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Mean field games with congestion

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Abstract

We consider a class of systems of time dependent partial differential equations which arise in mean field type models with congestion. The systems couple a backward viscous Hamilton–Jacobi equation and a forward Kolmogorov equation both posed in $(0, T) \times (\mathbb{R}^N/\mathbb{Z}^N)$. Because of congestion and by contrast with simpler cases, the latter system can never be seen as the optimality conditions of an optimal control problem driven by a partial differential equation. The Hamiltonian vanishes as the density tends to $+\infty$ and may not even be defined in the regions where the density is zero. After giving a suitable definition of weak solutions, we prove the existence and uniqueness results of the latter under rather general assumptions. No restriction is made on the horizon T.

Keywords: Mean field games; Congestion models; Local coupling; Existence and uniqueness; Weak solutions

1. Introduction

Recently, an important research activity on mean field games (MFGs for short) has been initiated since the pioneering works [17–19] of Lasry and Lions: it aims at studying the asymptotic behavior of stochastic differential games (Nash equilibria) as the number n of agents tends to infinity. In these models, it is assumed that the agents are all identical and that an individual agent can hardly influence the outcome of the game. Moreover, each individual strategy is influenced by some averages of functions of the states of the other agents. In the limit when $n \to +\infty$, a given agent feels the presence of the others through the statistical distribution of the states. Since perturbations of a single agent's strategy does not influence the statistical states distribution, the latter acts as a parameter in the control problem to be solved by each agent. When the dynamics of the agents are independent stochastic processes, MFGs naturally lead to a coupled system of two partial differential equations (PDEs for short), a forward Kolmogorov or Fokker–Planck equation and a backward Hamilton–Jacobi–Bellman equation, see for example (1.1) below.

The theory of MFGs allows one to model congestion effects, i.e. situations in which the cost of displacement of the agents increases in those regions where the density is large. MFGs models including congestion were introduced and studied in [20]. A typical such model leads to the following system of PDEs:

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$$\begin{cases} -\partial_t u - \nu\Delta u + \frac{1}{\beta}\frac{|Du|^\beta}{(m+\mu)^\alpha} = F(t,x,m)\,, & (t,x)\in(0,T)\times\Omega \\ \partial_t m - \nu\Delta m - \mathrm{div}(m\frac{|Du|^{\beta-2}Du}{(m+\mu)^\alpha}) = 0\,, & (t,x)\in(0,T)\times\Omega \\ m(0,x)=m_0(x)\,,\ u(T,x)=G(x,m(T))\,, & x\in\Omega\,, \end{cases} \tag{1.1}$$

with $\nu > 0$, $\alpha > 0$, $\beta \in (1,2]$, $\mu \in \mathbb{R}$ with either $\mu > 0$ or $\mu = 0$. System (1.1) must generally be complemented with suitable boundary conditions on $(0,T)\times\partial\Omega$, but we will avoid the additional technical difficulties coming from the latter by focusing on the case when Ω is the flat torus, i.e. $\Omega = \mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ and all the data are periodic.

Loosely speaking, (1.1) describes the optimization over a stochastic dynamics defined on a standard probability space $(\mathcal{X}, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$

$$dX_t = w_t\,dt + \sqrt{2\nu}\,dB_t$$

where B_t is a N-dimensional Brownian motion, with a cost criterion given by

$$\inf_{w_t}\left[\mathbb{E}\int_0^T\left\{c_\beta\ (m_t+\mu)^\gamma\ |w_t|^{\frac{\beta}{\beta-1}} + F(t,X_t,m_t)\right\}dt + \mathbb{E}(G(X_T,m_T))\right], \tag{1.2}$$

where $\gamma = \frac{\alpha}{\beta-1}$ and c_β is a suitable normalization constant. In the viewpoint of the generic agent, $m_t = m(t,X_t)$ is meant to represent the distribution law of the states, however in the optimization process it is just, a priori, a given frozen density function. The mean field game equilibrium is next given through a fixed point scheme, by requiring, a posteriori, that m_t coincides with the law of the optimal process X_t.

In [20], P.-L. Lions put the stress on general structural conditions yielding the uniqueness for the following MFG systems with local coupling:

$$-\partial_t u - \nu\Delta u + H(t,x,m,Du) = F(t,x,m)\,,\quad (t,x)\in(0,T)\times\Omega \tag{1.3}$$

$$\partial_t m - \nu\Delta m - \mathrm{div}(mH_p(t,x,m,Du)) = 0\,,\quad (t,x)\in(0,T)\times\Omega \tag{1.4}$$

$$m(0,x)=m_0(x)\,,\ u(T,x)=G(x,m(T))\,,\quad x\in\Omega \tag{1.5}$$

namely that F and G be increasing w.r.t. m and that the following matrix be positive semidefinite:

$$\begin{pmatrix} -\frac{2}{m}\frac{\partial H}{\partial m}(t,x,m,p) & \frac{\partial}{\partial m}\nabla_p^T H(t,x,m,p) \\ \frac{\partial}{\partial m}\nabla_p H(t,x,m,p) & 2D^2_{p,p}H(t,x,m,p) \end{pmatrix} \geq 0 \tag{1.6}$$

for all $x \in \Omega$, $m > 0$ and $p \in \mathbb{R}^N$. Since (1.1) is equivalent to (1.3)–(1.5) with $H(t,x,m,p) = \frac{1}{\beta}\frac{|p|^\beta}{(m+\mu)^\alpha}$, (1.6) becomes in this case

$$\alpha \leq \frac{4(\beta-1)}{\beta}. \tag{1.7}$$

In the present work, we will show that this hypothesis yields both the existence and the uniqueness of weak solutions.

Except for situations in which special tricks may be applied (stationary problems and quadratic Hamiltonian, see [13]), the existence of classical solutions of suitable generalizations of (1.1) seems difficult to obtain, because generally neither upper bounds on m nor strict positivity of m are known unless one restricts the growth conditions for the nonlinearities and assumes that the time horizon T is small, see [14,15] (see also [12] for the stationary case). Therefore, in order to get at a sufficiently general result, we aim at proving the existence and uniqueness of suitably defined weak solutions.

For MFG models without congestion, the first results on the existence of weak solutions were supplied in [19]. Besides, as already observed in [17–19], in the easiest cases, the system of PDEs can be seen as the optimality conditions of a problem of optimal control driven by a PDE: in such cases, a pair of primal-dual optimization problems can be introduced, leading to a suitable weak formulation for which there exists a unique solution, see [9], where possibly degenerate diffusions are dealt with. A striking fact is that in general, MFGs with congestion cannot be cast into an optimal control problem driven by a PDE, by contrast with simpler cases. For MFG systems (1.3)–(1.5) with H independent of m, a complete analysis is available in [24], which contains in particular new results on weak solutions of Fokker–Planck equations, and an answer to the delicate question of the uniqueness of weak solutions of

MFG systems. Proving uniqueness of weak solutions is difficult for the following reason: to compare two solutions (u, m) and $(\tilde{u}, \tilde{m})$ of (1.1), the main idea is to test the Bellman equations by $m - \tilde{m}$ and the Kolmogorov equations by $u - \tilde{u}$ and sum the resulting identities. While this is of course permitted for classical solutions, special care is needed for weak solutions, because the PDEs only hold in the distributional sense. The present work will borrow several ideas and results from [24].

Let us mention that other kinds of models, which may also include congestion, are obtained by assuming that all the agents use the same distributed feedback strategy and by passing to the limit as $N \to \infty$ before optimizing the common feedback. Given a common feedback strategy, the asymptotics are given by the McKean–Vlasov theory, [21]: the dynamics of a given agent is found by solving a stochastic differential equation with coefficients depending on a mean field, namely the statistical distribution of the states, which may also affect the objective function. Since the feedback strategy is common to all agents, perturbations of the latter affect the mean field. Then, letting each player optimize its objective function amounts to solving a control problem driven by the McKean–Vlasov dynamics. The latter is named control of McKean–Vlasov dynamics by R. Carmona and F. Delarue [11,10] and mean field type control by A. Bensoussan et al., [5,6]. By contrast with MFGs, this genuine connection to optimal control problems driven by a PDE makes it possible to use techniques from the calculus of variations and define a suitable notion of weak solutions for which existence and uniqueness can be proved, see [3].

Finally, let us mention that numerical methods and simulations for MFGs and mean field type control with congestion are dealt with in [1,2].

The present paper deals with the existence and uniqueness of weak solutions to a class of MFG systems which generalize (1.1) including the congestion effects in the structure conditions of the Hamiltonian function. Making reference to the model example (1.1), we will consider both the case when $\mu > 0$ and the case when $\mu = 0$. Let us notice that the congestion effect is essentially contained in the behavior of the Hamiltonian for *large values of* m, so both cases could be considered as congestion models, since the difference between $\mu > 0$ and $\mu = 0$ will be important in those regions of vanishing density. Hereafter, the case that $\mu = 0$ will be referred to as the case of *singular congestion*, since in this case the Hamiltonian may not even be defined at $m = 0$. In the latter case, the Lagrangian vanishes at $m = 0$, and this vanishing of the cost criterion will lead to a slightly incomplete information on the equation solved by the value function, and therefore to a relaxed formulation of the notion of weak solution.

The work is organized as follows: in Section 2 we set the problem and introduce the assumptions that will be used throughout the paper. In Section 3 we define the notions of weak solutions and state the main existence and uniqueness results: as explained above, we will give a different definition of solution in the case $\mu = 0$, in which special care is required because the Hamiltonian has a singularity at $m = 0$. In Section 4, we consider non-singular Hamiltonians ($\mu > 0$) and give the first steps of the existence proof which consist of studying a sequence of suitably regularized problems. Section 5 contain lemmas which will turn crucial for proving uniqueness but also for identifying the limit of the aforementioned regularized problems. In Section 6, in the case $\mu > 0$, we conclude the proof of existence and obtain uniqueness under suitable additional assumptions. Section 7 is devoted to the singular case $\mu = 0$.

2. Running assumptions

Let us consider the general MFG system (1.3)–(1.5). Different types of boundary conditions could be considered but for simplicity we neglect this point by assuming that the equation takes place in a standard flat torus $\Omega = \mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ and all functions are $\mathbb{Z}^N$-periodic in x. We define $Q_T = (0, T) \times \Omega$.

In (1.3)–(1.5), the function $H(t, x, m, p)$ is assumed to be measurable with respect to (t, x), continuous with respect to m and C^1 with respect to p. The notation H_p stands for $\frac{\partial H(t,x,m,p)}{\partial p}$. We also assume that H is a convex function of p.

We modify the structure growth conditions introduced in [19] to take into account the congestion factor. So, we will assume that H satisfies, for some positive β and α

$$H(t, x, m, 0) \le 0\,, \tag{2.1}$$

$$H(t, x, m, p) \ge c_0 \frac{|p|^\beta}{(m+\mu)^\alpha} - c_1 (1 + m^{\frac{\alpha}{\beta-1}})\,, \tag{2.2}$$

$$|H_p(t, x, m, p)| \le c_2 (1 + \frac{|p|^{\beta-1}}{(m+\mu)^\alpha})\,, \tag{2.3}$$

$$H_p(t,x,m,p)\cdot p \geq (1+\sigma)\,H(t,x,m,p) - c_3\,(1+m^{\frac{\alpha}{\beta-1}}), \tag{2.4}$$

for a.e. $(t,x)\in Q_T$ and every $(m,p)\in\mathbb{R}_+\times\mathbb{R}^N$, where $\sigma, c_0,\dots,c_3$ are positive constants and $\mu\in\mathbb{R}$, $\mu\geq 0$. For the sake of clarity, we will separately consider the two cases $\mu>0$ and $\mu=0$.

Notice that (2.1) can be alternatively rephrased as: $H(t,x,m,0)$ is bounded from above; indeed, if $H(t,x,m,0)\leq K$ for some $K>0$, we can reduce to (2.1) by changing u into $u+K(T-t)$.

As for the ranges of β,α in the previous assumptions, we will assume that

$$1<\beta\leq 2, \qquad 0<\alpha\leq \frac{4(\beta-1)}{\beta}. \tag{2.5}$$

The first condition on β is meant to exclude both the case of linear or sub linear growth and the case of super quadratic growth. Actually, if $\beta\leq 1$, then the bound on H_p would imply that m is bounded, and this would allow for a huge simplification in the system (at least if $\mu>0$), in particular the difficult issues coming from the congestion effect would be mostly circumvented and this case would be rather similar to the results which already exist in the literature. The super quadratic case is different in nature since one cannot rely any more on classical tools for parabolic equations and a different approach should be employed, as in the first order case. Since this would bring us too far for the present paper, we decided to assume that $\beta\leq 2$.

As for (2.5), this structural condition seems natural in order to have a completely well-posed formulation in the congestion model. As already mentioned, this limitation is indeed required in order to obtain uniqueness, as pointed out by P.L. Lions [20], when he discussed general conditions for the uniqueness of mean field games systems. In addition, this condition also seems to be useful at other stages of our existence proof. The upper bound required on α can be interpreted as a tolerable growth condition on the cost function which penalizes the motion in the zones where the distribution density is high (since we have $\gamma=\frac{\alpha}{\beta-1}$ in the Lagrangian model (1.2)).

Finally, the case $\alpha=0$ is excluded in the above conditions since this is already treated in [24].

Remark 2.1. Straightforward calculus leads to $4/\beta'\leq\beta$: from this and (2.5), we see that $\alpha\leq\beta$, and that $\alpha=\beta$ can occur only if $\beta=\alpha=2$.

Notice that combining (2.1) and the convexity of H in the p variable yields that

$$H_p(t,x,m,p)\cdot p - H(t,x,m,p)\geq -H(t,x,m,0)\geq 0. \tag{2.6}$$

Moreover, since $1<\beta\leq 2$, and from (2.2)–(2.4), we can deduce that there exist nonnegative constants C_0, C_1, C_2 such that

$$m^{\frac{\alpha}{\beta-1}+1}\left(|H_p|^2 - C_0\right)\leq C_1 m\left\{H_p\cdot p - H + C_2\right\}. \tag{2.7}$$

Using generic non-negative constants C_0, C_1 that can change from line to line, (2.7) is obtained as follows:

$$\begin{aligned}
m^{\frac{\alpha}{\beta-1}}|H_p|^2 &\leq C_1\left(m^{\frac{\alpha}{\beta-1}} + \frac{|p|^{2(\beta-1)}}{(m+\mu)^{2\alpha-\frac{\alpha}{\beta-1}}}\right)\\
&\leq C_1 m^{\frac{\alpha}{\beta-1}} + c_0\frac{|p|^{\beta}}{(m+\mu)^{\alpha}} + C_0(m+\mu)^{\frac{\alpha}{\beta-1}}\\
&\leq H + C_1(m^{\frac{\alpha}{\beta-1}}+1)\\
&\leq \sigma^{-1}(H_p\cdot p - H) + C_1 + C_0 m^{\frac{\alpha}{\beta-1}}
\end{aligned}$$

where we have used (2.3) in the first line, $1<\beta\leq 2$ and Young's inequality in the second line, (2.2) in the third line and (2.4) in the fourth line.

We also stress that, if H satisfies (2.1)–(2.4), then $H+b\cdot p$ satisfies the same conditions for any bounded vector field b. Hence the addition of bounded drift terms in the dynamics is permitted by the above setting of assumptions.

Remark 2.2. Let us compare the above assumptions with previous settings used for congestion models in mean field games. In [14], congestion models are presented starting from the Lagrangian function

$$L(t,x,m,q) = m^\gamma L_0(t,x,q-b(t,x)) + F(t,x,m), \quad \text{with} \quad L_0(t,x,q) = a(t,x)(1+|q|^2)^{\frac{\beta'}{2}},$$

where $\beta' = \beta/(\beta-1)$, a is a smooth positive function and b is a smooth velocity field. This leads to the system

$$\begin{cases} -\partial_t u - \nu\Delta u + m^\gamma H(t,x,\frac{Du}{m^\gamma}) = F(t,x,m)\,, & (t,x)\in(0,T)\times\Omega \\ \partial_t m - \nu\Delta m - \mathrm{div}(mH_p(t,x,\frac{Du}{m^\gamma})) = 0\,, & (t,x)\in(0,T)\times\Omega \\ m(0,x) = m_0(x)\,,\ u(T,x) = G(x,m(T))\,, & x\in\Omega \end{cases}$$

where

$$H(t,x,p) = H_0(t,x,p) + b(t,x)\cdot p, \quad \text{and} \quad H_0(t,x,p) = \sup_q -q\cdot p - L_0(t,x,q).$$

If we set $\alpha = \gamma(\beta-1)$, we recover our assumptions if $\gamma \le 4/\beta$ which is equivalent to saying that $\nabla_p H_0(t,x,p)\cdot p - H_0(t,x,p) \ge \frac{\gamma}{4} p^T D_{pp} H_0(t,x,p)p$ for all $p \ne 0$, as assumed in [14].

Finally, let us make precise the assumptions on the functions F, G appearing in (1.3)–(1.5). The function F is assumed to be measurable with respect to $(t,x)\in Q_T$ and continuous with respect to m. In addition, we assume that there exists a nondecreasing function $f(s)$ such that $f(s)s$ is convex and

$$\lambda f(m) - \kappa \le F(t,x,m) \le \frac{1}{\lambda} f(m) + \kappa \quad \forall m\in\mathbb{R}_+\,, \tag{2.8}$$

for some real numbers $\lambda, \kappa > 0$.

The terminal cost G is assumed to be measurable with respect to $x\in\Omega$, continuous with respect to m and such that

$$m \mapsto G(x,m) \qquad \text{is nondecreasing.} \tag{2.9}$$

In addition, as for F, we assume that there exists a function $g(s)$ such that $g(s)s$ is convex and

$$\lambda g(m) - \kappa \le G(x,m) \le \frac{1}{\lambda} g(m) + \kappa \quad \forall m\in\mathbb{R}_+\,, \tag{2.10}$$

with for example the same constants $\lambda, \kappa > 0$ as in (2.8).

Let us notice that, as a consequence of assumptions (2.8) and (2.10), the functions F, G are bounded below, namely there exists a constant $c_4\in\mathbb{R}$ such that

$$F(t,x,m) \ge c_4\,, \quad \forall\, m\in\mathbb{R}_+\,,\ \text{a.e. } (t,x)\in Q_T\,, \tag{2.11}$$

and

$$G(x,m) \ge c_4\,, \quad \forall\, m\in\mathbb{R}_+\,,\ \text{a.e. } x\in\Omega\,. \tag{2.12}$$

Assumptions (2.1)–(2.5) and (2.8)–(2.10) are the structural conditions under which we will prove the existence of weak solutions to the MFG system with congestion. Eventually, we will need a further condition for H and F in order to have uniqueness. This will be a natural reformulation of condition (1.6). Precisely, we assume the following:

$$\begin{aligned} &\text{for any } m\ge 0,\ z\ge -m,\ p,r\in\mathbb{R}^N \text{ such that } (z,r)\ne(0,0), \text{ setting} \\ &\quad m_s = m+sz \text{ and } p_s = p+sr \text{ for } s\in[0,1], \text{ then the function} \\ &\qquad\qquad h: s\mapsto -zH(t,x,m_s,p_s) + m_s H_p(t,x,m_s,p_s)\cdot r + zF(t,x,m_s) \\ &\text{is (strictly) increasing on } [0,1]. \end{aligned} \tag{2.13}$$

Remark 2.3. Note that if $H(t,x,m,p) = (\mu+m)^{-\alpha}|p|^\beta$, with (α,β) satisfying (2.5) and if F is nondecreasing with respect to m, then h defined in (2.13) is increasing. Condition (2.5) is thus compatible with the assumption yielding uniqueness.

Finally, we assume $m_0\in C(\Omega)$, $m_0\ge 0$ and, in order to be consistent with the interpretation of m, we fix the normalization condition $\int_\Omega m_0 = 1$.

3. Main results

3.1. Case of non-singular congestion

We assume here that conditions (2.1)–(2.4) hold with $\mu > 0$. Let us first make our notion of weak solution precise.

Definition 3.1. A pair $(u, m) \in L^1(Q_T) \times L^1(Q_T)$ is a weak solution to (1.3)–(1.5) if

(i) $m \in C([0, T]; L^1(\Omega))$, $m \geq 0$, $u \in L^q(0, T; W^{1,q}(\Omega))$ for every $q < \frac{N+2}{N+1}$,

(ii)

$$F(t, x, m)m \in L^1(Q_T), \qquad G(x, m(T))m(T) \in L^1(\Omega),$$
$$m \frac{|Du|^\beta}{(m+\mu)^\alpha} \in L^1(Q_T), \qquad \frac{|Du|^\beta}{(m+\mu)^\alpha} \in L^1(Q_T), \qquad m \in L^{1+\frac{\alpha}{\beta-1}}(Q_T)$$

(iii) $u \in L^\infty(0, T; L^1(\Omega))$ is bounded below and is a solution of the Bellman equation in the sense of distributions:

$$\begin{aligned} &\int_0^T\int_\Omega u\,\varphi_t\,dxdt - \nu\int_0^T\int_\Omega u\,\Delta\varphi\,dxdt + \int_0^T\int_\Omega H(t, x, m, Du)\varphi\,dxdt \\ &\qquad = \int_0^T\int_\Omega F(t, x, m)\varphi\,dxdt + \int_\Omega G(x, m(T))\varphi(T)\,dx \end{aligned} \tag{3.1}$$

for every $\varphi \in C_c^\infty((0, T] \times \Omega)$.

(iv) m is a solution of the Kolmogorov equation:

$$\int_0^T\int_\Omega m\left\{-\varphi_t - \nu\Delta\varphi + H_p(t, x, m, Du)D\varphi\right\}dxdt = \int_\Omega m_0\,\varphi(0)\,dx \tag{3.2}$$

for every $\varphi \in C_c^\infty([0, T) \times \Omega)$.

Remark 3.2. We notice that, on account of conditions (ii), (3.1) implies that $u \in L^1(Q_T)$ solves, in weak sense,

$$\begin{cases} -\partial_t u - \nu\Delta u = f(t, x), & (t, x) \in (0, T) \times \Omega \\ u(T, x) = g(x), & x \in \Omega \end{cases}$$

for some $f \in L^1(Q_T)$, $g \in L^1(\Omega)$. It follows that $u \in C^0([0, T]; L^1(\Omega))$ and, in addition, u is a renormalized solution of the same equation (see e.g. [24, Proposition 2.1]). Indeed, the condition that u belongs to $L^\infty(0, T; L^1(\Omega))$ is actually redundant in (iii), since it follows from the above result.

As stated in the previous remark, a solution u of equation (3.1) belongs to $C^0([0, T]; L^1(\Omega))$. However we also need to work with merely *subsolutions* of the same equation, for which this kind of continuity may not hold. We recall a lemma, whose proof can be found in [9,3], which establishes continuity for subsolutions in a weaker sense.

Lemma 3.3. *Let $u \in L^\infty(0, T; L^1(\Omega))$ satisfy, for some function $h \in L^1(Q_T)$ and $k \in L^1(\Omega)$,*

$$\int_0^T\int_\Omega u\,\varphi_t\,dxdt - \nu\int_0^T\int_\Omega u\,\Delta\varphi\,dxdt \leq \int_0^T\int_\Omega h\varphi\,dxdt + \int_\Omega k(x)\varphi(T, x)\,dx,$$

for every nonnegative function $\varphi \in C_c^\infty((0, T] \times \Omega)$.

Then for any Lipschitz continuous map $\xi : \Omega \to \mathbb{R}$, the map $t \mapsto \int_\Omega \xi(x)u(t, x)dx$ has a BV representative on $[0, T]$. Moreover, if we note $\int_\Omega \xi(x)u(t^+, x)dx$ its right limit at $t \in [0, T)$, then the map $\xi \mapsto \int_\Omega \xi(x)u(t^+, x)dx$ can be extended to a bounded linear form on $C(\Omega)$.

Remark 3.4. As a consequence of Lemma 3.3, for any subsolution u of (3.1) we can define $u(0^+)$ as a bounded Radon measure on Ω. For simplicity, we note $u(0) = u(0^+)$.

Remark 3.5. A weaker definition than Definition 3.1 is possible, by replacing (ii) by $G(x, m(T)) \in L^1(\Omega)$, $F(t, x, m) \in L^1(Q_T)$, $H(t, x, m, Du) \in L^1(Q_T)$ and $m\, H_p(t, x, m, Du) \in L^1(Q_T)$. In this case, we would need to derive firstly the regularity (ii) in order to prove uniqueness and this could be itself a delicate step. It does not seem restrictive to ask directly (ii) in the formulation since such regularity is actually obtained in the existence result.

Finally, let us state below our main result, where we prove existence and uniqueness of weak solutions under the above assumptions.

Theorem 3.6. *Under Assumptions (2.1)–(2.4) with $\mu > 0$, (2.5) and (2.8)–(2.10), there exists a weak solution of problem (1.3)–(1.5).*

Furthermore, if the assumption (2.13) is satisfied, then there is a unique weak solution of problem (1.3)–(1.5).

3.2. Case of singular congestion

Here we consider the limit case $\mu = 0$ in assumptions (2.2)–(2.3), which corresponds to the singular congestion model $H = \frac{|Du|^\beta}{m^\alpha}$. Due to the singularity, we will only find a relaxed solution.

Before defining weak solutions, we notice that we can always change $H(t, x, m, p)$ into $H(t, x, m, p) + c_1$ and u into $u - c_1(T - t)$; therefore, in view of (2.2), we may assume without restriction that

$$H(t, x, 0, p) \geq 0. \tag{3.3}$$

This allows us to define weak solutions as follows:

Definition 3.7. A pair $(u, m) \in L^1(Q_T) \times L^1(Q_T)_+$ is a weak solution to (1.3)–(1.5) if

(i) $m \in C([0, T]; L^1(\Omega))$, $u \in L^q(0, T; W^{1,q}(\Omega))$ for every $q < \frac{N+2}{N+1}$,

(ii)

$$\begin{aligned} &F(t, x, m)m \in L^1(Q_T), \qquad G(x, m(T))m(T) \in L^1(\Omega), \\ &m\, \mathbb{1}_{m>0} \frac{|Du|^\beta}{m^\alpha} \in L^1(Q_T), \qquad \mathbb{1}_{m>0} \frac{|Du|^\beta}{m^\alpha} \in L^1(Q_T), \qquad m \in L^{1+\frac{\alpha}{\beta-1}}(Q_T), \\ &Du = 0 \quad \text{a.e. in } \{m = 0\}, \end{aligned} \tag{3.4}$$

(iii) $u \in L^\infty(0, T; L^1(\Omega))$ is bounded below and is a subsolution of the Bellman equation:

$$\begin{aligned} &\int_0^T\!\!\int_\Omega u\, \varphi_t\, dxdt - \nu \int_0^T\!\!\int_\Omega u\, \Delta\varphi\, dxdt + \int_0^T\!\!\int_\Omega H(t, x, m, Du)\mathbb{1}_{\{m>0\}}\varphi\, dxdt \\ &\qquad \leq \int_0^T\!\!\int_\Omega F(t, x, m)\varphi\, dxdt + \int_\Omega G(x, m(T))\varphi(T)\, dx \end{aligned} \tag{3.5}$$

for every nonnegative $\varphi \in C_c^\infty((0, T] \times \Omega)$.

(iv) m is a solution of the Kolmogorov equation:

$$\int_0^T\!\!\int_\Omega m \left\{-\varphi_t - \nu\Delta\varphi + H_p(t, x, m, Du)\mathbb{1}_{\{m>0\}}\, D\varphi\right\} dxdt = \int_\Omega m_0\, \varphi(0)\, dx \tag{3.6}$$

for every $\varphi \in C_c^\infty([0, T) \times \Omega)$.

(v) The following identity is satisfied:

$$\int_\Omega m_0\, u(0)\, dx = \int_\Omega G(x, m(T))\, m(T)\, dx + \int_0^T\int_\Omega F(t,x,m) m\, dx dt + \int_0^T\int_\Omega m\left[H_p(t,x,m,Du)\cdot Du - H(t,x,m,Du)\right]\mathbb{1}_{\{m>0\}} dx dt \tag{3.7}$$

where the first term is understood as the trace of $\int_\Omega u(t)\, m_0\, dx$ in $BV(0,T)$, in view of Lemma 3.3.

Notice that, if $\alpha \le 1$, we actually have $m^{1-\alpha}|Du|^\beta \in L^1(Q_T)$ and the restriction to the set $\{m > 0\}$ in (3.6)–(3.7) is not needed. However, for the case $\alpha > 1$ this restriction applies.

We state below the existence and uniqueness result which we obtain for the case of singular congestion.

Theorem 3.8. *Assume that (2.1)–(2.4) hold with $\mu = 0$ and with either $\beta < 2$ and $0 < \alpha \le \frac{4(\beta-1)}{\beta}$ or $\beta = 2$ and $0 < \alpha < 2$, and that (2.8)–(2.10) hold true (assuming furthermore the nonrestrictive condition (3.3)). Then there exists a weak solution of problem (1.3)–(1.5).*

Furthermore, if condition (2.13) holds true for any $m > 0$, $z > -m$, $p, r \in \mathbb{R}^N$ such that $(z, r) \ne (0, 0)$, and if

$$m > 0 \Rightarrow F(t,x,m) > F(t,x,0), \tag{3.8}$$

then there is a unique weak solution of problem (1.3)–(1.5).

4. Non-singular congestion: approximations of (1.3)–(1.5)

For $\mu > 0$, we consider the following system of PDEs:

$$-\partial_t u^\epsilon - \nu\Delta u^\epsilon + H(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon) = F^\epsilon(t,x,m^\epsilon), \quad (t,x)\in(0,T)\times\Omega \tag{4.1}$$

$$\partial_t m^\epsilon - \nu\Delta m^\epsilon - \operatorname{div}(m^\epsilon H_p(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon)) = 0, \quad (t,x)\in(0,T)\times\Omega \tag{4.2}$$

$$m^\epsilon(0,x) = m_0^\epsilon(x), \; u^\epsilon(T,x) = G^\epsilon(x, m^\epsilon(T)), \quad x\in\Omega \tag{4.3}$$

where $T_{1/\epsilon}m = \min(m, 1/\epsilon)$, $F^\epsilon(t,x,m) = \rho^\epsilon \star F(t,\cdot,\rho^\epsilon\star m))(x)$, $G^\epsilon(x,m) = \rho^\epsilon\star G(\cdot,\rho^\epsilon\star m))(x)$, $m_0^\epsilon = \rho^\epsilon\star m_0$. Here $\star$ denotes the convolution with respect to the spatial variable and ρ^ϵ is a standard symmetric mollifier, i.e. $\rho^\epsilon(x) = \frac{1}{\epsilon^N}\rho(\frac{x}{\epsilon})$ for a nonnegative function $\rho \in C_c^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N}\rho(x)dx = 1$.

Lemma 4.1. *There exists a weak solution (u^ϵ, m^ϵ) of (4.1)–(4.3) such that $u^\epsilon \in C^{1+\alpha,1/2+\alpha/2}(\bar{Q}_T)$, $m^\epsilon \in C^{\alpha,\alpha/2}(\bar{Q}_T)$, and $m^\epsilon \ge 0$, $\int_\Omega m^\epsilon(t,x)dx = \int_\Omega m_0^\epsilon(x)dx$, $\forall t$.*

Proof. We set $X = \left\{m \in C(\bar{Q}_T) : m \ge 0 \text{ and } \int_\Omega m(t,x)dx = \int_\Omega m_0(x)dx,\ \forall t\right\}$. For any $m \in X$, the boundary value problem

$$-\partial_t u - \nu\Delta u + H(t,x,T_{1/\epsilon}m, Du) = F^\epsilon(t,x,m), \quad (t,x)\in(0,T)\times\Omega \tag{4.4}$$

$$u(T,x) = G^\epsilon(x, m(T)), \quad x\in\Omega \tag{4.5}$$

has a unique solution $u \in L^\infty(0,T;W^{1,\infty}(\Omega))$ (see e.g. [16], Chapter 5, Theorem 6.3) and $\|u\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}$ is bounded independently of m in X. This follows e.g. from the $C^{1,\alpha}$ estimates in [16], Chapter 5, Theorem 3.1, since F^ϵ is bounded and $|H(t,x,T_{1/\epsilon}m, Du)| \le C_\epsilon(1 + |Du|^2)$ for some constant $C_\epsilon > 0$. In addition, the map $m \mapsto u$ is continuous from X to $L^\infty(0,T;W^{1,\infty}(\Omega))$.

Thus, the boundary value problem

$$\partial_t\tilde{m} - \nu\Delta\tilde{m} - \operatorname{div}(\tilde{m}H_p(t,x,T_{1/\epsilon}m^\epsilon, Du)) = 0, \quad (t,x)\in(0,T)\times\Omega$$

$$\tilde{m}(0,x) = m_0^\epsilon(x), \quad x\in\Omega$$

has a unique solution $\tilde{m} \in X \cap C^{\alpha,\alpha/2}(\bar{Q}_T)$, which is bounded in $C^{\alpha,\alpha/2}(\bar{Q}_T)$ uniformly with respect to $m \in X$, see e.g. [16], Chapter 3, Theorem 10.1. Moreover the map $m \mapsto \tilde{m}$ is continuous from X to $C^{\alpha,\alpha/2}(\bar{Q}_T)$.

Hence, Schauder's theorem implies that the map $m \mapsto \tilde{m}$ has a fixed point m^ϵ. □

Lemma 4.2. *Let (u^ϵ, m^ϵ) be a solution of system (4.1)–(4.3). Then*

$$u^\epsilon(t,x) \geq c_4, \tag{4.6}$$

$$\int_\Omega G^\epsilon(x, m^\epsilon(T))m^\epsilon(T) + \int_0^T \int_\Omega F^\epsilon(t,x,m^\epsilon)m^\epsilon\,dxdt + \|(T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}+1}\|_{L^{\frac{N+2}{N}}(Q_T)} \leq C, \tag{4.7}$$

$$\int_0^T \int_\Omega m^\epsilon \left\{H_p(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon)\cdot Du^\epsilon - H(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon)\right\} dxdt \leq C, \tag{4.8}$$

$$\int_0^T \int_\Omega \frac{|Du^\epsilon|^\beta}{(T_{1/\epsilon}m^\epsilon + \mu)^\alpha}\,dxdt + \int_0^T \int_\Omega m^\epsilon \frac{|Du^\epsilon|^\beta}{(T_{1/\epsilon}m^\epsilon + \mu)^\alpha}\,dxdt \leq C, \tag{4.9}$$

$$\|u^\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \tag{4.10}$$

for a positive constant $C = C(T, H, F, G, \|m_0\|_\infty)$ and where c_4 is the constant appearing in (2.11) and (2.12).

Proof. We multiply the equation of u^ϵ by m^ϵ, the equation of m^ϵ by u^ϵ, integrate by parts some terms and subtract; we get

$$\begin{aligned}&\int_\Omega u^\epsilon(0)m_0^\epsilon\,dx - \int_\Omega G^\epsilon(x,m^\epsilon(T))m^\epsilon(T) - \int_0^T\int_\Omega m^\epsilon H_p(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon)\cdot Du^\epsilon\,dxdt\\ &+ \int_0^T\int_\Omega m^\epsilon H(t,x,T_{1/\epsilon}m^\epsilon,Du^\epsilon)\,dxdt = \int_0^T\int_\Omega F^\epsilon(t,x,m^\epsilon)m^\epsilon\,dxdt,\end{aligned}$$

which implies that

$$\begin{aligned}&\int_\Omega G^\epsilon(x,m^\epsilon(T))m^\epsilon(T) + \int_0^T\int_\Omega F^\epsilon(t,x,m^\epsilon)m^\epsilon\,dxdt + \int_\Omega u^\epsilon(0)^- m_{0\epsilon}\,dx\\ &+ \int_0^T\int_\Omega m^\epsilon\left\{H_p(t,x,T_{1/\epsilon}m^\epsilon,Du^\epsilon)\cdot Du^\epsilon - H(t,x,T_{1/\epsilon}m^\epsilon,Du^\epsilon)\right\}dxdt\\ &\leq C\,\|m_0\|_{L^\infty(\Omega)}\int_\Omega [u^\epsilon(0)]^+\,dx\,.\end{aligned} \tag{4.11}$$

By using (2.1) and $F, G \geq c_4$, we see by comparison that $u^\epsilon(t) \geq c_4 + c_4(T-t)$, and therefore that there exists an absolute constant C such that

$$[u^\epsilon(0)]^- \leq C. \tag{4.12}$$

Integrating (4.1) and using (2.2) and (4.12), we obtain that

$$\begin{aligned}
&\int_\Omega [u^\epsilon(0)]^+ dx + c_0 \int_0^T\int_\Omega \frac{|Du^\epsilon|^\beta}{(T_{1/\epsilon}m^\epsilon+\mu)^\alpha} dxdt \\
&\le c_1 \int_0^T\int_\Omega (1+(T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}})dxdt + \int_0^T\int_\Omega F^\epsilon(t,x,m^\epsilon)dxdt + \int_\Omega G^\epsilon(x,m^\epsilon(T))dx + C \\
&= c_1 \int_0^T\int_\Omega (1+(T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}})dxdt + \int_0^T\int_\Omega F(t,x,\rho^\epsilon\star m^\epsilon)dxdt + \int_\Omega G(x,\rho^\epsilon\star m^\epsilon(T))dx + C
\end{aligned} \tag{4.13}$$

From Assumption (2.8) we see that, for any choice of $L>0$,

$$F(t,x,m) \le f_L(t,x) + \frac{m}{L}(F(t,x,m)+|c_4|), \tag{4.14}$$

where c_4 is the constant in (2.11) and $f_L(t,x) = \max_{0\le m\le L} F(t,x,m) \le \frac{1}{\lambda}\max_{0\le m\le L} f(m)+\kappa$, from (2.8).

Similarly, $G(x,m) =\le g_L(x) + \frac{m}{L}(G(x,m)+|c_4|)$ where $g_L(x) = \max_{0\le m\le L} G(x,m) \le \frac{1}{\lambda}\max_{0\le m\le L} g(m)+\kappa$, from (2.10). Therefore, (4.13) implies

$$\begin{aligned}
&\int_\Omega [u^\epsilon(0)]^+ dx + c_0 \int_0^T\int_\Omega \frac{|Du^\epsilon|^\beta}{(T_{1/\epsilon}m^\epsilon+\mu)^\alpha} dxdt \le c_1 \int_0^T\int_\Omega (1+(T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}})dxdt \\
&\quad + \frac{1}{L}\left(\int_0^T\int_\Omega m^\epsilon F^\epsilon(t,x,m^\epsilon)dxdt + \int_\Omega m^\epsilon(T) G^\epsilon(x,m^\epsilon(T))dx\right) + C_L,
\end{aligned} \tag{4.15}$$

where C_L is a constant depending on L. Therefore, by choosing L sufficiently large (depending on $\|m_0\|_{L^\infty(\Omega)}$) we can combine (4.11) and (4.15) and obtain

$$\begin{aligned}
&\int_\Omega G^\epsilon(x,m^\epsilon(T))m^\epsilon(T) + \int_0^T\int_\Omega F^\epsilon(t,x,m^\epsilon)m^\epsilon\, dxdt + \int_0^T\int_\Omega \frac{|Du^\epsilon|^\beta}{(T_{1/\epsilon}m^\epsilon+\mu)^\alpha} dxdt \\
&+ \int_0^T\int_\Omega m^\epsilon \left\{H_p(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon)\cdot Du^\epsilon - H(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon)\right\} dxdt \\
&\le C + C\int_0^T\int_\Omega (T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}} dxdt\,.
\end{aligned} \tag{4.16}$$

We now use (4.2) multiplied by $(T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}}$. We get

$$\begin{aligned}
&\iint \partial_t m^\epsilon (T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}}\, dxdt + \frac{\nu\alpha}{\beta-1}\iint_{m^\epsilon<1/\epsilon} |Dm^\epsilon|^2 (m^\epsilon)^{\frac{\alpha}{\beta-1}-1}\, dxdt \\
&= -\frac{\alpha}{\beta-1}\iint_{m^\epsilon<1/\epsilon} (m^\epsilon)^{\frac{\alpha}{\beta-1}} H_p(t,x,m^\epsilon, Du^\epsilon)\cdot Dm^\epsilon\, dxdt\,.
\end{aligned}$$

Calling $I_\epsilon(z) = \int_0^z (T_{1/\epsilon}y)^{\frac{\alpha}{\beta-1}}dy$, we see that

$$\int_0^t\int_\Omega \partial_t m^\epsilon (T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}}\, dxds = \int_\Omega I_\epsilon(m^\epsilon(t))dx - \int_\Omega I_\epsilon(m_0^\epsilon)dx.$$

Therefore, by Young's inequality

$$\begin{aligned}&\sup_t \int_\Omega I_\epsilon(m^\epsilon(t))dx - \int_\Omega I_\epsilon(m_0^\epsilon)dx + \iint_{m^\epsilon<1/\epsilon} |Dm^\epsilon|^2 (m^\epsilon)^{\frac{\alpha}{\beta-1}-1}\,dxdt \\ &\le C \iint_{m^\epsilon<1/\epsilon} (m^\epsilon)^{\frac{\alpha}{\beta-1}+1} |H_p(t,x,m^\epsilon,Du^\epsilon)|^2\,dxdt\,.\end{aligned} \tag{4.17}$$

Since $I_\epsilon(z) \ge \frac{1}{\frac{\alpha}{\beta-1}+1}(T_{1/\epsilon}z)^{\frac{\alpha}{\beta-1}+1}$ we use (2.7) to obtain

$$\begin{aligned}&\sup_{t\in(0,T)} \int_\Omega (T_{1/\epsilon}m^\epsilon(t))^{\frac{\alpha}{\beta-1}+1}\,dx + \int_0^T\int_\Omega \left|DT_{1/\epsilon}m^\epsilon\right|^2 (T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}-1}\,dxdt \\ &\le C \iint_{m^\epsilon<1/\epsilon} m^\epsilon \left\{H_p(t,x,m^\epsilon,Du^\epsilon)\cdot Du^\epsilon - H(t,x,m^\epsilon,Du^\epsilon)\right\} dxdt \\ &\quad + C\left[1 + \iint_{m^\epsilon<1/\epsilon} (m^\epsilon)^{\frac{\alpha}{\beta-1}+1}\,dxdt\right] + C\int_\Omega (m_0^\epsilon)^{\frac{\alpha}{\beta-1}+1} dx \\ &\le C\int_0^T\int_\Omega m^\epsilon \left\{H_p(t,x,T_{1/\epsilon}m^\epsilon,Du^\epsilon)\cdot Du^\epsilon - H(t,x,T_{1/\epsilon}m^\epsilon,Du^\epsilon)\right\} dxdt \\ &\quad + C\left[1 + \int_0^T\int_\Omega (T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}+1}\,dxdt\right] + C\int_\Omega (m_0^\epsilon)^{\frac{\alpha}{\beta-1}+1} dx,\end{aligned} \tag{4.18}$$

where the last inequality is a consequence of (2.6). On the other hand, the left-hand side of (4.18) can be estimated by Gagliardo–Nirenberg interpolation inequality, which implies

$$\begin{aligned}&\|(T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}+1}\|_{L^{\frac{N+2}{N}}(Q_T)} \\ &\le C\left[\sup_t \int_\Omega (T_{1/\epsilon}m^\epsilon(t))^{\frac{\alpha}{\beta-1}+1}\,dx + \int_0^T\int_\Omega |DT_{1/\epsilon}m^\epsilon|^2 (T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}-1}\,dxdt\right].\end{aligned}$$

Then, when comparing with the right-hand side of (4.18), and recalling that m^ϵ is already bounded in L^1, we conclude that

$$\begin{aligned}&\|(T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}+1}\|_{L^{\frac{N+2}{N}}(Q_T)} \\ &\le C\int_0^T\int_\Omega m^\epsilon \left\{H_p(t,x,T_{1/\epsilon}m^\epsilon,Du)\cdot Du^\epsilon - H(t,x,T_{1/\epsilon}m^\epsilon,Du^\epsilon)\right\} dxdt + C\,,\end{aligned}$$

where C also depends on m_0. We use now this information in (4.16) and we deduce that

$$\begin{aligned}&\int_\Omega G^\epsilon(x,m^\epsilon(T))m^\epsilon(T) + \int_0^T\int_\Omega F^\epsilon(t,x,m^\epsilon)m^\epsilon\,dxdt + \|(T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}+1}\|_{L^{\frac{N+2}{N}}(Q_T)} \\ &\le C + C\int_0^T\int_\Omega (T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}}\,dxdt\,.\end{aligned}$$

The latter inequality implies that its right hand side is bounded uniformly in ϵ, and we obtain (4.7). From (4.16), we also obtain (4.8) and we estimate the first term in (4.9).

Finally, from (4.17), we see that $\sup_t \int_\Omega I_\epsilon(m^\epsilon(t))dx \le C$ which implies that

$$\sup_t \int_\Omega m^\epsilon(t)(T_{1/\epsilon}m^\epsilon(t))^{\frac{\alpha}{\beta-1}}\,dx \le C. \tag{4.19}$$

Then using (2.2) and (2.4), (4.8) and (4.19) imply that

$$\int_0^T\int_\Omega m^\epsilon \frac{|Du^\epsilon|^\beta}{(\mu + T_{1/\epsilon}m^\epsilon)^\alpha}dxdt \le C, \tag{4.20}$$

which completes (4.9). In particular, we also deduce that $\int_0^T\int_\Omega m^\epsilon \frac{|Du^\epsilon|^\beta}{(\mu+m^\epsilon)^\alpha}dxdt \le C$.

Finally, integrating (4.1) in $(t,T)\times\Omega$, we get that

$$\begin{aligned}&\int_\Omega u^\epsilon(t,x)dx\\ &= \int_\Omega G(x,\rho^\epsilon \star m^\epsilon(T))dx + \int_t^T\int_\Omega F(t,x,\rho^\epsilon\star m^\epsilon(s))dxds - \int_t^T\int_\Omega H(s,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon)dxds.\end{aligned}$$

The last term is bounded above by C from (4.7) and (4.9). Let us deal with $\int_\Omega G(x,\rho^\epsilon\star m^\epsilon(T))dx$: from (2.12), $G(x,\rho^\epsilon\star m^\epsilon(T)) \le g_L(x) + \frac{1}{L}(\rho^\epsilon\star m^\epsilon(T)G(x,\rho^\epsilon\star m^\epsilon(T)) + |c_4|\rho^\epsilon\star m^\epsilon(T))$, where L is an arbitrary positive number and $g_L(x) = \max_{0\le m\le L} G(x,m) \le \frac{1}{\lambda}\max_{0\le m\le L} g(m) + \kappa$, from (2.10). Noting that $\int_\Omega \rho^\epsilon \star m^\epsilon(T)G(x,\rho^\epsilon\star m^\epsilon(T))dx = \int_\Omega m^\epsilon G^\epsilon(x,m^\epsilon)dx$, and that $\int_\Omega m^\epsilon(x) = \int_\Omega m_0(x)$, we deduce from (4.7) that $\int_\Omega G(x,\rho^\epsilon\star m^\epsilon(T))dx \le C$, for some constant C independent of ϵ and μ.

The same argument can be used for proving that $\int_t^T\int_\Omega F(t,x,\rho^\epsilon\star m^\epsilon(s))dxds \le C$. Thus

$$\int_\Omega u^\epsilon(t,x)dx \le C.$$

Combining this with the lower bound on u^ϵ already obtained, we obtain (4.10). □

Remark 4.3. Note that the constant C appearing in Lemma 4.2 does not depend on μ.

Lemma 4.4. *For each $0<\epsilon<1$, there exist two functions w^ϵ and z^ϵ defined on Q_T such that*

$$\left|m^\epsilon H_p(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon)\right| \le c_2 m^\epsilon + w^\epsilon + \sqrt{m^\epsilon}z^\epsilon,$$

where the family (w^ϵ) is bounded in $L^{\beta'}(Q_T)$, the family (z^ϵ) is bounded in $L^2(Q_T)$ and, in addition, relatively compact if $\beta<2$.

Proof. From Remark 2.1, we know that $\alpha\le\beta$. Therefore, if $\alpha\ge1$, then $\alpha'\ge\beta'$. This yields

$$(1-\alpha)\beta' \ge -\alpha, \qquad \text{for any } \alpha \text{ such that } 0<\alpha\le\frac{4(\beta-1)}{\beta}, \tag{4.21}$$

because (4.21) is trivial if $0<\alpha\le1$.

From (2.3), we see that $\left|m^\epsilon H_p(t,x,T_{1/\epsilon}m^\epsilon,Du^\epsilon)\right| \le c_2 m^\epsilon\,(1+\frac{|Du^\epsilon|^{\beta-1}}{(T_{1/\epsilon}m^\epsilon+\mu)^\alpha}) \le c_2(m^\epsilon + A_1 + A_2)$, where $A_1 = \mathbb{1}_{\{T_{1/\epsilon}m^\epsilon+\mu\le1\}}\frac{|Du^\epsilon|^{\beta-1}}{(T_{1/\epsilon}m^\epsilon+\mu)^{\alpha-1}}$ and $A_2 = \mathbb{1}_{\{T_{1/\epsilon}m^\epsilon+\mu\ge1\}}m^\epsilon\frac{|Du^\epsilon|^{\beta-1}}{(T_{1/\epsilon}m^\epsilon+\mu)^\alpha}$.

Let us first deal with A_1: (4.21) implies that $A_1^{\beta'} \le \mathbb{1}_{\{T_{1/\epsilon}m^\epsilon+\mu\le1\}}\frac{|Du^\epsilon|^\beta}{(T_{1/\epsilon}m^\epsilon+\mu)^\alpha} \le \frac{|Du^\epsilon|^\beta}{(T_{1/\epsilon}m^\epsilon+\mu)^\alpha}$. Therefore, from (4.9), $w^\epsilon = c_2A_1$ is bounded in $L^{\beta'}(Q_T)$ uniformly with respect to ϵ (and to μ).

Next, we see that $A_2 \leq \sqrt{m^\epsilon} z^\epsilon$ where $z^\epsilon = \mathbb{1}_{\{T_{1/\epsilon} m^\epsilon + \mu \geq 1\}} \sqrt{m^\epsilon} \frac{|Du^\epsilon|^{\beta-1}}{(T_{1/\epsilon} m^\epsilon + \mu)^\alpha}$. Then, by Hölder inequality, for any $E \subseteq Q_T$ we have

$$\begin{aligned}
\iint_E (z^\epsilon)^2 dxdt &= \iint_E \mathbb{1}_{\{T_{1/\epsilon} m^\epsilon + \mu \geq 1\}} m^\epsilon \frac{|Du^\epsilon|^{2\beta-2}}{(T_{1/\epsilon} m^\epsilon + \mu)^{2\alpha}} dxdt \\
&\leq \left(\iint_E m^\epsilon \frac{|Du^\epsilon|^\beta}{(T_{1/\epsilon} m^\epsilon + \mu)^\alpha} dxdt \right)^{\frac{2}{\beta'}} \left(\iint_E \mathbb{1}_{\{T_{1/\epsilon} m^\epsilon + \mu \geq 1\}} \frac{m^\epsilon}{(T_{1/\epsilon} m^\epsilon + \mu)^{\alpha\left(1+\frac{\beta'}{\beta'-2}\right)}} dxdt \right)^{1-\frac{2}{\beta'}} \\
&\leq \left(\iint_E m^\epsilon \frac{|Du^\epsilon|^\beta}{(T_{1/\epsilon} m^\epsilon + \mu)^\alpha} dxdt \right)^{\frac{2}{\beta'}} \left(\iint_E m^\epsilon dxdt \right)^{1-\frac{2}{\beta'}} .
\end{aligned} \tag{4.22}$$

We immediately deduce from (4.9) that z^ϵ is bounded in $L^2(Q_T)$. In addition, for any $k < \frac{1}{\epsilon}$ we have

$$\iint_E m^\epsilon dxdt \leq k|E| + \frac{1}{k^{\frac{\alpha}{\beta-1}}} \iint_E m^\epsilon \, (T_{1/\epsilon} m^\epsilon)^{\frac{\alpha}{\beta-1}} \leq k|E| + \frac{C}{k^{\frac{\alpha}{\beta-1}}}$$

where we used (4.19). This implies that for ϵ small enough,

$$\iint_E m^\epsilon dxdt \leq C|E|^{\frac{\alpha}{\alpha+\beta-1}}$$

and so we additionally deduce from (4.22) that z^ϵ is equi-integrable in $L^2(Q_T)$ provided $\beta < 2$. □

We now collect some compactness properties for the family (u^ϵ, m^ϵ), which are mostly borrowed from [24].

Proposition 4.5.

(1) *The family (m^ϵ) is relatively compact in $L^1(Q_T)$ and in $C([0,T]; W^{-1,r}(\Omega))$ for some $r > 1$.*
(2) *There exist $m \in C([0,T]; L^1(\Omega))$ and $u \in L^\infty(0,T; L^1(\Omega)) \cap L^q(0,T; W^{1,q}(\Omega))$ for all $q < \frac{N+2}{N+1}$ such that after the extraction of a subsequence (not relabeled), $m^\epsilon \to m$ in $L^1(Q_T)$ and almost everywhere, $u^\epsilon \to u$ and $Du^\epsilon \to Du$ in $L^1(Q_T)$ and almost everywhere. Moreover, u and m satisfy*

$$\int_0^T \int_\Omega \frac{|Du|^\beta}{(m+\mu)^\alpha} dxdt + \int_0^T \int_\Omega m \frac{|Du|^\beta}{(m+\mu)^\alpha} dxdt \leq C, \tag{4.23}$$

$$\int_\Omega G(x, m(T)) m(T) \, dx + \int_0^T \int_\Omega m F(t,x,m) \, dxdt + \|m^{\frac{\alpha}{\beta-1}+1}\|_{L^{\frac{N+2}{N}}(Q_T)} \leq C, \tag{4.24}$$

for some $C = C(T, H, F, G, \|m_0\|_\infty)$ (independent of μ).
(3) *$m^\epsilon(t) \to m(t)$ weakly in $L^1(\Omega)$, for every $t \in [0,T]$, and (3.2) holds for any $\varphi \in C_c^\infty([0,T) \times \Omega)$. If in addition $\beta < 2$, then $m^\epsilon \to m$ in $C([0,T]; L^1(\Omega))$.*

Proof. Proceeding as in the end of the proof of Lemma 4.2, we see that $(F^\epsilon(t,x,m^\epsilon))_\epsilon$ is bounded in $L^1(Q_T)$, $(u^\epsilon(t=T))_\epsilon$ is bounded in $L^1(\Omega)$ and that $H(t,x,T_{1/\epsilon} m, Du^\epsilon)$ is bounded in $L^1(Q_T)$ uniformly in ϵ. Therefore, $(-\partial_t u^\epsilon - \nu \Delta u^\epsilon)_\epsilon$ is bounded in $L^1(Q_T)$ and $(u^\epsilon(t=T))_\epsilon$ is bounded in $L^1(\Omega)$. The compactness of u^ϵ and Du^ϵ in $L^1(Q_T)$ (and almost everywhere) follow from classical results on the heat equation with L^1 data, see e.g. [7,8]: there exist $u \in L^\infty(0,T; L^1(\Omega)) \cap L^q((0,T); W^{1,q}(\Omega))$ for every $q < \frac{N+2}{N+1}$, such that after the extraction of a subsequence $u^\epsilon \to u$ and $Du^\epsilon \to Du$ in $L^1(Q_T)$ and almost everywhere.

As far as m^ϵ is concerned, thanks to Lemma 4.4 we can apply the estimates and the compactness results in [24, Theorem 6.1]. In particular, this implies the point (1) in the statement and so, up to extraction of a subsequence, the convergence of m^ϵ in $L^1(Q_T)$ and almost everywhere.

The bounds (4.23) and (4.24) follow from (4.7), (4.9) and Fatou's lemma, keeping in mind that the constant C in the estimates of Lemma 4.2 does not depend on μ.

To prove that m is a solution of (3.2), we observe that $m^\epsilon H_p(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon)$ strongly converges in $L^1(Q_T)$. Indeed, the decomposition provided by Lemma 4.4 implies that this term is equi-integrable, so using the almost everywhere convergence and Vitali's theorem, we obtain that $m^\epsilon H_p(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon) \to m H_p(t,x,m,Du)$ in $L^1(Q_T)$. As a consequence, (3.2) holds for any $\varphi \in C_c^\infty([0,T)\times\Omega)$.

Finally, we have that $m^\epsilon(t)$ is bounded in $L\log L(\Omega)$ uniformly in time (see e.g. [24]), which means that it is equi-integrable in $L^1(\Omega)$. By Dunford–Pettis theorem, it is weakly relatively compact in $L^1(\Omega)$, and since we already know that it converges to $m(t)$ in $W^{-1,r}(\Omega)$, we conclude that it also converges weakly in $L^1(\Omega)$ to the same limit. In fact, if $\beta < 2$, from Lemma 4.4 we have the strong convergence of the terms w^ϵ, z^ϵ in $L^2(Q_T)$. This means that we can apply directly Theorem 6.1 in [24] to deduce the strong convergence of m^ϵ in $C([0,T];L^1(\Omega))$. □

Henceforth, we let (u,m) as well as the convergent subsequence (u^ϵ, m^ϵ) be given by Proposition 4.5. Let us notice that the weak convergence of $m^\epsilon(T)$ in $L^1(\Omega)$ easily yields that $\rho^\epsilon \star m^\epsilon(T)$ also converges to $m(T)$ weakly in $L^1(\Omega)$. In particular, it is well-known that the sequence $\rho^\epsilon \star m^\epsilon(T)$ generates a family of parametrized Young measures $\{\nu_x\} \in \mathcal{P}(\mathbb{R})$, see e.g. [4,22]. This means that ν_x is a probability measure for a.e. $x \in \Omega$, the mapping $x \mapsto \nu_x$ is weakly-$*$ measurable and, for a subsequence (not relabeled),

$$f(x, \rho^\epsilon \star m^\epsilon(T)) \rightharpoonup \int_{\mathbb{R}} f(x,\lambda)d\nu_x(\lambda) \qquad \text{weakly in } L^1(\Omega) \tag{4.25}$$

for every Carathéodory function $f(x,s)$ such that $f(x,\rho^\epsilon \star m^\epsilon(T))$ is equi-integrable in $L^1(\Omega)$.

Thanks to the Young measures, we can initially identify the limit of $u^\epsilon(T)$ and give a first description, by now in a relaxed sense, of the equation satisfied by u.

Lemma 4.6. *Let (u,m) be given by Proposition 4.5. Then*

$$\begin{aligned}\int_0^T\int_\Omega u\,\varphi_t\,dxdt - \nu\int_0^T\int_\Omega u\,\Delta\varphi\,dxdt + \int_0^T\int_\Omega H(t,x,m,Du)\varphi\,dxdt \\ \le \int_0^T\int_\Omega F(t,x,m)\varphi\,dxdt + \int_\Omega\int_{\mathbb{R}} G(x,\lambda)d\nu_x(\lambda)\,\varphi(T)\,dx\end{aligned} \tag{4.26}$$

for every nonnegative function $\varphi \in C_c^\infty((0,T]\times\Omega)$.

In addition, we have that $\int_{\mathbb{R}} G(x,\lambda)\lambda\,d\nu_x(\lambda) \in L^1(\Omega)$, $G(x,m(T))m(T) \in L^1(\Omega)$ and

$$\int_\Omega G(x,m(T))m(T)\,dx \le C \tag{4.27}$$

for some $C = C(T,H,F,G,\|m_0\|_\infty)$ (independent of μ).

Proof. From (4.7) and the convergence of m^ϵ, we see that $(T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{\beta-1}}$ converges in $L^1(Q_T)$. Using this observation and (2.2), we can apply Fatou's lemma and obtain that, for any $\varphi \in C_c^\infty((0,T]\times\Omega)$, $\varphi \ge 0$,

$$\int_0^T\int_\Omega H(t,x,m,Du)\varphi(t,x)dxdt \le \liminf_{\epsilon\to 0}\int_0^T\int_\Omega H(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon)\varphi(t,x)dxdt.$$

Moreover, from (4.7) and (2.8), and the definition of F^ϵ, we see that $F^\epsilon(t,x,m^\epsilon)$ is equi-integrable in Q_T. Therefore, from Vitali's theorem,

$$\lim_{\epsilon\to 0}\int_0^T\int_\Omega F^\epsilon(t,x,m^\epsilon)\varphi(t,x)dxdt = \int_0^T\int_\Omega F(t,x,m)\varphi(t,x)dxdt. \tag{4.28}$$

Let us deal with the boundary condition at $t = T$: again from (4.7), the definition of G^ϵ and (2.10), we deduce that $G(x, \rho^\epsilon \star m^\epsilon(T))$ is equi-integrable, so by the properties of Young measures we obtain that

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega u^\epsilon(T,x)\varphi(T,x,)dx &= \lim_{\epsilon\to 0}\int_\Omega G(x,\rho^\epsilon\star m^\epsilon(T))\,\rho^\epsilon\star\varphi(T,x)dx\\ &= \int_\Omega\int_{\mathbb{R}} G(x,\lambda)d\nu_x(\lambda)\varphi(T,x)dx.\end{aligned} \tag{4.29}$$

Combining the latter three points yields (4.26).

Finally, we notice that the bound $\int_\Omega G^\epsilon(x,m^\epsilon(T))m^\epsilon(T) \le C$ given by (4.7) implies that both $G(x,m(T))m(T) \in L^1(\Omega)$ and $\int_\Omega\int_{\mathbb{R}} G(x,\lambda)\lambda\, d\nu_x(\lambda) \in L^1(\Omega)$; indeed, on the one hand, thanks to (2.10) we have

$$\int_\Omega g(\rho^\epsilon\star m^\epsilon(T))\rho^\epsilon\star m^\epsilon(T)\,dx \le C$$

so by convexity of $g(s)s$ and weak convergence of $\rho^\epsilon \star m^\epsilon(T)$ we deduce that $g(m(T))m(T) \in L^1(\Omega)$, hence $G(x,m(T))m(T) \in L^1(\Omega)$ again by (2.10), and estimate (4.27) holds. On the other hand, by properties of Young measures we have

$$T_k(G(x,\rho^\epsilon\star m^\epsilon(T)))\rho^\epsilon\star m^\epsilon(T) \rightharpoonup \int_{\mathbb{R}} T_k(G(x,\lambda))\lambda\, d\nu_x(\lambda)$$

where $T_k(s) = \min(s,k)$. Hence

$$\int_\Omega\int_{\mathbb{R}} T_k(G(x,\lambda))\lambda\, d\nu_x(\lambda)\,dx = \lim_{\epsilon\to 0}\int_\Omega T_k(G(x,\rho^\epsilon\star m^\epsilon(T)))\rho^\epsilon\star m^\epsilon(T)\,dx \le C.$$

By monotone convergence, letting $k\to\infty$ we deduce that $\int_{\mathbb{R}} G(x,\lambda)\lambda\, d\nu_x(\lambda) \in L^1(\Omega)$. □

Lemma 4.7. *Let (m^ϵ, u^ϵ) be given by Proposition 4.5. Possibly after another extraction of a subsequence, $u^\epsilon|_{t=0}$ converges weakly * to a bounded measure χ on Ω. Moreover $u(0)$ is a well defined Radon measure on Ω and $u(0) \ge \chi$.*

Proof. From (4.10), we know that $u^\epsilon(0)$ is bounded in $L^1(\Omega)$. Therefore, there exists some bounded measure χ such that, up to a subsequence, $u^\epsilon(0)$ tends to χ weakly $*$. Moreover, from Lemma 3.3, Remark 3.4 and Lemma 4.6, $u(0)$ is a well defined Radon measure.

Let us prove that $\chi \le u(0)$. Consider $\varphi(t,x) = \psi(x)(1 - t/\eta)_+$, where ψ is any nonnegative smooth function defined on Ω. Taking φ as a test-function in (4.1), we obtain that

$$\begin{aligned}&\int_0^T\int_\Omega u^\epsilon\varphi_t\,dxdt - \nu\int_0^T\int_\Omega u^\epsilon\,\Delta\varphi\,dxdt + \int_0^T\int_\Omega H^\epsilon(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon)\varphi\,dxdt\\ &\quad= \int_0^T\int_\Omega F^\epsilon(t,x,m^\epsilon)\varphi\,dxdt - \int_\Omega \varphi(0,x)u^\epsilon(0,x)dx,\end{aligned}$$

which implies by using Fatou's lemma as in the proof of Proposition 4.5 that

$$
\begin{aligned}
\int_0^T\int_\Omega u\,\varphi_t\,dxdt - \nu\int_0^T\int_\Omega u\,\Delta\varphi\,dxdt + \int_0^T\int_\Omega H(t,x,m,Du)\varphi\,dxdt \\
\le \int_0^T\int_\Omega F(t,x,m)\varphi\,dxdt - \langle\chi,\psi\rangle .
\end{aligned}
\tag{4.30}
$$

On the other hand, as $\eta\to 0$, $\int_0^T\int_\Omega u\,\varphi_t\,dxdt = -\frac{1}{\eta}\int_0^\eta\int_\Omega u\,\psi(x)\,dxdt$ tends to $-\langle u(0),\psi\rangle$. Passing to the limit in (4.30) as $\eta\to 0$, we obtain that $\langle u(0),\psi\rangle \ge \langle\chi,\psi\rangle$, which is the desired result. □

5. Crossed energy inequality

We prove here the main step towards the uniqueness result.

Lemma 5.1. *Assume that H satisfies assumptions (2.2)–(2.3) with $\mu>0$, and that (2.8)–(2.10) hold true.*

Let $u\in L^\infty(0,T;L^1(\Omega))$ satisfy (4.26) for some family of probability measures $\{\nu_x\}$ (weakly−∗ measurable w.r.t. x) such that

$$
\int_{\mathbb{R}} G(x,\lambda)\lambda\,d\nu_x(\lambda)\in L^1(\Omega) \tag{5.1}
$$

and let $m\in C^0([0,T];L^1(\Omega))$ be a solution of (1.4), and assume that (u,m) satisfy (ii) in Definition 3.1. Then we have

$$
\begin{aligned}
\langle\tilde m_0\,,\,u(0)\rangle \le \int_\Omega\int_{\mathbb{R}} G(x,\lambda)\,d\nu_x(\lambda)\,\tilde m(T)\,dx + \int_0^T\int_\Omega F(t,x,m)\tilde m\,dxdt \\
+\int_0^T\int_\Omega \left[\tilde m\,H_p(t,x,\tilde m,D\tilde u)\cdot Du - \tilde m\,H(t,x,m,Du)\right]dxdt
\end{aligned}
\tag{5.2}
$$

for any couple $(\tilde u,\tilde m)$ satisfying the same conditions as (u,m).

Remark 5.2. The above Lemma applies in particular if $\nu_x=\delta_{m(T,x)}$, in which case u is simply a subsolution of (1.3) and condition (5.1) is equivalent to $G(x,m(T))m(T)\in L^1(\Omega)$.

Proof. Let $\rho_\delta(\cdot)$ be a sequence of standard symmetric mollifiers in $\mathbb{R}^N$ and set

$$
\tilde m_\delta(t,x)=\tilde m(t)\star\rho_\delta := \int_{\mathbb{R}^N}\tilde m(t,y)\rho_\delta(x-y)\,dy\,.
$$

Notice in particular that $\tilde m_\delta, D\tilde m_\delta\in L^\infty(Q_T)$ since $\tilde m\in L^\infty(0,T;L^1(\Omega))$. We also take a sequence of 1-d mollifiers $\xi_\varepsilon(t)$ such that $\mathrm{supp}(\xi_\varepsilon)\subset(-\varepsilon,0)$, and we set

$$
\tilde m_{\delta,\varepsilon} := \int_0^T \xi_\varepsilon(s-t)\,\tilde m_\delta(s)ds\,.
$$

Notice that this function vanishes near $t=0$, so we can take it as test function in the inequality satisfied by u. We get

$$\int_0^T\int_\Omega u\,[\partial_t\tilde m_{\delta,\varepsilon}-\nu\Delta\tilde m_{\delta,\varepsilon}]\,dxdt+\int_0^T\int_\Omega H(t,x,m,Du)\tilde m_{\delta,\varepsilon}\,dxdt \\ \le \int_0^T\int_\Omega F(t,x,m)\tilde m_{\delta,\varepsilon}\,dxdt+\int_\Omega\int_{\mathbb R} G(x,\lambda)\,d\nu_x(\lambda)\tilde m_{\delta,\varepsilon}(T)\,dx \tag{5.3}$$

The first integral is equal to

$$\int_0^T\int_\Omega [-\partial_s u_{\delta,\varepsilon}-\nu\Delta u_{\delta,\varepsilon}]\tilde m(s,y)\,dsdy$$

where $u_{\delta,\varepsilon}(s,y)=\int_0^T\int_\Omega u(t,x)\xi_\varepsilon(s-t)\rho_\delta(x-y)\,dtdx$. Notice that this function vanishes near $s=T$. Thus using the equation of $\tilde m$ we have

$$\int_0^T\int_\Omega [-\partial_s u_{\delta,\varepsilon}-\nu\Delta u_{\delta,\varepsilon}]\tilde m(s,y)\,dsdy \\ =-\int_0^T\int_\Omega \tilde m(s,y)\tilde b(s,y)\cdot D_y u_{\delta,\varepsilon}\,dsdy+\int_\Omega \tilde m_0(y)u_{\delta,\varepsilon}(0)dy\,,$$

where $\tilde b=H_p(t,x,\tilde m,D\tilde u)$. We shift the convolution kernels from u to $\tilde m$ in the right-hand side and we use this equality in (5.3). We get

$$-\int_0^T\int_\Omega Du\cdot\tilde w_{\delta,\varepsilon}\,dtdx+\int_0^T\int_\Omega H(t,x,m,Du)\tilde m_{\delta,\varepsilon}\,dxdt+\int_\Omega(\tilde m_0\star\rho_\delta)\int_0^T u(t)\xi_\varepsilon(-t)\,dt\,dx \\ \le \int_0^T\int_\Omega F(t,x,m)\tilde m_{\delta,\varepsilon}\,dxdt+\int_\Omega\int_{\mathbb R} G(x,\lambda)\,d\nu_x(\lambda)\tilde m_{\delta,\varepsilon}(T)\,dx \tag{5.4}$$

where we denote $\tilde w_\delta=[(\tilde b\tilde m)\star\rho_\delta]$ and $\tilde w_{\delta,\varepsilon}=\int_0^T\tilde w_\delta(s)\,\xi_\varepsilon(s-t)\,ds$.

Since now, we distinguish between the cases $\alpha\le 1$ and $\alpha>1$. Recall that $\alpha\le\frac{4(\beta-1)}{\beta}$ is always assumed, hence the latter case only happens for $1<\frac{4(\beta-1)}{\beta}$, i.e. $\beta>\frac43$.

(A) Case when $\alpha\le 1$. Let us deal with the first two integrals in (5.4). By assumption (2.3) we have

$$|(\tilde m\,\tilde b)\star\rho_\delta|\le c_2\tilde m_\delta+c_2\left(\left(\tilde m\,\frac{|D\tilde u|^\beta}{(\tilde m+\mu)^\alpha}\right)\star\rho_\delta\right)^{\frac1{\beta'}}\left(\frac{\tilde m}{(\tilde m+\mu)^\alpha}\star\rho_\delta\right)^{\frac1\beta} \\ \le c_2\tilde m_\delta+c_2\left(\left(\tilde m\,\frac{|D\tilde u|^\beta}{(\tilde m+\mu)^\alpha}\right)\star\rho_\delta\right)^{\frac1{\beta'}}(\tilde m\star\rho_\delta)^{\frac{1-\alpha}{\beta}}$$

where we used $\alpha\le 1$ in the latter inequality. Therefore, using also (2.2) we estimate

$$H(t,x,m,Du)\tilde m_\delta(s)-Du\cdot\tilde w_\delta(s)\ge c_0\tilde m_\delta(s)\frac{|Du|^\beta}{(m+\mu)^\alpha}-c_1\tilde m_\delta(s)(1+m^{\frac{\alpha}{\beta-1}}) \\ -c_2\tilde m_\delta(s)|Du|-c_2|Du|\left(\left(\tilde m(s)\,\frac{|D\tilde u(s)|^\beta}{(\tilde m(s)+\mu)^\alpha}\right)\star\rho_\delta\right)^{\frac1{\beta'}}\tilde m_\delta(s)^{\frac{1-\alpha}{\beta}}$$

which yields, by Young's inequality,

$$\begin{aligned} H(t,x,m,Du)\tilde m_\delta(s) - Du\cdot \tilde w_\delta(s) &\geq \frac{c_0}{2}\tilde m_\delta(s)\frac{|Du|^\beta}{(m+\mu)^\alpha} - c\,\tilde m_\delta(s)(1+m^{\frac{\alpha}{\beta-1}}) \\ &- \sigma\,|Du|^\beta \tilde m_\delta(s)^{1-\alpha} - C_\sigma\left(\tilde m(s)\,\frac{|D\tilde u(s)|^\beta}{(\tilde m(s)+\mu)^\alpha}\right)\star\rho_\delta\,. \end{aligned} \tag{5.5}$$

Since $\alpha \leq 1$, choosing $\sigma < \frac{c_0}{2}$ we have

$$\sigma\,\tilde m_\delta^{1-\alpha} < \frac{c_0}{2}\frac{\tilde m_\delta}{(m+\mu)^\alpha} + \frac{c_0}{2}(m+\mu)^{1-\alpha}\,,$$

while

$$\tilde m_\delta(1+m^{\frac{\alpha}{\beta-1}}) \leq c\,(1+\tilde m_\delta^{1+\frac{\alpha}{\beta-1}} + m^{1+\frac{\alpha}{\beta-1}})\,.$$

So we conclude from (5.5) that

$$\begin{aligned} H(t,x,m,Du)\tilde m_\delta(s) - Du\cdot\tilde w_\delta(s) &\geq -c\,(\tilde m_\delta(s)^{1+\frac{\alpha}{\beta-1}} + m^{1+\frac{\alpha}{\beta-1}}) \\ &- c\,(m+\mu)^{1-\alpha}\,|Du|^\beta - c\left(\tilde m(s)\,\frac{|D\tilde u(s)|^\beta}{(\tilde m(s)+\mu)^\alpha}\right)\star\rho_\delta\,. \end{aligned}$$

Notice that condition (ii) in Definition 3.1 implies that $(m+\mu)^{1-\alpha}|Du|^\beta$ belongs to $L^1(Q_T)$. Due to the summability of all the above terms, we are allowed to use Fatou's lemma and deduce

$$\begin{aligned} &\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega [H(t,x,m,Du)\tilde m_{\delta,\varepsilon} - Du\cdot\tilde w_{\delta,\varepsilon}]\,dxdt \\ &= \liminf_{\varepsilon\to 0}\int_0^T\int_\Omega\int_0^T [H(t,x,m,Du)\tilde m_\delta(s) - Du\cdot\tilde w_\delta(s)]\xi_\varepsilon(s-t)\,dsdxdt \\ &\qquad \geq \int_0^T\int_\Omega [H(t,x,m,Du)\tilde m_\delta - Du\cdot\tilde w_\delta]\,dxdt\,, \end{aligned}$$

and in the same way

$$\begin{aligned} &\liminf_{\delta\to 0}\int_0^T\int_\Omega [H(t,x,m,Du)\tilde m_\delta - Du\cdot\tilde w_\delta]\,dxdt \\ &\qquad \geq \int_0^T\int_\Omega [H(t,x,m,Du)\tilde m - Du\cdot H_p(t,x,\tilde m,D\tilde u)\tilde m]\,dxdt\,. \end{aligned} \tag{5.6}$$

Now we consider the remaining terms in (5.4), in particular the terms at $t=0$ and $t=T$. For $t=T$, we have

$$\begin{aligned} \int_\Omega\int_{\mathbb{R}} G(x,\lambda)\,d\nu_x(\lambda)\tilde m_{\delta,\varepsilon}(T)\,dx &= \int_\Omega\int_{\mathbb{R}} G(x,\lambda)\,d\nu_x(\lambda)\tilde m_\delta(T)\,dx \\ &+ \int_\Omega\int_0^T\int_{\mathbb{R}} G(x,\lambda)\,d\nu_x(\lambda)[\tilde m_\delta(s) - \tilde m_\delta(T)]\xi_\varepsilon(s-T)\,dsdx\,. \end{aligned}$$

Since $\tilde m \in C^0([0,T];L^1(\Omega))$ implies that $\int_0^T [\tilde m_\delta(s) - \tilde m_\delta(T)]\xi_\varepsilon(s-T)\,ds$ converges to zero uniformly as $\varepsilon \to 0$, the last integral will vanish as $\varepsilon \to 0$.

For $t=0$, we recall that $\int_\Omega u(t)\,(\tilde m_0 \star \rho_\delta)dx$ has a trace at $t=0$ from Lemma 3.3, and this trace is also continuous as $\delta \to 0$ since $\tilde m_0$ is continuous. Therefore, as $\varepsilon \to 0$ we obtain from (5.4)

$$
\begin{aligned}
&\int_0^T\int_\Omega [H(t,x,m,Du)\tilde m_\delta - Du\cdot \tilde w_\delta]\,dxdt + \int_\Omega (\tilde m_0\star\rho_\delta)u(0)\,dx \\
&\quad\le \int_0^T\int_\Omega F(t,x,m)\tilde m_\delta\,dxdt + \int_\Omega\int_{\mathbb{R}} G(x,\lambda)\,d\nu_x(\lambda)\tilde m_\delta(T)\,dx
\end{aligned}
\tag{5.7}
$$

We apply assumptions (2.8) and (2.10) to deal with the last two terms. Indeed, if c_4 is the constant in (2.11), we have $c_4\tilde m_\delta \le F(t,x,m)\tilde m_\delta \le (F(m)+|c_4|)\tilde m_\delta$ implies that

$$
\begin{aligned}
c_4\tilde m_\delta \le F(t,x,m)\tilde m_\delta &\le (F(t,x,m)+|c_4|)m\mathbb{1}_{m\ge\tilde m_\delta} + \left(\frac{1}{\lambda}f(m)+\kappa+|c_4|\right)\tilde m_\delta\mathbb{1}_{m<\tilde m_\delta}\\
&\le (F(t,x,m)+|c_4|)m + \left(\frac{1}{\lambda}f(\tilde m_\delta)+\kappa+|c_4|\right)\tilde m_\delta\\
&\le F(t,x,m)m + \frac{1}{\lambda}f(\tilde m_\delta)\tilde m_\delta + C(m+\tilde m_\delta).
\end{aligned}
$$

Hence

$$
\begin{aligned}
c_4\tilde m_\delta \le F(t,x,m)\tilde m_\delta &\le F(t,x,m)m + \frac{1}{\lambda}f(\tilde m_\delta)\tilde m_\delta + C\,(m+\tilde m_\delta)\\
&\le F(t,x,m)m + \frac{1}{\lambda}(f(m)m)\star\rho_\delta + C\,(m+\tilde m_\delta),
\end{aligned}
$$

and since $F(t,x,m)m \in L^1(Q_T)$ (and the same holds for $f(m)m$ by (2.8)), we conclude that the sequence $F(t,x,m)\tilde m_\delta$ is bounded above and below by convergent sequences in $L^1(Q_T)$, which allows us to pass to the limit as $\delta\to 0$. We proceed similarly for the term with G. Indeed, by (2.9),

$$
\int_{\mathbb{R}} G(x,\lambda)\,d\nu_x(\lambda)\tilde m_\delta(T) \le \int_{\mathbb{R}} G(x,\lambda)\lambda\,d\nu_x(\lambda) + G(x,\tilde m_\delta(T))\tilde m_\delta(T) \quad \text{for a.e. } x\in\Omega,
$$

and using (2.10) we deduce

$$
\begin{aligned}
\int_{\mathbb{R}} G(x,\lambda)\,d\nu_x(\lambda)\tilde m_\delta(T) &\le \int_{\mathbb{R}} G(x,\lambda)\lambda\,d\nu_x(\lambda) + \frac{1}{\lambda}g(\tilde m_\delta(T))\tilde m_\delta(T) + C\,\tilde m_\delta(T)\\
&\le \int_{\mathbb{R}} G(x,\lambda)\lambda\,d\nu_x(\lambda) + \frac{1}{\lambda}(g(\tilde m(T))\tilde m(T))\star\rho_\delta + C\,\tilde m_\delta(T)\,.
\end{aligned}
$$

Thanks to (5.1) and since $G(x,\tilde m(T))\tilde m(T)\in L^1(\Omega)$ (and the same holds for $g(\tilde m(T))\tilde m(T)$ by (2.10)), we can handle this term too. Finally, passing to the limit in (5.7) and using also (5.6), we obtain (5.2).

(B) Case when $\alpha > 1$**.** First recall from Remark 2.1 that $\alpha\le\beta$. Moreover, since $1\ge\beta-1$, we also have $\alpha>\beta-1$, i.e. $\beta-\alpha<1$.

We estimate now differently the first two terms in (5.4). First of all, by Young's inequality, we have (omitting to write that $\tilde m_\delta$, $D\tilde u$ are evaluated at s and m, Du at t)

$$
Du\cdot\tilde w_\delta \le \sigma\,\frac{\max(m+\mu,\tilde m_\delta)}{\tilde m_\delta^{\frac{\beta-\alpha}{\beta-1}}}|\tilde w_\delta|^{\beta'} + C_\sigma\,\frac{\tilde m_\delta^{\beta-\alpha}}{\max(m+\mu,\tilde m_\delta)^{\beta-1}}|Du|^\beta
$$

and using $\beta\ge\alpha$ and $\alpha>1$ this yields

$$
Du\cdot\tilde w_\delta \le \sigma\,\frac{m+\mu+\tilde m_\delta}{\tilde m_\delta^{\frac{\beta-\alpha}{\beta-1}}}|\tilde w_\delta|^{\beta'} + C_\sigma\,(m+\mu)^{1-\alpha}|Du|^\beta\,. \tag{5.8}
$$

By (2.3),

$$
\begin{aligned}
|(\tilde m\,\tilde b)\star\rho_\delta| &\le c_2\tilde m_\delta + c_2\left(\tilde m\,\frac{|D\tilde u|^{\beta-1}}{(\tilde m+\mu)^\alpha}\right)\star\rho_\delta\\
&\le c_2\tilde m_\delta + c_2\left(\left(\frac{|D\tilde u|^{\beta}}{(\tilde m+\mu)^\alpha}\right)\star\rho_\delta\right)^{\frac{1}{\beta'}}\left(\frac{\tilde m^\beta}{(\tilde m+\mu)^\alpha}\star\rho_\delta\right)^{\frac{1}{\beta}}\\
&\le c_2\tilde m_\delta + c_2\left(\left(\frac{|D\tilde u|^{\beta}}{(\tilde m+\mu)^\alpha}\right)\star\rho_\delta\right)^{\frac{1}{\beta'}}\tilde m_\delta^{\frac{\beta-\alpha}{\beta}}
\end{aligned}
$$

where we used $\beta-\alpha\le 1$ in the last inequality. We deduce that

$$
\frac{|\tilde w_\delta|^{\beta'}}{\tilde m_\delta^{\frac{\beta-\alpha}{\beta-1}}} \le C\tilde m_\delta^{\frac{\alpha}{\beta-1}} + C\left(\frac{|D\tilde u|^{\beta}}{(\tilde m+\mu)^\alpha}\right)\star\rho_\delta\,. \tag{5.9}
$$

Combining this information with (5.8), we deduce that

$$
\begin{aligned}
Du\cdot\tilde w_\delta &\le \sigma\, C[m+\mu+\tilde m_\delta]\left(\frac{|D\tilde u|^{\beta}}{(\tilde m+\mu)^\alpha}\right)\star\rho_\delta\\
&\quad + C\,[1+m^{1+\frac{\alpha}{\beta-1}}+\tilde m_\delta^{1+\frac{\alpha}{\beta-1}}] + C_\sigma\,(m+\mu)^{1-\alpha}|Du|^\beta\,.
\end{aligned}\tag{5.10}
$$

We first use this inequality to get

$$
\begin{aligned}
Du\cdot\tilde w_{\delta,\varepsilon} &\le \sigma\, C\int_0^T [m+\mu+\tilde m_\delta]\left(\frac{|D\tilde u|^{\beta}}{(\tilde m+\mu)^\alpha}\right)\star\rho_\delta\,\xi_\varepsilon(s-t)ds\\
&\quad + C[1+m^{1+\frac{\alpha}{\beta-1}}+\int_0^T \tilde m_\delta^{1+\frac{\alpha}{\beta-1}}\xi_\varepsilon(s-t)ds] + C_\sigma\,(m+\mu)^{1-\alpha}|Du|^\beta\,.
\end{aligned}
$$

Since $\int_\Omega m(t,x)dx = \int_\Omega m_0(x)dx$ and $\frac{|D\tilde u|^\beta}{(\tilde m+\mu)^\alpha}\in L^1(Q_T)$,

$$
\int_0^T m(t)\left(\left(\frac{|D\tilde u(s)|^{\beta}}{(\tilde m(s)+\mu)^\alpha}\right)\star\rho_\delta\right)\xi_\varepsilon(s-t)ds \quad\overset{\varepsilon\to 0}{\to}\quad m(t)\left(\frac{|D\tilde u(t)|^{\beta}}{(\tilde m(t)+\mu)^\alpha}\right)\star\rho_\delta \qquad \text{in } L^1(Q_T).
$$

Using also that $\tilde m_\delta(s)$ is continuous, we deduce that $Du\cdot\tilde w_{\delta,\varepsilon}$ is dominated by a L^1-convergent sequence, so we are allowed to take $\varepsilon\to 0$ in (5.4), obtaining

$$
\begin{aligned}
&\int_0^T\int_\Omega [H(t,x,m,Du)\tilde m_\delta - Du\cdot\tilde w_\delta]\,dxdt + \int_\Omega(\tilde m_0\star\rho_\delta)u(0)\,dx\\
&\quad\le \int_0^T\int_\Omega F(t,x,m)\tilde m_\delta\,dxdt + \int_\Omega\int_{\mathbb R} G(x,\lambda)d\nu_x(\lambda)\tilde m_\delta(T)\,dx\,.
\end{aligned}\tag{5.11}
$$

Thanks to (5.10), and since the right-hand side is bounded as in the previous case, the above inequality also implies

$$
\begin{aligned}
\int_0^T\int_\Omega H(t,x,m,Du)\tilde m_\delta\,dxdt &\le C + \sigma\, C\int_0^T\int_\Omega [m+\mu+\tilde m_\delta]\left(\frac{|D\tilde u|^{\beta}}{(\tilde m+\mu)^\alpha}\right)\star\rho_\delta\,dxdt\\
&\quad + C\int_0^T\int_\Omega [1+m^{1+\frac{\alpha}{\beta-1}}+\tilde m_\delta^{1+\frac{\alpha}{\beta-1}}]\,dxdt + C_\sigma\int_0^T\int_\Omega (m+\mu)^{1-\alpha}|Du|^\beta\,dxdt\,.
\end{aligned}
$$

On account of the bounds on $m, \tilde m$ in $L^{1+\frac{\alpha}{\beta-1}}(Q_T)$ and using the properties (ii) in Definition 3.1, the last line of the above inequality is uniformly bounded. Therefore, using (2.2) we deduce that

$$\int_0^T\int_\Omega \tilde m_\delta \frac{|Du|^\beta}{(m+\mu)^\alpha}\,dxdt \le C + \sigma\, C \int_0^T\int_\Omega m_\delta \frac{|D\tilde u|^\beta}{(\tilde m+\mu)^\alpha}\,dxdt + \sigma\, C \int_0^T\int_\Omega \tilde m_\delta \left(\frac{|D\tilde u|^\beta}{(\tilde m+\mu)^\alpha}\right)\star\rho_\delta\,dxdt\,,$$

where $m_\delta = m\star\rho_\delta$. We obtain a similar inequality reversing the roles of (u, m) and $(\tilde u, \tilde m)$, and by addition we get

$$\begin{aligned}
&\int_0^T\int_\Omega m_\delta \frac{|D\tilde u|^\beta}{(\tilde m+\mu)^\alpha}\,dxdt + \int_0^T\int_\Omega \tilde m_\delta \frac{|Du|^\beta}{(m+\mu)^\alpha}\,dxdt\\
&\quad\le C + \sigma\, C\left\{\int_0^T\int_\Omega m_\delta \frac{|D\tilde u|^\beta}{(\tilde m+\mu)^\alpha}\,dxdt + \int_0^T\int_\Omega \tilde m_\delta \frac{|Du|^\beta}{(m+\mu)^\alpha}\,dxdt\right\}\\
&\quad + \sigma\, C\left\{\int_0^T\int_\Omega \tilde m_\delta \left(\frac{|D\tilde u|^\beta}{(\tilde m+\mu)^\alpha}\right)\star\rho_\delta\,dxdt + \int_0^T\int_\Omega m_\delta \left(\frac{|Du|^\beta}{(m+\mu)^\alpha}\right)\star\rho_\delta\,dxdt\right\},
\end{aligned}$$

hence choosing σ sufficiently small we conclude that

$$\begin{aligned}
&\int_0^T\int_\Omega m_\delta \frac{|D\tilde u|^\beta}{(\tilde m+\mu)^\alpha}\,dxdt + \int_0^T\int_\Omega \tilde m_\delta \frac{|Du|^\beta}{(m+\mu)^\alpha}\,dxdt\\
&\quad\le C + C\left\{\int_0^T\int_\Omega \tilde m_\delta \left(\frac{|D\tilde u|^\beta}{(\tilde m+\mu)^\alpha}\right)\star\rho_\delta\,dxdt + \int_0^T\int_\Omega m_\delta \left(\frac{|Du|^\beta}{(m+\mu)^\alpha}\right)\star\rho_\delta\,dxdt\right\}.
\end{aligned}\tag{5.12}$$

We wish now to estimate last terms in the above inequality. To this purpose, we use (5.11) with $\tilde m$ replaced by m_δ (and so $\tilde w$ by $(mH_p(t,x,Du))\star\rho_\delta$). Since the terms with F and G can be estimated as before, we get

$$\int_0^T\int_\Omega m_\delta\,[H(t,x,m,Du)\star\rho_\delta]\,dxdt \le \int_0^T\int_\Omega [Du\star\rho_\delta]\cdot[(mH_p(t,x,Du))\star\rho_\delta]\,dxdt + C\,.$$

Therefore, using (2.2) we obtain

$$\begin{aligned}
c_0\int_0^T\int_\Omega m_\delta[\left(\frac{|Du|^\beta}{(m+\mu)^\alpha}\right)\star\rho_\delta]\,dxdt &\le \int_0^T\int_\Omega [Du\star\rho_\delta]\cdot[(mH_p(t,x,Du))\star\rho_\delta]\,dxdt + C\\
&\le c_2\int_0^T\int_\Omega |Du\star\rho_\delta|\left(m\frac{|Du|^{\beta-1}}{(m+\mu)^\alpha}\right)\star\rho_\delta\,dxdt + C\int_0^T\int_\Omega |Du\star\rho_\delta|\,m_\delta\,dxdt\\
&\le c_2\int_0^T\int_\Omega |Du\star\rho_\delta|\left(\left(\frac{|Du|^\beta}{(m+\mu)^\alpha}\right)\star\rho_\delta\right)^{\frac{1}{\beta'}} m_\delta^{\frac{\beta-\alpha}{\beta}}\,dxdt + C\int_0^T\int_\Omega |Du\star\rho_\delta|\,m_\delta\,dxdt\\
&\le \frac{c_0}{2}\int_0^T\int_\Omega (m+\mu)\star\rho_\delta\left(\frac{|Du|^\beta}{(m+\mu)^\alpha}\right)\star\rho_\delta\,dxdt
\end{aligned}$$

$$+ C \int_0^T \int_\Omega |Du \star \rho_\delta|^\beta \, ((m+\mu)\star\rho_\delta)^{1-\alpha} \, dxdt \; + C \int_0^T \int_\Omega 1 + m_\delta^{1+\frac{\alpha}{\beta-1}} \, dxdt \, .$$

Hence we conclude that

$$\int_0^T \int_\Omega m_\delta \left(\frac{|Du|^\beta}{(m+\mu)^\alpha}\right) \star \rho_\delta \, dxdt \le C \int_0^T \int_\Omega |Du \star \rho_\delta|^\beta \, ((m+\mu)\star\rho_\delta)^{1-\alpha} \, dxdt \; + C \, .$$

Now we observe that the function $(m,\xi) \mapsto (m+\mu)^{1-\alpha}|\xi|^\beta$ is convex as a function of two variables, therefore

$$|Du \star \rho_\delta|^\beta \, ((m+\mu)\star\rho_\delta)^{1-\alpha} \le \left((m+\mu)^{1-\alpha}|Du|^\beta\right) \star \rho_\delta$$

which is bounded since $(m+\mu)^{1-\alpha}|Du|^\beta \in L^1(Q_T)$. We deduce the uniform bound (with respect to δ)

$$\int_0^T \int_\Omega m_\delta \left(\frac{|Du|^\beta}{(m+\mu)^\alpha}\right) \star \rho_\delta \, dxdt \le C \, ,$$

and similarly for $\tilde m$. Going back to (5.12), this also implies the bound

$$\int_0^T \int_\Omega m_\delta \frac{|D\tilde u|^\beta}{(\tilde m+\mu)^\alpha} \, dxdt + \int_0^T \int_\Omega \tilde m_\delta \, \frac{|Du|^\beta}{(m+\mu)^\alpha} \, dxdt \le C \, .$$

Now, reasoning as in (5.8)–(5.10),

$$\begin{aligned} \int\int_E |Du \cdot \tilde w_\delta| &\le \left(\int\int_E \frac{(m+\mu+\tilde m_\delta)}{\tilde m_\delta^{\frac{\beta-\alpha}{\beta-1}}} \, |\tilde w_\delta|^{\beta'}\right)^{\frac{1}{\beta'}} \left(\int\int_E (m+\mu)^{1-\alpha}|Du|^\beta\right)^{\frac{1}{\beta}} \\ &\le \left(C + \int\int_E [m+\mu+\tilde m_\delta] \left(\frac{|D\tilde u|^\beta}{(\tilde m+\mu)^\alpha}\right) \star \rho_\delta\right)^{\frac{1}{\beta'}} \left(\int\int_E (m+\mu)^{1-\alpha}|Du|^\beta\right)^{\frac{1}{\beta}} . \end{aligned}$$

The bounds previously established yield

$$\int\int_E |Du \cdot \tilde w_\delta| \le C \left(\int\int_E (m+\mu)^{1-\alpha}|Du|^\beta\right)^{\frac{1}{\beta}}$$

and since the set E is arbitrary and $(m+\mu)^{1-\alpha}|Du|^\beta \in L^1(Q_T)$, we deduce the equi-integrability of $Du \cdot \tilde w_\delta$. Finally, this allows us to pass to the limit in (5.11) and to obtain (5.2). □

A similar Lemma holds for the case of singular congestion, suitably adapted to the formulation of this case.

Lemma 5.3. *Assume that H satisfies assumptions (2.2)–(2.3) with $\mu = 0$, and that (2.8)–(2.10) hold true. Let $u \in L^\infty(0,T;L^1(\Omega))$ satisfy*

$$\begin{aligned} &\int_0^T \int_\Omega u \, \varphi_t \, dxdt - \nu \int_0^T \int_\Omega u \, \Delta\varphi \, dxdt + \int_0^T \int_\Omega H(t,x,m,Du) \mathbb{1}_{\{m>0\}} \, \varphi \, dxdt \\ &\quad \le \int_0^T \int_\Omega F(t,x,m)\varphi \, dxdt + \int_\Omega \int_{\mathbb{R}} G(x,\lambda) d\nu_x(\lambda) \, \varphi(T) \, dx \end{aligned}$$

for some family of probability measures $\{\nu_x\}$ (weakly−∗ measurable w.r.t. x) such that (5.1) is satisfied, and let $m \in C^0([0,T]; L^1(\Omega))$ be a solution of (3.6). Assume that (u,m) satisfy the conditions (ii) in Definition 3.7. Then

$$\begin{aligned}\langle \tilde m_0 , u(0)\rangle \le &\int_\Omega\int_{\mathbb{R}} G(x,\lambda)d\nu_x(\lambda)\,\tilde m(T)\,dx + \int_0^T\int_\Omega F(t,x,m)\tilde m\,dxdt \\ &+ \int_0^T\int_\Omega \left[\tilde m\, H_p(t,x,\tilde m, D\tilde u)\cdot Du - \tilde m\, H(t,x,m,Du)\right] \mathbb{1}_{\{m>0\,,\,\tilde m>0\}} dxdt\end{aligned} \tag{5.13}$$

for any couple $(\tilde u, \tilde m)$ satisfying the same conditions as (u,m).

Proof. The proof follows the same lines as that of Lemma 5.1.

(A) Case when $\alpha \le 1$. Proceeding as in the proof of Lemma 5.1, we obtain the counterparts of (5.3)

$$\begin{aligned}&\int_0^T\int_\Omega u\,[\partial_t \tilde m_{\delta,\varepsilon} - \nu\Delta \tilde m_{\delta,\varepsilon}]\,dxdt + \int_0^T\int_\Omega H(t,x,m,Du)\tilde m_{\delta,\varepsilon}\,\mathbb{1}_{\{m>0\}}\,dxdt \\ &\quad\le \int_0^T\int_\Omega F(t,x,m)\tilde m_{\delta,\varepsilon}\,dxdt + \int_\Omega\int_{\mathbb{R}} G(x,\lambda)d\nu_x(\lambda)\tilde m_{\delta,\varepsilon}(T)\,dx\end{aligned} \tag{5.14}$$

and of (5.4):

$$\begin{aligned}&-\int_0^T\int_\Omega Du\cdot \tilde w_{\delta,\varepsilon}\,dtdx + \int_0^T\int_\Omega H(t,x,m,Du)\tilde m_{\delta,\varepsilon}\,\mathbb{1}_{\{m>0\}}\,dxdt + \int_\Omega (m_0\star\rho_\delta)\int_0^T u(t)\xi_\varepsilon(-t)\,dt\,dx \\ &\quad\le \int_0^T\int_\Omega F(t,x,m)\tilde m_{\delta,\varepsilon}\,dxdt + \int_\Omega\int_{\mathbb{R}} G(x,\lambda)d\nu_x(\lambda)\tilde m_{\delta,\varepsilon}(T)\,dx\end{aligned} \tag{5.15}$$

where we denote $\tilde w_\delta = [(\tilde m\, H_p(t,x,\tilde m, D\tilde u)\,\mathbb{1}_{\{\tilde m>0\}})\star\rho_\delta]$ and $\tilde w_{\delta,\varepsilon} = \int_0^T \tilde w_\delta(s)\,\xi_\varepsilon(s-t)\,ds$. Note that $\tilde w_\delta = [(\tilde m\, H_p(t,x,\tilde m, D\tilde u))\star\rho_\delta]$ if $\alpha<1$. Set $\tilde b = H_p(t,x,\tilde m, D\tilde u)\,\mathbb{1}_{\{\tilde m>0\}}$. We deduce from Assumption (2.3) that

$$\begin{aligned}|(\tilde m\,\tilde b)\star\rho_\delta| &\le c_2\tilde m_\delta + c_2\left(\left(\tilde m\,\frac{|D\tilde u|^\beta}{\tilde m^\alpha}\mathbb{1}_{\{\tilde m>0\}}\right)\star\rho_\delta\right)^{\frac{1}{\beta'}}\left(\frac{\tilde m}{\tilde m^\alpha}\star\rho_\delta\right)^{\frac{1}{\beta}} \\ &\le c_2\tilde m_\delta + c_2\left(\left(\tilde m^{1-\alpha}\,|D\tilde u|^\beta\mathbb{1}_{\{\tilde m>0\}}\right)\star\rho_\delta\right)^{\frac{1}{\beta'}}(\tilde m\star\rho_\delta)^{\frac{1-\alpha}{\beta}}\end{aligned}$$

where we used $\alpha\le 1$ in the latter inequality. Therefore, using also (2.2) we estimate

$$\begin{aligned}&H(t,x,m,Du)\mathbb{1}_{\{m>0\}}\tilde m_\delta(s) - Du\cdot\tilde w_\delta(s) \ge c_0\tilde m_\delta(s)\frac{|Du|^\beta}{m^\alpha}\mathbb{1}_{\{m>0\}} - c_1\tilde m_\delta(s)(1+m^{\frac{\alpha}{\beta-1}}) \\ &- c_2\tilde m_\delta(s)|Du| - c_2|Du|\left(\left(\tilde m(s)^{1-\alpha}\,|D\tilde u(s)|^\beta\mathbb{1}_{\{\tilde m(s)>0\}}\right)\star\rho_\delta\right)^{\frac{1}{\beta'}}\tilde m_\delta(s)^{\frac{1-\alpha}{\beta}}.\end{aligned} \tag{5.16}$$

We now use the fact that $Du=0$ a.e. in $\{m=0\}$. From Young inequalities, we deduce

$$c_2\tilde m_\delta(s)|Du| = c_2\tilde m_\delta(s)|Du|\mathbb{1}_{\{m>0\}} \le \frac{c_0}{2}\tilde m_\delta(s)\frac{|Du|^\beta}{m^\alpha}\mathbb{1}_{\{m>0\}} + C\tilde m_\delta(s)m^{\frac{\alpha}{\beta-1}}, \tag{5.17}$$

and

$$\begin{aligned}&c_2|Du|\left(\left(\tilde m(s)^{1-\alpha}\,|D\tilde u(s)|^\beta\mathbb{1}_{\{\tilde m(s)>0\}}\right)\star\rho_\delta\right)^{\frac{1}{\beta'}}\tilde m_\delta(s)^{\frac{1-\alpha}{\beta}} \\ &\le \sigma|Du|^\beta\mathbb{1}_{\{m>0\}}\tilde m_\delta(s)^{1-\alpha} + C_\sigma\left(\tilde m(s)^{1-\alpha}\,|D\tilde u(s)|^\beta\mathbb{1}_{\{\tilde m(s)>0\}}\right)\star\rho_\delta,\end{aligned} \tag{5.18}$$

and if $0 < \sigma \le c_0/2$,

$$\sigma \mathbb{1}_{\{m>0\}} \tilde m_\delta(s)^{1-\alpha} \le \frac{c_0}{2}\frac{\tilde m_\delta(s)}{m^\alpha}\mathbb{1}_{\{m>0\}} + \frac{c_0}{2} m^{1-\alpha}\mathbb{1}_{\{m>0\}}, \tag{5.19}$$

because $0 < \alpha \le 1$. Finally,

$$\tilde m_\delta(s) m^{\frac{\alpha}{\beta-1}} \le C\left(1 + m^{1+\frac{\alpha}{\beta-1}} + \tilde m_\delta(s)^{1+\frac{\alpha}{\beta-1}}\right). \tag{5.20}$$

Combining (5.16) with (5.17)–(5.20), we get

$$H(t,x,m,Du)\tilde m_\delta(s) - Du\cdot \tilde w_\delta(s) \ge -C\left(1 + \tilde m_\delta(s)^{1+\frac{\alpha}{\beta-1}} + m^{1+\frac{\alpha}{\beta-1}}\right)$$
$$- \frac{c_0}{2} m^{1-\alpha}|Du|^\beta \mathbb{1}_{\{m>0\}} - c\left(\tilde m(s)^{1-\alpha}|D\tilde u(s)|^\beta \mathbb{1}_{\{\tilde m(s)>0\}}\right)\star \rho_\delta .$$

The conclusion of the proof is exactly as for Lemma 5.1.

(B) Case when $\alpha > 1$. We start from (5.15), and proceed differently for the first two terms. From (2.3) we have $|\tilde b| \le c_2(1 + |D\tilde u|^{\beta-1}\tilde m^{-\alpha}\mathbb{1}_{\{\tilde m>0\}})$, so we estimate $\tilde w_\delta$ as in (5.9) of Lemma 5.1 and we get

$$\frac{|\tilde w_\delta|}{\tilde m_\delta^{\frac{\beta-\alpha}{\beta-1}}} \le C\tilde m_\delta^{\frac{\alpha}{\beta-1}} + C\left(\frac{|D\tilde u|^\beta}{\tilde m^\alpha}\mathbb{1}_{\{\tilde m>0\}}\right)\star\rho_\delta.$$

Since $Du = 0$ a.e. if $m = 0$, we can use (5.8) with $\mu = 0$ and, similarly as in Lemma 5.1, we get at

$$\begin{aligned}Du\cdot\tilde w_\delta &= Du\cdot \frac{\tilde w_\delta}{\tilde m_\delta^{\frac{\beta-\alpha}{\beta}}}\tilde m_\delta^{\frac{\beta-\alpha}{\beta}}\mathbb{1}_{\{m>0\}}\\ &\le \sigma C(\tilde m_\delta + m\mathbb{1}_{\{m>0\}})\left(\frac{|D\tilde u|^\beta}{\tilde m^\alpha}\mathbb{1}_{\{\tilde m>0\}}\right)\star\rho_\delta + C\left(1 + m^{1+\frac{\alpha}{\beta-1}} + \tilde m_\delta^{1+\frac{\alpha}{\beta-1}}\right)\\ &\quad + C_\sigma m^{1-\alpha}\mathbb{1}_{\{m>0\}}|Du|^\beta.\end{aligned}$$

Thanks to this estimate, we can proceed exactly as in the proof of Lemma 5.1 (when $\alpha > 1$), and obtain

$$\begin{aligned}&\int_0^T\int_\Omega [H(t,x,m,Du)\mathbb{1}_{\{m>0\}}\tilde m_\delta - Du\cdot\tilde w_\delta]\,dxdt + \int_\Omega (\tilde m_0\star\rho_\delta)u(0)\,dx\\ &\le \int_0^T\int_\Omega F(t,x,m)\tilde m_\delta\,dxdt + \int_\Omega\int_{\mathbb{R}} G(x,\lambda)d\nu_x(\lambda)\tilde m_\delta(T)\,dx,\end{aligned} \tag{5.21}$$

and, after some steps,

$$\begin{aligned}&\int_0^T\int_\Omega m_\delta\frac{|D\tilde u|^\beta}{\tilde m^\alpha}\mathbb{1}_{\{\tilde m>0\}}\,dxdt + \int_0^T\int_\Omega \tilde m_\delta\frac{|Du|^\beta}{m^\alpha}\mathbb{1}_{\{m>0\}}\,dxdt\\ &\le C + C\left\{\int_0^T\int_\Omega \tilde m_\delta\left(\frac{|D\tilde u|^\beta}{\tilde m^\alpha}\mathbb{1}_{\{\tilde m>0\}}\right)\star\rho_\delta\,dxdt + \int_0^T\int_\Omega m_\delta\left(\frac{|Du|^\beta}{m^\alpha}\mathbb{1}_{\{m>0\}}\right)\star\rho_\delta\,dxdt\right\}.\end{aligned} \tag{5.22}$$

To estimate the last two terms in (5.22), we use (5.21) where we replace $\tilde m$ with m_δ and proceed as above,

$$\begin{aligned}c_0\int_0^T\int_\Omega m_\delta\left(\frac{|Du|^\beta\mathbb{1}_{\{m>0\}}}{m^\alpha}\right)\star\rho_\delta\,dxdt &\le \int_0^T\int_\Omega Du\star\rho_\delta\cdot(mH_p(t,x,Du)\mathbb{1}_{\{m>0\}})\star\rho_\delta\,dxdt + C\\ &\le c_2\int_0^T\int_\Omega |Du\star\rho_\delta|\left(m^{1-\alpha}|Du|^{\beta-1}\mathbb{1}_{\{m>0\}}\right)\star\rho_\delta\,dxdt + C\end{aligned}$$

$$\leq c_2 \int_0^T \int_\Omega |Du \star \rho_\delta| \left(\left(\frac{|Du|^\beta}{m^\alpha} \mathbb{1}_{\{m>0\}} \right) \star \rho_\delta \right)^{\frac{1}{\beta'}} (m \star \rho_\delta)^{\frac{\beta-\alpha}{\beta}} \, dxdt + C$$

Let us observe that $m \star \rho_\delta = 0 \Rightarrow Du \star \rho_\delta = 0$ since $Du = 0$ at almost every point where $m = 0$. This means that last integral in the above inequality is restricted to the set $\{m \star \rho_\delta > 0\}$ (even in the limiting case $\beta = \alpha$). By Young's inequality, we deduce that

$$\int_0^T \int_\Omega m_\delta \left(\frac{|Du|^\beta}{m^\alpha} \mathbb{1}_{\{m>0\}} \right) \star \rho_\delta \, dxdt \leq C \int_0^T \int_\Omega |Du \star \rho_\delta|^\beta \, (m \star \rho_\delta)^{1-\alpha} \mathbb{1}_{\{m \star \rho_\delta > 0\}} \, dxdt \ + C. \tag{5.23}$$

Let us define the convex and lower semi-continuous function Ψ on $\mathbb{R}^N \times \mathbb{R}$ by

$$\Psi(p, m) = \begin{cases} m^{1-\alpha} |p|^\beta & \text{if} \quad m > 0, \\ 0 & \text{if} \quad m = 0 \text{ and } p = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

This implies that

$$\begin{aligned} (m \star \rho_\delta)^{1-\alpha} \, |Du \star \rho_\delta|^\beta \, \mathbb{1}_{\{m \star \rho_\delta > 0\}} &= \Psi(Du \star \rho_\delta, m \star \rho_\delta) \\ &\leq \Psi(Du, m) \star \rho_\delta \\ &= \left(m^{1-\alpha} \, |Du|^\beta \, \mathbb{1}_{\{m>0\}} \right) \star \rho_\delta \end{aligned}$$

and the latter function is bounded in $L^1(Q_T)$ by assumption. We deduce from (5.23) the uniform bound (with respect to δ)

$$\int_0^T \int_\Omega m_\delta \left(\frac{|Du|^\beta}{m^\alpha} \mathbb{1}_{\{m>0\}} \right) \star \rho_\delta \, dxdt \leq C,$$

and the same holds replacing m with $\tilde{m}$ and u with $\tilde{u}$. Going back to (5.22), we obtain

$$\int_0^T \int_\Omega m_\delta \frac{|D\tilde{u}|^\beta}{\tilde{m}^\alpha} \mathbb{1}_{\{\tilde{m}>0\}} \, dxdt + \int_0^T \int_\Omega \tilde{m}_\delta \, \frac{|Du|^\beta}{m^\alpha} \mathbb{1}_{\{m>0\}} \, dxdt \leq C,$$

and we conclude as in the proof of Lemma 5.1. □

6. Existence and uniqueness for non-singular congestion

Our first goal is to prove the existence of a weak solution and, to this purpose, we now show the strong convergence of $G^\epsilon(x, m^\epsilon(T))$ in $L^1(\Omega)$.

Lemma 6.1. *Consider a subsequence (u^ϵ, m^ϵ) converging to (u, m) as in Proposition 4.5. We have that $G^\epsilon(x, m^\epsilon(T))$ converges to $G(x, m(T))$ in $L^1(\Omega)$ and (u, m) satisfies the energy identity:*

$$\begin{aligned} &\int_0^T \int_\Omega m \left(H_p(t, x, m, Du) \cdot Du - H(t, x, m, Du) \right) \, dxdt \\ &\quad + \int_\Omega m(T) G(x, m(T)) dx + \int_0^T \int_\Omega m F(t, x, m) \, dxdt = \langle u(0), m_0 \rangle. \end{aligned} \tag{6.1}$$

Proof. We start from the energy identity for the solution of (4.1)–(4.3):

$$\int_0^T\int_\Omega m^\epsilon\left(H_p(t,x,T_{1/\epsilon}m^\epsilon,Du^\epsilon)\cdot Du^\epsilon - H(t,x,T_{1/\epsilon}m^\epsilon,Du^\epsilon)\right)dxdt \\ + \int_\Omega m^\epsilon(T)G^\epsilon(x,m^\epsilon(T))dx + \int_0^T\int_\Omega m^\epsilon F^\epsilon(t,x,m^\epsilon)\,dxdt = \int_\Omega u^\epsilon(0)m_0^\epsilon\,dx. \tag{6.2}$$

Thanks to (2.6) and (2.11), Fatou's lemma implies that

$$\liminf_{\epsilon\to 0}\int_0^T\int_\Omega m^\epsilon\left(H_p(t,x,T_{1/\epsilon}m^\epsilon,Du^\epsilon)\cdot Du^\epsilon - H(t,x,T_{1/\epsilon}m^\epsilon,Du^\epsilon)\right)dxdt \\ \geq \int_0^T\int_\Omega m\left(H_p(t,x,m,Du)\cdot Du - H(t,x,m,Du)\right)dxdt,$$

and

$$\liminf_{\epsilon\to 0}\int_0^T\int_\Omega m^\epsilon F^\epsilon(t,x,m^\epsilon)\,dxdt \geq \int_0^T\int_\Omega mF(t,x,m)\,dxdt.$$

From Lemma 4.7 we can assume that $u^\epsilon|_{t=0}$ converges weakly * to a bounded measure χ on Ω, and $\chi \leq u(0)$. Since $m_0 \in C(\Omega)$, we deduce that

$$\lim_{\epsilon\to 0}\int_\Omega u^\epsilon(0)m_0^\epsilon\,dx = \langle\chi, m_0\rangle \leq \langle u(0), m_0\rangle\,.$$

Combining the informations above, we get from (6.2)

$$\limsup_{\epsilon\to 0}\int_\Omega m^\epsilon(T)G^\epsilon(x,m^\epsilon(T))dx \leq \langle u(0),m_0\rangle - \int_0^T\int_\Omega mF(t,x,m)\,dxdt \\ - \int_0^T\int_\Omega m\left(H_p(t,x,m,Du)\cdot Du - H(t,x,m,Du)\right)dxdt$$

We now use (5.2) in Lemma 5.1 with $\tilde{m} = m$ and $\tilde{u} = u$, and we get

$$\limsup_{\epsilon\to 0}\int_\Omega m^\epsilon(T)G^\epsilon(x,m^\epsilon(T))dx \leq \langle u(0),m_0\rangle - \int_0^T\int_\Omega mF(t,x,m)\,dxdt \\ - \int_0^T\int_\Omega m\left(H_p(t,x,m,Du)\cdot Du - H(t,x,m,Du)\right)dxdt \\ \leq \int_\Omega\int_{\mathbb{R}} m(T)G(x,\lambda)d\nu_x(\lambda)dx\,. \tag{6.3}$$

We now use the monotonicity of G in order to get the strong convergence of $G^\epsilon(x,m^\epsilon(T))$. Indeed, if we set $\hat{m}^\epsilon := \rho^\epsilon \star m^\epsilon(T)$ and $T_k(r) = \min(r,k)$,

$$\int_\Omega \left[T_k(G(x,\hat m^\epsilon)) - T_k(G(x,m(T)))\right][\hat m^\epsilon - m(T)]dx$$
$$\le \int_\Omega G(x,\hat m^\epsilon)\hat m^\epsilon\, dx - \int_\Omega T_k(G(x,\hat m^\epsilon))m(T)\, dx - \int_\Omega T_k(G(x,m(T)))\,[\hat m^\epsilon - m(T)]dx\,.$$

Since $\hat m^\epsilon$ weakly converges to $m(T)$, the last term vanishes as $\epsilon \to 0$. The first one is estimated by (6.3), while for the second we can use (4.25) with $f(x,\cdot) = T_k(G(x,\cdot))$. Therefore, we get

$$\limsup_{\epsilon\to 0} \int_\Omega \left[T_k(G(x,\hat m^\epsilon)) - T_k(G(x,m(T)))\right][\hat m^\epsilon - m(T)]dx$$
$$\le \int_\Omega\int_{\mathbb{R}} G(x,\lambda)m(T)d\nu_x(\lambda)\, dx - \int_\Omega\int_{\mathbb{R}} T_k(G(x,\lambda))m(T)d\nu_x(\lambda)\, dx\,.$$

Letting $k\to\infty$ we conclude that

$$\limsup_{k\to\infty}\ \limsup_{\epsilon\to 0} \int_\Omega \left[T_k(G(x,\hat m^\epsilon)) - T_k(G(x,m(T)))\right][\hat m^\epsilon - m(T)]dx = 0\,. \tag{6.4}$$

We claim now that, as a consequence of (6.4), $G(x,\hat m^\epsilon)$ converges to $G(x,m(T))$ almost everywhere in Ω, at least for a subsequence. Indeed, we observe that

$$\int_\Omega \frac{[G(x,\hat{m^\varepsilon}) - G(x,m(T))]\,[\hat m^\varepsilon - m(T)]}{1+\hat m^\varepsilon + m(T)}dx \le \int_\Omega \left[T_k(G(x,\hat m^\epsilon)) - T_k(G(x,m(T)))\right][\hat m^\epsilon - m(T)]dx$$
$$+\int_\Omega |G(x,\hat m^\epsilon)|\,\mathbb{1}_{\{G(x,\hat m^\epsilon)>k\}}\, dx + \int_\Omega |G(x,m(T))|\,\mathbb{1}_{\{G(x,m(T))>k\}}\, dx$$

and last two terms tend to zero as $k\to\infty$ uniformly with respect to ε. So we have

$$\limsup_{\epsilon\to 0} \int_\Omega \frac{[G(x,\hat{m^\varepsilon}) - G(x,m(T))]\,[\hat m^\varepsilon - m(T)]}{1+\hat m^\varepsilon + m(T)}dx$$
$$\le \limsup_{\epsilon\to 0} \int_\Omega \left[T_k(G(x,\hat m^\epsilon)) - T_k(G(x,m(T)))\right][\hat m^\epsilon - m(T)]dx$$
$$+\sup_\varepsilon \int_\Omega |G(x,\hat m^\epsilon)|\,\mathbb{1}_{\{G(x,\hat m^\epsilon)>k\}}\, dx + \int_\Omega |G(x,m(T))|\,\mathbb{1}_{\{G(x,m(T))>k\}}\, dx$$

hence, using (6.4) and letting $k\to\infty$, the right-hand side vanishes. We deduce that

$$\limsup_{\epsilon\to 0} \int_\Omega \frac{[G(x,\hat{m^\varepsilon}) - G(x,m(T))]\,[\hat m^\varepsilon - m(T)]}{1+\hat m^\varepsilon + m(T)}dx = 0$$

which means, since G is monotone, the L^1 convergence of the integrand function. We deduce that, up to subsequences,

$$\frac{[G(x,\hat{m^\varepsilon}) - G(x,m(T))]\,[\hat m^\varepsilon - m(T)]}{1+\hat m^\varepsilon + m(T)} \to 0 \quad \text{a.e.}$$

and this readily yields the a.e. convergence of $G(x,\hat m^\epsilon)$ to $G(x,m(T))$ in Ω.

Finally, since $G(x,\hat m^\epsilon)$ is also equi-integrable by (4.7), it is therefore convergent in $L^1(\Omega)$. As a consequence, the L^1 convergence of $G^\varepsilon(x,m^\epsilon(T))$ towards $G(x,m(T))$ is established. In addition, we now deduce that

$$\int_{\mathbb{R}} G(x,\lambda)d\nu_x(\lambda) = G(x,m(T))\,.$$

We insert this information in the right-hand side of (6.3), moreover now we can use Fatou's lemma in the left-hand side and we conclude with the identity

$$\int_\Omega G(x,m(T))m(T)dx = \langle u(0), m_0\rangle - \int_0^T\int_\Omega mF(t,x,m)\, dxdt$$

$$- \int_0^T \int_\Omega m \left(H_p(t,x,m,Du) \cdot Du - H(t,x,m,Du) \right) dxdt$$

which is (6.1). □

Remark 6.2. The monotonicity of G is only used in the above lemma in order to obtain the strong L^1 convergence of $G^\varepsilon(x, m^\epsilon(T))$. If $q < 2$, this condition would not be required since one already knows from Proposition 4.5 that $m^\epsilon(T)$ strongly converges in L^1 to $m(T)$.

We can finally conclude with the existence result.

Theorem 6.3. *Consider a subsequence (u^ϵ, m^ϵ) converging to (u,m) as in Proposition 4.5. Then (u,m) is a weak solution of (1.3).*

Proof. With the results proved in §4, there only remains to pass to the limit in the Hamilton–Jacobi equation and show that (1.3) holds.

By Lemma 6.1, $G^\epsilon(x, m^\epsilon(T)) \to G(x, m(T))$ in $L^1(\Omega)$. We also know that $F^\epsilon(t,x,m^\epsilon) \to F(t,x,m)$ in $L^1(Q_T)$ (see the proof of Lemma 4.6). Now we observe that assumptions (2.1)–(2.3) imply that H satisfies

$$c_0 \frac{|p|^\beta}{(m+\mu)^\alpha} - c_1 (1 + m^{\frac{\alpha}{\beta-1}}) \le H(t,x,m,p) \le c \left(|p| + \frac{|p|^\beta}{(m+\mu)^\alpha} \right) \le c \left(\frac{|p|^\beta}{(m+\mu)^\alpha} + 1 + m^{\frac{\alpha}{\beta-1}} \right).$$

Therefore, up to addition of a L^1-convergent sequence, the Hamiltonian

$$(t,x,p) \mapsto H(t,x,T_{1/\epsilon} m^\epsilon(t,x), p)$$

is nonnegative and has natural growth. Since u^ϵ is bounded below, without loss of generality we can assume that the solution u^ϵ is also nonnegative. It is therefore possible to apply (a straightforward adaptation of) the result in [23, Theorem 3.1], in order to deduce that $H(t,x,T_{1/\epsilon} m^\epsilon, Du^\epsilon) \to H(t,x,m,Du)$ in $L^1(Q_T)$, and we can pass to the limit in (4.1)–(4.3) and obtain (3.1). □

In order to prove the uniqueness of weak solutions, the main step was given by Lemma 5.1. We need however a counterpart which ensures that *any* weak solution satisfies the energy equality (6.1). To this purpose, we follow an argument developed in [25].

Lemma 6.4. *Let (u,m) be any weak solution of (1.3)–(1.5). Then the energy identity (6.1) holds true.*

Proof. By Lemma 5.1, we already know that

$$\int_\Omega m_0 u(0)\, dx \le \int_\Omega G(x, m(T))\, m(T)\, dx + \int_0^T \int_\Omega F(t,x,m) m \, dxdt$$
$$+ \int_0^T \int_\Omega \left[m\, H_p(t,x,m,Du) \cdot Du - m\, H(t,x,m,Du) \right] dxdt$$

where we used the fact that $u \in C^0([0,T]; L^1(\Omega))$. In order to prove the reverse inequality, let $u_k := \min(u,k)$. Since $-u_t - \nu \Delta u \in L^1(Q_T)$, by Kato's inequality,

$$-(u_k)_t - \nu \Delta u_k + H(t,x,m,Du) \mathbb{1}_{\{u<k\}} \ge F(t,x,m) \mathbb{1}_{\{u<k\}}.$$

Since $m_t - \nu \Delta m - \operatorname{div}(m\, b) = 0$ for some b such that $m|b|^2 \in L^1(Q_T)$, by [24, Theorem 3.6] we know that m is also a renormalized solution, hence it satisfies

$$S_n(m)_t - \nu \Delta S_n(m) - \operatorname{div}(S_n'(m) m\, H_p(t,x,m,Du)) = \omega_n$$

where $S_n(m)$ is a suitable C^1 truncation (i.e. $S_n(r) = nS(r/n)$, for some S compactly supported in $[-2, 2]$ such that $S = 1$ in $[-1, 1]$) and where $\omega_n \to 0$ in $L^1(Q_T)$. Notice that both u_k and $S_n(m)$ are bounded functions and belong to $L^2(0, T; H^1) \cap C^0([0, T]; L^1(\Omega))$. By using u_k in the equation of $S_n(m)$ we get

$$\int_\Omega S_n(m_0)u_k(0)\,dx - \int_\Omega u_k(T)\,S_n(m(T))\,dx \ge \int_0^T\int_\Omega F(t,x,m)\mathbb{1}_{\{u<k\}}S_n(m)\,dxdt$$
$$+\int_0^T\int_\Omega \left[S_n'(m)m\,H_p(t,x,m,Du)\cdot Du - S_n(m)H(t,x,m,Du)\right]\mathbb{1}_{\{u<k\}}\,dxdt - \int_0^T\int_\Omega \omega_n\,u_k\,dxdt\,.$$

Letting first $n \to \infty$, the last term vanishes since u_k is bounded. The regularity of weak solutions allows us to pass to the limit in the other terms (recall that $u(T) = G(x, m(T))$) and we obtain

$$\int_\Omega m_0\,u_k(0)\,dx - \int_\Omega u_k(T)\,m(T)\,dx \ge \int_0^T\int_\Omega F(t,x,m)\mathbb{1}_{\{u<k\}}m\,dxdt$$
$$+\int_0^T\int_\Omega \left[m\,H_p(t,x,m,Du)\cdot Du - mH(t,x,m,Du)\right]\mathbb{1}_{\{u<k\}}\,dxdt\,.$$

Finally, letting $k \to \infty$ we deduce the desired inequality and we conclude that (6.1) holds true. □

Lemma 6.5. *Under all the assumptions made in Theorem 3.6, there is a unique weak solution of (1.3)–(1.5).*

Proof. Let (u, m) and $(\tilde u, \tilde m)$ be two weak solutions of (1.3)–(1.5). By Lemma 6.4, they both satisfy the energy identity (6.1), so using also Lemma 5.1 overall we know that

$$\int_\Omega m_0\,u(0)\,dx \le \int_\Omega G(x,m(T))\,\tilde m(T)\,dx + \int_0^T\int_\Omega F(t,x,m)\tilde m\,dxdt$$
$$+\int_0^T\int_\Omega \left[\tilde m\,H_p(t,x,\tilde m,D\tilde u)\cdot Du - \tilde m\,H(t,x,m,Du)\right]dxdt,$$

$$\int_\Omega m_0\,\tilde u(0)\,dx \le \int_\Omega G(x,\tilde m(T))\,m(T)\,dx + \int_0^T\int_\Omega F(t,x,\tilde m)m\,dxdt$$
$$+\int_0^T\int_\Omega \left[m\,H_p(t,x,m,Du)\cdot D\tilde u - m\,H(t,x,\tilde m,D\tilde u)\right]dxdt,$$

$$\int_\Omega m_0\,u(0)\,dx = \int_\Omega G(x,m(T))\,m(T)\,dx + \int_0^T\int_\Omega F(t,x,m)m\,dxdt$$
$$+\int_0^T\int_\Omega \left[m\,H_p(t,x,m,Du)\cdot Du - m\,H(t,x,m,Du)\right]dxdt,$$

$$\int_\Omega m_0\,\tilde u(0)\,dx = \int_\Omega G(x,\tilde m(T))\,\tilde m(T)\,dx + \int_0^T\int_\Omega F(t,x,\tilde m)\tilde m\,dxdt$$

$$+\int_0^T\int_\Omega \left[\tilde m\, H_p(t,x,\tilde m, D\tilde u)\cdot D\tilde u - m\, H(t,x,\tilde m, D\tilde u)\right] dxdt,$$

and therefore

$$\begin{aligned}
0 \ge &\int_\Omega (G(m(T)) - G(\tilde m(T)))\,(m(T)-\tilde m(T))\,dx + \int_0^T\int_\Omega (F(t,x,m)-F(t,x,\tilde m))(m-\tilde m)\,dxdt\\
&+\int_0^T\int_\Omega \tilde m\left[H(t,x,m,Du) - H(t,x,\tilde m, D\tilde u) - H_p(t,x,\tilde m,D\tilde u)\cdot(Du - D\tilde u)\right]dxdt\\
&+\int_0^T\int_\Omega m\left[H(t,x,\tilde m,D\tilde u) - H(t,x,m,Du) - H_p(t,x,m,Du)\cdot(D\tilde u - Du)\right]dxdt.
\end{aligned}\tag{6.5}$$

Let us define

$$\begin{aligned}
E(t,x,m_1,p_1,m_2,p_2) = &-(H(t,x,m_1,p_1) - H(t,x,m_2,p_2))(m_1-m_2)\\
&+\left(m_1 H_p(t,x,m_1,p_1) - m_2 H_p(t,x,m_2,p_2)\right)(p_1-p_2)\\
&+(F(t,x,m_1)-F(t,x,m_2))(m_1-m_2)
\end{aligned}$$

and $r = p_2 - p_1$, $p_s = p_1 + sr$, $z = m_2 - m_1$, $m_s = m_1 + sz$. From the assumptions, the function $h : s \mapsto -zH(t,x,m_s,p_s) + m_s H_p(t,x,m_s,p_s)\cdot r + zF(t,x,m_s)$ is increasing on $[0,1]$; this yields that $E(t,x,m_1,p_1,m_2,p_2) \ge 0$ for all $m_1, m_2 \ge 0$ and all $p_1, p_2 \in \mathbb{R}^N$ and that $E(t,x,m_1,p_1,m_2,p_2) = 0$ if and only if $m_1 = m_2$ and $p_1 = p_2$. Thus, (6.5) and the assumptions imply

$$\begin{aligned}
&\int_\Omega (G(x,m(T)) - G(x,\tilde m(T)))\,(m(T)-\tilde m(T))\,dx = 0,\\
&\int_0^T\int_\Omega E(t,x,m,Du,\tilde m, D\tilde u)\,dxdt = 0.
\end{aligned}\tag{6.6}$$

From (6.6), we deduce that $m = \tilde m$ and $Du = D\tilde u$ a.e. in Q_T, and that $G(\cdot, m(T,\cdot)) = G(\cdot,\tilde m(T,\cdot))$ a.e. in Ω. We then deduce that, in a weak sense, $(u-\tilde u)_t = 0$ and $u(T) = \tilde u(T)$, hence $u = \tilde u$ a.e. in Q_T. □

7. Existence and uniqueness for singular congestion

We consider the limit case when $\mu = 0$; we aim at proving the existence and uniqueness of a weak solution of (1.3)–(1.5) as defined in Definition 3.7. Recall that it is not restrictive to assume (3.3). To prove existence, we consider the weak solutions (u^μ, m^μ) of

$$-\partial_t u^\mu - \nu\Delta u^\mu + H(t,x,\mu+m^\mu, Du^\mu) = F(t,x,m^\mu)\,,\qquad (t,x)\in(0,T)\times\Omega \tag{7.1}$$

$$\partial_t m^\mu - \nu\Delta m^\mu - \operatorname{div}(m^\mu H_p(t,x,\mu+m^\mu,Du^\mu)) = 0\,,\qquad (t,x)\in(0,T)\times\Omega \tag{7.2}$$

$$m^\mu(0,x) = m_0(x)\,,\ \ u^\mu(T,x) = G(x,m^\mu(T))\,,\qquad x\in\Omega \tag{7.3}$$

for $\mu > 0$ tending to 0. By (4.23), (4.24), we already know that the following estimates hold

$$\int_\Omega G(x,m^\mu(T))m^\mu(T)dx + \int_0^T\int_\Omega F(t,x,m^\mu)m^\mu\,dxdt + \|(m^\mu)^{\frac{\alpha}{\beta-1}+1}\|_{L^{\frac{N+2}{N}}(Q_T)} \le C, \tag{7.4}$$

$$\int_0^T\int_\Omega m^\mu\left\{H_p(t,x,\mu+m^\mu,Du^\mu)\cdot Du^\mu - H(t,x,\mu+m^\mu,Du^\mu)\right\}dxdt \le C, \tag{7.5}$$

$$\int_0^T\int_\Omega \frac{|Du^\mu|^\beta}{(m^\mu+\mu)^\alpha}\,dxdt + \int_0^T\int_\Omega m^\mu \frac{|Du^\mu|^\beta}{(m^\mu+\mu)^\alpha}\,dxdt \le C, \tag{7.6}$$

for some C independent of μ.

With the same proof as for Lemma 4.4, we obtain:

Lemma 7.1. *For each $0<\mu<1$, the following functions defined on Q_T, $w^\mu = \mathbb{1}_{\{m^\mu+\mu\le 1\}} m^\mu \frac{|Du^\mu|^{\beta-1}}{(\mu+m^\mu)^\alpha}$ and $z^\mu = \mathbb{1}_{\{m^\mu+\mu>1\}}\sqrt{m^\mu}\frac{|Du^\mu|^{\beta-1}}{(\mu+m^\mu)^\alpha}$, are such that*

$$\left|m^\mu H_p(t,x,\mu+m^\mu,Du^\mu)\right| \le c_2(m^\mu + w^\mu + \sqrt{m^\mu}z^\mu),$$

and the family (w^μ) is bounded in $L^{\beta'}(Q_T)$, the family z^μ is bounded in $L^2(Q_T)$ and also relatively compact if $\beta<2$.

Remark 7.2. In fact, the estimate of Lemma 7.1 shows that $m^\mu H_p(t,x,\mu+m^\mu,Du^\mu)$ is actually bounded in $L^{1+\epsilon}(Q_T)$ for some $\epsilon>0$. This is obvious for m^μ from estimate (7.4) and so for w^μ, which is bounded in $L^{\beta'}(Q_T)$. As for the term $\sqrt{m^\mu}z^\mu$, we see that, for $0<\epsilon<1$

$$\int_0^T\int_\Omega (\sqrt{m^\mu}z^\mu)^{1+\epsilon}dxdt \le \left(\int_0^T\int_\Omega (z^\mu)^2 dxdt\right)^{\frac{1+\epsilon}{2}} \left(\int_0^T\int_\Omega (m^\mu)^{\frac{1+\epsilon}{1-\epsilon}}dxdt\right)^{\frac{1-\epsilon}{2}}$$

so using the bound of z^μ in $L^2(Q_T)$ and the estimate (7.4) for m^μ, the right-hand side is bounded as soon as ϵ is sufficiently small.

Lemma 7.3.

(1) *The family (m^μ) is relatively compact in $L^1(Q_T)$ and in $C([0,T];W^{-1,r}(\Omega))$ for some $r>1$.*
(2) *There exist $m\in L^1(Q_T)$ and $u\in L^q(0,T;W^{1,q}(\Omega))$ for any $q<\frac{N+2}{N+1}$ such that, after the extraction of a subsequence (not relabeled), $m^\mu\to m$ in $L^1(Q_T)$ and almost everywhere, $u^\mu\to u$ and $Du^\mu\to Du$ in $L^1(\Omega)$ and almost everywhere. Moreover,*

$$\int_0^T\int_\Omega m^{-\alpha}|Du|^\beta \mathbb{1}_{\{m>0\}}\,dxdt + \int_0^T\int_\Omega m^{1-\alpha}|Du|^\beta\mathbb{1}_{\{m>0\}}\,dxdt < +\infty, \tag{7.7}$$

$$\int_0^T\int_\Omega F(x,m)m\,dxdt + \|m^{\frac{\alpha}{\beta-1}+1}\|_{L^{\frac{N+2}{N}}(Q_T)} < \infty, \tag{7.8}$$

and $Du=0$ a.e. in $\{m=0\}$.
(3) *If $\beta\le 2$ and $\alpha<1$, then $m^\mu H_p(t,x,\mu+m^\mu,Du^\mu)\to mH_p(t,x,m,Du)$ in $L^1(Q_T)$. If $\beta<2$ and $1\le\alpha\le\frac{4(\beta-1)}{\beta}$ or $\beta=2$ and $1\le\alpha<2$, then $m^\mu H_p(t,x,\mu+m^\mu,Du^\mu)\rightharpoonup mH_p(t,x,m,Du)\mathbb{1}_{\{m>0\}}$ weakly in $L^1(Q_T)$.*
(4) *$m\in C^0([0,T];L^1(\Omega))$ and $m^\mu(t)$ converges to $m(t)$ weakly in $L^1(\Omega)$ for any $t\in[0,T]$. If $\beta<2$, then $m^\mu\to m$ in $C([0,T];L^1(\Omega))$. Finally, (3.6) holds for any $\varphi\in C_c^\infty([0,T)\times\Omega)$.*

Proof. The compactness of m^μ, u^μ and Du^μ are established exactly as in Proposition 4.5. Then we obtain (7.7) by using (7.6) and Fatou's Lemma in the set $\{m>0\}$. We obtain (7.8) as an easy consequence of (7.4). If Du did not vanish a.e. in $\{m=0\}$, there would exist $\rho>0$ and a measurable subset E of $\{m=0\}$ in which $|Du|>\rho$. Then, by Fatou's lemma, $\int_E (m^\mu+\mu)^{-\alpha}|Du^\mu|^\beta dxdt$ would tend to $+\infty$, in contradiction with (7.6). Therefore $Du=0$ a.e. in $\{m=0\}$.

To pass to the limit in $m^\mu H_p(t,x,\mu+m^\mu,Du^\mu)$, we first observe that this sequence is equi-integrable in $L^1(Q_T)$ as a consequence of Remark 7.2. Let us now consider the case when $\alpha<1$: from (2.3) with $\mu=0$, $(t,x,m,p)\mapsto$

$mH_p(t,x,m,p)$ is well defined in $\overline{Q_T}\times[0,+\infty)\times\mathbb{R}^N$, and $m^\mu H_p(t,x,\mu+m^\mu,Du^\mu)\to mH_p(t,x,m,Du)$ almost everywhere. Then by Vitali's theorem, $m^\mu H_p(t,x,\mu+m^\mu,Du^\mu)\to mH_p(t,x,m,Du)$ in $L^1(Q_T)$.

If $\beta<2$ and $1\le\alpha\le\frac{4(\beta-1)}{\beta}$ or $\beta=2$ and $1\le\alpha<2$, then naming $\xi^\mu=m^\mu H_p(t,x,\mu+m^\mu,Du^\mu)$ for brevity, we proceed through the following steps:

(1) By Remark 7.2, we can extract a subsequence, not relabeled, such that $\xi^\mu\rightharpoonup\xi$ in $L^{1+\epsilon}(Q_T)$ weak.
(2) For any bounded and smooth function ϕ, Lebesgue theorem implies that $\phi(m^\mu)\to\phi(m)$ in $L^p(Q_T)$ for any $p\ge1$. Therefore, $\xi^\mu\phi(m^\mu)\rightharpoonup\xi\phi(m)$ in $L^1(Q_T)$ weak.
(3) If ϕ is also supported in $[\delta,+\infty)$ for some positive δ, then using the almost everywhere convergence and Vitali's theorem, $\xi^\mu\phi(m^\mu)\to mH_p(t,x,m,Du)\phi(m)$ in $L^1(Q_T)$. This implies that $(\xi-mH_p(t,x,m,Du))\phi(m)=0$, and that ξ coincides with $mH_p(t,x,m,Du)$ at almost every (t,x) such that $m(t,x)>0$.
(4) We notice that, by definition of ξ^μ and assumption (2.3),

$$\int_0^T\int_\Omega|\xi^\mu\phi(m^\mu)|dxdt\le c_2\int_0^T\int_\Omega m^\mu|\phi(m^\mu)|dxdt$$
$$+c_2\left(\int_0^T\int_\Omega(m^\mu)^{\beta-\alpha}|\phi(m^\mu)|dxdt\right)^{\frac{1}{\beta}}\left(\int_0^T\int_\Omega\frac{|Du^\mu|^\beta}{(\mu+m^\mu)^\alpha}|\phi(m^\mu)|dxdt\right)^{\frac{1}{\beta'}}$$

and (7.6) implies that

$$\int_0^T\int_\Omega|\xi^\mu\phi(m^\mu)|dxdt\le c_2\int_0^T\int_\Omega m^\mu|\phi(m^\mu)|dxdt+C\left(\int_0^T\int_\Omega(m^\mu)^{\beta-\alpha}|\phi(m^\mu)|dxdt\right)^{\frac{1}{\beta}}.$$

Then using the weak convergence of $\xi^\mu\phi(m^\mu)$ in $L^1(Q_T)$ and the fact that $0<\beta-\alpha\le1$, we can deduce

$$\int_0^T\int_\Omega|\xi\phi(m)|dxdt\le\liminf_{\mu\to0}\left(c_2\int_0^T\int_\Omega m^\mu|\phi(m^\mu)|dxdt+C\left(\int_0^T\int_\Omega(m^\mu)^{\beta-\alpha}|\phi(m^\mu)|dxdt\right)^{\frac{1}{\beta}}\right)$$
$$=c_2\int_0^T\int_\Omega m|\phi(m)|dxdt+C\left(\int_0^T\int_\Omega m^{\beta-\alpha}|\phi(m)|dxdt\right)^{\frac{1}{\beta}}.$$

Taking now $\phi(m)=\exp(-Km)$, and letting K tend to $+\infty$ yields that $\int_{\{m=0\}}|\xi|dxdt=0$, and that $\xi=0$ a.e. in $\{m=0\}$. Notice that we used $\alpha<\beta$ in this step, in particular if $\beta=2$ we needed the restriction $\alpha<2$.
(5) Collecting the above results, we obtain that $\xi=mH_p(t,x,m,Du)\mathbb{1}_{\{m>0\}}$.

Finally, the weak convergence of $m^\mu(t)$ in $L^1(Q_T)$ can be justified as in Proposition 4.5 (as well as the strong $C^0([0,T];L^1(\Omega))$ convergence if $\beta<2$), and obtaining equation (3.6) is a consequence of the previous points. □

We conclude the existence part by showing that u satisfies (3.5) and the energy equality holds true.

Proposition 7.4. *Assume that $\beta<2$ and $0<\alpha\le\frac{4(\beta-1)}{\beta}$ or $\beta=2$ and $0<\alpha<2$. Let (u,m) be given by Lemma 7.3. Then inequality (3.5) holds true for every nonnegative $\varphi\in C_c^\infty((0,T]\times\Omega)$.*

Moreover, (u,m) satisfies the energy identity (3.7).

Proof. From (3.1) and (3.3), we deduce that for every nonnegative $\varphi \in C_c^\infty((0,T]\times\Omega)$,

$$\int_0^T\int_\Omega u^\mu \varphi_t\,dxdt - \nu\int_0^T\int_\Omega u^\mu\,\Delta\varphi\,dxdt + \int_0^T\int_\Omega H(t,x,\mu+m^\mu,Du^\mu)\varphi\mathbb{1}_{\{m>0\}}\,dxdt \le \int_0^T\int_\Omega F(t,x,m^\mu)\varphi\,dxdt + \int_\Omega G(x,m^\mu(T))\varphi(T)\,dx. \tag{7.9}$$

By using the a.e. convergence of $H(t,x,\mu+m^\mu,Du^\mu)\mathbb{1}_{\{m>0\}}$ to $H(t,x,m,Du)\mathbb{1}_{\{m>0\}}$ and Fatou's lemma,

$$\int_0^T\int_\Omega H(t,x,m,Du)\varphi(t,x)\mathbb{1}_{\{m>0\}}dxdt \le \liminf_{\mu\to 0}\int_0^T\int_\Omega H(t,x,\mu+m^\mu,Du^\mu)\mathbb{1}_{\{m>0\}}\varphi(t,x)dxdt.$$

By properties of Young measures, and since $G(x,m^\mu(T))$ is equi-integrable,

$$\lim_{\mu\to 0}\int_\Omega G(x,m^\mu(T))\varphi(T)\,dx = \int_\Omega\int_{\mathbb{R}} G(x,\lambda)d\nu_x(\lambda)\varphi(T)\,dx\,.$$

We easily pass to the limit in the other terms of (7.9) since u^μ and $F(t,x,m^\mu)$ converge in $L^1(\Omega)$. So we end up with

$$\int_0^T\int_\Omega u\,\varphi_t\,dxdt - \nu\int_0^T\int_\Omega u\,\Delta\varphi\,dxdt + \int_0^T\int_\Omega H(t,x,m,Du)\varphi\mathbb{1}_{\{m>0\}}\,dxdt \le \int_0^T\int_\Omega F(t,x,m)\varphi\,dxdt + \int_\Omega\int_{\mathbb{R}} G(x,\lambda)d\nu_x(\lambda)\varphi(T)\,dx\,. \tag{7.10}$$

Thanks to (7.10), the conclusion of Lemma 4.7 is also true. Therefore,

$$\lim_{\mu\to 0}\int_\Omega u^\mu(0)m_0\,dx \le \langle u(0), m_0\rangle\,.$$

Starting now from the energy identity for the solution of (1.3)–(1.5) with $\mu>0$:

$$\int_0^T\int_\Omega m^\mu\left(H_p(t,x,\mu+m^\mu,Du^\mu)\cdot Du^\mu - H(t,x,\mu+m^\mu,Du^\mu)\right)dxdt$$
$$+\int_\Omega m^\mu(T)G(x,m^\mu(T))dx + \int_0^T\int_\Omega m^\mu F(t,x,m^\mu)\,dxdt \quad = \int_\Omega u^\mu(0)m_0\,dx$$

using Fatou's lemma thanks to (2.6) and (2.11), we obtain

$$\limsup_{\mu\to 0}\int_\Omega G(x,m^\mu(T))m^\mu(T)dx \le \langle u(0),m_0\rangle - \int_0^T\int_\Omega mF(t,x,m)\,dxdt$$
$$-\int_0^T\int_\Omega \mathbb{1}_{\{m>0\}}m\left(H_p(t,x,m,Du)\cdot Du - H(t,x,m,Du)\right)dxdt.$$

Reasoning exactly as in the previous section, from the estimate (7.4) and assumption (2.10) we obtain both $G(x,m(T))m(T)\in L^1(\Omega)$ and $\int_{\mathbb{R}} G(x,\lambda)\lambda\,d\nu_x(\lambda)\in L^1(\Omega)$. Therefore, we are allowed to use (5.13) in Lemma 5.3 with $\tilde m = m$ and $\tilde u = u$, and we get

$$\limsup_{\mu\to 0}\int_\Omega G(x,m^\mu(T))m^\mu(T)dx \le \langle u(0),m_0\rangle - \int_0^T\int_\Omega mF(t,x,m)\,dxdt$$
$$-\int_0^T\int_\Omega \mathbb{1}_{\{m>0\}} m\left(H_p(t,x,m,Du)\cdot Du - H(t,x,m,Du)\right)dxdt$$
$$\le \int_\Omega\int_{\mathbb{R}} G(x,\lambda)d\nu_x(\lambda)\,m(T)dx.$$

Now we proceed as in Lemma 6.1 using the monotonicity of G and we conclude that $G(x,m^\mu(T))\to G(x,m(T))$ in $L^1(\Omega)$ and the energy identity holds true. Coming back to (7.10), now we know that $\int_{\mathbb{R}} G(x,\lambda)d\nu_x(\lambda) = G(x,m(T))$ and so (3.5) is proven. □

We conclude with the uniqueness of the weak solution. To this purpose, since u is now only a subsolution, we will need a stronger version of the energy equality for the uniqueness argument to succeed.

Lemma 7.5. *Let (u,m) be a weak solution of (1.3)–(1.5). Then $um\in L^1(Q_T)$ and*

$$\begin{aligned}\int_\Omega m(t)\,u(t)\,dx &= \int_\Omega G(x,m(T))\,m(T)\,dx + \int_t^T\int_\Omega F(s,x,m)m\,dxds\\ &+\int_t^T\int_\Omega \left[m\,H_p(s,x,m,Du)\cdot Du - m\,H(s,x,m,Du)\right]\mathbb{1}_{\{m>0\}}dxds\end{aligned}\tag{7.11}$$

for almost every $t\in(0,T)$.

Proof. Let us set $u_k := (u-k)^+$. We first observe that, whenever u is a subsolution, then u_k is also a subsolution of a similar problem, namely

$$\int_0^T\int_\Omega u_k\,\varphi_t\,dxdt - \nu\int_0^T\int_\Omega u_k\,\Delta\varphi\,dxdt + \int_0^T\int_\Omega H(t,x,m,Du)\mathbb{1}_{\{m>0\}}\,\mathbb{1}_{\{u>k\}}\,\varphi\,dxdt$$
$$\le \int_0^T\int_\Omega F(t,x,m)\mathbb{1}_{\{u>k\}}\,\varphi\,dxdt + \int_\Omega (G(x,m(T))-k)^+\,\varphi(T)\,dx$$

Henceforth, we can proceed exactly as in Lemma 5.3 with $\tilde m = m$ and we obtain (recall that $m_\delta(t,x) = (m(t)\star\rho_\delta)(x)$)

$$\begin{aligned}&\int_t^T\int_\Omega [\mathbb{1}_{\{m>0\}}H(s,x,m,Du)m_\delta - Du\cdot w_\delta]\,\mathbb{1}_{\{u>k\}}\,dxds + \int_\Omega m_\delta(t)\,u_k(t)\,dx\\ &\le \int_t^T\int_\Omega F(s,x,m)m_\delta\,\mathbb{1}_{\{u>k\}}\,dxds + \int_\Omega (G(x,m(T))-k)^+\,m_\delta(T)\,dx\end{aligned}\tag{7.12}$$

which holds for almost every t since $m_\delta\in L^\infty(Q_T)$ (it is indeed even continuous) and $u_k\in L^1(Q_T)$ and is bounded below. Starting from (7.12) and following the same steps as in Lemma 5.3, we obtain that

$$\int_\Omega m_\delta(t)\,u_k(t)\,dx \le \int_t^T\int_\Omega \phi_\delta\,\mathbb{1}_{\{u>k\}}\,dxds + \int_\Omega \chi_\delta\,\mathbb{1}_{\{G(x,m(T))>k\}}\,dx$$

for some sequences ϕ_δ, χ_δ which are convergent, and therefore equi-integrable, in $L^1(Q_T)$ and in $L^1(\Omega)$ respectively. This implies that

$$\sup_\delta \int_\Omega m_\delta(t)\, u_k(t)\, dx \overset{k\to\infty}{\to} 0\,.$$

Since $u - u_k$ is bounded, we know that $m_\delta(u-u_k)$ strongly converges in $L^1(Q_T)$ as $\delta \to 0$. Therefore, a standard argument allows us to conclude that

$$m_\delta\, u \to mu \qquad \text{in } L^1(Q_T). \tag{7.13}$$

This means that we can repeat the argument of Lemma 5.3 integrating either in $(0,t)$ or in (t,T) obtaining both

$$\int_0^t\int_\Omega [H(s,x,m,Du)m_\delta - Du\cdot w_\delta]\mathbb{1}_{\{m>0\}}\, dxds + \langle m_\delta(0)\, u(0)\rangle\, dx$$
$$\le \int_0^t\int_\Omega F(s,x,m)m_\delta\, dxds + \int_\Omega u(t)\, m_\delta(t)\, dx$$

and

$$\int_t^T\int_\Omega [H(s,x,m,Du)m_\delta - Du\cdot w_\delta]\mathbb{1}_{\{m>0\}}\, dxds + \int_\Omega m_\delta(t)\, u(t)\, dx$$
$$\le \int_t^T\int_\Omega F(s,x,m)m_\delta\, dxds + \int_\Omega G(x,m(T))\, m_\delta(T)\, dx$$

for almost every $t \in (0,T)$. Passing to the limit as $\delta \to 0$ is now allowed thanks to (7.13), besides the arguments already given in Lemma 5.3. Finally, as $\delta \to 0$ we obtain both

$$\langle m_0\, , u(0)\rangle\, dx \le \int_\Omega u(t)\, m(t)\, dx + \int_0^t\int_\Omega F(s,x,m)m\, dxds$$
$$+ \int_0^t\int_\Omega \left[m\, H_p(s,x,m,Du)\cdot Du - m\, H(s,x,m,Du)\right] \mathbb{1}_{\{m>0\}} dxds$$

and

$$\int_\Omega m(t)\, u(t)\, dx \le \int_\Omega G(x,m(T))\, m(T)\, dx + \int_t^T\int_\Omega F(s,x,m)m\, dxds$$
$$+ \int_t^T\int_\Omega \left[m\, H_p(s,x,m,Du)\cdot Du - m\, H(s,x,m,Du)\right] \mathbb{1}_{\{m>0\}} dxds\,.$$

On account of the energy equality (3.7) which holds by definition of weak solution, we conclude that (7.11) holds true, for almost every $t \in (0,T)$. □

Lemma 7.6. *Under all the assumptions made in Theorem 3.8, there is a unique weak solution of (1.3)–(1.5).*

Proof. Let (u,m) and $(\tilde u, \tilde m)$ be two weak solutions of (1.3)–(1.5). From Lemma 5.3 and Proposition 7.4, we know that

$$\int_\Omega m_0\, u(0)\, dx \le \int_\Omega G(x, m(T))\, \tilde m(T)\, dx + \int_0^T\int_\Omega F(t,x,m)\tilde m\, dxdt$$
$$+ \int_0^T\int_\Omega \left[\tilde m\, H_p(t,x,\tilde m, D\tilde u)\cdot Du - \tilde m\, H(t,x,m,Du)\right] \mathbb{1}_{\{m>0\,,\, \tilde m>0\}} dxdt,$$

$$\int_\Omega m_0\, \tilde u(0)\, dx \le \int_\Omega G(x, \tilde m(T))\, m(T)\, dx + \int_0^T\int_\Omega F(t,x,\tilde m) m\, dxdt$$
$$+ \int_0^T\int_\Omega \left[m\, H_p(t,x,m, Du)\cdot D\tilde u - m\, H(t,x,\tilde m,D\tilde u)\right] \mathbb{1}_{\{m>0\,,\, \tilde m>0\}} dxdt,$$

$$\int_\Omega m_0\, u(0)\, dx = \int_\Omega G(x, m(T))\, m(T)\, dx + \int_0^T\int_\Omega F(t,x,m) m\, dxdt$$
$$+ \int_0^T\int_\Omega \left[m\, H_p(t,x,m, Du)\cdot Du - m\, H(t,x,m,Du)\right] \mathbb{1}_{\{m>0\}} dxdt,$$

$$\int_\Omega m_0\, \tilde u(0)\, dx = \int_\Omega G(x, \tilde m(T))\, \tilde m(T)\, dx + \int_0^T\int_\Omega F(t,x,\tilde m)\tilde m\, dxdt$$
$$+ \int_0^T\int_\Omega \left[\tilde m\, H_p(t,x,\tilde m, D\tilde u)\cdot D\tilde u - m\, H(t,x,\tilde m,D\tilde u)\right] \mathbb{1}_{\{\tilde m>0\}} dxdt,$$

and therefore

$$0 \ge \int_\Omega (G(x,m(T)) - G(x,\tilde m(T)))\,(m(T)-\tilde m(T))\, dx + \int_0^T\int_\Omega (F(t,x,m)-F(t,x,\tilde m))(m-\tilde m)\, dxdt$$
$$+ \int_0^T\int_\Omega \mathbb{1}_{\{m>0\,,\,\tilde m>0\}}\tilde m \left[H(t,x,m,Du) - H(t,x,\tilde m, D\tilde u) - H_p(t,x,\tilde m, D\tilde u)\cdot (Du - D\tilde u)\right] dxdt$$
$$+ \int_0^T\int_\Omega \mathbb{1}_{\{m>0\,,\,\tilde m>0\}} m \left[H(t,x,\tilde m,D\tilde u) - H(t,x,m, Du) - H_p(t,x,m, Du)\cdot (D\tilde u - Du)\right] dxdt$$
$$+ \int_0^T\int_\Omega \mathbb{1}_{\{m>0\,,\,\tilde m=0\}} m \left[-H(t,x,m,Du) + H_p(t,x,m,Du)\cdot Du\right] dxdt$$
$$+ \int_0^T\int_\Omega \mathbb{1}_{\{\tilde m>0\,,\,m=0\}} \tilde m \left[-H(t,x,\tilde m,D\tilde u) + H_p(t,x,\tilde m,D\tilde u)\cdot D\tilde u\right] dxdt. \tag{7.14}$$

Note that from (2.1) and the convexity of $H(t,x,m,p)$ w.r.t p,

$$\tilde m > 0 \quad \Rightarrow \quad -H(t,x,\tilde m, D\tilde u) + H_p(t,x,\tilde m, D\tilde u)\cdot D\tilde u \ge -H(t,x,\tilde m,0) \ge 0, \tag{7.15}$$
$$m > 0 \quad \Rightarrow \quad -H(t,x,m, Du) + H_p(t,x,m, Du)\cdot Du \ge -H(t,x,m,0) \ge 0. \tag{7.16}$$

Therefore all integrals in (7.14) involve nonnegative functions, which then must be zero almost everywhere. We deduce that

$$\int_\Omega (G(m(T)) - G(\tilde m(T)))\,(m(T) - \tilde m(T))\,dx = 0,$$

$$\int_0^T \int_\Omega \mathbb{1}_{\{m>0\,,\,\tilde m>0\}} E(t,x,m,Du,\tilde m,D\tilde u)\,dxdt = 0,$$

$$\int_0^T \int_\Omega \mathbb{1}_{\{m=0,\tilde m>0\}} \tilde m(-H(t,x,\tilde m,D\tilde u) + H_p(t,x,\tilde m,D\tilde u)\cdot D\tilde u + F(t,x,\tilde m) - F(t,x,0))dxdt = 0,$$

$$\int_0^T \int_\Omega \mathbb{1}_{\{\tilde m=0,m>0\}} m(-H(t,x,m,Du) + H_p(t,x,m,Du)\cdot Du + F(t,x,m) - F(t,x,0)) = 0, \tag{7.17}$$

where E has been defined in the proof of Lemma 6.5. From the last two equations in (7.17), (7.15)–(7.16) and (3.8), we deduce that the sets $\{\tilde m = 0, m > 0\}$ and $\{m = 0, \tilde m > 0\}$ have measure 0. From the second equation in (7.17) and the assumptions made for uniqueness (which apply to the function E), as in Lemma 6.5 we deduce that $m = \tilde m$ and $Du = D\tilde u$ a.e. in $\{m > 0, \tilde m > 0\}$. Combining all the previous observations, $m = \tilde m$ almost everywhere in Q_T. Using also that $Du = 0$ a.e. in $\{m = 0\}$ and $D\tilde u = 0$ a.e. in $\{\tilde m = 0\}$, we deduce that $Du = D\tilde u$ a.e. in Q_T. Therefore, $u - \tilde u = \int_\Omega (u(t) - \tilde u(t))dx$. By applying Lemma 7.5 to u and $\tilde u$, subtracting the two equalities on account of the above informations we finally get

$$\int_\Omega m(t)(u - \tilde u)(t)\,dx = 0\,.$$

Since $u - \tilde u$ is only time-dependent, this implies $u = \tilde u$ a.e. in Q_T. □

Conflict of interest statement

There is no conflict of interest.

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