} \otimes \frac{p + s'q'}{|p + s'q'|} + s'\frac{\sigma(|p + s'q'|)}{|p + s'q'|} I_n,\\
\omega_{s'}^\pm &= \left(\sigma'(|p + s'q'|) - \frac{\sigma(|p + s'q'|)}{|p + s'q'|}\right)\left(\frac{p + s'q'}{|p + s'q'|} \cdot q'\right)\frac{p + s'q'}{|p + s'q'|} + \frac{\sigma(|p + s'q'|)}{|p + s'q'|} q' \pm \gamma'.
\end{aligned}$$

Here the prime only in σ' denotes the derivative. For notational simplicity, we write $(\gamma', q', s'; p, \beta) = (\gamma_0', q_0', s_0'; p_0, \beta_0)$. Applying suitable elementary row operations, as $s' < 0$, we have

$$\begin{aligned}
D_{(\gamma',q',s')}F &\to \begin{pmatrix} -s' I_n & M_{s'} & \omega_{s'}^- \\ O & M_1 - \frac{1}{s'} M_{s'} & -\frac{1}{s'}\omega_{s'}^- \\ 0 & \gamma' + \frac{q_1'}{s'}(M_{s'})^1 + \cdots + \frac{q_n'}{s'}(M_{s'})^n & \frac{1}{s'} q' \cdot \omega_{s'}^- \end{pmatrix}\\
&\to \begin{pmatrix} -s' I_n & M_{s'} & \omega_{s'}^- \\ O & s' M_1 - M_{s'} & -\omega_{s'}^- \\ 0 & s'\gamma' + q_1'(M_{s'})^1 + \cdots + q_n'(M_{s'})^n & q' \cdot \omega_{s'}^- \end{pmatrix},
\end{aligned}$$

where O is the $n\times n$ zero matrix, and $(M_{s'})^i$ is the ith row of $M_{s'}$. Since $|q'|=t_0$, $\gamma'\cdot q'=0$, and $s_-(r_1)<|p+s'q'|<s_-(r_2)$, we have

$$q'\cdot\omega^-_{s'}=\left(\sigma'(|p+s'q'|)-\frac{\sigma(|p+s'q'|)}{|p+s'q'|}\right)\left(\frac{p+s'q'}{|p+s'q'|}\cdot q'\right)^2+\frac{\sigma(|p+s'q'|)}{|p+s'q'|}t_0^2$$
$$=t_0^2\left(\cos^2\theta'\sigma'(|p+s'q'|)+(1-\cos^2\theta')\frac{\sigma(|p+s'q'|)}{|p+s'q'|}\right)>0,$$

where $\theta'\in[0,\pi]$ is the angle between $p+s'q'$ and q'. Observe here that the forward part of σ in the definition of F_- becomes essential to guarantee that $\sigma'(|p+s'q'|)>0$. After some elementary column operations to the last matrix from the above row operations, we obtain

$$D_{(\gamma',q',s')}F\;\to\;\begin{pmatrix}-s'I_n & M_{s'}-N_{s'} & \omega^-_{s'}\\ O & s'M_1-M_{s'}+N_{s'} & -\omega^-_{s'}\\ 0 & 0 & q'\cdot\omega^-_{s'}\end{pmatrix},$$

where the jth column of $N_{s'}\in\mathbb{M}^{n\times n}$ is $\frac{s'\gamma'_j+q'\cdot(M_{s'})_j}{q'\cdot\omega^-_{s'}}\omega^-_{s'}$. So $D_{(\gamma',q',s')}F$ is invertible if and only if the $n\times n$ matrix $M_1-\frac{1}{s'}M_{s'}+\frac{1}{s'}N_{s'}$ is invertible. We compute

$$M_1-\frac{1}{s'}M_{s'}+\frac{1}{s'}N_{s'}=\left(\sigma'(|p+q'|)-\frac{\sigma(|p+q'|)}{|p+q'|}\right)\frac{p+q'}{|p+q'|}\otimes\frac{p+q'}{|p+q'|}$$
$$+\frac{\sigma(|p+q'|)}{|p+q'|}I_n-\left(\sigma'(|p+s'q'|)-\frac{\sigma(|p+s'q'|)}{|p+s'q'|}\right)\frac{p+s'q'}{|p+s'q'|}\otimes\frac{p+s'q'}{|p+s'q'|}-\frac{\sigma(|p+s'q'|)}{|p+s'q'|}I_n$$
$$+\frac{1}{q'\cdot\omega^-_{s'}}\omega^-_{s'}\otimes\left[\gamma'+\left(\sigma'(|p+s'q'|)-\frac{\sigma(|p+s'q'|)}{|p+s'q'|}\right)\left(\frac{p+s'q'}{|p+s'q'|}\cdot q'\right)\frac{p+s'q'}{|p+s'q'|}+\frac{\sigma(|p+s'q'|)}{|p+s'q'|}q'\right]$$
$$=(a_1-a_{s'})I_n+(b_1-a_1)\frac{p+q'}{|p+q'|}\otimes\frac{p+q'}{|p+q'|}-(b_{s'}-a_{s'})\frac{p+s'q'}{|p+s'q'|}\otimes\frac{p+s'q'}{|p+s'q'|}+\frac{1}{q'\cdot\omega^-_{s'}}\omega^-_{s'}\otimes\omega^+_{s'},$$

and set (with an assumption $a_1\neq a_{s'}$)

$$B:=\frac{1}{a_1-a_{s'}}(M_1-\frac{1}{s'}M_{s'}+\frac{1}{s'}N_{s'})$$
$$=I_n+\frac{b_1-a_1}{a_1-a_{s'}}\frac{p+q'}{|p+q'|}\otimes\frac{p+q'}{|p+q'|}-\frac{b_{s'}-a_{s'}}{a_1-a_{s'}}\frac{p+s'q'}{|p+s'q'|}\otimes\frac{p+s'q'}{|p+s'q'|}+\frac{1}{(a_1-a_{s'})q'\cdot\omega^-_{s'}}\omega^-_{s'}\otimes\omega^+_{s'},$$

where

$$a_{s'}=\frac{\sigma(|p+s'q'|)}{|p+s'q'|},\quad b_{s'}=\sigma'(|p+s'q'|);$$

then $D_{(\gamma',q',s')}F$ is invertible if and only if the matrix $B\in\mathbb{M}^{n\times n}$ is invertible.

4. To close the arguments in Step 2 and thus to finish the proof, we choose a suitable $l_2=l_{r_2}\in(0,r_2)$ (**Case I**), $l_2=l_{r_2}\in(\sigma(s_2),r_2)$ (**Case II**), depending on r_2, in such a way that for any $r_1\in(l_2,r_2)$, the matrix B, determined through Steps 2 and 3 for any given $(p_0,\beta_0)\in\mathcal{S}=\mathcal{S}_{r_1,r_2}$, is invertible.

First, by Hypothesis (PM) or (H), $\tilde r_2\in(0,r_2)$ (**Case I**), $\tilde r_2\in(\sigma(s_2),r_2)$ (**Case II**) can be chosen close enough to r_2 so that

$$\frac{\sigma(k)}{k}<\frac{\sigma(l)}{l}\qquad\forall l\in[s_-(\tilde r_2),s_-(r_2)],$$
$$\forall k\in[s_+(r_2),s_+(\tilde r_2)]\ (\textbf{Case I}),\quad \forall k\in[s_+(\tilde r_2),s_+(r_2)]\ (\textbf{Case II}).$$

Then define a real-valued continuous function (to express the determinant of the matrix B from Step 3)

$$\mathrm{DET}(u,v,q,\gamma)=\det\Big(I_n+\frac{\sigma'(|u|)-\frac{\sigma(|u|)}{|u|}}{\frac{\sigma(|u|)}{|u|}-\frac{\sigma(|v|)}{|v|}}\frac{u}{|u|}\otimes\frac{u}{|u|}-\frac{\sigma'(|v|)-\frac{\sigma(|v|)}{|v|}}{\frac{\sigma(|u|)}{|u|}-\frac{\sigma(|v|)}{|v|}}\frac{v}{|v|}\otimes\frac{v}{|v|}$$
$$+\frac{1}{(\frac{\sigma(|u|)}{|u|}-\frac{\sigma(|v|)}{|v|})((\sigma'(|v|)-\frac{\sigma(|v|)}{|v|})(\frac{v}{|v|}\cdot q)^2+\frac{\sigma(|v|)}{|v|})}\Big((\sigma'(|v|)-\frac{\sigma(|v|)}{|v|})(\frac{v}{|v|}\cdot q)\frac{v}{|v|}$$
$$+\frac{\sigma(|v|)}{|v|}q-\gamma\Big)\otimes\Big((\sigma'(|v|)-\frac{\sigma(|v|)}{|v|})(\frac{v}{|v|}\cdot q)\frac{v}{|v|}+\frac{\sigma(|v|)}{|v|}q+\gamma\Big)\Big)$$

on the compact set $\mathcal{M}$ of points $(u,v,q,\gamma)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}^{n-1}\times\mathbb{R}^n$ with

$$|v|\in[s_-(\tilde r_2),s_-(r_2)],\ \ |\gamma|\le 1,$$
$$|u|\in[s_+(r_2),s_+(\tilde r_2)]\ \ \textbf{(Case I)},\ \ |u|\in[s_+(\tilde r_2),s_+(r_2)]\ \ \textbf{(Case II)}.$$

With $\bar k=s_+(r_2)$ and $\bar l=s_-(r_2)$, for each $q\in\mathbb{S}^{n-1}$,

$$\mathrm{DET}(\bar kq,\bar lq,q,0)=\det\Big(I_n+\frac{\sigma'(\bar k)-\frac{\sigma(\bar k)}{\bar k}+\frac{\sigma(\bar l)}{\bar l}}{\frac{\sigma(\bar k)}{\bar k}-\frac{\sigma(\bar l)}{\bar l}}q\otimes q\Big)\ne 0,$$

since $\sigma'(\bar k)\ne 0$ and hence the fraction in front of $q\otimes q$ is different from -1. So

$$d:=\min_{q\in\mathbb{S}^{n-1}}|\mathrm{DET}(\bar kq,\bar lq,q,0)|>0.$$

Next, choose a number $\delta>0$ such that for all $(u,v,q,\gamma),(\tilde u,\tilde v,\tilde q,\tilde\gamma)\in\mathcal{M}$ with $|u-\tilde u|$, $|v-\tilde v|$, $|q-\tilde q|$, $|\gamma-\tilde\gamma|<\delta$, we have

$$|\mathrm{DET}(u,v,q,\gamma)-\mathrm{DET}(\tilde u,\tilde v,\tilde q,\tilde\gamma)|<d/2. \tag{5.10}$$

Let $l_2\in(\tilde r_2,r_2)$ be sufficiently close to r_2 so that for all $r_1\in(l_2,r_2)$,

$$h_1(s_-(r_2),s_+(r_2),r_2;s_-(r_2)-s_-(r_1),s_+(r_1)-s_+(r_2),r_2-r_1)<\tau,\ \textbf{(Case I)}$$
$$h_2(s_-(r_2),s_+(r_2),r_2;s_-(r_2)-s_-(r_1),s_+(r_2)-s_+(r_1),r_2-r_1)<\tau,\ \textbf{(Case II)}$$

where h_i's are the functions in Theorem 5.5, and

$$\tau:=\min\{\delta,\delta(s_+(r_2)-s_-(r_2))/4\},\ \ \textbf{(Case I)}$$
$$\tau:=\min\{\delta,\delta(s_2-s_1)/4\}.\ \textbf{(Case II)}$$

Now, fix any $r_1\in(l_2,r_2)$, and let B be the $n\times n$ matrix determined through Steps 2 and 3 in terms of any given $(p_0,\beta_0)\in\mathcal{S}=\mathcal{S}_{r_1,r_2}$. Let $p_+=p_0+t_0q_0$ and $p_-=p_0+s_0q_0$ from Step 2; then $p_\pm$ and $A(p_\pm)$ fulfill the conditions in Theorem 5.5. So this theorem implies that there exists a vector $\zeta^0\in\mathbb{S}^{n-1}$ such that

$$\max\{|p_-^0-p_-|,|p_+^0-p_+|,|A(p_-^0)-A(p_-)|,|A(p_+^0)-A(p_+)|\}<\tau,$$

where $p_+^0=\bar k\zeta^0$, $p_-^0=\bar l\zeta^0$, and $A(p_\pm^0)=r_2\zeta^0$. Using (5.7) and (5.8),

$$|p_+-\bar k\zeta^0|<\delta,\ \ |p_--\bar l\zeta^0|<\delta,$$
$$|q_0-\zeta^0|=|\frac{p_+-p_-}{t_0-s_0}-\zeta^0|\le\frac{|(p_+-p_-)-(\bar k-\bar l)\zeta^0|+|(\bar k-\bar l)-(t_0-s_0)|}{t_0-s_0}$$
$$\le\frac{2\tau+||p_+^0-p_-^0|-|p_+-p_-||}{t_0-s_0}<\frac{4\tau}{t_0-s_0}<\delta,$$
$$|\gamma_0|=|\frac{A(p_+)-A(p_-)}{t_0-s_0}|\le\frac{|A(p_+)-A(p_+^0)|+|A(p_-^0)-A(p_-)|}{t_0-s_0}<\delta.$$

Since $\det(B)=\mathrm{DET}(p_+,p_-,q_0,\gamma_0)$ and $|\mathrm{DET}(\bar k\zeta^0,\bar l\zeta^0,\zeta^0,0)|\ge d$, it follows from (5.10) that

$$|\det(B)|>d/2>0.$$

The proof is now complete. □

5.2. Relaxation of $\nabla\omega(z) \in K_0$

The following result is important for the convex integration with linear constraint; see [24, Lemma 5.7].

Lemma 5.7. *Let $\lambda_1, \lambda_2 > 0$ and $\eta_1 = -\lambda_1\eta,\ \eta_2 = \lambda_2\eta$ with*

$$\eta = \begin{pmatrix} q & b \\ \frac{1}{b}\gamma \otimes q & \gamma \end{pmatrix}, \quad |q| = 1,\ \gamma \cdot q = 0,\ b \neq 0.$$

Let $G \subset \mathbb{R}^{n+1}$ be a bounded domain. Then for each $\epsilon > 0$, there exists a function $\omega = (\varphi, \psi) \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{1+n})$ with $\operatorname{supp}(\omega) \subset\subset G$ that satisfies the following properties:

(a) $\operatorname{div}\psi = 0$ *in G,*
(b) $|\{z \in G \mid \nabla\omega(z) \notin \{\eta_1,\ \eta_2\}\}| < \epsilon$,
(c) $\operatorname{dist}(\nabla\omega(z), [\eta_1, \eta_2]) < \epsilon$ *for all $z \in G$,*
(d) $\|\omega\|_{L^\infty(G)} < \epsilon$,
(e) $\int_{\mathbb{R}^n} \varphi(x,t)\,dx = 0$ *for all $t \in \mathbb{R}$.*

We now state the relaxation theorem for homogeneous differential inclusion $\nabla\omega(z) \in K_0$ in a form that is more suitable for later use; we restrict the inclusion only to the diagonal components.

Theorem 5.8. *Let $0 < r_2 < \sigma(s_0)$ (**Case I**), $\sigma(s_2) < r_2 < \sigma(s_1)$ (**Case II**), and let $l_2 = l_{r_2} \in (0, r_2)$ (**Case I**), $l_2 = l_{r_2} \in (\sigma(s_2), r_2)$ (**Case II**) be some number, depending on r_2, from Theorem 5.6. Let $l_2 < r_1 < r_2$, and let $\mathcal{K}$ be a compact subset of $\mathcal{S} = \mathcal{S}_{r_1,r_2}$. Let $\tilde{Q} \times \tilde{I}$ be a box in $\mathbb{R}^{n+1}$. Then, given any $\epsilon > 0$, there exists a $\delta > 0$ such that for each box $Q \times I \subset \tilde{Q} \times \tilde{I}$, point $(p, \beta) \in \mathcal{K}$, and number $\rho > 0$ sufficiently small, there exists a function $\omega = (\varphi, \psi) \in C_c^\infty(Q \times I; \mathbb{R}^{1+n})$ satisfying the following properties:*

(a) $\operatorname{div}\psi = 0$ *in $Q \times I$,*
(b) $(p' + D\varphi(z), \beta' + \psi_t(z)) \in \mathcal{S}$ *for all $z \in Q \times I$ and $|(p', \beta') - (p, \beta)| \le \delta$,*
(c) $\|\omega\|_{L^\infty(Q\times I)} < \rho$,
(d) $\int_{Q\times I} |\beta + \psi_t(z) - A(p + D\varphi(z))|dz < \epsilon|Q \times I|/|\tilde{Q} \times \tilde{I}|$,
(e) $\int_{Q\times I} \operatorname{dist}((p + D\varphi(z), \beta + \psi_t(z)), \mathcal{A})dz < \epsilon|Q \times I|/|\tilde{Q} \times \tilde{I}|$,
(f) $\int_Q \varphi(x,t)dx = 0$ *for all $t \in I$,*
(g) $\|\varphi_t\|_{L^\infty(Q\times I)} < \rho$,

where $\mathcal{A} = \mathcal{A}_{r_1,r_2} \subset \mathbb{R}^{n+n}$ is the set defined by

$$\mathcal{A} = \{(p', A(p')) \mid |p'| \in [s_-(r_1), s_-(r_2)] \cup [s_+(r_2), s_+(r_1)]\}, \quad (\textbf{Case I})$$
$$\mathcal{A} = \{(p', A(p')) \mid |p'| \in [s_-(r_1), s_-(r_2)] \cup [s_+(r_1), s_+(r_2)]\}. \quad (\textbf{Case II})$$

Proof. By Theorem 5.6, there exist finitely many open balls $\mathcal{B}_1, \cdots, \mathcal{B}_N \subset\subset \mathcal{S}$ covering $\mathcal{K}$ and C^1 functions $q_i : \bar{\mathcal{B}}_i \to \mathbb{S}^{n-1}$, $\gamma_i : \bar{\mathcal{B}}_i \to \mathbb{R}^n$, $t_{i,\pm} : \bar{\mathcal{B}}_i \to \mathbb{R}$ $(1 \le i \le N)$ with $\gamma_i \cdot q_i = 0$ and $t_{i,-} < 0 < t_{i,+}$ on $\bar{\mathcal{B}}_i$ such that for each $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0)$ with $(p, \beta) \in \bar{\mathcal{B}}_i$, we have

$$\xi + t_{i,\pm}\eta_i \in F_\pm,$$

where $t_{i,\pm} = t_{i,\pm}(p,\beta)$, $\eta_i = \begin{pmatrix} q_i(p,\beta) & b \\ \frac{1}{b}\gamma_i(p,\beta) \otimes q_i(p,\beta) & \gamma_i(p,\beta) \end{pmatrix}$, and $b \neq 0$ is arbitrary.

Let $1 \le i \le N$. We write $\xi_i = \xi_i(p,\beta) = \begin{pmatrix} p & 0 \\ O & \beta \end{pmatrix} \in R(F_0)$ for $(p,\beta) \in \bar{\mathcal{B}}_i \subset \mathcal{S}$, where O is the $n \times n$ zero matrix. We omit the dependence on $(p,\beta) \in \bar{\mathcal{B}}_i$ in the following whenever it is clear from the context. Given any $\rho > 0$, we choose a constant b_i with

$$0 < b_i < \min_{\bar{\mathcal{B}}_i} \frac{\rho}{t_{i,+} - t_{i,-}}.$$

With this choice of $b = b_i$, let η_i be defined on $\bar{\mathcal{B}}_i$ as above. Then

$$\xi_{i,\pm} = \begin{pmatrix} p_{i,\pm} & c_{i,\pm} \\ B_{i,\pm} & \beta_{i,\pm} \end{pmatrix} := \xi_i + t_{i,\pm}\eta_i \in F_\pm,$$
$$\xi_i = \lambda_i \xi_{i,+} + (1-\lambda_i)\xi_{i,-}, \quad \lambda_i = \frac{-t_{i,-}}{t_{i,+} - t_{i,-}} \in (0,1) \quad \text{on } \bar{\mathcal{B}}_i.$$

By the definition of $R(F_0)$, on $\bar{\mathcal{B}}_i$, both $\xi^\tau_{i,-} = \tau\xi_{i,+} + (1-\tau)\xi_{i,-}$ and $\xi^\tau_{i,+} = (1-\tau)\xi_{i,+} + \tau\xi_{i,-}$ belong to $R(F_0)$ for all $\tau \in (0,1)$. Let $0 < \tau < \min_{1\le j\le N} \min_{\bar{\mathcal{B}}_j} \min\{\lambda_j, 1-\lambda_j\} \le \frac{1}{2}$ be a small number to be selected later. Let $\lambda'_i = \frac{\lambda_i - \tau}{1-2\tau}$ on $\bar{\mathcal{B}}_i$. Then $\lambda'_i \in (0,1)$, $\xi_i = \lambda'_i \xi^\tau_{i,+} + (1-\lambda'_i)\xi^\tau_{i,-}$ on $\bar{\mathcal{B}}_i$. Moreover, on $\bar{\mathcal{B}}_i$, $\xi^\tau_{i,+} - \xi^\tau_{i,-} = (1-2\tau)(\xi_{i,+} - \xi_{i,-})$ is rank-one, $[\xi^\tau_{i,-}, \xi^\tau_{i,+}] \subset (\xi_{i,-}, \xi_{i,+}) \subset R(F_0)$, and

$$c\tau \le |\xi^\tau_{i,+} - \xi_{i,+}| = |\xi^\tau_{i,-} - \xi_{i,-}| = \tau|\xi_{i,+} - \xi_{i,-}| = \tau(t_{i,+} - t_{i,-})|\eta_i| \le C\tau,$$

where $C = \max_{1\le j\le N} \max_{\bar{\mathcal{B}}_j}(t_{j,+} - t_{j,-})|\eta_j| \ge \min_{1\le j\le N} \min_{\bar{\mathcal{B}}_j}(t_{j,+} - t_{j,-})|\eta_j| = c > 0$. By continuity, $H_\tau = \bigcup_{(p,\beta)\in\bar{\mathcal{B}}_j, 1\le j\le N} [\xi^\tau_{j,-}(p,\beta), \xi^\tau_{j,+}(p,\beta)]$ is a compact subset of $R(F_0)$, where $R(F_0)$ is open in the space

$$\Sigma_0 = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \,\middle|\, \operatorname{tr} B = 0 \right\},$$

by Lemma 5.3 and Theorem 5.6. So $d_\tau = \operatorname{dist}(H_\tau, \partial|_{\Sigma_0} R(F_0)) > 0$, where $\partial|_{\Sigma_0}$ is the relative boundary in Σ_0.

Let $\eta_{i,1} = -\lambda_{i,1}\eta_i = -\lambda'_i(1-2\tau)(t_{i,+} - t_{i,-})\eta_i$, $\eta_{i,2} = \lambda_{i,2}\eta_i = (1-\lambda'_i)(1-2\tau)(t_{i,+} - t_{i,-})\eta_i$ on $\bar{\mathcal{B}}_i$, where $\lambda_{i,1} = \tau(-t_{i,+}) + (1-\tau)(-t_{i,-}) > 0$, $\lambda_{i,2} = (1-\tau)t_{i,+} + \tau t_{i,-} > 0$ on $\bar{\mathcal{B}}_i$, and $\tau > 0$ is so small that

$$\min_{1\le j\le N} \min_{\bar{\mathcal{B}}_j} \lambda_{j,k} > 0 \ \ (k = 1, 2).$$

Applying Lemma 5.7 to matrices $\eta_{i,1} = \eta_{i,1}(p,\beta)$, $\eta_{i,2} = \eta_{i,2}(p,\beta)$ with a fixed $(p,\beta) \in \bar{\mathcal{B}}_i$ and a given box $G = Q \times I$, we obtain that for each $\rho > 0$, there exist a function $\omega = (\varphi, \psi) \in C^\infty_c(Q\times I; \mathbb{R}^{1+n})$ and an open set $G_\rho \subset\subset Q \times I$ satisfying the following conditions:

$$\begin{cases} (1) \ \operatorname{div}\psi = 0 \text{ in } Q\times I, \\ (2) \ |(Q\times I)\setminus G_\rho| < \rho \xi_i + \nabla\omega(z) \in \{\xi^\tau_{i,-}, \ \xi^\tau_{i,+}\} \text{ for all } z \in G_\rho, \\ (3) \ \xi_i + \nabla\omega(z) \in [\xi^\tau_{i,-}, \xi^\tau_{i,+}]_\rho \text{ for all } z \in Q\times I, \\ (4) \ \|\omega\|_{L^\infty(Q\times I)} < \rho, \\ (5) \ \int_Q \varphi(x,t)\,dx = 0 \text{ for all } t \in I, \\ (6) \ \|\varphi_t\|_{L^\infty(Q\times I)} < 2\rho, \end{cases} \tag{5.11}$$

where $[\xi^\tau_{i,-}, \xi^\tau_{i,+}]_\rho$ denotes the ρ-neighborhood of closed line segment $[\xi^\tau_{i,-}, \xi^\tau_{i,+}]$. Here, from (5.11.3), condition (5.11.6) follows as

$$|\varphi_t| < |c_{i,+} - c_{i,-}| + \rho = (t_{i,+} - t_{i,-})|b_i| + \rho < 2\rho \quad \text{in } Q\times I.$$

Note (a), (c), (f), and (g) follow from (5.11), where 2ρ in (5.11.6) can be adjusted to ρ as in (g). By the uniform continuity of A on the set $J = \{p' \in \mathbb{R}^n \mid |p'| \le s_+(r_1)$ (**Case I**), $|p'| \le s_+(r_2)$ (**Case II**)$\}$, we can find a $\delta' > 0$ such that $|A(p') - A(p'')| < \frac{\epsilon}{3|\tilde{Q}\times\tilde{I}|}$ whenever $p', p'' \in J$ and $|p' - p''| < \delta'$. We then choose a $\tau > 0$ so small that

$$C\tau < \delta', \ \ C|\tilde{Q}\times\tilde{I}|\tau < \frac{\epsilon}{3}.$$

Next, we choose a $\delta > 0$ such that $\delta < \frac{d_\tau}{2}$. If $0 < \rho < \delta$, then by (5.11.1) and (5.11.3), for all $z \in Q\times I$ and $|(p',\beta') - (p,\beta)| \le \delta$,

$$\xi_i(p',\beta') + \nabla\omega(z) \in \Sigma_0, \quad \operatorname{dist}(\xi_i(p',\beta') + \nabla\omega(z), H_\tau) < d_\tau,$$

and so $\xi_i(p', \beta') + \nabla\omega(z) \in R(F_0)$, that is, $(p' + D\varphi(z), \beta' + \psi_t(z)) \in \mathcal{S}$. Thus (b) holds for all $0 < \rho < \delta$. In particular, $(p + D\varphi(z), \beta + \psi_t(z)) \in \mathcal{S}$ and so $|p + D\varphi(z)| \le s_+(r_1)$ (**Case I**), $|p + D\varphi(z)| \le s_+(r_2)$ (**Case II**) and $|\beta + \psi_t(z)| \le r_2$ for all $z \in Q \times I$, by (i) of Theorem 5.6. Thus

$$\int_{Q\times I} |\beta + \psi_t - A(p + D\varphi)|dz$$

$$\le \int_{G_\rho} |\beta + \psi_t - A(p + D\varphi)|dz + (r_2 + M_\sigma)\rho \le |Q \times I| \max\{|\beta^\tau_{i,\pm} - A(p^\tau_{i,\pm})|\} + (r_2 + M_\sigma)\rho$$

$$\le C|Q \times I|\tau + |Q \times I| \max\{|A(p_{i,\pm}) - A(p^\tau_{i,\pm})|\} + (r_2 + M_\sigma)\rho \le \frac{2\epsilon|Q \times I|}{3|\tilde{Q} \times \tilde{I}|} + (r_2 + M_\sigma)\rho,$$

where $\xi^\tau_{i,\pm} = \begin{pmatrix} p^\tau_{i,\pm} & c^\tau_{i,\pm} \\ B^\tau_{i,\pm} & \beta^\tau_{i,\pm} \end{pmatrix}$ and $M_\sigma = \sigma(s_0)$ (**Case I**), $M_\sigma = \sigma(s_2^*)$ (**Case II**). Thus, (d) holds for all $\rho > 0$ satisfying $(r_2 + M_\sigma)\rho < \frac{\epsilon|Q\times I|}{3|\tilde{Q}\times\tilde{I}|}$. Similarly,

$$\int_{Q\times I} \operatorname{dist}((p + D\varphi(z), \beta + \psi_t(z)), \mathcal{A})dz$$

$$\le \int_{G_\rho} \max |(p^\tau_{i,\pm}, \beta^\tau_{i,\pm}) - (p_{i,\pm}, \beta_{i,\pm})|dz + 2(r_2 + M_\sigma)\rho$$

$$\le C|Q \times I|\tau + 2(r_2 + M_\sigma)\rho \le \frac{\epsilon|Q \times I|}{3|\tilde{Q} \times \tilde{I}|} + 2(r_2 + M_\sigma)\rho;$$

therefore, (e) also holds for all such $\rho > 0$ for (d).

We have verified (a)–(g) for any $(p, \beta) \in \bar{\mathcal{B}}_i$ and $1 \le i \le N$, where $\delta > 0$ is independent of the index i. Since $\mathcal{B}_1, \cdots, \mathcal{B}_N$ cover $\mathcal{K}$, the proof is now complete. □

6. Boundary function Φ and the admissible set $\mathcal{U}$

Assume Ω and u_0 satisfy (1.6) and (2.1).

6.1. *Boundary function* Φ

We first construct a suitable boundary function $\Phi = (u^*, v^*)$ in each case of the profile $\sigma(s)$. Recall that the goals of **Cases I** and **II** are to prove Theorems 3.1 and 3.5 under Hypotheses (PM) and (H), respectively.

Case I:. In this case, Ω is assumed to be convex. Let $0 < r = r_2 < \sigma(M_0)$, and let $l = l_{r_2} \in (0, r)$ be some number determined by Theorem 5.6. Choose any $\tilde{r} = r_1 \in (l, r)$.

Case II:. In this case, we assume $|Du_0(x_0)| \in (s_1^*, s_2^*)$ for some $x_0 \in \Omega$. Let $\sigma(s_2) < r = r_2 < \sigma(M_0')$, and let $l = l_{r_2} \in (\sigma(s_2), r)$ be some number determined by Theorem 5.6. Choose any $\tilde{r} = r_1 \in (l, r)$.

We now apply Lemma 4.2 (**Case I**), Lemma 4.3 (**Case II**) to determine functions $\tilde{\sigma}$, $\tilde{f} \in C^3([0, \infty))$ (**Case I**), $\tilde{\sigma}$, $\tilde{f} \in C^{1+\alpha}([0, \infty))$ (**Case II**) satisfying its conclusion. Also, let $\tilde{A}(p) = \tilde{f}(|p|^2)p$ $(p \in \mathbb{R}^n)$. Then

Lemma 6.1. *We have*

$$(p, \tilde{A}(p)) \in \mathcal{S} \quad \forall\, s_-(r_1) < |p| < s_+(r_2),$$

where $\mathcal{S} = \mathcal{S}_{r_1,r_2}$ *is the set in* Lemma 5.3.

Proof. Let $s = |p|$, $r = \tilde{\sigma}(s)$ and $\zeta = p/|p|$, so that $s_-(r_1) < s < s_+(r_2)$, $\zeta \in \mathbb{S}^{n-1}$ and $\tilde{A}(p) = r\zeta$. By Lemma 4.2 (**Case I**), Lemma 4.3 (**Case II**), $s_-(r) < s < s_+(r)$ and $r_1 < r < r_2$. Set $p_\pm = s_\pm(r)\zeta$ and $\beta_\pm = r\zeta$. Then $A(p_\pm) =$

$r\zeta = \beta_\pm$. Define $\xi = \begin{pmatrix} p & 0 \\ O & \tilde{A}(p) \end{pmatrix}$ and $\xi_\pm = \begin{pmatrix} p_\pm & 0 \\ O & \beta_\pm \end{pmatrix}$. Then $\xi = \lambda\xi_+ + (1-\lambda)\xi_-$ for some $0 < \lambda < 1$. Since $\xi_\pm \in F_\pm$ and $\operatorname{rank}(\xi_+ - \xi_-) = 1$, it follows from the definition of $R(F_0) = R(F_{r_1,r_2}(0))$ that $\xi \in (\xi_-, \xi_+) \subset R(F_0)$. Thus, by Lemma 5.3, $(p, \tilde{A}(p)) \in \mathcal{S}$. □

By Lemma 4.2 (**Case I**), Lemma 4.3 (**Case II**), equation $u_t = \operatorname{div}(\tilde{A}(Du))$ is uniformly parabolic. So by Theorem 4.1, the initial-Neumann boundary value problem

$$\begin{cases} u^*_t = \operatorname{div}(\tilde{A}(Du^*)) & \text{in } \Omega_T \\ \partial u^*/\partial \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T) \\ u^*(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{6.1}$$

admits a unique classical solution $u^* \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T)$; moreover, only in **Case I**, it satisfies

$$|Du^*(x,t)| \le M_0 \quad \forall (x,t) \in \Omega_T.$$

From conditions (1.6) and (2.1), we can find a function $h \in C^{2+\alpha}(\bar{\Omega})$ satisfying

$$\Delta h = u_0 \text{ in } \Omega, \quad \partial h/\partial \mathbf{n} = 0 \text{ on } \partial\Omega.$$

Let $v_0 = Dh \in C^{1+\alpha}(\bar{\Omega}; \mathbb{R}^n)$ and define, for $(x,t) \in \Omega_T$,

$$v^*(x,t) = v_0(x) + \int_0^t \tilde{A}(Du^*(x,s))\,ds. \tag{6.2}$$

Then it is easily seen that $\Phi := (u^*, v^*) \in C^1(\bar{\Omega}_T; \mathbb{R}^{1+n})$ satisfies (2.2); that is,

$$\begin{cases} u^*(x,0) = u_0(x) \ (x \in \Omega), \\ \operatorname{div} v^* = u^* \ \text{ a.e. in } \Omega_T, \\ v^*(\cdot,t)\cdot \mathbf{n}|_{\partial\Omega} = 0 \ \forall\, t \in [0,T], \end{cases} \tag{6.3}$$

and so Φ is a boundary function for the initial datum u_0.

Next, let

$$\mathcal{F} = \{(p, A(p)) \mid |p| \in [0, s_-(r_1)]\}, \quad (\textbf{Case I})$$
$$\mathcal{F} = \left\{(p, A(p)) \mid |p| \in [0, s_-(r_1)] \cup [s_+(r_2), \max\{M, s_2^*\}]\right\}, \quad (\textbf{Case II})$$

where $M = \|Du^*\|_{L^\infty(\Omega_T)}$ in **Case II**. Then we have the following:

Lemma 6.2.

$$(Du^*(x,t), v^*_t(x,t)) \in \mathcal{S} \cup \mathcal{F} \quad \forall\,(x,t) \in \Omega_T.$$

Proof. Let $(x,t) \in \Omega_T$ and $p = Du^*(x,t)$.

Case I: In this case, $|p| \le M_0 < s_+(r_2)$. If $|p| \le s_-(r_1)$, then $\tilde{A}(p) = A(p)$ and hence by (6.2)

$$(Du^*(x,t), v^*_t(x,t)) = (p, \tilde{A}(p)) = (p, A(p)) \in \mathcal{F}.$$

If $s_-(r_1) < |p| \le M_0$, then by Lemma 6.1 and (6.2)

$$(Du^*(x,t), v^*_t(x,t)) = (p, \tilde{A}(p)) \in \mathcal{S}.$$

Case II: If $|p| \le s_-(r_1)$ or $s_+(r_2) \le |p| \le M$, then $\tilde{A}(p) = A(p)$ and hence as above

$$(Du^*(x,t), v^*_t(x,t)) = (p, \tilde{A}(p)) = (p, A(p)) \in \mathcal{F}.$$

If $s_-(r_1) < |p| < s_+(r_2)$, then as above

$$(Du^*(x,t), v^*_t(x,t)) = (p, \tilde{A}(p)) \in \mathcal{S}.$$

Therefore, for both cases, $(Du^*, v^*_t) \in \mathcal{S} \cup \mathcal{F}$ in Ω_T. □

Definition 6.3. We say a function u is *piecewise* C^1 in Ω_T and write $u \in C^1_{piece}(\Omega_T)$ if there exists a sequence of disjoint open sets $\{G_j\}_{j=1}^\infty$ in Ω_T such that

$$u \in C^1(\bar{G}_j) \ \forall j \in \mathbb{N}, \quad |\Omega_T \setminus \cup_{j=1}^\infty G_j| = 0.$$

It is then necessary to have $|\partial G_j| = 0 \, \forall j \in \mathbb{N}$.

6.2. Selection of interface

To separate the space–time domain Ω_T into the classical and micro-oscillatory parts for Lipschitz solutions, we assume the following.

Case I. Observe that

$$|\{(x,t) \in \Omega_T \,|\, |Du^*(x,t)| = s_-(\bar{r})\}| > 0$$

for at most countably many $\bar{r} \in (0, \tilde{r})$. We fix any $\bar{r} \in (0, \tilde{r})$ with

$$|\{(x,t) \in \Omega_T \,|\, |Du^*(x,t)| = s_-(\bar{r})\}| = 0,$$

and let

$$\Omega_T^1 = \{(x,t) \in \Omega_T \,|\, |Du^*(x,t)| < s_-(\bar{r})\},$$
$$\Omega_T^2 = \{(x,t) \in \Omega_T \,|\, |Du^*(x,t)| > s_-(\bar{r})\},$$

so that Ω_T^1 and Ω_T^2 are disjoint open subsets of Ω_T whose union has full measure $|\Omega_T|$. Define

$$\Omega_0^{\bar{r}} = \{(x,0) \,|\, x \in \Omega, \ |Du_0(x)| < s_-(\bar{r})\};$$

then $\Omega_0^{\bar{r}} \subset \partial\Omega_T^1$.

Case II. Observe that

$$|\{(x,t) \in \Omega_T \,|\, |Du^*(x,t)| = s_-(\bar{r}_j)\}| > 0 \quad (j = 1,3)$$

for at most countably many $\bar{r}_1 \in (\sigma(s_2), \tilde{r})$, $\bar{r}_3 \in (r, \sigma(s_1))$. We fix any two $\bar{r}_1 \in (\sigma(s_2), \tilde{r})$, $\bar{r}_3 \in (r, \sigma(s_1))$ such that

$$|\{(x,t) \in \Omega_T \,|\, |Du^*(x,t)| = s_-(\bar{r}_j)\}| = 0, \quad (j = 1,3)$$

and let

$$\Omega_T^1 = \{(x,t) \in \Omega_T \,|\, |Du^*(x,t)| < s_-(\bar{r}_1)\},$$
$$\Omega_T^3 = \{(x,t) \in \Omega_T \,|\, |Du^*(x,t)| > s_+(\bar{r}_3)\},$$

and

$$\Omega_T^2 = \{(x,t) \in \Omega_T \,|\, s_-(\bar{r}_1) < |Du^*(x,t)| < s_+(\bar{r}_3)\},$$

so that these are disjoint open subsets of Ω_T whose union has full measure $|\Omega_T|$. Define

$$\Omega_0^{\bar{r}_1} = \{(x,0) \,|\, x \in \Omega, \ |Du_0(x)| < s_-(\bar{r}_1)\},$$
$$\Omega_0^{\bar{r}_3} = \{(x,0) \,|\, x \in \Omega, \ |Du_0(x)| > s_+(\bar{r}_3)\};$$

then $\Omega_0^{\bar{r}_j} \subset \partial\Omega_T^j$ $(j = 1,3)$.

6.3. The admissible set $\mathcal{U}$

Let $m = \|u_t^*\|_{L^\infty(\Omega_T)} + 1$. We finally define the admissible set $\mathcal{U}$ and approximating sets $\mathcal{U}_\epsilon$ ($\epsilon > 0$) as follows. Recall $0 < \bar{r} < \tilde{r} = r_1 < r = r_2 < \sigma(s_0)$ (**Case I**), $\sigma(s_2) < \bar{r}_1 < \tilde{r} = r_1 < r = r_2 < \bar{r}_3 < \sigma(s_1)$ (**Case II**).

Case I.

$$\begin{aligned}\mathcal{U} = \Big\{ & u \in C^1_{piece} \cap W^{1,\infty}_{u^*}(\Omega_T) \,\Big|\, u \equiv u^* \text{ in } \Omega_T^1,\ \|u_t\|_{L^\infty(\Omega_T)} < m,\\ & \exists\, v \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T;\mathbb{R}^n) \text{ such that } v \equiv v^* \text{ in } \Omega_T^1,\\ & \operatorname{div} v = u \text{ and } (Du, v_t) \in \mathcal{S} \cup \mathcal{F} \text{ a.e. in } \Omega_T, \text{ and}\\ & (Du, v_t) \in \mathcal{S} \cup \mathcal{B} \text{ a.e. in } \Omega_T^2 \Big\},\\ \mathcal{U}_\epsilon = \Big\{ & u \in \mathcal{U} \,\Big|\, \exists\, v \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T;\mathbb{R}^n) \text{ such that } v \equiv v^* \text{ in } \Omega_T^1,\\ & \operatorname{div} v = u \text{ and } (Du, v_t) \in \mathcal{S} \cup \mathcal{F} \text{ a.e. in } \Omega_T,\\ & (Du, v_t) \in \mathcal{S} \cup \mathcal{B} \text{ a.e. in } \Omega_T^2,\ \int_{\Omega_T} |v_t - A(Du)|\,dxdt \le \epsilon|\Omega_T|, \text{ and}\\ & \int_{\Omega_T^2} \operatorname{dist}((Du, v_t), \mathcal{C})\,dxdt \le \epsilon|\Omega_T^2| \Big\},\end{aligned}$$

where

$$\begin{aligned}\mathcal{B} &= \mathcal{B}_{\bar{r},\tilde{r},r} = \big\{(p, A(p)) \,\big|\, |p| \in [s_-(\bar{r}), s_-(\tilde{r})]\big\},\\ \mathcal{C} &= \mathcal{C}_{\bar{r},\tilde{r},r} = \big\{(p, A(p)) \,\big|\, |p| \in [s_-(\bar{r}), s_-(r)] \cup [s_+(r), s_+(\tilde{r})]\big\}.\end{aligned}$$

Case II.

$$\begin{aligned}\mathcal{U} = \Big\{ & u \in C^1_{piece} \cap W^{1,\infty}_{u^*}(\Omega_T) \,\Big|\, u \equiv u^* \text{ in } \Omega_T^1 \cup \Omega_T^3,\ \|u_t\|_{L^\infty(\Omega_T)} < m,\\ & \exists\, v \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T;\mathbb{R}^n) \text{ such that } v \equiv v^* \text{ in } \Omega_T^1 \cup \Omega_T^3,\\ & \operatorname{div} v = u \text{ and } (Du, v_t) \in \mathcal{S} \cup \mathcal{F} \text{ a.e. in } \Omega_T, \text{ and}\\ & (Du, v_t) \in \mathcal{S} \cup \mathcal{B} \text{ a.e. in } \Omega_T^2 \Big\},\\ \mathcal{U}_\epsilon = \Big\{ & u \in \mathcal{U} \,\Big|\, \exists\, v \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T;\mathbb{R}^n) \text{ such that } v \equiv v^* \text{ in } \Omega_T^1 \cup \Omega_T^3,\\ & \operatorname{div} v = u \text{ and } (Du, v_t) \in \mathcal{S} \cup \mathcal{F} \text{ a.e. in } \Omega_T,\\ & (Du, v_t) \in \mathcal{S} \cup \mathcal{B} \text{ a.e. in } \Omega_T^2,\ \int_{\Omega_T} |v_t - A(Du)|\,dxdt \le \epsilon|\Omega_T|, \text{ and}\\ & \int_{\Omega_T^2} \operatorname{dist}((Du, v_t), \mathcal{C})\,dxdt \le \epsilon|\Omega_T^2| \Big\},\end{aligned}$$

where

$$\begin{aligned}\mathcal{B} &= \mathcal{B}_{\bar{r}_1,\tilde{r},r,\bar{r}_3} = \big\{(p, A(p)) \,\big|\, |p| \in [s_-(\bar{r}_1), s_-(\tilde{r})] \cup [s_+(r), s_+(\bar{r}_3)]\big\},\\ \mathcal{C} &= \mathcal{C}_{\bar{r}_1,\tilde{r},r,\bar{r}_3} = \big\{(p, A(p)) \,\big|\, |p| \in [s_-(\bar{r}_1), s_-(r)] \cup [s_+(\tilde{r}), s_+(\bar{r}_3)]\big\}.\end{aligned}$$

Observe that for both cases, $\mathcal{A} \cup \mathcal{B} = \mathcal{C}$ and $\mathcal{B} \subset \mathcal{F}$, where $\mathcal{A}$ is as in Theorem 5.8. Also for both cases, as in the proof of Lemma 6.2, it is easy to check that $(Du^*, v_t^*) \in \mathcal{S} \cup \mathcal{B}$ in Ω_T^2.

Note that one more requirement is imposed on the elements of $\mathcal{U}_\epsilon$ in both cases than in the general density approach in Subsection 2.2. As we will see later, such a smallness condition on the distance integral is designed to extract the micro-structured ramping of Lipschitz solutions with alternate gradients whose magnitudes lie in two disjoint (possibly very small) compact intervals; this occurs only in Ω_T^2, and the solutions are classical elsewhere.

Remark 6.4. Summarizing the above, we have constructed a boundary function $\Phi = (u^*, v^*) \in C^1(\bar{\Omega}_T;\mathbb{R}^{1+n})$ for the initial datum u_0 with $u^* \in \mathcal{U}$; so $\mathcal{U}$ is non-empty. Also $\mathcal{U}$ is a bounded subset of $W^{1,\infty}_{u^*}(\Omega_T)$, since $\mathcal{S} \cup \mathcal{F}$ is bounded and $\|u_t\|_{L^\infty(\Omega_T)} < m$ for all $u \in \mathcal{U}$. Moreover, by (i) of Theorem 5.6 and the definition of $\mathcal{F}$, for each $u \in \mathcal{U}$, its corresponding vector function v satisfies $\|v_t\|_{L^\infty(\Omega_T)} \le r = r_2$ (**Case I**), $\|v_t\|_{L^\infty(\Omega_T)} \le \max\{r = r_2, \sigma(M_0)\}$ (**Case II**); this bound in each case plays the role of a fixed number $R > 0$ in the general density approach in Subsection 2.2. Finally, note that $s_-(r_1) < |Du^*| < s_+(r_2)$ on some non-empty open subset of Ω_T and so $\tilde{A}(Du^*) \ne A(Du^*)$ on a subset of Ω_T with positive measure; hence u^* itself is not a Lipschitz solution to (1.2).

In view of the general existence theorem, Theorem 2.2, it only remains to prove the L^∞-density of $\mathcal{U}_\epsilon$ in $\mathcal{U}$ towards the existence of infinitely many Lipschitz solutions for both cases; this core subject is carried out in the next section.

7. Density of $\mathcal{U}_\epsilon$ in $\mathcal{U}$: final step for the proofs of Theorems 3.1 and 3.5

In this section, we follow Section 6 to complete the proofs of Theorems 3.1 and 3.5.

7.1. *The density property*

The density theorem below is the last preparation for both cases.

Theorem 7.1. *For each $\epsilon > 0$, $\mathcal{U}_\epsilon$ is dense in $\mathcal{U}$ under the L^∞-norm.*

Proof. Let $u \in \mathcal{U}$, $\eta > 0$. The goal is to construct a function $\tilde{u} \in \mathcal{U}_\epsilon$ such that $\|\tilde{u} - u\|_{L^\infty(\Omega_T)} < \eta$. For clarity, we divide the proof into several steps.

1. Note that $u \equiv u^*$ in Ω_T^1 (**Case I**), in $\Omega_T^1 \cup \Omega_T^3$ (**Case II**), $\|u_t\|_{L^\infty(\Omega_T)} < m - \tau_0$ for some $\tau_0 > 0$, and there exists a vector function $v \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T; \mathbb{R}^n)$ such that $v \equiv v^*$ in Ω_T^1 (**Case I**), in $\Omega_T^1 \cup \Omega_T^3$ (**Case II**), $\operatorname{div} v = u$ and $(Du, v_t) \in \mathcal{S} \cup \mathcal{F}$ a.e. in Ω_T, and $(Du, v_t) \in \mathcal{S} \cup \mathcal{B}$ a.e. in Ω_T^2. Since both u and v are piecewise C^1 in Ω_T, there exists a sequence of disjoint open sets $\{G_j\}_{j=1}^\infty$ in Ω_T^2 with $|\partial G_j| = 0$ such that

$$u \in C^1(\bar{G}_j),\ v \in C^1(\bar{G}_j; \mathbb{R}^n)\ \forall j \ge 1, \quad |\Omega_T^2 \setminus \cup_{j=1}^\infty G_j| = 0.$$

2. Let $j \in \mathbb{N}$ be fixed. Note that $(Du(z), v_t(z)) \in \bar{\mathcal{S}} \cup \mathcal{B}$ for all $z = (x,t) \in G_j$ and that $H_j := \{z \in G_j \mid (Du(z), v_t(z)) \in \partial\mathcal{S}\}$ is a (relatively) closed set in G_j with measure zero. So $\tilde{G}_j := G_j \setminus H_j$ is an open subset of G_j with $|\tilde{G}_j| = |G_j|$, and $(Du(z), v_t(z)) \in \mathcal{S} \cup \mathcal{B}$ for all $z \in \tilde{G}_j$.

3. For each $\tau > 0$, let

$$\mathcal{G}_\tau = \{(p,\beta) \in \mathcal{S} \mid \operatorname{dist}((p,\beta), \partial\mathcal{S}) > \tau,\ \operatorname{dist}((p,\beta), \mathcal{C}) > \tau\};$$

then

$$\mathcal{S} = (\cup_{\tau>0}\mathcal{G}_\tau) \cup \{(p,\beta) \in \mathcal{S} \mid \operatorname{dist}((p,\beta), \mathcal{C}) = 0\}.$$

Since $\mathcal{B} \subset \mathcal{C}$, we have

$$\int_{\tilde{G}_j} |v_t(z) - A(Du(z))|\, dz = \lim_{\tau\to 0^+} \int_{\{z\in\tilde{G}_j \mid (Du(z),v_t(z))\in\mathcal{G}_\tau\}} |v_t(z) - A(Du(z))|\, dz,$$

$$\int_{\tilde{G}_j} \operatorname{dist}((Du(z), v_t(z)), \mathcal{C})\, dz = \lim_{\tau\to 0^+} \int_{\{z\in\tilde{G}_j \mid (Du(z),v_t(z))\in\mathcal{G}_\tau\}} \operatorname{dist}((Du(z), v_t(z)), \mathcal{C})\, dz;$$

thus we can find a $\tau_j > 0$ such that $|\partial O_j| = 0$,

$$\int_{F_j} |v_t(z) - A(Du(z))|\, dz < \frac{\epsilon}{3\cdot 2^j}|\Omega_T^2|, \tag{7.1}$$

and

$$\int_{F_j} \operatorname{dist}((Du(z), v_t(z)), \mathcal{C})\, dz < \frac{\epsilon}{3\cdot 2^j}|\Omega_T^2|, \tag{7.2}$$

where $F_j = \{z \in \tilde{G}_j \mid (Du(z), v_t(z)) \notin \mathcal{G}_{\tau_j}\}$ and $O_j = \tilde{G}_j \setminus F_j$ is open. Let J be the set of all indices $j \in \mathbb{N}$ with $O_j \neq \emptyset$. Then for $j \notin J$, $F_j = \tilde{G}_j$.

4. We now fix a $j \in J$. Note that $O_j = \{z \in \tilde{G}_j \mid (Du(z), v_t(z)) \in \mathcal{G}_{\tau_j}\}$ and that $\mathcal{K}_j := \bar{\mathcal{G}}_{\tau_j}$ is a compact subset of $\mathcal{S}$. Let $\tilde{Q} \subset \mathbb{R}^n$ be a box with $\Omega \subset \tilde{Q}$ and $\tilde{I} = (0, T)$. Applying Theorem 5.8 to box $\tilde{Q} \times \tilde{I}$, $\mathcal{K}_j \subset\subset \mathcal{S} = \mathcal{S}_{r_1,r_2}$, and $\epsilon' = \frac{\epsilon|\Omega_T|}{12}$, we obtain a constant $\delta_j > 0$ that satisfies the conclusion of the theorem. By the uniform continuity of A on compact subsets of $\mathbb{R}^n$, we can find a $\theta = \theta_{\epsilon,\bar{s}} > 0$ such that

$$|A(p) - A(p')| < \frac{\epsilon}{12} \tag{7.3}$$

whenever $|p|, |p'| \le \bar{s}$ and $|p - p'| \le \theta$, where the number $\bar{s} > 0$ will be chosen later. Also by the uniform continuity of u, v and their gradients on $\bar{G}_j$, there exists a $\nu_j > 0$ such that

$$|u(z) - u(z')| + |\nabla u(z) - \nabla u(z')| + |v(z) - v(z')| + |\nabla v(z) - \nabla v(z')| < \min\{\frac{\delta_j}{2}, \frac{\epsilon}{12}, \theta\} \tag{7.4}$$

whenever $z, z' \in \bar{G}_j$ and $|z - z'| \le \nu_j$. We now cover O_j (up to measure zero) by a sequence of disjoint boxes $\{Q^i_j \times I^i_j\}_{i=1}^\infty$ in O_j with center z^i_j and diameter $l^i_j < \nu_j$.

5. Fix an $i \in \mathbb{N}$ and write $w = (u, v)$, $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} = \nabla w(z^i_j) = \begin{pmatrix} Du(z^i_j) & u_t(z^i_j) \\ Dv(z^i_j) & v_t(z^i_j) \end{pmatrix}$. By the choice of $\delta_j > 0$ in Step 4 via Theorem 5.8, since $Q^i_j \times I^i_j \subset \tilde{Q} \times \tilde{I}$ and $(p, \beta) \in \mathcal{K}_j$, for all sufficiently small $\rho > 0$, there exists a function $\omega^i_j = (\varphi^i_j, \psi^i_j) \in C^\infty_c(Q^i_j \times I^i_j; \mathbb{R}^{1+n})$ satisfying

(a) $\operatorname{div} \psi^i_j = 0$ in $Q^i_j \times I^i_j$,
(b) $(p' + D\varphi^i_j(z), \beta' + (\psi^i_j)_t(z)) \in \mathcal{S}$ for all $z \in Q^i_j \times I^i_j$ and all $|(p', \beta') - (p, \beta)| \le \delta_j$,
(c) $\|\omega^i_j\|_{L^\infty(Q^i_j \times I^i_j)} < \rho$,
(d) $\int_{Q^i_j \times I^i_j} |\beta + (\psi^i_j)_t(z) - A(p + D\varphi^i_j(z))| dz < \epsilon' |Q^i_j \times I^i_j| / |\tilde{Q} \times \tilde{I}|$,
(e) $\int_{Q^i_j \times I^i_j} \operatorname{dist}((p + D\varphi^i_j(z), \beta + (\psi^i_j)_t(z)), \mathcal{A}) dz < \epsilon' |Q^i_j \times I^i_j| / |\tilde{Q} \times \tilde{I}|$,
(f) $\int_{Q^i_j} \varphi^i_j(x, t) dx = 0$ for all $t \in I^i_j$,
(g) $\|(\varphi^i_j)_t\|_{L^\infty(Q^i_j \times I^i_j)} < \rho$,

where the set $\mathcal{A} = \mathcal{A}_{r_1,r_2} \subset \mathbb{R}^{n+n}$ is as in Theorem 5.8. Here, we let $0 < \rho \le \min\{\tau_0, \frac{\delta_j}{2C}, \frac{\epsilon}{12C}, \eta\}$, where $C_n > 0$ is the constant in Theorem 4.4 and C is the product of C_n and the sum of the lengths of all sides of $\tilde{Q}$. From $\varphi^i_j|_{\partial(Q^i_j \times I^i_j)} \equiv 0$ and (f), we can apply Theorem 4.4 to φ^i_j on $Q^i_j \times I^i_j$ to obtain a function $g^i_j = \mathcal{R}\varphi^i_j \in C^1(\overline{Q^i_j \times I^i_j}; \mathbb{R}^n) \cap W^{1,\infty}_0(Q^i_j \times I^i_j; \mathbb{R}^n)$ such that $\operatorname{div} g^i_j = \varphi^i_j$ in $Q^i_j \times I^i_j$ and

$$\|(g^i_j)_t\|_{L^\infty(Q^i_j \times I^i_j)} \le C \|(\varphi^i_j)_t\|_{L^\infty(Q^i_j \times I^i_j)} \le \frac{\delta_j}{2} \quad \text{(by (g))} \tag{7.5}$$

6. As $|v_t - A(Du)|$, $\operatorname{dist}((Du, v_t), \mathcal{C}) \in L^\infty(\Omega_T)$, we can select a finite index set $\mathcal{I} \subset J \times \mathbb{N}$ such that

$$\int\limits_{\bigcup_{(j,i) \in (J \times \mathbb{N}) \setminus \mathcal{I}} Q^i_j \times I^i_j} |v_t(z) - A(Du(z))| dz \le \frac{\epsilon}{3} |\Omega^2_T|, \tag{7.6}$$

$$\int\limits_{\bigcup_{(j,i) \in (J \times \mathbb{N}) \setminus \mathcal{I}} Q^i_j \times I^i_j} \operatorname{dist}((Du(z), v_t(z)), \mathcal{C}) dz \le \frac{\epsilon}{3} |\Omega^2_T|. \tag{7.7}$$

We finally define

$$(\tilde{u}, \tilde{v}) = (u, v) + \sum_{(j,i)\in\mathcal{I}} \chi_{Q^i_j \times I^i_j}(\varphi^i_j, \psi^i_j + g^i_j) \quad \text{in } \Omega_T.$$

As a side remark, note here that only *finitely* many functions $(\varphi^i_j, \psi^i_j + g^i_j)$ are disjointly patched to the original (u, v) to obtain a new function $(\tilde{u}, \tilde{v})$ towards the goal of the proof. The reason for using only finitely many pieces of gluing is due to the lack of control over the spatial gradients $D(\psi^i_j + g^i_j)$, and overcoming this difficulty is at the heart of this paper.

7. Let us finally check that $\tilde{u}$ together with $\tilde{v}$ indeed gives the desired result. By construction, it is clear that $\tilde{u} \in C^1_{piece} \cap W^{1,\infty}_{u^*}(\Omega_T)$, $\tilde{v} \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T; \mathbb{R}^n)$ and that $\tilde{u} = u = u^*$ and $\tilde{v} = v = v^*$ in Ω^1_T (**Case I**), in $\Omega^1_T \cup \Omega^3_T$ (**Case II**). By the choice of ρ in (g) as $\rho \le \tau_0$, we have $\|\tilde{u}_t\|_{L^\infty(\Omega_T)} < m$. Next, let $(j, i) \in \mathcal{I}$, and observe that for $z \in Q^i_j \times I^i_j$, with $(p, \beta) = (Du(z^i_j), v_t(z^i_j)) \in \mathcal{G}_{\tau_j}$, since $|z - z^i_j| < l^i_j < \nu_j$, it follows from (7.4) and (7.5) that

$$|(Du(z), v_t(z) + (g^i_j)_t(z)) - (p, \beta)| \le \delta_j,$$

and so $(D\tilde{u}(z), \tilde{v}_t(z)) \in \mathcal{S}$ from (b) above. From (a) and $\operatorname{div} g^i_j = \varphi^i_j$, for $z \in Q^i_j \times I^i_j$,

$$\operatorname{div} \tilde{v}(z) = \operatorname{div}(v + \psi^i_j + g^i_j)(z) = u(z) + 0 + \varphi^i_j(z) = \tilde{u}(z).$$

Therefore, $\tilde{u} \in \mathcal{U}$. Next, observe

$$\begin{aligned}
\int_{\Omega_T} |\tilde{v}_t - A(D\tilde{u})|dz &= \int_{\Omega_T\setminus\Omega^2_T} |v^*_t - \tilde{A}(Du^*)|dz + \int_{\Omega^2_T} |\tilde{v}_t - A(D\tilde{u})|dz = \int_{\Omega^2_T} |\tilde{v}_t - A(D\tilde{u})|dz \\
&= \int_{\cup_{j\in\mathbb{N}} F_j} |v_t - A(Du)|dz + \int_{\cup_{(j,i)\in(J\times\mathbb{N})\setminus\mathcal{I}} Q^i_j\times I^i_j} |v_t - A(Du)|dz \\
&\quad + \int_{\cup_{(j,i)\in\mathcal{I}} Q^i_j\times I^i_j} |\tilde{v}_t - A(D\tilde{u})|dz =: I^1_1 + I^1_2 + I^1_3,
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega^2_T} \operatorname{dist}((D\tilde{u}, \tilde{v}_t), \mathcal{C})dz &= \int_{\cup_{j\in\mathbb{N}} F_j} \operatorname{dist}((Du, v_t), \mathcal{C})dz + \int_{\cup_{(j,i)\in(J\times\mathbb{N})\setminus\mathcal{I}} Q^i_j\times I^i_j} \operatorname{dist}((Du, v_t), \mathcal{C})dz \\
&\quad + \int_{\cup_{(j,i)\in\mathcal{I}} Q^i_j\times I^i_j} \operatorname{dist}((D\tilde{u}, \tilde{v}_t), \mathcal{C})dz =: I^2_1 + I^2_2 + I^2_3.
\end{aligned}$$

From (7.1), (7.2), (7.6) and (7.7), we have $I^k_1 + I^k_2 \le \frac{2\epsilon}{3}|\Omega^2_T|$ $(k = 1, 2)$. Note that for $(j, i) \in \mathcal{I}$ and $z \in Q^i_j \times I^i_j$, from (7.4), (7.5) and (g),

$$\begin{aligned}
|\tilde{v}_t(z) - A(D\tilde{u}(z))| &= |v_t(z) + (\psi^i_j)_t(z) + (g^i_j)_t(z) - A(Du(z) + D\varphi^i_j(z))| \\
&\le |v_t(z) - v_t(z^i_j)| + |v_t(z^i_j) + (\psi^i_j)_t(z) - A(Du(z^i_j) + D\varphi^i_j(z))| \\
&\quad + |(g^i_j)_t(z)| + |A(Du(z^i_j) + D\varphi^i_j(z)) - A(Du(z) + D\varphi^i_j(z))| \\
&\le \frac{\epsilon}{6} + |v_t(z^i_j) + (\psi^i_j)_t(z) - A(Du(z^i_j) + D\varphi^i_j(z))| \\
&\quad + |A(Du(z^i_j) + D\varphi^i_j(z)) - A(Du(z) + D\varphi^i_j(z))|.
\end{aligned}$$

Similarly, since $\mathcal{A} \subset \mathcal{C}$, we have

$$\begin{aligned}
&\operatorname{dist}(\,(D\tilde{u}(z), \tilde{v}_t(z)), \mathcal{C}) \\
&\quad \le \frac{\epsilon}{4} + \operatorname{dist}((Du(z^i_j) + D\varphi^i_j(z), v_t(z^i_j) + (\psi^i_j)_t(z)), \mathcal{C}) \\
&\quad \le \frac{\epsilon}{4} + \operatorname{dist}((Du(z^i_j) + D\varphi^i_j(z), v_t(z^i_j) + (\psi^i_j)_t(z)), \mathcal{A}).
\end{aligned}$$

From (b) and (i) of Theorem 5.6, we have $|Du(z_j^i) + D\varphi_j^i(z)| \le s_+(r_1) =: \bar{s}$ (**Case I**), $|Du(z_j^i) + D\varphi_j^i(z)| \le s_+(r_2) =: \bar{s}$ (**Case II**). As $(D\tilde{u}(z), \tilde{v}_t(z)) \in \mathcal{S}$, we also have $|Du(z) + D\varphi_j^i(z)| = |D\tilde{u}(z)| \le \bar{s}$, and by (7.4), $|Du(z_j^i) - Du(z)| < \theta$. From (7.3), we thus have

$$|A(Du(z_j^i) + D\varphi_j^i(z)) - A(Du(z) + D\varphi_j^i(z))| < \frac{\epsilon}{12}.$$

Integrating the two inequalities above over $Q_j^i \times I_j^i$, we now obtain from (d) and (e), respectively, that

$$\int_{Q_j^i \times I_j^i} |\tilde{v}_t(z) - A(D\tilde{u}(z))| dz \le \frac{\epsilon}{4}|Q_j^i \times I_j^i| + \frac{\epsilon|\Omega_T|}{12}\frac{|Q_j^i \times I_j^i|}{|\tilde{Q} \times \tilde{I}|} \le \frac{\epsilon}{3}|Q_j^i \times I_j^i|,$$

$$\int_{Q_j^i \times I_j^i} \mathrm{dist}((D\tilde{u}(z), \tilde{v}_t(z)), \mathcal{C}) dz \le \frac{\epsilon}{4}|Q_j^i \times I_j^i| + \frac{\epsilon|\Omega_T|}{12}\frac{|Q_j^i \times I_j^i|}{|\tilde{Q} \times \tilde{I}|} \le \frac{\epsilon}{3}|Q_j^i \times I_j^i|;$$

thus $I_3^k \le \frac{\epsilon}{3}|\Omega_T^2|$, and so $I_1^k + I_2^k + I_3^k \le \epsilon|\Omega_T^2|$, where $k = 1, 2$. Therefore, $\tilde{u} \in \mathcal{U}_\epsilon$. Lastly, from (c) with $\rho \le \eta$ and the definition of $\tilde{u}$, we have $\|\tilde{u} - u\|_{L^\infty(\Omega_T)} < \eta$.

The proof is now complete. □

7.2. *Completion of the proof of Theorems 3.1 and 3.5*

Unless specifically distinguished, the proof below is common for both **Case I:** Theorem 3.1 and **Case II:** Theorem 3.5.

Proof of Theorems 3.1 and 3.5. We return to Section 6. As outlined in Remark 6.4, Theorem 7.1 and Theorem 2.2 together give infinitely many Lipschitz solutions u to problem (1.2).

We now follow the proof of Theorem 2.2 for detailed information on such a Lipschitz solution $u \in \mathcal{G}$ to (1.2). Here Du is the a.e.-pointwise limit of some sequence Du_j, where the sequence $u_j \in \mathcal{U}_{1/j}$ converges to u in $L^\infty(\Omega_T)$. Since $u_j \equiv u^*$ in Ω_T^1 (**Case I**), in $\Omega_T^1 \cup \Omega_T^3$ (**Case II**), we also have $u \equiv u^* \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T^1)$ (**Case I**), $u \equiv u^* \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T^1 \cup \bar{\Omega}_T^3)$ (**Case II**) so that

$$u_t = \mathrm{div}(A(Du)) \text{ and } |Du| < s_-(\bar{r}) \text{ in } \Omega_T^1, \quad \textbf{(Case I)}$$
$$u_t = \mathrm{div}(A(Du)) \text{ in } \Omega_T^1 \cup \Omega_T^3,\ |Du| < s_-(\bar{r}_1) \text{ in } \Omega_T^1, \text{ and } |Du| > s_+(\bar{r}_3) \text{ in } \Omega_T^3. \quad \textbf{(Case II)}$$

Note $(v_j)_t \rightharpoonup v_t$ in $L^2(\Omega_T; \mathbb{R}^n)$, where v_j is the corresponding vector function to u_j and $v \in W^{1,2}((0,T); L^2(\Omega; \mathbb{R}^n))$. From (2.8), we can even deduce that $(v_j)_t \to v_t$ pointwise a.e. in Ω_T. On the other hand, from the definition of $\mathcal{U}_{1/j}$,

$$\int_{\Omega_T^2} \mathrm{dist}((Du_j, (v_j)_t), \mathcal{C})\, dxdt \le \frac{1}{j}|\Omega_T^2| \to 0 \text{ as } j \to \infty;$$

thus $(Du, v_t) \in \mathcal{C}$ a.e. in Ω_T^2, yielding $|S| + |L| = |\Omega_T^2|$.

For the remaining assertions, we separate the proof for each case.

Case I. If $|L| = 0$, then $|Du| \le s_-(r)$ a.e. in Ω_T; so u is a Lipschitz solution to (6.1) with monotone flux $\tilde{A}(p)$. Thus, by Proposition 2.3, we have $u = u^*$ in Ω_T. This contradicts the fact that $\|Du^*(\cdot, 0)\|_{L^\infty(\Omega)} = \|Du_0\|_{L^\infty(\Omega)} = M_0 > s_-(r)$. Thus $|L| > 0$.

Case II. Suppose $|L| = 0$. Then

$$|Du| \in [0, s_-(r)] \cup [s_+(\bar{r}_3), \infty) \text{ a.e. in } \Omega_T.$$

Now, modify the profile $\sigma(s)$ so as to obtain a function $\bar{\sigma} \in C^{1+\alpha}([0,\infty))$ satisfying

$$\begin{cases} \bar{\sigma}(s) = \sigma(s), & s \in [0, s_-(r)] \cup [s_+(\bar{r}_3), \infty), \\ \bar{\theta} \le \bar{\sigma}'(s) \le \bar{\Theta}, & 0 \le s < \infty, \end{cases}$$

for some constants $\bar{\Theta} \geq \bar{\theta} > 0$, and let $\bar{f}(s) = \bar{\sigma}(\sqrt{s})/\sqrt{s}$ $(s > 0)$, $\bar{f}(0) = f(0)$, and $\bar{A}(p) = \bar{f}(|p|^2)p$ $(p \in \mathbb{R}^n)$. Then the functions $u \in \mathcal{G}$ are all Lipschitz solutions to the problem of type (1.2) with monotone flux $\bar{A}(p)$, a contradiction to the uniqueness by Proposition 2.3. Thus $|L| > 0$.

Next, suppose $|S| = 0$. Then

$$|Du| \in [0, s_-(\tilde{r}_1)] \cup [s_+(\tilde{r}), \infty) \text{ a.e. in } \Omega_T.$$

So we get a contradiction similarly as above by obtaining a function $\hat{\sigma} \in C^{1+\alpha}([0,\infty))$ satisfying

$$\begin{cases} \hat{\sigma}(s) = \sigma(s), & s \in [0, s_-(\tilde{r}_1)] \cup [s_+(\tilde{r}), \infty), \\ \hat{\theta} \leq \hat{\sigma}'(s) \leq \hat{\Theta}, & 0 \leq s < \infty, \end{cases}$$

for some constants $\hat{\Theta} \geq \hat{\theta} > 0$. Thus $|S| > 0$. □

7.3. *Proof of Corollary 3.6*

Recall that this corollary is under **Case II:** Hypothesis (H). Let $u_0 \in C^{2+\alpha}(\bar{\Omega})$ with $Du_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$. The existence of infinitely many Lipschitz solutions to problem (1.2) when $|Du_0(x_0)| \in (s_1^*, s_2^*)$ for some $x_0 \in \Omega$ is simply the result of Theorem 3.5. So we cover the other possibilities here.

Assume $\|Du_0\|_{L^\infty(\Omega)} \leq s_1^*$. Fix any two numbers $\sigma(s_2) < r_1 < r_2 < \sigma(s_1)$, and let $\tilde{\sigma}, \tilde{f} \in C^{1+\alpha}([0,\infty))$ be some functions from Lemma 4.3. Using the flux $\tilde{A}(p) = \tilde{f}(|p|^2)p$, Theorem 4.1 gives a unique solution $u^* \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega}_T)$ to problem (4.3). If $|Du^*|$ stays on or below the threshold s_1^* in Ω_T, then u^* itself is a Lipschitz solution to (1.2). Otherwise, set $\bar{s} = \frac{s_1^* + s_-(r_1)}{2}$ and choose a point $(\bar{x}, \bar{t}) \in \Omega_T$ such that

$$|Du^*| \leq \bar{s} \text{ in } \Omega \times (0, \bar{t}), \quad |Du^*(\bar{x}, \bar{t})| \in (s_1^*, s_2^*).$$

Regarding $u_1(\cdot) := u^*(\cdot, \bar{t}) \in C^{2+\alpha}(\bar{\Omega})$, satisfying $Du_1 \cdot \mathbf{n}|_{\partial\Omega} = 0$, as a new initial datum, it follows from Theorem 3.5 that problem (1.2), with the initial datum u_1 at time $t = \bar{t}$, admits infinitely many Lipschitz solutions $\bar{u}$ in $\Omega \times (\bar{t}, T)$. Then the patched functions $u = \chi_{\Omega \times (0,\bar{t})} u^* + \chi_{\Omega \times [\bar{t}, T)} \bar{u}$ in Ω_T become Lipschitz solutions to the original problem (1.2).

Lastly, assume $\min_{\bar{\Omega}} |Du_0| \geq s_2^*$. Let r_1, r_2 and $\tilde{A}(p)$ be as above, and let u^* be the solution to (4.3) corresponding to this flux $\tilde{A}(p)$. If $|Du^*|$ stays on or above the threshold s_2^* in Ω_T, then u^* itself is a Lipschitz solution to (1.2). Otherwise, set $\bar{s} = \frac{s_2^* + s_+(r_2)}{2}$ and choose a point $(\bar{x}, \bar{t}) \in \Omega_T$ such that

$$|Du^*| \geq \bar{s} \text{ in } \Omega \times (0, \bar{t}), \quad |Du^*(\bar{x}, \bar{t})| \in (s_1^*, s_2^*).$$

Then we can do the obvious as above to obtain infinitely many Lipschitz solutions u to (1.2), and the proof is complete.

7.4. *Co-existence of radial and non-radial solutions*

We assume that $\Omega = B_R(0)$ is an open ball in $\mathbb{R}^n$ and that the flux $A(p)$ fulfills Hypothesis (PM). Assume the initial datum $u_0 \in C^{2+\alpha}(\bar{\Omega})$ satisfies the compatibility condition

$$A(Du_0) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

A function u defined in Ω_T (Ω, resp.) is *radial* if $u(x,t) = u(y,t)$ $\forall x, y \in \Omega$, $|x| = |y|$, $\forall t \in (0,T)$ ($u(x) = u(y)$ $\forall x, y \in \Omega$, $|x| = |y|$, resp.).

We now state and prove Theorem 1.3 as the following result.

Theorem 7.2. *Assume in addition that u_0 is non-constant and radial. Then there are infinitely many radial and non-radial Lipschitz solutions to (1.2).*

Proof. The existence of infinitely many *radial* Lipschitz solutions to (1.2) has been established in the authors' recent paper [23]. We now prove the existence of infinitely many *non-radial* Lipschitz solutions to (1.2). In the current situation, it is easy to see that the function $u^* \in \mathcal{U}$ constructed in Section 6 is radial in Ω_T. Our strategy is to imitate the procedure of the density proof above to the function (u^*, v^*). We choose a space–time box in Ω_T having positive distance from the central axis $\{0\} \times [0, T]$ of Ω_T, where v_t^* is sufficiently away from $A(Du^*)$ in L^1-sense. Then as in Steps 4 to 6 of the proof of Theorem 7.1, we perform the surgery on (u^*, v^*) only in the box to obtain a function (u_{nr}^*, v_{nr}^*) with membership $u_{nr}^* \in \mathcal{U}$ maintained. Such a surgery breaks down the radial symmetry of u^*; hence, u_{nr}^* is non-radial. Note also that this u_{nr}^* is not a Lipschitz solution to (1.2). Suppose there are only finitely many (possibly no) non-radial Lipschitz solutions to (1.2). Since the Lipschitz solutions to (1.2) are dense in $\mathcal{U}$ under the L^∞-norm, this forces that the non-radial function $u_{nr}^* \in \mathcal{X}$ must be the L^∞-limit of some sequence of distinct radial functions in $\mathcal{G}$ in the context of the proof of Theorem 2.2, yielding u_{nr}^* must be radial, a contradiction. Therefore, there are infinitely many non-radial Lipschitz solutions to (1.2). □

Conflict of interest statement

The authors declare that they have no conflict of interest.

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