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# Resolvent and spectral measure on non-trapping asymptotically hyperbolic manifolds III: Global-in-time Strichartz estimates without loss

# Résolvante et mesure spectrale sur des variétés non-captives asymptotiquement hyperboliques III: Estimations de Strichartz sans perte et globales en temps

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**Abstract**

In the present paper, we investigate global-in-time Strichartz estimates without loss on non-trapping asymptotically hyperbolic manifolds. Due to the hyperbolic nature of such manifolds, the set of admissible pairs for Strichartz estimates is much larger than usual. These results generalize the works on hyperbolic space due to Anker–Pierfelice and Ionescu–Staffilani. However, our approach is to employ the spectral measure estimates, obtained in the author's joint work with Hassell, to establish the dispersive estimates for truncated/microlocalized Schrödinger propagators as well as the corresponding energy estimates. Compared with hyperbolic space, the crucial point here is to cope with the conjugate points on the manifold. Additionally, these Strichartz estimates are applied to the $L^2$ well-posedness and $L^2$ scattering for nonlinear Schrödinger equations with power-like nonlinearity and small Cauchy data.


**Résumé**

Dans cet article, nous examinons les estimations de Strichartz sans perte et globales en temps, définies sur des variétés non-captives asymptotiquement hyperboliques. De par la nature hyperbolique de ces variétés, l'ensemble des paires admissibles pour les estimations de Strichartz est beaucoup plus grand que d'ordinaire. Ces résultats généralisent les travaux menés par Anker–Pierfelice et Ionescu–Staffilani sur les espaces hyperboliques. Toutefois, notre approche utilise ici les estimations de mesures

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spectrales obtenues par l'auteur en collaboration avec Hassell afin d'établir les estimations de dispersion pour des propagateurs de Schrödinger tronqués ou micro-localisés ainsi que les estimations d'énergie correspondantes. À la différence des espaces hyperboliques, l'élément crucial est ici de gérer les points conjugués de la variété. Enfin, ces estimations de Strichartz sont appliquées au caractère bien posé dans $L^2$ et à la diffusion $L^2$ pour les équations de Schrödinger avec des non-linearités de type puissance et des données initiales petites.


*Keywords:* Asymptotically hyperbolic manifolds; Spectral measure; Dispersive estimates; Strichartz estimates

## 1. Introduction

This paper, following the author's joint works [11] and [12] with Andrew Hassell, is the last in a series of papers concerning the analysis of the resolvent family and spectral measure for the Laplacian on non-trapping asymptotically hyperbolic manifolds. The present paper is devoted to the application of the spectral measure to Schrödinger equations.

We investigate the Cauchy problem of the nonlinear Schrödinger equation

$$\begin{cases} i\frac{\partial}{\partial t}u(t,z) + \Delta u(t,z) = F(u(t,z)) \\ u(0,z) = f(z) \end{cases}, \tag{1.1}$$

on an $n+1$-dimensional asymptotically hyperbolic manifold $X$ (see Section 2 for the definition). Here the nonlinear term is power-like, i.e. $F$ satisfies

$$|F(u)| \le C|u|^\gamma \quad \text{and} \quad |F(u) - F(v)| \le C(|u|^{\gamma-1} + |v|^{\gamma-1})|u-v|,$$

for $1 < \gamma \le 1 + 4/(n+1)$. This sort of nonlinear dispersive equation cuts an important figure in mathematics and physics. A fundamental problem is the well-posedness of the equation. We say the equation (1.1) is globally well-posed in $L^2$ if for any subset $B$ of $L^2$ there exists a subspace $A$, continuously embedded into $C(\mathbb{R}_+; L^2(X))$ such that (1.1) has a unique solution in $A$ for any initial data $f$ and the map from $B$ to $A$ is continuous. Another interesting question is that how the solution of (1.1) behaves as time goes to infinity. One can understand that in terms of $L^2$ scattering, by which we mean the solution of (1.1) converges to the solution of the corresponding homogeneous linear equation in $L^2(X)$ sense as time goes to $\pm\infty$. More precisely, for any solution $u$ of (1.1) there exists scattering data $u_\pm$ such that

$$\|u - u_\pm\|_{L^2_z} \longrightarrow 0, \quad \text{as } t \to \pm\infty.$$

By the classical theories of well-posedness and scattering for Schrödinger equations (see for example [10,31]), these problems usually reduce to the so-called 'Strichartz estimates' for the linear Schrödinger equation

$$\begin{cases} i\frac{\partial}{\partial t}u + \Delta u = F(t,z) \\ u(0,z) = f(z) \end{cases}. \tag{1.2}$$

It has been deeply studied on Euclidean space (see [23,15,24]) as well as on manifolds (see [5–8]). We shall prove global-in-time Strichartz estimates without loss in asymptotically hyperbolic geometry.

**Theorem 1** *(Strichartz estimates). Suppose $(X, g)$ is an $(n+1)$-dimensional non-trapping asymptotically hyperbolic manifold with no resonance at the bottom of spectrum. For any admissible pairs $(q, r)$ and $(\tilde{q}', \tilde{r}')$ satisfying*

$$\frac{2}{q} + \frac{n+1}{r} \ge \frac{n+1}{2}, \quad q \ge 2, \quad r > 2, \quad (q,r) \ne (2, \infty), \tag{1.3}$$

*we have the inhomogeneous Strichartz estimates*

$$\|u\|_{L^q_t L^r_z(\mathbb{R}\times X)} \le C\big(\|f\|_{L^2(X)} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_z(\mathbb{R}\times X)}\big), \tag{1.4}$$

*provided $f$ and $F$ are both orthogonal to the eigenfunctions of $\Delta$.*

This result readily applies to $L^2$ well-posedness and $L^2$ scattering for the nonlinear Schrödinger equation (1.1). We obtain

**Theorem 2** *($L^2$ well-posedness and $L^2$ scattering). Suppose $(X, g)$ is a manifold as in Theorem 1 and there are no eigenvalues of $\Delta$. Given $\gamma \in (1, 1 + 4/(n+1)]$ and a small Cauchy data $f \in L^2(X)$, the nonlinear Schrödinger equation (1.1) is globally well-posed in $L^2(X)$, whilst for any solution $u(t, z)$ there exists $u_\pm \in L^2(X)$ such that*

$$\|u(t) - e^{it\Delta}u_\pm\|_{L^2(X)} \longrightarrow 0, \quad \text{as } t \to \pm\infty.$$

Assuming Theorem 1, the proof of Theorem 2 is standard. The well-posedness is given by a contraction mapping theorem method with the global-in-time Strichartz estimates, whilst the scattering part is simply given by a quick application of Cauchy criterion. We omit the proof and refer the reader to [1], because their argument works verbatim in the asymptotically hyperbolic settings.

Return to the Strichartz estimates. By the classical approach formulated by Kato [23], Ginibre and Velo [15], Keel and Tao [24] etc., it is sufficient to prove energy estimates

$$\|e^{it\Delta} f\|_{L^2} \le \|f\|_{L^2}$$

and dispersive estimates

$$\|e^{i(t-s)\Delta} f\|_\infty \le |t-s|^{-(n+1)/2}\|f\|_{L^1} \tag{1.5}$$

for the Schrödinger propagator on Euclidean space. However, unlike on Euclidean space, dispersive estimates in the above form is too strong to hold on asymptotically hyperbolic manifolds.

First of all, unlike (1.5), we don't have dispersive estimates uniformly in time on hyperbolic space $\mathbb{H}^{n+1}$ with $n > 2$. Actually, Anker and Pierfelice [1] and independently Ionescu and Staffilani [22] proved the following dispersive estimates

$$\left|\operatorname{Ker} e^{it\Delta_{\mathbb{H}^{n+1}}}\right| \le C \begin{cases} t^{-3/2}\big(1 + d(z, z')\big)e^{-nd(z,z')/2} & \text{if } t \ge 1 + d(z, z') \\ t^{-(n+1)/2}\big(1 + d(z, z')\big)^{n/2}e^{-nd(z,z')/2} & \text{if } t \le 1 + d(z, z') \end{cases} \tag{1.6}$$

on real hyperbolic space $\mathbb{H}^{n+1}$. Similar results on convex co-compact hyperbolic manifolds were proved by Burq, Guillarmou and Hassell [9]. The spectral theorem gives

$$e^{it\Delta} = e^{itn^2/4}\int_0^\infty e^{it\boldsymbol{\lambda}^2}\, dE_{\sqrt{(\Delta - n^2/4)_+}}(\boldsymbol{\lambda}, z, z').$$

So we can explain (1.6) via the spectral measure $dE_{\sqrt{(\Delta-n^2/4)_+}}$. On the one hand, the nonuniformity of (1.6) in time actually results from the discrepancy of powers in the spectral measure estimates on $\mathbb{H}^{n+1}$. We shall see in Section 2, Section 5 and 6 that the quicker growth for large $\boldsymbol{\lambda}$ and the slower decay for small $\boldsymbol{\lambda}$ creates the discrepancy of the powers of $t$. Consequently, the long time dispersive estimates on $\mathbb{H}^{n+1}$ is at a lower speed than on Euclidean space, though the short time estimates are the same with Euclidean case. On the other hand, we also get something better. Near the spatial infinity, the spectral measure $dE_{\sqrt{(\Delta-n^2/4)_+}}$ as well as the Schrödinger propagator $e^{it\Delta}$ gains an exponential decay factor. We can crudely interpret that as follows. Near the spatial infinity the conformal metric creates an exponentially growing volume. Since the spectral measure is globally $L^2$ integrable, it must decay exponentially to cancel the exponential growth volume as $d(z, z')$ goes to infinity. Not only does this long distance exponential decay compensate the low speed for long times but it also gives better Strichartz estimates. One may note a distinctive phenomenon that the admissible set (1.3) is much wider than the set of sharp Schrödinger admissible pairs on Euclidean space, which satisfy

$$\frac{2}{q} + \frac{n+1}{r} = \frac{n+1}{2}, \quad q, r \ge 2, \quad (q, r) \ne (2, \infty).$$

Banica, Carles and Staffilani [3] first observed this while studying the radial solution of the Schrödinger equations on $\mathbb{H}^{n+1}$. Inspired by that, Anker and Pierfelice [1] and independently Ionescu and Staffilani [22] proved the Strichartz estimates on $\mathbb{H}^{n+1}$ with such admissibility condition.

More generally, if we study an asymptotically hyperbolic manifold with conjugate points, what kind of dispersive estimate could we get instead? The Schwartz kernel of the spectral measure at high energy is a Lagrangian distribution microlocally supported on the geodesic flow-out. Then the difficulty is that there will be no global expression for the geodesic distance function. Because of that, neither (1.5) nor (1.6) will hold. Alternatively, one can microlocalize the spectral measure around the diagonal with a pair of pseudodifferential operators $(Q_k, Q_k^*)$. Consequently we can get some near-diagonal estimates for the spectral measure, which give some sort of dispersive estimate for corresponding microlocalized Schrödinger propagators. It is sufficient to establish the Strichartz estimates. Guillarmou, Hassell and Sikora [19], Hassell and Zhang [21], applied this technique to the spectral measure and the Schrödinger propagator on asymptotically conic (Euclidean) manifolds.

Due to above distinctive geometric and spectral properties on asymptotically hyperbolic manifolds, we will integrate some existing techniques to prove Theorem 1. First of all, we primarily follow the standard argument due to Kato, Ginibre–Velo and Keel–Tao. However, as mentioned before, their method doesn't exploit the distinctive phenomena of hyperbolic type spaces, including the spatial decay at infinity and the low speed at long times. Therefore, borrowing the trick of Anker–Pierfelice and Ionescu–Staffilani, we split the time-space norm of the solution in temporal variables as well as the Schrödinger propagator in spectral parameter. Also inspired by the microlocalization argument of Guillarmou–Hassell–Sikora and Hassell–Zhang, we establish the microlocalized dispersive estimates in Proposition 6 and in Proposition 9 to cope with the conjugate points.

The geometric microlocal technique used for the spectral measure does require the non-trapping condition and conformal compactness on the space. Bouclet [4] investigates the local-in-time homogeneous Strichartz estimates without loss on more general asymptotically hyperbolic manifolds without taking these advantages. The author constructed a parametrix for Schrödinger propagators. However, the issue here is that the error may be difficult to control as time goes to infinity. For the consideration of long time behaviour, one needs an exact spectral measure or propagator (a function of spectral measure) in these estimates. In the joint work of Hassell and the present author [12], we studied the spectral measure estimates on asymptotically hyperbolic manifolds, which enables us to study the global-in-time Strichartz estimates.

The paper is organized as follows. First of all, we shall review the asymptotically hyperbolic manifolds and the spectral measure in Section 2. Based on the spectral measure estimates, we introduce the microlocalized/truncated expressions of Schrödinger propagators. We then turn to the proof of $L^2$-energy estimates. In Section 5 and Section 6, we establish the dispersive estimates for microlocalized/truncated propagators. The Strichartz estimates will be proved in the last two sections.

## 2. Spectral measure on asymptotically hyperbolic manifolds

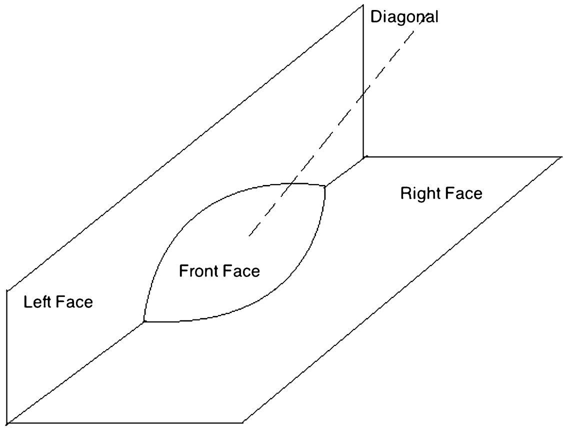
A conformally compact manifold $X$ is an $(n+1)$-dimensional manifold with boundary $\partial X$, compact closure $\bar{X}$ and endowed with a Riemannian metric $g$ which extends smoothly to its closure. One can write

$$g = \frac{dx^2}{x^2} + \frac{h(x, y, dx, dy)}{x^2},$$

where $x$ is a boundary defining function, and $h$ is a metric on the boundary but depending parametrically on $x$. Mazzeo [26] showed $g$ is complete and its sectional curvature approaches $-|dx|^2_{x^2 g}$ as it approaches the boundary. In particular, $g$ is said to be asymptotically hyperbolic if $-|dx|^2_{x^2 g} = -1$.

Let $\Delta$ be the Laplacian, on $(n+1)$-dimensional non-trapping asymptotically hyperbolic manifold $X$. As on $\mathbb{H}^{n+1}$, the continuous spectrum of $\Delta$ is contained in $[n^2/4, \infty)$. Additionally, Mazzeo [27] showed the point spectrum is contained in $(0, n^2/4)$.

Mazzeo and Melrose [28] constructed the resolvent $(\Delta - \boldsymbol{\sigma}(n - \boldsymbol{\sigma}))^{-1}$ on asymptotically hyperbolic manifolds for fixed parameter $\boldsymbol{\sigma}$ and proved it has a meromorphic extension except at points $(n+1)/2 - \mathbb{Z}_+$. The resolvent they construct is a 0-pseudo differential operator plus a smooth function on the 0-blown up double space $X \times_0 X$ (or $X_0^2$ for short), where the space $X \times_0 X$ is obtained by blowing up the boundary of the diagonal $\partial\text{diag} = \{(0, y, 0, y)\} \in X^2$.




We also refer the reader to Guillarmou [17] for more information about the poles $(n+1)/2 - \mathbb{Z}_+$ as well as a nice review of the Mazzeo–Melrose resolvent construction. From here on, we will work on $X_0^2$ instead of $X^2$ for the nice expression of the resolvent and the spectral measure. A very important feature of $X_0^2$ is that the front face is a bundle with fibres similar to hyperbolic space. Therefore hyperbolic space is a good model for asymptotically hyperbolic manifolds. Apart from these, we also get the following useful asymptotic expansion of the geodesic distance function near the boundaries of $X_0^2$

$$d(z,z') = -\log\rho_L - \log\rho_R + b(z,z'),$$

where $\rho_L$ and $\rho_R$ are boundary defining functions of the left and the right faces respectively, $b$ is a uniformly bounded function on $X_0^2$. See [12, Proposition 10]. In particular, $b(z,z')$ is smooth on $X_0^2$ in the case of asymptotically hyperbolic manifolds of Cartan–Hadamard type. This was observed on hyperbolic spaces and proved on asymptotically hyperbolic manifolds by Melrose, Sá Barreto and Vasy [29]. On general asymptotically hyperbolic manifolds, arising conjugate points ruin the smoothness of $b$ but we still get the boundedness of $b$. Consequently, we have the asymptotic

$$e^{-d(z,z')} \approx \rho_L\rho_R, \quad \text{if } \rho_L\rho_R \text{ is small.} \tag{2.1}$$

Additionally, Melrose, Sá Barreto and Vasy [29], Wang [32], and Hassell and the present author [11] constructed the semiclassical resolvent at high energy (near the infinity of the spectrum). Specifically, the high energy resolvent defined on $X_0^2$ is a 0-pseudo differential operator plus a Fourier integral operator microlocally supported on the union of the diagonal conormal bundle and its bicharacteristic flow-out. To avoid unnecessary technical details, we wouldn't repeat the bulky theories about 0-calculus, blow-up, flow-out, Lagrangian distribution, intersecting Lagrangian etc., but refer the readers to [28,29,11].

Based on the results of the resolvent, Hassell and the author [12] studied the spectral measure on asymptotically hyperbolic manifolds, via Stone's formula

$$2\pi i\, dE_{\mathcal{L}}(\boldsymbol{\lambda}) = R_{\mathcal{L}}(\boldsymbol{\lambda}+i0) - R_{\mathcal{L}}(\boldsymbol{\lambda}-i0),$$

provided $\boldsymbol{\lambda}$ is in the continuous spectrum of a self-adjoint operator $\mathcal{L}$.[1] Since the spectral measure is defined on the continuous spectrum $(n^2/4, \infty)$, we are in particular concerned about the asymptotic behaviour around two endpoints $n^2/4$ (low energy) and $\infty$ (high energy) respectively. Because of the absence of embedded eigenvalues, the intermediate values can be estimated in either way.

On the one hand, the spectral measure $dE_P(\boldsymbol{\lambda})$[2] with $P = \sqrt{(\Delta - n^2/4)_+}$ for small $\boldsymbol{\lambda}$ has a similar structure to the resolvent near the bottom of the spectrum. As on $\mathbb{H}^{n+1}$, it is convenient to assume the smoothness of the resolvent at the bottom of the spectrum to gain the asymptotic of spectral measure. We say there is no resonance at the bottom

1 We use Greek letters $\lambda, \mu, \zeta$ to denote the phase variables on cotangent bundle, respectively bold Greek letters $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\zeta}$ to denote spectral parameters.

2 Here we use another spectral parameter $\boldsymbol{\lambda} \in [0,\infty)$ with $\boldsymbol{\sigma} = n/2 \pm i\boldsymbol{\lambda}$.

of the continuous spectrum if the resolvent is analytic at $n^2/4$.[3] With this hypotheses of analyticity at $n^2/4$, we [12] deduce, from the resolvent of Mazzeo and Melrose, that

$$dE_P(\boldsymbol{\lambda})(z,z') = \boldsymbol{\lambda}\Big((\rho_L\rho_R)^{n/2+i\boldsymbol{\lambda}}a(\boldsymbol{\lambda},z,z') - (\rho_L\rho_R)^{n/2-i\boldsymbol{\lambda}}a(-\boldsymbol{\lambda},z,z')\Big) \quad \text{when } \boldsymbol{\lambda} < 1, \tag{2.2}$$

where $a \in C^\infty([0,1]_{\boldsymbol{\lambda}^{-1}} \times X_0^2)$. A quick corollary of this result is that

$$|dE_P(\boldsymbol{\lambda})(z,z')| \le C\boldsymbol{\lambda}^2(1+d(z,z'))e^{-nd(z,z')/2}. \tag{2.3}$$

One may note the spectral measure doesn't vanish as rapidly as its counterparts on some other spaces do at low energy. For example, Guillarmou, Hassell and Sikora [19], Hassell and Zhang [21] showed on $n+1$-dimensional asymptotically Euclidean manifolds there is a pseudodifferential operator partition of unity

$$I = \sum_{i=1}^{N} Q_i$$

such that the kernel of the microlocalized spectral measure reads

$$Q_i dE_P Q_i^*(\boldsymbol{\lambda},z,z') = \boldsymbol{\lambda}^n e^{i\boldsymbol{\lambda}d(z,z')}a_+(\boldsymbol{\lambda},z,z') + \boldsymbol{\lambda}^n e^{-i\boldsymbol{\lambda}d(z,z')}a_-(\boldsymbol{\lambda},z,z') + b(\boldsymbol{\lambda},z,z'), \tag{2.4}$$

where the derivatives of $a_\pm$ and $b$ obey

$$\left|\frac{d^\alpha}{d\boldsymbol{\lambda}^\alpha}a_\pm(\boldsymbol{\lambda},z,z')\right| \le C\boldsymbol{\lambda}^{-\alpha}\big(1+\boldsymbol{\lambda}d(z,z')\big)^{-n/2},$$
$$\left|\frac{d^\alpha}{d\boldsymbol{\lambda}^\alpha}b(\boldsymbol{\lambda},z,z')\right| \le C\boldsymbol{\lambda}^{-\alpha}\big(1+\boldsymbol{\lambda}d(z,z')\big)^{-N} \quad \text{for any } N \in \mathbb{Z}_+.$$

We remark that one could remove the diagonal microlocalization $(Q_i, Q_i^*)$ in case there are no conjugate points on the manifold; however it is necessary for general settings. Apart from the 3-dimensional space where $n=2$, the spectral measure on asymptotically hyperbolic manifolds is unable to provide such decay for low energies. Nonetheless, the property (2.1) for large distance on asymptotically hyperbolic manifolds compensates the lack of decay with an exponential vanishing at spatial infinity.

On the other hand, the spectral measure for large $\boldsymbol{\lambda}$ shares a microlocal structure with the resolvent at high energies. Suppose we have local coordinates $\{(x, y_1, \dots, y_n)\}$ near $\partial X$ and local coordinates $\{(z_1, \dots, z_{n+1})\}$ away from $\partial X$. The 0-cotangent bundle ${}^0T^*X^\circ$, introduced by Mazzeo and Melrose [28], is a vector bundle with sections

$$\lambda\frac{dx}{x} + \mu_1\frac{dy_1}{x} + \cdots + \mu_n\frac{dy_n}{x} \qquad \text{near } \partial X$$
$$\zeta_1 dz_1 + \cdots + \zeta_{n+1}dz_{n+1} \qquad \text{away from } \partial X.$$

Recall from [11] that the microlocal support (or wavefront set) of the high energy resolvent is the diagonal conormal bundle $N^*\mathrm{diag} \subset {}^0TX_0^2$ and its bicharacteristic flow-out $\Lambda$, which is contained in ${}^0SX^\circ \times {}^0SX^\circ$, where

$${}^0S^*X^\circ = \{|\zeta|^2 = 1 \text{ or } |\lambda|^2 + |\mu|^2 = 1\} \subset {}^0TX^\circ.$$

By Stone's formula, the spectral measure is microlocally supported on $\Lambda$, while the singularity at $N^*\mathrm{diag}$ cancels out by the subtraction between the outgoing resolvent and the incoming resolvent. Therefore the spectral measure is a Fourier integral operator associated with the Lagrangian $\Lambda$. Apart from the boundary behaviour, this Lagrangian structure on asymptotically hyperbolic manifolds is analogous to the asymptotically Euclidean case. So we can gain similar spectral measure estimates at high energies to (2.4).

To state the microlocalized spectral measure estimates explicitly, let us recall the partition of unity on ${}^0T^*X^\circ$ in [12]. First of all, we take $Q_0$ microlocally supported away from the spherical bundle ${}^0S^*X^\circ$, say $\{|\zeta|^2 >$

[3] Intriguingly, it is still unknown that what geometric conditions amounts to the analyticity of the resolvent at the bottom of spectrum. However, there are some sufficiency results. For instance Guillarmou and Qing [20] shows that the largest real scattering pole of $(\Delta - \boldsymbol{\sigma}(n-\boldsymbol{\sigma}))^{-1}$ on an $n+1$-dimensional conformally compact Einstein manifold $(X,g)$ is less than $n/2-1$ if and only if the conformal infinity of $(X,g)$ is of positive Yamabe type, where $n>1$.

$3/2$ or $|\lambda|^2 + |\mu|^2 > 3/2\}$, which contains the wavefront set of the spectral measure. On the other hand, we divide the interval $(-3/2, 3/2)$ into a union of intervals $I_1, \dots, I_{N_1}$ with overlapping interiors, and with diameter $\leq \delta$, which is a sufficiently small number, whilst each $I_i$ intersects only $I_{i-1}$ and $I_{i+1}$. We also take a small strip neighbourhood of the boundary such that the sectional curvature is negative; in the meantime, we divide the 0-cotangent bundle over this strip into a union of small slices $B_1, \dots, B_{N_1}$ such that every $B_i \subset \{\lambda \in I_i\}$. Then we have 0th-order pseudodifferential operators $Q_1, \dots, Q_{N_1}$ supported on them respectively. Next, we divide the remaining region into the union of small balls $B_{N_1+1}, \dots, B_{N_2}$ with diameter $\leq \eta$, which is also sufficiently small, and have $Q_{N_1+1}, \dots, Q_{N_2}$ supported on them respectively. With this partition, we have the estimates for the microlocalized spectral measure.

**Proposition 3** *([12]). One can choose a pseudodifferential operator partition of unity*

$$Id = \sum_{k=0}^{N} Q_k(\boldsymbol{\lambda}),$$

*where $Q_k$ for $k \neq 0$ is supported around the spherical bundle, such that $Q_k$ for any $k$ are uniformly ($L^2$-)bounded over $\boldsymbol{\lambda}$ and the kernel of the microlocalized spectral measure reads*

$$Q_k(\boldsymbol{\lambda}) dE_P(\boldsymbol{\lambda}) Q_k^*(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^n e^{i\boldsymbol{\lambda} d(z,z')} a_+(\boldsymbol{\lambda}) + \boldsymbol{\lambda}^n e^{-i\boldsymbol{\lambda} d(z,z')} a_-(\boldsymbol{\lambda}) + O(\boldsymbol{\lambda}^{-\infty}), \quad \textit{for large } \boldsymbol{\lambda},$$

*where $a_\pm$ are defined on the forward and backward bicharacteristic flow respectively and satisfying*

$$\frac{d^j}{d\boldsymbol{\lambda}^j} a_\pm(\boldsymbol{\lambda}) = \begin{cases} O\Big(\boldsymbol{\lambda}^{-j}\big(1 + \boldsymbol{\lambda} d(z,z')\big)^{-n/2}\Big) & , \quad \textit{if } d(z,z') \textit{ is small} \\ O\Big(\boldsymbol{\lambda}^{-n/2-j} e^{-nd(z,z')/2}\Big) & , \quad \textit{if } d(z,z') \textit{ is large} \end{cases}.$$

*Here we mean by $f = O(\boldsymbol{\lambda}^{-\infty})$ that for any $N \in \mathbb{Z}_+$*

$$f \in \boldsymbol{\lambda}^{-N} C^\infty([0,1]_{\boldsymbol{\lambda}^{-1}} \times X_0^2).$$

This result is actually better than (2.4) for large $\boldsymbol{\lambda}$. Not only does it give the same growth rate in $\boldsymbol{\lambda}$, but there is also a spatial exponential decay.

Moreover, the restriction theorem (in the sense of Stein and Tomas)

$$\|dE_P(\boldsymbol{\lambda})\|_{L^p \to L^{p'}} \leq C\boldsymbol{\lambda}^{(n+1)(1/p-1/p')-1} \quad \text{where } p \in [1, 2(n+2)/(n+4)],$$

at high energies on non-trapping asymptotically hyperbolic manifolds follows from the above spectral measure estimates. It is well-known that Strichartz [30] insightfully points out the deep relationship between Strichartz estimates and restriction theorem. It motivates us to show the Strichartz estimates from these spectral measure estimates, which are sufficient to give restriction theorem. In fact, combining Strichartz estimates and dispersive estimates in this paper with our previous results of resolvent in [11], spectral measure with applications to restriction theorem and spectral multiplier in [12], we have elucidated the following diagram on non-trapping asymptotically hyperbolic manifolds.

| | | |
|---|---|---|
| | | $L^p$ boundedness<br>of spectral multiplier |
| | | ↑ |
| Strichartz estimate<br>of Schrödinger equation | ← | Restriction theorem<br>of spectral measure |
| ↑ | | ↑ |
| Dispersive estimate<br>of Schrödinger propagator | ← | Pointwise estimates<br>for spectral measure |
| | | ↑ |
| | | Resolvent construction<br>near continuous spectrum |

## 3. Schrödinger propagators via spectral measure

The spectral theorem of projection valued measure form for unbounded self-adjoint operators gives the following expression of Schrödinger propagators $e^{it\Delta}$ via spectral measure,

$$e^{it\Delta} = e^{itn^2/4} \int_0^\infty e^{it\boldsymbol{\lambda}^2} dE_P(\boldsymbol{\lambda}).$$

We remark that since we assume $f$ and $F$ are orthogonal to the eigenfunction spaces the discrete terms don't show up.

We are motivated to employ the spectral measure estimates, including the microlocalized form at high energy (say $\boldsymbol{\lambda} > 1$) together with the global form at low energy (say $\boldsymbol{\lambda} < 1$), to estimate the Schrödinger propagator. As seen on Euclidean space or asymptotically conic manifolds, the spectral measure behaves uniformly on the full continuous spectrum, for example see [19]. However, comparing (2.2) and Proposition 3, one can see the discrepancy of the order of $\boldsymbol{\lambda}$ between low energies and high energies on $n+1$-dimensional asymptotically hyperbolic manifolds for $n+1>3$. We thus have to split up the propagator to remedy the discrepancy.

One may pick two smooth bump functions $\chi_{\text{low}}$ supported in $[0, 2)$ and $\chi_\infty$ supported in $(1, \infty)$ such that $\chi_{\text{low}} + \chi_\infty = 1$ and split the propagator as

$$U(t) = \int_0^\infty e^{it\boldsymbol{\lambda}^2} \chi_{\text{low}}(\boldsymbol{\lambda})\, dE_P(\boldsymbol{\lambda}) + \int_0^\infty e^{it\boldsymbol{\lambda}^2} \chi_\infty(\boldsymbol{\lambda})\, dE_P(\boldsymbol{\lambda}).$$

In accordance with Proposition 3, we also will have to microlocalize the spectral measure at high energies by a family of semiclassical pseudodifferential operators $\{Q_k\}_0^N$ as follows

$$U_k(t) = \int_0^\infty e^{it\boldsymbol{\lambda}^2} \chi_\infty(\boldsymbol{\lambda}) Q_k(\boldsymbol{\lambda})\, dE_P(\boldsymbol{\lambda}).$$

In summary, we truncate and microlocalize the propagator and gain the following decomposition

$$e^{it\Delta - itn^2/4} = U_{\text{low}}(t) + \sum_{k=0}^N U_k(t).$$

Returning to the Cauchy problem (1.2), the solution $u$ is given by Duhamel's formula

$$u(t,z) = e^{it\Delta} f(z) - i \int_0^t e^{i(t-s)\Delta} F(s,z)\, ds.$$

To prove the Strichartz estimates, we shall invoke Keel–Tao bilinear approach. In our case, one may reduce to the energy estimates and dispersive estimates for the following bilinear propagators

$$U_{\text{low}}(t) U_{\text{low}}^*(s) = \int_0^\infty e^{i(t-s)\boldsymbol{\lambda}} \chi_{\text{low}}^2(\boldsymbol{\lambda}) dE_P(\boldsymbol{\lambda}), \tag{3.1}$$

$$U_k(t) U_k^*(s) = \int_0^\infty e^{i(t-s)\boldsymbol{\lambda}} \chi_\infty^2(\boldsymbol{\lambda}) Q_k(\boldsymbol{\lambda}) dE_P(\boldsymbol{\lambda}) Q_k^*(\boldsymbol{\lambda}), \tag{3.2}$$

$$U_j(t) U_k^*(s) = \int_0^\infty e^{i(t-s)\boldsymbol{\lambda}} \chi_\infty^2(\boldsymbol{\lambda}) Q_j(\boldsymbol{\lambda}) dE_P(\boldsymbol{\lambda}) Q_k^*(\boldsymbol{\lambda}). \tag{3.3}$$

Here (3.1), (3.2) and (3.3) follow from [21, Lemma 5.3]. Aside from these, we also remark the estimates for $U_{\text{low}}(t)U_k^*(s)$ or $U_k(t)U_{\text{low}}^*(s)$ are the same with $U_{\text{low}}(t)U_{\text{low}}^*(s)$.

In the next three sections, we prove the energy estimates for them in Proposition 4 and dispersive estimates for (3.1) and (3.2) in Proposition 6 and for (3.3) in Proposition 9 respectively.

## 4. Energy estimates for Schrödinger propagators at high energy

We shall prove the $L^2$-boundedness of microlocalized/truncated Schrödinger propagators. More precisely,

**Proposition 4** *(Energy estimates). The propagator $e^{it\Delta}$, low energy truncated propagator $U_{low}$, microlocalized high energy truncated propagators $U_0$ and $U_k$ for $k = 1, 2, \dots$ are all $L^2$-bounded.*

**Proof.**[4] The boundedness of $e^{it\Delta}$ and $U_{\text{low}}$ is clear. Since the entire cut off propagator at high energy is of course $L^2$-bounded, we can ignore the $k = 0$ term but only consider $U_k$ for $k = 1, 2, \dots$.

Our main tool is almost orthogonality lemma established by Cotlar, Knapp and Stein, see for example [16, p. 620].

**Lemma 5** *(Almost orthogonality). Let $\{T_j\}_{j\in\mathbb{Z}}$ be a family of bounded operators on Hilbert space $H$ obeying*

$$\|T_j^*T_k\|_{H\to H} + \|T_jT_k^*\|_{H\to H} \le \gamma(j-k) \quad \text{for any } j,k \in \mathbb{Z},$$

*where the function $\gamma : \mathbb{Z} \to \mathbb{R}^+$ satisfies $\sum_{j\in\mathbb{Z}} \sqrt{\gamma(j)} < \infty$. Then linear operator $T$, the limit of $\sum_{|j|<N} T_j$ in the norm topology of $H$ as $N$ goes to infinity, is $H$-bounded.*

First of all, the propagators are well-defined on $L^2$ if the integrand is supported on a compact subset of $(0, \infty)$ in $\boldsymbol{\lambda}$ as the pseudodifferential operator would be $L^2$-bounded uniformly with respect to $\boldsymbol{\lambda}$. We want to extend the well-definedness to entire positive half real line by almost orthogonality.

The strategy is to get a decomposition of the microlocalized propagator such that every term is an integral of a compactly supported function with respect to the microlocalized spectral measure, and then show the almost orthogonality of the decomposition required in Lemma 5.

First of all, we take the decomposition with a compactly supported smooth function $\psi \in C_c^\infty[1/2, 2]$ valued in $[0, 1]$ such that

$$\sum_j \psi\left(\frac{\boldsymbol{\lambda}}{2^j}\right) = 1.$$

Then we define

$$\begin{aligned}U_{i,j}(t) &= \int_0^\infty e^{it\boldsymbol{\lambda}^2}\chi_\infty(\boldsymbol{\lambda})\psi\left(\frac{\boldsymbol{\lambda}}{2^j}\right)Q_i(\boldsymbol{\lambda})\,dE_P(\boldsymbol{\lambda})\\ &= -\int_0^\infty \frac{d}{d\boldsymbol{\lambda}}\left(e^{it\boldsymbol{\lambda}^2}\chi_\infty(\boldsymbol{\lambda})\psi\left(\frac{\boldsymbol{\lambda}}{2^j}\right)Q_i(\boldsymbol{\lambda})\right)E_P(\boldsymbol{\lambda})\end{aligned}$$

and calculate as follows

$$\begin{aligned}U_{i,j}(t)U_{i,k}^*(t) = \iint &\frac{d}{d\boldsymbol{\lambda}}\left(e^{it\boldsymbol{\lambda}^2}\chi_\infty(\boldsymbol{\lambda})\psi\left(\frac{\boldsymbol{\lambda}}{2^j}\right)Q_i(\boldsymbol{\lambda})\right)E_P(\boldsymbol{\lambda})\\ &\times E_P(\boldsymbol{\mu})\frac{d}{d\boldsymbol{\mu}}\left(e^{-it\boldsymbol{\mu}^2}\chi_\infty(\boldsymbol{\mu})\psi\left(\frac{\boldsymbol{\mu}}{2^k}\right)Q_i^*(\boldsymbol{\mu})\right)d\boldsymbol{\lambda}d\boldsymbol{\mu}\end{aligned}$$

[4] This proof is essentially due to Hassell and Zhang [21] in case of asymptotically Euclidean manifolds, as only minor modifications are needed here. But we give the detailed proof for the self-containedness of the paper.

$$= \iint_{\boldsymbol{\lambda} \leq \boldsymbol{\mu}} \frac{d}{d\boldsymbol{\lambda}} \bigg( e^{it\boldsymbol{\lambda}^2} \chi_\infty(\boldsymbol{\lambda}) \chi_\infty(\boldsymbol{\lambda}) \psi\Big(\frac{\boldsymbol{\lambda}}{2^j}\Big) Q_i(\boldsymbol{\lambda}) \bigg) E_P(\boldsymbol{\lambda}) \times \frac{d}{d\boldsymbol{\mu}} \bigg( e^{-it\boldsymbol{\mu}^2} \chi_\infty(\boldsymbol{\mu}) \psi\Big(\frac{\boldsymbol{\mu}}{2^k}\Big) Q_i^*(\boldsymbol{\mu}) \bigg) d\boldsymbol{\lambda} d\boldsymbol{\mu}$$
$$+ \iint_{\boldsymbol{\mu} \leq \boldsymbol{\lambda}} \frac{d}{d\boldsymbol{\lambda}} \bigg( e^{it\boldsymbol{\lambda}^2} \chi_\infty(\boldsymbol{\lambda}) \psi\Big(\frac{\boldsymbol{\lambda}}{2^j}\Big) Q_i(\boldsymbol{\lambda}) \bigg) E_P(\boldsymbol{\mu}) \times \frac{d}{d\boldsymbol{\mu}} \bigg( e^{-it\boldsymbol{\mu}^2} \chi_\infty(\boldsymbol{\mu}) \psi\Big(\frac{\boldsymbol{\mu}}{2^k}\Big) Q_i^*(\boldsymbol{\mu}) \bigg) d\boldsymbol{\lambda} d\boldsymbol{\mu}$$

We then perform integration by parts and get

$$U_{i,j}(t) U_{i,k}^*(t) = \int \frac{d}{d\boldsymbol{\lambda}} \bigg( e^{it\boldsymbol{\lambda}^2} \chi_\infty(\boldsymbol{\lambda}) \psi\Big(\frac{\boldsymbol{\lambda}}{2^j}\Big) Q_i(\boldsymbol{\lambda}) \bigg) E_P(\boldsymbol{\lambda}) \times \bigg( - e^{-it\boldsymbol{\lambda}^2} \chi_\infty(\boldsymbol{\lambda}) \psi\Big(\frac{\boldsymbol{\lambda}}{2^k}\Big) Q_i^*(\boldsymbol{\lambda}) \bigg) d\boldsymbol{\lambda}$$
$$+ \int \bigg( - e^{it\boldsymbol{\mu}^2} \chi_\infty(\boldsymbol{\mu}) \psi\Big(\frac{\boldsymbol{\mu}}{2^j}\Big) Q_i(\boldsymbol{\mu}) \bigg) E_P(\boldsymbol{\mu}) \times \frac{d}{d\boldsymbol{\mu}} \bigg( e^{-it\boldsymbol{\mu}^2} \chi_\infty(\boldsymbol{\mu}) \psi\Big(\frac{\boldsymbol{\mu}}{2^k}\Big) Q_i^*(\boldsymbol{\mu}) \bigg) d\boldsymbol{\mu}$$
$$= \int \chi_\infty^2(\boldsymbol{\lambda}) \psi\Big(\frac{\boldsymbol{\lambda}}{2^j}\Big) \psi\Big(\frac{\boldsymbol{\lambda}}{2^k}\Big) Q_i(\boldsymbol{\lambda}) dE_P(\boldsymbol{\lambda}) Q_i^*(\boldsymbol{\lambda})$$
$$= \int \frac{d}{d\boldsymbol{\lambda}} \bigg( \chi_\infty^2(\boldsymbol{\lambda}) \psi\Big(\frac{\boldsymbol{\lambda}}{2^j}\Big) \psi\Big(\frac{\boldsymbol{\lambda}}{2^k}\Big) Q_i(\boldsymbol{\lambda}) \bigg) E_P(\boldsymbol{\lambda}) Q_i^*(\boldsymbol{\lambda})$$
$$+ \int \chi_\infty^2(\boldsymbol{\lambda}) \psi\Big(\frac{\boldsymbol{\lambda}}{2^j}\Big) \psi\Big(\frac{\boldsymbol{\lambda}}{2^k}\Big) Q_i(\boldsymbol{\lambda}) E_P(\boldsymbol{\lambda}) \frac{d}{d\boldsymbol{\lambda}} Q_i^*(\boldsymbol{\lambda}).$$

As implied, $U_{i,j}(t)U_{i,k}^*(t)$ is indeed $t$-independent. Therefore, we shall prove the $L^2$-boundedness for all $t$ via $U_{i,j}(0)U_{i,k}^*(0)$, which equals

$$\iint E_P(\boldsymbol{\lambda}) \frac{d}{d\boldsymbol{\lambda}} \bigg( \chi_\infty(\boldsymbol{\lambda}) \psi\Big(\frac{\boldsymbol{\lambda}}{2^j}\Big) Q_i^*(\boldsymbol{\lambda}) \bigg) \frac{d}{d\boldsymbol{\mu}} \bigg( Q_i(\boldsymbol{\mu}) \psi\Big(\frac{\boldsymbol{\mu}}{2^k}\Big) \chi_\infty(\boldsymbol{\mu}) \bigg) E_P(\boldsymbol{\mu}) \, d\boldsymbol{\lambda} d\boldsymbol{\mu}.$$

We claim $U_{i,j}(0)U_{i,k}^*(0)$ obeys the almost orthogonality estimate

$$\|U_{i,j}(0)U_{i,k}^*(0)\|_{L^2 \to L^2} \leq C 2^{-|j-k|}.$$

In light of the $L^2$-boundedness of spectral projection, it suffices to prove

$$\frac{d}{d\boldsymbol{\lambda}} \bigg( \chi_\infty(\boldsymbol{\lambda}) \psi\Big(\frac{\boldsymbol{\lambda}}{2^j}\Big) Q_i^*(\boldsymbol{\lambda}) \bigg) \frac{d}{d\boldsymbol{\mu}} \bigg( Q_i(\boldsymbol{\mu}) \psi\Big(\frac{\boldsymbol{\mu}}{2^k}\Big) \chi_\infty(\boldsymbol{\mu}) \bigg) \leq C 2^{-|j-k|}.$$

We denote the operators in the parentheses $Q_{i,j}^*(\boldsymbol{\lambda})$ and $Q_{i,k}(\boldsymbol{\mu})$ respectively. We write the product of the two as

$$Q_{i,j}^*(\boldsymbol{\lambda}) Q_{i,k}(\boldsymbol{\mu}) = \boldsymbol{\lambda}^{n+1} \boldsymbol{\mu}^{n+1} \iiint e^{i\boldsymbol{\lambda}(z-z'')\cdot\zeta/x''} q_{i,j}(z'', \zeta, \boldsymbol{\lambda}) \times e^{i\boldsymbol{\mu}(z''-z')\cdot\zeta'/x''} q_{i,k}(z'', \zeta', \boldsymbol{\mu}) \, d\zeta d\zeta' dz'',$$

away from $\partial X$ or

$$Q_{i,j}^*(\boldsymbol{\lambda}) Q_{i,k}(\boldsymbol{\mu}) = \boldsymbol{\lambda}^{n+1} \boldsymbol{\mu}^{n+1} \iiint e^{i\boldsymbol{\lambda}((x-x'')\lambda + (y-y'')\cdot\mu)/x''} q_{i,j}(x'', y'', \lambda, \mu, \boldsymbol{\lambda}) \times e^{i\boldsymbol{\mu}((x''-x')\lambda' + (y''-y')\cdot\mu')/x''} q_{i,k}(x'', y'', \lambda', \mu', \boldsymbol{\mu}) \, d\lambda d\mu d\lambda' d\mu' dx'' dy'',$$

near $\partial X$. The second case is indeed the same with the first, as one can denote $(x, y)$ by $(z_1, \dots, z_n)$ and $(\lambda, \mu)$ by $(\zeta_1, \dots, \zeta_n)$. Furthermore, one may assume $j > k$, equivalent to $\boldsymbol{\lambda} > \boldsymbol{\mu}$, due to the symmetry. We insert a differential operator $ix''\zeta \cdot \partial_{z''}/(\boldsymbol{\lambda}|\zeta|^2)$, to which $e^{i\boldsymbol{\lambda}(z-z'')\cdot\zeta/x''}$ is invariant, and make integration by parts.

$$\begin{aligned}
&\boldsymbol{\lambda}^{-n-1}\boldsymbol{\mu}^{-n-1}Q_{i,j}^*(\boldsymbol{\lambda})Q_{i,k}(\boldsymbol{\mu})\\
&= \iiint \frac{ix''\zeta\cdot\partial_{z''}}{\boldsymbol{\lambda}|\zeta|^2}\Big(e^{i\boldsymbol{\lambda}(z-z'')\cdot\zeta/x''}\Big)q_{i,j}(z'',\zeta,\boldsymbol{\lambda})e^{i\boldsymbol{\mu}(z''-z')\cdot\zeta'/x''}q_{i,k}(z'',\zeta',\boldsymbol{\mu})\,d\zeta d\zeta' dz''\\
&= \frac{\boldsymbol{\mu}}{\boldsymbol{\lambda}}\iiint e^{i\boldsymbol{\lambda}(z-z'')\cdot\zeta/x''}\frac{\zeta\cdot\zeta'}{|\zeta|^2}e^{i\boldsymbol{\mu}(z''-z')\cdot\zeta'/x''}q_{i,j}(z'',\zeta,\boldsymbol{\lambda})q_{i,k}(z'',\zeta',\boldsymbol{\mu})\,d\zeta d\zeta' dz''\\
&\quad -\frac{x''}{\boldsymbol{\lambda}}\iiint ie^{i\boldsymbol{\lambda}(z-z'')\cdot\zeta/x''}e^{i\boldsymbol{\mu}(z''-z')\cdot\zeta'/x''}\frac{\zeta}{|\zeta|^2}\cdot\partial_{z''}\Big(q_{i,j}(z'',\zeta,\boldsymbol{\lambda})q_{i,k}(z'',\zeta',\boldsymbol{\mu})\Big)\,d\zeta d\zeta' dz''
\end{aligned}$$

Because $i \neq 0$, $Q_i$ is microlocally supported around the spherical bundle, namely, $|\zeta| \approx |\zeta'| \approx 1$. Therefore, using the $L^2$-boundedness of semiclassical pseudodifferential operators and noting $\boldsymbol{\lambda}, \boldsymbol{\mu} \geq 1$ on the support of the high energy cut-off function $\chi_\infty$, we deduce

$$\big\| Q_{i,j}^*(\boldsymbol{\lambda})Q_{i,k}(\boldsymbol{\mu})\big\|_{L^2\to L^2} \leq C\frac{\boldsymbol{\mu}+x''}{\boldsymbol{\lambda}} \leq C\frac{\boldsymbol{\mu}}{\boldsymbol{\lambda}} \leq C2^{-|j-k|},$$

which proves the almost orthogonality for $t = 0$. Almost orthogonality lemma then gives that $\sum_{|j|\leq l} U_{i,j}^*(0)$ strongly converges in $L^2$, that is,

$$\lim_{l\to\infty}\sup_{m>l}\Big\| \sum_{l\leq |j|\leq m} U_{i,j}^*(0)f\Big\|_{L^2}^2 = 0.$$

We now extend this conclusion to any $t$. Given $f \in L^2$, we want to have

$$\lim_{l\to\infty}\sup_{m>l}\Big\| \sum_{l\leq |j|\leq m} U_{i,j}^*(t)f\Big\|_{L^2}^2 = \lim_{l\to\infty}\sup_{m>l}\sum_{l\leq |j|,|j'|\leq m}\langle U_{i,j}(t)U_{i,j'}^*(t)f, f\rangle = 0.$$

In fact, it is easy to reduce the convergence for general $t$ to the case $t = 0$ by the time independence of the operator $U_{i,j}(t)U_{i,j}^*(t)$

$$\lim_{l\to\infty}\sup_{m>l}\sum_{l\leq |j|,|j'|\leq m}\langle U_{i,j}(t)U_{i,j'}^*(t)f, f\rangle = \lim_{l\to\infty}\sup_{m>l}\sum_{l\leq |j|,|j'|\leq m}\langle U_{i,j}(0)U_{i,j'}^*(0)f, f\rangle = 0$$

Finally, noting

$$\|U_i^*(t)\|^2 \leq \lim_{l\to\infty}\Big\| \sum_{|j|\leq l} U_{i,j}^*(t)\Big\|^2,$$

we conclude that $U_i(t)$ is uniformly bounded on $L^2$. □

## 5. Dispersive estimates for Schrödinger propagators I

In this section we establish the dispersive estimates for diagonal microlocalized/truncated Schrödinger propagators. We shall show

**Proposition 6** *(Dispersive estimates I). The long time dispersive estimates for the microlocalized Schrödinger propagators at high energy*

$$\Big|\int_0^\infty e^{it\boldsymbol{\lambda}^2}\chi_\infty^2(\boldsymbol{\lambda})\Big(Q_k(\boldsymbol{\lambda})dE_P(\boldsymbol{\lambda})Q_k^*(\boldsymbol{\lambda})\Big)(z,z')\,d\boldsymbol{\lambda}\Big| \leq C|t|^{-\infty}e^{-nd(z,z')/2} \tag{5.1}$$

*hold, provided* $t > 1 + d(z, z')$. *The low energy truncated propagator obeys*

$$\left| \int_0^\infty e^{it\lambda^2} \chi_{low}^2(\lambda) dE_P(\lambda, z, z')\, d\lambda \right| \le C|t|^{-3/2}\big(1 + d(z, z')\big)e^{-nd(z,z')/2} \tag{5.2}$$

*for all times. On the other hand, we have short time dispersive estimates for the high energy truncated propagator microlocalized near the diagonal*

$$\left| \int_0^\infty e^{it\lambda^2} \chi_\infty^2(\lambda) \Big(Q_k(\lambda) dE_P(\lambda) Q_k^*(\lambda)\Big)(z, z')\, d\lambda \right| \le C|t|^{-(n+1)/2}(1 + d(z, z'))^{n/2} e^{-nd(z,z')/2} \tag{5.3}$$

*provided* $t < 1 + d(z, z')$.

**Remark 7.** If we work on a manifold without conjugate points, this result will reduce to the dispersive estimates (1.6) on hyperbolic space, where the microlocalization is needless. Moreover, for short time estimates, say $t < 1 + d(z, z')$, we can combine (5.2) and (5.3).

**Proof of (5.1).** Let us look at the long time dispersion first. Because we want to use stationary phase estimates, we have to split the amplitude of the microlocalized propagator into functions compactly supported in $\lambda$. To do so, we select a bump function $\phi \in C_c^\infty[1/2, 2]$ such that $\sum_j \phi(2^{-j}\lambda) = 1$ and let $\phi_0(\lambda) = \sum_{j\le 0} \phi(2^{-j}\lambda)$. Then the Schrödinger propagator is decomposed as $I_0 + \sum_{j>0} I_j$, which is

$$I_0 = \int_0^\infty e^{it\lambda^2} \chi_\infty^2(\lambda) Q_k(\lambda) dE_P(\lambda) Q_k^*(\lambda) \phi_0(\lambda)\, d\lambda$$

$$I_j = \int_0^\infty e^{it\lambda^2} \chi_\infty^2(\lambda) Q_k(\lambda) dE_P(\lambda) Q_k^*(\lambda) \phi(2^{-j}\lambda)\, d\lambda.$$

• Case 1: $d(z, z') \le 1$

As $t$ goes to infinity, the phase function is $\lambda^2$ which is clearly non-degenerate at the stationary point $\lambda = 0$.

Noting 0 is not on the support of $\chi_\infty$, we have $I_0 = O(t^{-\infty})e^{-nd(z,z')/2}$. On the other hand, noting the phase function of the $I_j$ terms are non-stationary, we deduce

$$\begin{aligned}\sum_{j>0} |I_j| &= \sum_{j>0} \left| \int_0^\infty \left(\frac{1}{2it\lambda}\frac{d}{d\lambda}\right)^N (e^{it\lambda^2}) \chi_\infty^2(\lambda) Q_k(\lambda) dE_P(\lambda) Q_k^*(\lambda) \phi(2^{-j}\lambda)\, d\lambda \right| \\ &\le C \sum_{j>0} t^{-N} e^{-nd(z,z')/2} \int_{2^{j-1}}^{2^{j+1}} \lambda^{n-2N}\, d\lambda \le C t^{-N} e^{-nd(z,z')/2}.\end{aligned}$$

Since $N$ is arbitrary, we get the estimate.

• Case 2: $d(z, z') \ge 1$

Since $d(z, z')$ goes to $\infty$ as well as $t$, the phase function consists of not only $\lambda^2$ but also some other term coming from the spectral measure. The outgoing and incoming parts of the spectral measure contribute the oscillatory terms $e^{-i\lambda d(z,z')}$ and $e^{i\lambda d(z,z')}$ respectively. So the new phase function will be $t\lambda^2 \mp \lambda d(z, z')$. In the incoming case, such phase function isn't stationary. Then we can select the bump function $\phi$ as above and get compactly supported amplitudes. Noting the support of $\phi_0$ isn't intersected with $\chi_\infty$, we can obtain the dispersive estimates by running the same argument of non-stationary phase and integration by parts

$$\sum_{j>0}|I_j| = \sum_{j>0}\left|\int_0^\infty \left(\frac{1}{2it\lambda + id(z,z')}\frac{d}{d\lambda}\right)^N (e^{it\lambda^2+id(z,z')\lambda})\tilde{a}_+(\lambda)\phi(2^{-j}\lambda)\,d\lambda\right|$$
$$\leq C\sum_{j>0} t^{-N}\int_{2^{j-1}}^{2^{j+1}} \lambda^{n-2N} e^{-nd(z,z')/2}\,d\lambda \leq Ct^{-N}e^{-nd(z,z')/2},$$

for any large $N$. We remark that $\tilde{a}_\pm = a_\pm\chi_\infty^2$ with $a_\pm$ as in Proposition 3. Since $\chi_\infty(\lambda)\equiv 1$ for large $\lambda$, $\tilde{a}_\pm$ obeys the same estimates with $a_\pm$.

On the other hand, the phase function $t\lambda^2 - \lambda d(z,z')$ is stationary at $\lambda = d(z,z')/(2t)$. Nonetheless $d(z,z')/(2t) < 1$ doesn't lie on the support of $\chi_\infty$ either, we thus can prove the dispersive estimates by the same argument. The proof is now complete. □

**Proof of (5.2).** It can be deduced from the results of the spectral measure at low energy. We make a change of variable and get

$$U_{\text{low}} = \int_0^\infty e^{it\lambda^2}\chi_{\text{low}}^2(\lambda)dE_P(\lambda,z,z')\,d\lambda$$
$$= t^{-1/2}\int_0^\infty e^{i\lambda^2}\chi_{\text{low}}^2(t^{-1/2}\lambda)dE_P(t^{-1/2}\lambda,z,z')\,d\lambda.$$

We decompose the LHS as $I_0 + I_\infty$, where

$$I_0 = t^{-1/2}\int_0^1 e^{i\lambda^2}\chi_{\text{low}}^2(t^{-1/2}\lambda)dE_P(t^{-1/2}\lambda,z,z')\,d\lambda$$
$$I_\infty = t^{-1/2}\int_1^\infty e^{i\lambda^2}\chi_{\text{low}}^2(t^{-1/2}\lambda)dE_P(t^{-1/2}\lambda,z,z')\,d\lambda.$$

We use (2.3) for low energies to estimate $I_0$ as follows

$$|I_0| = t^{-1/2}\left|\int_0^1 e^{i\lambda^2}\chi_{\text{low}}^2(t^{-1/2}\lambda)dE_P(t^{-1/2}\lambda,z,z')\,d\lambda\right|$$
$$\leq t^{-1/2}\int_0^1 (t^{-1/2}\lambda)^2(1+d(z,z'))e^{-nd(z,z')/2}\,d\lambda$$
$$\leq Ct^{-3/2}(1+d(z,z'))e^{-nd(z,z')/2}.$$

On the other hand, we shall invoke (2.2) for low energies and perform integration by parts on $I_\infty$.

• Case 1: $t^{1/2} > 1 + d(z,z')$

We plug in the low energy spectral measure (2.2) and derive that

$$I_\infty = t^{-1/2}\int_1^\infty e^{i\lambda^2}\chi_{\text{low}}^2(t^{-1/2}\lambda)dE_P(t^{-1/2}\lambda,z,z')\,d\lambda$$
$$= \frac{t^{-1}(\rho_L\rho_R)^{n/2}}{-8i}\int_1^\infty e^{i\lambda^2}\chi_{\text{low}}^2(t^{-1/2}\lambda)\lambda$$

$$\times\Big((\rho_L\rho_R)^{it^{-1/2}\lambda}a(t^{-1/2}\lambda)-(\rho_L\rho_R)^{-it^{-1/2}\lambda}a(-t^{-1/2}\lambda)\Big)\,d\lambda,$$

with a smooth function $a$ supported on $[0,1]$. Noting $\chi_{\text{low}}(\lambda)\equiv 1$ on $[0,1]$, that is the support of $a(\lambda)$, we perform integration by parts and get

$$\begin{aligned}I_\infty &= \frac{t^{-1}(\rho_L\rho_R)^{n/2}}{-8i}\int_1^\infty\left(\frac{1}{\lambda}\frac{d}{d\lambda}\right)^3\left(e^{i\lambda^2}\right)\chi^2_{\text{low}}(t^{-1/2}\lambda)\lambda\\ &\quad\times\Big((\rho_L\rho_R)^{it^{-1/2}\lambda}a(t^{-1/2}\lambda)-(\rho_L\rho_R)^{-it^{-1/2}\lambda}a(-t^{-1/2}\lambda)\Big)\,d\lambda\\ &= I_{\infty,1}+I_{\infty,2}\end{aligned}$$

where we write

$$\begin{aligned}I_{\infty,1} &= \frac{t^{-1}(\rho_L\rho_R)^{n/2}}{8i}\left(\frac{1}{\lambda}\frac{d}{d\lambda}\right)^2\left(e^{i\lambda^2}\right)\chi_{\text{low}}(t^{-1/2}\lambda)\\ &\quad\times\Big((\rho_L\rho_R)^{it^{-1/2}\lambda}a(t^{-1/2}\lambda)-(\rho_L\rho_R)^{-it^{-1/2}\lambda}a(-t^{-1/2}\lambda)\Big)\Big|_{\lambda=1}\\ I_{\infty,2} &= \frac{t^{-3/2}(\rho_L\rho_R)^{n/2}}{8i}\int_1^\infty\left(\frac{1}{\lambda}\frac{d}{d\lambda}\right)^2\left(e^{i\lambda^2}\right)\chi_{\text{low}}(t^{-1/2}\lambda)\\ &\quad\times\Big((\rho_L\rho_R)^{it^{-1/2}\lambda}a'(t^{-1/2}\lambda)+(\rho_L\rho_R)^{-it^{-1/2}\lambda}a'(-t^{-1/2}\lambda)\\ &\quad+i(\rho_L\rho_R)^{it^{-1/2}\lambda}\ln(\rho_L\rho_R)a(t^{-1/2}\lambda)+i(\rho_L\rho_R)^{-it^{-1/2}\lambda}\ln(\rho_L\rho_R)a(-t^{-1/2}\lambda)\Big)\,d\lambda.\end{aligned}$$

We now use (2.1) to estimate $I_{\infty,1}$. If $t<M$ provided $M$ is sufficiently large,

$$|I_{\infty,1}|\le t^{-1}e^{-nd(z,z')/2}\le CM^{1/2}t^{-3/2}e^{-nd(z,z')/2}.$$

On the other hand, if $t>M$ (i.e. $t^{-1/2}$ is very small), we then use the smoothness of $a$ at 0 and obtain

$$\Big((\rho_L\rho_R)^{it^{-1/2}}a(t^{-1/2})-(\rho_L\rho_R)^{-it^{-1/2}}a(-t^{-1/2})\Big)\le Ct^{-1/2}.$$

Consequently, we obtain that

$$|I_{\infty,1}|\le Ct^{-3/2}e^{-nd(z,z')/2}.$$

For $I_{\infty,2}$, by (2.1), we observe the part of the integrand contained in the parentheses is bounded by $C(1+d(z,z'))$. We make integration by parts two more times and get

$$\begin{aligned}|I_{\infty,2}| &\le Ct^{-3/2}(1+d(z,z'))e^{-nd(z,z')/2}\\ &\quad\times\left(\int_1^\infty\lambda^{-4}\,d\lambda+t^{-1/2}(1+d(z,z'))\int_1^\infty\lambda^{-3}\,d\lambda+t^{-1}(1+d(z,z')^2)\int_1^\infty\lambda^{-2}\,d\lambda\right).\end{aligned}$$

Noting $1+d(z,z')\le t^{1/2}$, we conclude that

$$|I_{\infty,2}|\le Ct^{-3/2}(1+d(z,z'))e^{-nd(z,z')/2}.$$

- Case 2: $1<t^{1/2}<1+d(z,z')$

We shall estimate the following integrals instead

$$I_{\infty,+} = t^{-1/2} e^{-n\tilde{d}} \int_1^\infty e^{i\lambda^2 + i\tilde{d}t^{-1/2}\lambda} \lambda a(t^{-1/2}\lambda)\, d\lambda$$

$$I_{\infty,-} = t^{-1/2} e^{-n\tilde{d}} \int_1^\infty e^{i\lambda^2 - i\tilde{d}t^{-1/2}\lambda} \lambda a(-t^{-1/2}\lambda)\, d\lambda,$$

where $\tilde{d} = \ln(\rho_L \rho_R)$ and $a \in C_c^\infty[0,1]$. Without loss of generality, we assume $\tilde{d} \geq 0$. By (2.1), $\tilde{d}$ is an approximation of the geodesic distance function. The term $I_{\infty,+}$ is easier, since the first derivative of the phase $2\lambda + \tilde{d}t^{-1/2}$ is not vanishing. So we directly adopt the standard integration by parts argument as follows. First, we insert an invariant operator

$$|I_{\infty,+}| \leq t^{-1} e^{-n\tilde{d}} \left| \int_1^\infty \frac{1}{2\lambda + \tilde{d}t^{-1/2}} \frac{\partial}{\partial \lambda} e^{i\lambda^2 + i\tilde{d}t^{-1/2}\lambda} \cdot \lambda a(t^{-1/2}\lambda)\, d\lambda \right|.$$

Then we perform integration by parts on the integral and get

$$\frac{a(t^{-1/2}) e^{i + it^{-1/2}\tilde{d}}}{2 + t^{-1/2}\tilde{d}} + \int_1^\infty e^{i\lambda^2 + i\tilde{d}t^{-1/2}\lambda} \left( \frac{\tilde{d}t^{-1/2} a(t^{-1/2}\lambda)}{(2\lambda + \tilde{d}t^{-1/2})^2} + \frac{\lambda t^{-1/2} a'(t^{-1/2}\lambda)}{2\lambda + \tilde{d}t^{-1/2}} \right) d\lambda$$

The boundary term and the first term of the integral is bounded by a constant, whilst the second term is yielded to

$$C \int_1^{t^{1/2}} \frac{\lambda t^{-1/2}}{2\lambda + t^{-1/2}\tilde{d}}\, d\lambda.$$

Also noting $t^{-1/2}\tilde{d} > C$, we conclude that

$$|I_{\infty,+}| \leq C t^{-3/2} e^{-nd(z,z')} (1 + d(z,z')).$$

We now estimate $I_{\infty,-}$. Since $2\lambda - t^{-1/2}\tilde{d}\lambda$ might vanish, we have to take a dyadic decomposition. To do so, we introduce a partition of unity $\sum_j \phi(2^{-j}\lambda) = 1$ with $\phi \in C_c^\infty[1/2, 2]$. We further denote

$$\psi_k(\lambda) = \phi\big(2^{-k} |2\lambda - t^{-1/2}\tilde{d}(z,z')|\big).$$

Then we have to estimate the following integrals

$$I^0_{\infty,-} = t^{-1} e^{-n\tilde{d}} \int_1^\infty e^{i\lambda^2 - it^{-1/2}\tilde{d}\lambda} \lambda a(t^{-1/2}\lambda) \sum_{k \leq 0} \psi_k(\lambda)\, d\lambda,$$

$$I^k_{\infty,-} = t^{-1} e^{-n\tilde{d}} \int_1^\infty e^{i\lambda^2 - it^{-1/2}\tilde{d}\lambda} \lambda a(t^{-1/2}\lambda) \psi_k(\lambda)\, d\lambda, \qquad k > 0.$$

We consider $I^0_{\infty,-}$ first. One can find a sufficiently large number $M$ such that for all $\lambda > M$ we have $\lambda \sim t^{-1/2}\tilde{d}$ if $|2\lambda - t^{-1/2}\tilde{d}| \leq 2$. Since the measure of the support of $\sum_{k \leq 0} \psi_k(\lambda)$ is smaller than 4, we thus get

$$\left| \int_1^M e^{i\lambda^2 - it^{-1/2}\tilde{d}\lambda} \lambda a(t^{-1/2}\lambda) \sum_{k \leq 0} \psi_k(\lambda)\, d\lambda \right| \leq C \leq C t^{-1/2} (1 + d(z,z')).$$

In the meantime, we have

$$\begin{aligned}
&\left|\int_M^\infty e^{i\lambda^2 - it^{-1/2}\tilde{d}\lambda}\lambda a(t^{-1/2}\lambda)\sum_{k\le 0}\psi_k(\lambda)\,d\lambda\right| \\
&\le t^{-1/2}\tilde{d}\left|\int_M^\infty e^{i\lambda^2 - it^{-1/2}\tilde{d}\lambda}\frac{\lambda a(t^{-1/2}\lambda)}{t^{-1/2}\tilde{d}}\sum_{k\le 0}\psi_k(\lambda)\,d\lambda\right| \\
&\le Ct^{-1/2}(1+d(z,z')).
\end{aligned}$$

On the other hand, we again use integration by parts $N$ times on $I^k_{\infty,-}$ for $k>0$.

$$\begin{aligned}
&\sum_{k>0}\left|\int_1^\infty e^{i\lambda^2 - it^{-1/2}\tilde{d}\lambda}\lambda a(t^{-1/2}\lambda)\psi_k(\lambda)\,d\lambda\right| \\
&\le \sum_{k>0}\left|\int_1^\infty \left(\frac{1}{2i\lambda - it^{-1/2}\tilde{d}}\frac{\partial}{\partial\lambda}\right)^N e^{i\lambda^2 - it^{-1/2}\tilde{d}\lambda}\lambda a(t^{-1/2}\lambda)\psi_k(\lambda)\,d\lambda\right| \\
&\le Ct^{-1/2}\tilde{d}\sum_{k>0}2^{-kN}\int_{|2\lambda - t^{-1/2}\tilde{d}|\sim 2^k}\lambda^{1-N}\,d\lambda \\
&\le Ct^{-1/2}(1+d(z,z')).
\end{aligned}$$

Plugging these estimates into $I^0_{\infty,-}$ and $I^k_{\infty,-}$ respectively, we conclude

$$|I_{\infty,-}| \le Ct^{-3/2}(1+d(z,z'))e^{-nd(z,z')/2}.$$

The proof is now complete. □

**Proof of (5.3).** Because of the distinction between the long and short distance, we discuss the two cases separately. In particular, the exponential decay is negligible in case of short distance, as $e^{-nd(z,z')/2}$ is bounded from below. The proof of (5.3) in case of small distance is the same with the proof of the dispersive estimates on asymptotically conic manifolds by Hassell and Zhang [21], as the spectral measure for small $d(z,z')$ and large $\lambda$ obeys the same estimates as on asymptotically conic manifolds. In fact, the idea for both long distance and short distance is to perform an appropriate dyadic decomposition over the value of the derivative of the phase function for an integration by parts argument. We only give the proof for long distance to see the more interesting exponential decay in $d(z,z')$.

First of all, we rescale the microlocalized high energy truncated propagator $U_k$ as follows

$$\begin{aligned}
U_k &= \int_0^\infty e^{it\lambda^2}\chi^2_\infty(\lambda)\Big(Q_k(\lambda)dE_P(\lambda)Q_k^*(\lambda)\Big)(z,z')\,d\lambda \\
&= t^{-1/2}\int_0^\infty e^{i\lambda^2}\chi^2_\infty(t^{-1/2}\lambda)\Big(Q_k dE_P Q_k^*\Big)(t^{-1/2}\lambda,z,z')\,d\lambda,
\end{aligned}$$

provided $t<1+d(z,z')$. Applying Proposition 3 for high energies, we write

$$U_k = t^{-(n+1)/2}T_+ + t^{-(n+1)/2}T_-, \tag{5.4}$$

where

$$\begin{aligned}
T_+ &= \int_0^\infty e^{i(\lambda^2 + t^{-1/2}\lambda d(z,z'))}\lambda^n\tilde{a}_+(t^{-1/2}\lambda,z,z')\,d\lambda \\
T_- &= \int_0^\infty e^{i(\lambda^2 - t^{-1/2}\lambda d(z,z'))}\lambda^n\tilde{a}_-(t^{-1/2}\lambda,z,z')\,d\lambda
\end{aligned}$$

with smooth function $\tilde{a}_\pm(\boldsymbol{\lambda}, z, z')$ on $[1, \infty) \times X_0^2$ obeying

$$\left|\frac{d^j}{d\boldsymbol{\lambda}^j}\tilde{a}_\pm(t^{-1/2}\boldsymbol{\lambda}, z, z')\right| = O\Big(t^{n/4}\boldsymbol{\lambda}^{-n/2-j}e^{-nd(z,z')/2}\Big) \quad \text{if } d(z,z') \text{ is large.}$$

Now it suffices to prove both $T_+$ and $T_-$ are bounded by $(1+d(z,z'))^{n/2}e^{-nd(z,z')/2}$.

We decompose the $T_+$ term further into $\sum_{j\geq 0} T_{j,+}$, where

$$T_{j,+} = \int_0^\infty e^{i(\boldsymbol{\lambda}^2+t^{-1/2}\boldsymbol{\lambda}d(z,z'))}\boldsymbol{\lambda}^n\tilde{a}_+(t^{-1/2}\boldsymbol{\lambda}, z, z')\phi(2^{-j}\boldsymbol{\lambda})\,d\boldsymbol{\lambda} \quad \text{for } j > 0$$

$$T_{0,+} = \int_0^\infty e^{i(\boldsymbol{\lambda}^2+t^{-1/2}\boldsymbol{\lambda}d(z,z'))}\boldsymbol{\lambda}^n\tilde{a}_+(t^{-1/2}\boldsymbol{\lambda}, z, z')(1-\sum_{j>0}\phi(2^{-j}\boldsymbol{\lambda}))\,d\boldsymbol{\lambda},$$

where we denote, by a partition of unity $\sum_j \phi(2^{-j}\boldsymbol{\lambda}) = 1$ with $\phi \in C_c^\infty[1/2, 2]$. It is clear that $T_{0,+}$ is bounded by $(1+d(z,z'))^{n/2}e^{-nd(z,z')/2}$ after a quick application of Proposition 3. On the other hand, for each $T_{j,+}$, the phase function of this oscillatory integral is actually non-stationary. One thus can insert a differential operator $N$ times leaving the exponential term invariant and take integration by parts

$$\begin{aligned}|T_{j,+}| &= \left|\int_0^\infty \left(\frac{1}{i(2\boldsymbol{\lambda}+t^{-1/2}d(z,z'))}\frac{\partial}{\partial\boldsymbol{\lambda}}\right)^N e^{i(\boldsymbol{\lambda}^2+t^{-1/2}\boldsymbol{\lambda}d(z,z'))}\boldsymbol{\lambda}^n\tilde{a}_+(t^{-1/2}\boldsymbol{\lambda}, z, z')\phi(2^{-j}\boldsymbol{\lambda})\,d\boldsymbol{\lambda}\right| \\ &\leq C\int_{|\boldsymbol{\lambda}|\sim 2^j} t^{n/4}e^{-nd(z,z')/2}\boldsymbol{\lambda}^{n/2-2N}\,d\boldsymbol{\lambda} \\ &\leq C(1+d(z,z'))^{n/2}e^{-nd(z,z')/2}\int_{|\boldsymbol{\lambda}|\sim 2^j}\boldsymbol{\lambda}^{n/2-2N}\,d\boldsymbol{\lambda}.\end{aligned}$$

The sum of $T_{j,+}$s in $j$ is clearly convergent if we take $N$ sufficiently large.

For the term $T_-$, the phase function may be stationary, we have to make a subtler decomposition. One may rewrite the integral as $T_- = \sum_{k\geq 0} T_{k,-}$, where

$$T_{0,-} = \int_0^\infty e^{i\boldsymbol{\lambda}^2 - it^{-1/2}\boldsymbol{\lambda}d(z,z')}\boldsymbol{\lambda}^n\tilde{a}_-(t^{-1/2}\boldsymbol{\lambda}, z, z')\sum_{k\leq 0}\psi_k(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}$$

$$T_{k,-} = \int_0^\infty e^{i\boldsymbol{\lambda}^2 - it^{-1/2}\boldsymbol{\lambda}d(z,z')}\boldsymbol{\lambda}^n\tilde{a}_-(t^{-1/2}\boldsymbol{\lambda}, z, z')\psi_k(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}, \quad k > 0$$

$$\psi_k(\boldsymbol{\lambda}) = \phi\Big(2^{-k}\big|2\boldsymbol{\lambda} - t^{-1/2}d(z,z')\big|\Big).$$

If we plug the estimates for $\tilde{a}_-$ in $T_{0,-}$, we will have $T_{0,-}$ bounded by

$$(1+d(z,z'))^{n/2}e^{-nd(z,z')/2}\int_0^\infty \frac{t^{n/4}}{(1+d(z,z'))^{n/2}\boldsymbol{\lambda}^{n/2}}\boldsymbol{\lambda}^n\sum_{k\leq 0}\psi_k(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}.$$

The latter integral is convergent. In fact, if $t^{-1/2}d(z,z')$ is bounded, $\boldsymbol{\lambda}$ will also be bounded, because of

$$\operatorname{supp}\left(\sum_{k\leq 0}\psi_k\right) = \{|2\boldsymbol{\lambda} - t^{-1/2}d(z,z')| \leq 2\}.$$

Therefore, the conditions of large distance (large $d(z,z')$), short time (small $t$), and high energy (large $\boldsymbol{\lambda}$), make the fraction in the integrand also bounded on the domain. So the $\boldsymbol{\lambda}$-integration is convergent. If $t^{-1/2}d(z,z')$ is large, the restriction $|2\boldsymbol{\lambda} - t^{-1/2}d(z,z')| \le 2$ from the support of $\sum_{k\le 0}\psi_k$ implies $\boldsymbol{\lambda} \sim t^{-1/2}d(z,z')$. Consequently, for any value of $t^{-1/2}d(z,z')$, we have

$$\frac{t^{n/4}}{(1+d(z,z'))^{n/2}\boldsymbol{\lambda}^{n/2}}\boldsymbol{\lambda}^n \le C$$

Then the integral $T_{0,-}$ is bounded by

$$C\int\limits_{\{2\boldsymbol{\lambda}-t^{-1/2}d(z,z')<2\}}\sum_{k<0}\psi_k\, d\boldsymbol{\lambda} \le C.$$

For the $T_{k,-}$ terms, since $|2\boldsymbol{\lambda} - t^{-1/2}d(z,z')| > 1$, namely the phase function is non-stationary, we can employ the integration by parts argument. We denote

$$L_- = \frac{1}{2\boldsymbol{\lambda} - t^{-1/2}d(z,z')}\frac{d}{d\boldsymbol{\lambda}}$$

and get the following estimates for $\sum_{k>0} T_{k,-}$

$$\begin{aligned}
&\sum_{k>0}\left|\int_0^\infty e^{i(\boldsymbol{\lambda}^2-t^{-1/2}d(z,z')\boldsymbol{\lambda})}\boldsymbol{\lambda}^n\tilde{a}_-\big(t^{-1/2}\boldsymbol{\lambda},z,z'\big)\psi_k(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}\right| \\
&\quad=\sum_{k>0}\left|\int_0^\infty L_-^N\Big(e^{i(\boldsymbol{\lambda}^2-t^{-1/2}d(z,z')\boldsymbol{\lambda})}\Big)\boldsymbol{\lambda}^n\tilde{a}_-\big(t^{-1/2}\boldsymbol{\lambda},z,z'\big)\psi_k(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}\right| \\
&\quad\le Ce^{-nd(z,z')/2}\sum_{k>0}2^{-kN}\int\limits_{|2\boldsymbol{\lambda}-t^{-1/2}d(z,z')|\sim 2^k}\boldsymbol{\lambda}^{n/2-N}t^{n/4}\,d\boldsymbol{\lambda} \\
&\quad\le C(1+d(z,z'))^{n/2}e^{-nd(z,z')/2},
\end{aligned}$$

provided $N$ is large enough. □

**Remark 8.** A key point of this integration by parts argument in this proof is that we can have a non-stationary phase function in the oscillatory integral. To get the dispersive estimates for the propagator $U_i(t)U_j^*(s)$, we will get a non-stationary phase function and run this argument again.

## 6. Dispersive estimates for Schrödinger propagators II

Because we will have to establish the retarded estimates for the Strichartz estimates, the off-diagonal microlocalized spectral measure $Q_j dE_P Q_k^*$ will confront us. Before stating off-diagonal microlocalized dispersive estimates, we have to define all relations of the microlocalization pairs $(Q_j, Q_k^*)$.

This was discussed by Guillarmou and Hassell [18] for Sobolev estimates, which is closely related to Strichartz estimates. Let us review their notions of outgoing/incoming relations. Suppose $g^t$ is the geodesic/bicharacteristic flow and $Q, Q'$ are two semiclassical pseudodifferential operators of semiclassical order 0 and differential order $-\infty$. We say $Q$ is not outgoing related to $Q'$ if the forward flowout $g^t(\mathrm{WF}_h(Q'))$ with $t \ge 0$ doesn't meet $\mathrm{WF}_h(Q)$, whilst $Q$ is not incoming related to $Q'$ if the backward flowout $g^t(\mathrm{WF}_h(Q'))$ with $t \le 0$ doesn't meet $\mathrm{WF}_h(Q)$. It is useful to note $Q$ not incoming related to $Q'$ is equivalent to $Q'$ not outgoing related to $Q$.

**Proposition 9** *(Dispersive estimates II). There is a refined pseudodifferential operator partition of unity $Id = \sum_{k=0}^N Q_k$ such that*

$$\left|\int_0^\infty e^{i(t-s)\lambda^2}\chi_\infty^2\Big(Q_j(\lambda)dE_P(\lambda)Q_k^*(\lambda)\Big)(z,z')\,d\lambda\right| \quad \text{for } |t-s|>1+d(z,z') \qquad (6.1)$$
$$\leq C|t-s|^{-\infty}e^{-nd(z,z')/2}$$

$$\left|\int_0^\infty e^{i(t-s)\lambda^2}\chi_\infty^2\Big(Q_j(\lambda)dE_P(\lambda)Q_k^*(\lambda)\Big)(z,z')\,d\lambda\right| \quad \text{for } |t-s|<1+d(z,z'), \qquad (6.2)$$
$$\leq C|t-s|^{-(n+1)/2}(1+d(z,z'))^{n/2}e^{-nd(z,z')/2}$$

*hold for all $t\neq s$ if $WF_h(Q_j)\cap WF_h(Q_k)\neq\emptyset$, for $t<s$ if $Q_j$ is not outgoing related to $Q_k$, for $s<t$ if $Q_j$ is not incoming related to $Q_k$.*

Before proving the dispersive estimates, we have to refine the microlocalization for spectral measure and categorize all microlocalization pairs $\{(Q_j, Q_k^*)\}_{j,k=1}^N$, where $Q_0$ is neglected as it is not on the semiclassical wavefront set of spectral measure.

**Lemma 10.** *The microlocalization pair $(Q_j, Q_k^*)$ with $j,k\geq 1$ must obey one of the following relations:*

(i) *$Q_j$ is not outgoing related to $Q_k$.*
(ii) *$Q_j$ is not incoming related to $Q_k$.*
(iii) *The off-diagonal microlocalized spectral measure at high energy takes the form*

$$Q_j(\lambda)\frac{d^j}{d\lambda^j}dE_P(\lambda)Q_k^*(\lambda) = e^{i\lambda d(z,z')}\lambda^n M_+ + e^{-i\lambda d(z,z')}\lambda^n M_- + O(\lambda^{-\infty}), \qquad (6.3)$$

*for large $\lambda$, where $M_\pm$ are defined on the forward and backward bicharacteristic flow respectively and satisfying*

$$\frac{d^jM_\pm}{d\lambda^j} = \begin{cases} O\Big(\lambda^{-j}\big(1+\lambda d(z,z')\big)^{-n/2}\Big) & , \quad \text{if } d(z,z') \text{ is small} \\ O\Big(\lambda^{-n/2-j}e^{-nd(z,z')/2}\Big) & , \quad \text{if } d(z,z') \text{ is large} \end{cases}.$$

**Proof.**[5] Recall in Section 2 we have taken a negatively curved strip neighbourhood of $\partial X$, say $\{x\leq 2\epsilon\}$. Also recall the cubes $Q_1,\ldots,Q_{N_1}$ contained in $B_1,\ldots,B_{N_1}$, affiliated with $I_1,\cdots,I_{N_1}$, over $\{x\leq\epsilon\}$ and $Q_{N_1+1},\ldots,Q_{N_2}$ supported on $B_{N_1+1},\ldots,B_{N_2}$ over $\{x>\epsilon\}$. Now we also assume $\{x\geq 2\epsilon\}$ is compact and geodesically convex. It is true as long as $\epsilon$ is sufficiently small.

Therefore there are the following cases of microlocalization:

(1) $(Q_j, Q_k^*)$ with $B_j\cap B_k\neq\emptyset$, $1\leq j\leq N_1$ and $1\leq k\leq N_1$
Since $B_j$ is intersected with $B_k$, then $I_j$ is intersected with $I_k$. One may prescribe $k=j+1$. Note both $I_j$ and $I_{j+1}$ are small subintervals in $[-3/2, 3/2]$ and they are contained in $I_j\cup I_{j+1}$. Then one can find a slice $B_{j,j+1}$ in $\{x<\epsilon\}\cap\{\lambda\in I_j\cup I_{j+1}\}$. Since $I_j\cup I_{j+1}$ is a small interval in $[-3/2, 3/2]$ too, we can find a pseudodifferential operator $Q_{j,j+1}$ microlocally supported on $B_{j,j+1}$ such that $Q_{j,j+1}dE_PQ_{j,j+1}^*$ satisfies (6.3). So does $Q_jdE_PQ_k^*$.
(2) $(Q_j, Q_k^*)$ with $B_j\cap B_k\neq\emptyset$, $N_1\leq j\leq N_2$ and $N_1\leq k\leq N_2$
Since the diameter of $B_j\cup B_k$ is bounded by a very small number, $B_j$ and $B_k$ are contained in a very small cube $B_{jk}$. Then $Q_{jk}dE_PQ_{jk}^*$, with $Q_{jk}$ microlocally supported on $B_{jk}$, satisfies (6.3). Consequently, so does $Q_jdE_PQ_k^*$.
(3) $(Q_j, Q_k^*)$ with $B_j\cap B_k\neq\emptyset$, $1\leq j\leq N_1$ and $N_1\leq k\leq N_2$
Since the diameter of $B_k$ is very small in the sense of Sasaki distance, we can narrow the range of $\lambda$ variable in $I_j$ and the range of $x$ variable in $\{x<2\epsilon\}$ such that both $B_k$ and $B_j$ are contained in a small slice $B_{jk}$ near

[5] The proof is essentially due to Guillarmou and Hassell [18].

the boundary with $\{\lambda \in I_j\}$. Then we again find a pseudodifferential operator $Q_{jk}$ microlocally supported on $B_{jk}$ such that $Q_{jk} dE_P Q_{jk}^*$ satisfies (6.3).

(4) $(Q_j, Q_k^*)$ with $B_j \cap B_k = \emptyset$, $1 \le j \le N_1$ and $1 \le k \le N_1$

Recall from [29] or [11] that the 0-Hamilton vector field with Hamiltonian $p = \lambda^2 + h(y, \lambda, \mu)$ on asymptotically hyperbolic manifold $X$ is

$$x \frac{\partial p}{\partial \lambda} \frac{\partial}{\partial x} + x \frac{\partial p}{\partial \mu} \cdot \frac{\partial}{\partial y} - \left( \mu \cdot \frac{\partial p}{\partial \mu} + x \frac{\partial p}{\partial x} \right) \frac{\partial}{\partial \lambda} + \left( \frac{\partial p}{\partial \lambda} \mu - x \frac{\partial p}{\partial y} \right) \cdot \frac{\partial}{\partial \mu}.$$

The variable $\lambda$, along the geodesic, decreases down to $-1$, in a small neighbourhood of the boundary.

Without loss of generality, one may assume $\inf(I_j) > \sup(I_k)$. Take a geodesic $\gamma(t)$ with $\gamma(0) \in B_k$. If $\gamma(t)$ stays in $\{x < \epsilon\}$ for $t \ge 0$, $\{\gamma(t) : t \ge 0\}$ will be disjoint from $B_j$, since $\lambda$ is nonincreasing along the forward bicharacteristic near the boundary. On the other hand, if $\gamma(t)$ goes beyond $\{x < \epsilon\}$ at time $t_2$ (i.e. $\gamma(t_2) \in \{x \ge \epsilon\}$), we have $\lambda(0) > 0$, hence $\inf(I_j) > \sup(I_k) > 0$. So we can find a maximal interval $(t_1, t_3)$ containing $t_2$ on $\{x \ge \epsilon\}$ such that $\lambda(t) > 0$ for all $t < t_1$ and $\lambda(t) < 0$ for all $t > t_3$, since $\lambda$ is nonincreasing in $\{x < \epsilon\}$. Consequently, $\gamma$ is disjoint from $B_j$ whenever $t > 0$: when $0 < t < t_1$ i.e. $\lambda < \lambda(0) < \inf(I_j)$; when $t_2 < t < \infty$ i.e. $\lambda < 0 < \inf(I_j)$.

(5) $(Q_j, Q_k^*)$ with $B_j \cap B_k = \emptyset$, $1 \le j \le N_1$ and $N_1 \le k \le N_2$

Take a geodesic $\gamma$ with $\gamma(0) \in B_j$. If $\sup I_j < 0$, then $x(t)$ is non-increasing namely $\gamma(t)$ will stay in $\{x < \epsilon\}$ for $t > 0$ and be disjoint from $B_k$. In the meantime, if $\inf I_j > 0$, $\gamma(t)$ will stay in $\{x < \epsilon\}$ for $t < 0$. If $0 \in I_j$ and $\lambda(t_0) = 0$, $x(t)$ is nonincreasing for all $t > t_0$, since $\lambda$ is non-positive afterwards. So $\gamma(t)$ will stay in $\{x < \epsilon\}$ for all $t > t_0$.

(6) $(Q_j, Q_k^*)$ with $B_j \cap B_k = \emptyset$, $N_1 \le j \le N_2$ and $N_1 \le k \le N_2$

Consider the function $(z, t) \to x(g^t)$. Since $dx(g^t(z))/dt \ne 0$ locally in $\{x > \epsilon/2\}$, we apply implicit function theorem and get an implicit function $t(z)$. We can find a time $t(z)$ such that $x(g^{t(z)}) = \epsilon/2$. Therefore, for any compact set $K \subset \{|\zeta| = 1\} \cap \{x > \epsilon/2\}$, there is a $T_+ > 0$ respectively $T_- < 0$ such that $g^t(K) \subset \{x < \epsilon/2\}$ for all $t > T_+$ respectively $t < T_-$. Assuming $B_j$ and $B_k$ are outgoing related and incoming related, we shall get a contradiction. Under this hypothesis and the compactness, there exist two sequences of points $\{z_l\} \subset \{x > \epsilon/2\}$ and $\{z_l'\} \subset \{x > \epsilon/2\}$ with two sequences of times $\{t_l : t_l < -\iota < 0\}$ and $\{t_l' : t_l' > \iota > 0\}$ both going to the same point $z \in \{z > \epsilon/2\}$ via the geodesic $g^t$, that is

$$\lim_{l \to \infty} g^{t_l}(z_l) = \lim_{l \to \infty} g^{t_l'}(z_l').$$

Since $T_- \le t < -\iota < \iota < t' \le T_+$, we can find accumulation points $t$ and $t'$ respectively. Then we have $g^t(z) = g^{t'}(z)$. It gives a periodic geodesic which contradicts the non-trapping condition. Therefore we have either $Q_j$ is not outgoing related to $Q_k$ or $Q_j$ is not incoming related to $Q_k$. □

With this lemma, we can prove the dispersive estimates for off-diagonal microlocalized high energy truncated Schrödinger propagators.

**Proof of Proposition 9.** It is proved by the argument of Proposition 6 with minor changes based on the classification of microlocalizations in Lemma 10.

If $Q_j dE_P(\boldsymbol{\lambda}) Q_k^*$ obeys (6.3), it means this operator satisfies the same estimates with the diagonal case as in Proposition 3. Therefore, we can get desired dispersive estimates by repeating the proof of Proposition 6.

If $Q_j$ and $Q_k$ are not outgoing related,[6] we claim the $U_j(t)U_k^*(s)$ for $t < s$ is a Fourier integral operator

$$\int_0^\infty \int_{\mathbb{R}^N} e^{i(t-s)\boldsymbol{\lambda}^2 + \boldsymbol{\lambda}\phi} a(h, z, z', \theta)\, d\theta\, d\boldsymbol{\lambda},$$

provided $\phi(z, z', \theta) < -\epsilon < 0$ is the phase function of the wavefront set $\Lambda$ of the spectral measure. Since we can always write the propagator in this integral form, the only point we need to justify is that $\phi(z, z', \theta) < -\epsilon < 0$.

[6] The proof is exactly the same in case $Q_j$ and $Q_k$ are not incoming related.

Recall that the forward bicharacteristic flow-out is that the flow-out of the Hamilton vector field of the metric function. By standard theory of Lagrangian distributions, the phase function $\phi$ can parametrize the forward flow-out in the following way that is $\Lambda^+$ is locally furnished coordinates

$$\{(z, \phi'_z) | \phi'_\theta = 0\}.$$

Hence phase function $\phi$ of forward bicharacteristic flow-out $\Lambda^+$ satisfies

$$\phi(z, z', \theta) = r(z, z') \geq d(z, z'), \quad \text{when } \phi'_\theta = 0$$

where $r$ is the curve length along the bicharacteristic and $d$ is the geodesic distance. Since $Q_j$ is not outgoing related to $Q_k$, i.e. the forward geodesic flow-out of $\mathrm{WF}_h Q_k^*$ doesn't meet $\mathrm{WF}_h Q_j$, they are connected by the backward flow-out, namely

$$\phi = -r(z, z') \leq -d(z, z') < 0.$$

The not outgoing relation gives a constantly negative sign of the phase function $\phi$ of microlocalized spectral measure $Q_j(\lambda) dE_P(\lambda) Q_k^*(\lambda)$. Since $t - s < 0$, the phase function of the propagator is negative. So it allows us to play the integration by parts argument in Proposition 6 by the differential operator

$$\frac{-i}{2\lambda - \phi/\sqrt{s-t}} \frac{\partial}{\partial \lambda}$$

to get the prove (6.2), instead of $-i/(2\lambda - t^{-1/2} d(z, z'))\partial_\lambda$ in the proof of (5.3). On the other hand, noting $\phi$ and $t - s$ have the same sign, namely the phase is non-stationary, we apply the rapid decay estimates, which readily shows (6.1). □

## 7. Strichartz estimates

We turn to proving Theorem 1.

First of all, we shall establish the Strichartz estimates

$$\|u\|_{L^q(\mathbb{R}, L^r(X))} \leq C \|f\|_{L^2(X)}$$

for the homogeneous equations (i.e. $F \equiv 0$). Recall the low energy truncated propagator and high energy microlocalized propagators. The solution $u$ of the homogeneous equation reads

$$e^{itn^2/4} u(t, x) = \left( U_{\mathrm{low}}(t) + \sum_{j=0}^{N} U_j(t) \right) f(z)$$

for $0 < t < 1$, where

$$U_{\mathrm{low}}(t) = \int_0^\infty e^{it\lambda^2} \chi_{\mathrm{low}}(\lambda)\, dE_P(\lambda) \quad \text{and} \quad U_j = \int_0^\infty e^{it\lambda^2} \chi_\infty(\lambda) Q_j(\lambda)\, dE_P(\lambda).$$

The Strichartz estimates for homogeneous equations

$$\|e^{itP^2} f(z)\|_{L^q_t L^r_z} \leq C \|f\|_{L^2_z}$$

are equivalent to

$$\left\| \int e^{-isP^2} G(s, z)\, ds \right\|_{L^2_z} \leq C \|G\|_{L^{q'}_t L^{r'}_z}.$$

Noting the decomposition

$$e^{i(-s)P^2} = U^*_{\mathrm{low}}(s) + \sum_{j=0}^{N} U_j^*(s),$$

it suffices to show

$$\left\| \int U_k^*(s)G(s,z)\,ds \right\|_{L_z^2} \le C\|G\|_{L_t^q L_z^r},$$

where $k \in \{0, 1, \dots, N, \text{low}\}$. By $TT^*$, it is equivalent to

$$\left\| \int \big(U_k(t)U_k^*(s)F(s,z)\big)ds \right\|_{L_t^q L_z^r} \le C\|F\|_{L_s^{q'} L_z^{r'}}.$$

One can split the left hand side by time. The long time part reduces to

$$\left( \int \left\| \int_{|t-s|\ge 1} U_k(t)U_k^*(s)F(s,z)\,ds \right\|_{L_z^r}^q dt \right)^{1/q}, \tag{7.1}$$

in the meantime, the short time part reduces to

$$\left( \int \left\| \int_{|t-s|\le 1} U_k(t)U_k^*(s)F(s,z)\,ds \right\|_{L_z^r}^q dt \right)^{1/q}. \tag{7.2}$$

To estimate these integrals, we need the following mapping properties of the propagators, which we shall prove in the last section:

**Lemma 11** *(Long times). Suppose* $|t-s| \ge 1$ *and* $2 < r, \tilde{r} \le \infty$. *Then the following inequalities hold*

$$\|U_{low}(t)U_{low}^*(s)\|_{L_z^{\tilde{r}'} \to L_z^r} \le C|t-s|^{-3/2} \tag{7.3}$$

$$\|U_j(t)U_k^*(s)\|_{L_z^{\tilde{r}'} \to L_z^r} \le C|t-s|^{-3/2} \tag{7.4}$$

*where the last one only holds for either* $t-s > 1$ *or* $s-t > 1$ *if* $j \ne k$ *and for both if* $j = k$.

**Lemma 12** *(Short times). Suppose* $0 < |t-s| < 1$ *and* $2 < r, \tilde{r} \le \infty$. *Then the following inequalities hold*

$$\|U_{low}(t)U_{low}^*(s)\|_{L_z^{\tilde{r}'} \to L_z^r} \le C|t-s|^{-\max\{1/2-1/r, 1/2-1/\tilde{r}\}(n+1)} \tag{7.5}$$

$$\|U_j(t)U_k^*(s)\|_{L_z^{\tilde{r}'} \to L_z^r} \le C|t-s|^{-\max\{1/2-1/r, 1/2-1/\tilde{r}\}(n+1)}, \tag{7.6}$$

*where the last one only holds for either* $0 < t-s < 1$ *or* $0 < s-t < 1$ *if* $j \ne k$ *and for both if* $j = k$.

Assuming these lemmas for the moment, we now continue the proof of Strichartz estimates.
We insert (7.4) and (7.3) into (7.1) and get

$$\begin{aligned} &\left( \int \left( \int_{|t-s|\ge 1} \left\| U_k(t)U_k^*(s)F(s,z) \right\|_{L_z^r} ds \right)^q dt \right)^{1/q} \\ &\le C\left( \int \left( \int_{|t-s|\ge 1} |t-s|^{-3/2} \|F(s,z)\|_{L_z^{r'}}\, ds \right)^q dt \right)^{1/q} \\ &\le \|F(s,z)\|_{L_s^{q'} L_z^{r'}}. \end{aligned}$$

We remark the kernel $|t-s|^{-3/2}\chi_{|t-s|>1}$ is integrable so it maps $L^{q'}(\mathbb{R})$ to $L^q(\mathbb{R})$ for any $q \ge 2$, where no admissibility is needed.

On the other hand, one can use the short time estimates (7.5) and (7.6). For $(q, r) \ne (2, 2(n+1)/(n-1))$, we invoke the admissibility condition (1.3) and Hardy–Littlewood–Sobolev inequality

$$\left(\int\left(\int_{|t-s|\le 1}\|U_k(t)U_k^*(s)F(s,z)\|_{L_z^r}\,ds\right)^q dt\right)^{1/q}$$
$$\le C\left(\int\left(\int_{|t-s|\le 1}\frac{1}{|t-s|^{(1/2-1/r)(n+1)}}\|F(s,z)\|_{L_z^{r'}}\,ds\right)^q dt\right)^{1/q}$$
$$\le C\left(\int\left(\int_{|t-s|\le 1}\frac{1}{|t-s|^{2/q}}\|F(s,z)\|_{L_z^{r'}}\,ds\right)^q dt\right)^{1/q}$$
$$\le C\|F(s,z)\|_{L_s^{q'}L_z^{r'}}.$$

Here the last inequality requires $q<2$, which is invalid for endpoints.

The short time endpoint estimates are proved via dispersive estimates and energy estimates by the standard Keel–Tao argument.

Next, following an argument from [21], we prove the inhomogeneous Strichartz estimates with the homogeneous estimates we have proved, that is

$$\|e^{itP^2}f(z)\|_{L_t^qL_z^r}\le C\|f\|_{L_z^2},$$

provided $(q,r)$ satisfies (1.3). These estimates are equivalent to

$$\left\|\int e^{i(t-s)P^2}F(s)\right\|_{L_s^qL_z^r}\le C\|F\|_{L_t^{\tilde q'}L_z^{\tilde r'}},$$

provided $(\tilde q,\tilde r)$ also satisfies (1.3). By Duhamel's formula, the desired inhomogeneous Strichartz estimates are equivalent to the retarded estimates

$$\left\|\int_{s<t} e^{i(t-s)P^2}F(s)\right\|_{L_s^qL_z^r}\le C\|F\|_{L_t^{\tilde q'}L_z^{\tilde r'}}. \tag{7.7}$$

For the non-endpoint case i.e. neither of $(q,r)$ and $(\tilde q,\tilde r)$ is $(2,2(n+1)/(n-1))$, the retarded estimates (7.7) follow immediately from Christ–Kiselev lemma:

**Lemma 13** *([13]). Let $X,Y$ be Banach spaces, let $I$ be a time interval, let $K\in C^0(I\times I)$ be a kernel taking values in the space bounded operators from $X$ to $Y$. Suppose that $1\le p<q\le\infty$ and*

$$\left\|\int_I K(t,s)f(s)\,ds\right\|_{L_t^q(I\to Y)}\le C\|f\|_{L_t^p(I\to X)}.$$

*Then one has*

$$\left\|\int_{\{s\in I:s<t\}} K(t,s)f(s)\,ds\right\|_{L_t^q(I\to Y)}\le C\|f\|_{L_t^p(I\to X)}.$$

On the other hand, in order to establish the endpoint inhomogeneous estimates, we end up with the following bilinear estimates

$$\iint_{s<t}\langle e^{i(t-s)P^2}F(s),G(t)\rangle\,dsdt\le C\|F\|_{L_t^2L_z^{r'}}\|G\|_{L_t^2L_z^{r'}}.$$

Plugging in the decomposition of the propagator, we have to establish the following estimates

$$\iint_{s<t}\langle U_j(t)U_k^*(s)F(s),G(t)\rangle\,dsdt\le C\|F\|_{L_t^2L_z^{r'}}\|G\|_{L_t^2L_z^{r'}} \tag{7.8}$$

$$\iint_{s<t} \langle U_{\text{low}}(t)U_{\text{low}}^*(s)F(s), G(t)\rangle\, dsdt \le C\|F\|_{L_t^2 L_z^{r'}} \|G\|_{L_t^2 L_z^{r'}} \tag{7.9}$$

$$\iint_{s<t} \langle U_{\text{low}}(t)U_k^*(s)F(s), G(t)\rangle\, dsdt \le C\|F\|_{L_t^2 L_z^{r'}} \|G\|_{L_t^2 L_z^{r'}}. \tag{7.10}$$

We want to use the standard Keel–Tao endpoint argument [24, Section 7]. So we have to establish the dispersive estimates and energy estimates for these propagators. The energy estimates are proved in Proposition 4. The dispersive estimates for $U_{\text{low}}(t)U_{\text{low}}^*(s)$ are proved in (5.2), while $U_{\text{low}}(t)U_k^*(s)$ actually satisfies (5.2) as well. Therefore (7.9) and (7.10) are proved by the standard Keel–Tao retarded estimates.

The tricky one is (7.8). According to Proposition 9, we only have the dispersive estimates for $U_j(t)U_k^*(s)$ when $t < s$ in the case of $Q_j$ is not outgoing related to $Q_k$, though (7.8) is proved as above for other cases. Namely, what we can prove by the Keel–Tao argument when $Q_j$ is not outgoing related to $Q_k$ is that

$$\iint_{t<s} \langle U_j(t)U_k^*(s)F(s), G(t)\rangle\, dsdt \le C\|F\|_{L_t^2 L_z^{r'}} \|G\|_{L_t^2 L_z^{r'}}.$$

Nevertheless, noting that the homogeneous Strichartz estimates, by duality, implies

$$\iint \langle U_j(t)U_k^*(s)F(s), G(t)\rangle\, dsdt \le C\|F\|_{L_t^2 L_z^{r'}} \|G\|_{L_t^2 L_z^{r'}}.$$

So we still obtain (7.8).

## 8. Mapping properties of Schrödinger propagators

It remains to prove Lemma 11 and Lemma 12.

**Proof of (7.5) and (7.6).** The short time behaviour (7.5) and (7.6) come from the interpolation among

$$\|\cdot\|_{L^1\to L^r} \le Ct^{-(n+1)/2} \quad \text{for any } r>2, \tag{8.1}$$

$$\|\cdot\|_{L^{r'}\to L^\infty} \le Ct^{-(n+1)/2} \quad \text{for any } r>2, \tag{8.2}$$

$$\|\cdot\|_{L^2\to L^2} \le C. \tag{8.3}$$

The last one (8.3) is indeed Proposition 4, while we shall prove (8.1) and (8.2), via dispersive estimates, by a comparison argument with hyperbolic space. The identical argument is used to show the restriction theorem in [12].

Observing the RHS of the short time dispersive estimates in Proposition 6, let us consider the kernel

$$K_t(z,z') = t^{-(n+1)/2}(1+d(z,z'))^{n/2} e^{-nd(z,z')/2}$$

instead of the propagators. We claim, for $r>2$,

$$\|K_t\|_{L^1\to L^r} = \sup_{z'} \|K_t\|_{L_z^r} \le t^{-(n+1)/2}.$$

Thinking of $K_t$ as a function supported on $X_0^2$, we decompose it as

$$K_t = K_t\cdot \chi_U + K_t\cdot(1-\chi_U),$$

where $U$ is a small neighbourhood of the front face.

The second part is proved by the fact that $d(z,z')$ is comparable to $-\log(xx')$ away from the front face. Then we have

$$\begin{aligned}\sup_{z'} \|K_t\|_{L_z^r} &= t^{-(n+1)/2} \sup_{z'} \left( \int (1+d(z,z'))^{nr/2} e^{-nrd(z,z')/2} dg_z \right)^{1/r} \\ &\le Ct^{-(n+1)/2} \sup_{x'} \int \big(-\log(xx')\big)^{nr/2} (xx')^{nr/2} \frac{dx}{x^n} \\ &\le Ct^{-(n+1)/2}.\end{aligned}$$

On the other hand, consider the spectral measure restricted to $U$ say $K_{t,U}(z,z') = K_t \cdot \chi_U$. Before looking into the specific estimate, we shall compare this region with hyperbolic space $\mathbb{H}^{n+1}$. To do so, one may further decompose the set $U$ into subsets $U_i$, where on each $U_i$, we have $x \le \epsilon$, $x' \le \epsilon$ and $d(y, y_i), d(y', y_i) \le \epsilon$ for some $y_i \in \partial X$ (where the distance is measured with respect to the metric $h(y, dy)$ on $\partial X$). Choose local coordinates $(x, y)$ on $X$, centred at $(0, y_i) \in \partial X$, covering the set $V_i = \{x \le \epsilon, d(y, y_i) \le \epsilon\}$, and use these local coordinates to define a map $\phi_i$ from $V_i$ to a neighbourhood $V_i'$ of $(0, 0)$ in hyperbolic space $\mathbb{H}^{n+1}$ using the upper half-space model (such that the map is the identity in the given coordinates). The map $\phi_i$ induces a diffeomorphism $\Phi_i$ from $U_i \subset X_0^2$ to a subset of $(\mathbb{H}^{n+1})_0^2$, the double space for $\mathbb{H}^{n+1}$, covering the set $x \le \epsilon$, $x' \le \epsilon$, $|y|, |y'| \le \epsilon$ in this space. Clearly, this map identifies $\rho_L$ and $\rho_R$ on $U_i$ with corresponding boundary defining functions for the left face and right face on $(\mathbb{H}^{n+1})_0^2$. We now reduce the kernel to

$$\phi_i \circ K_{t,U_i} \circ \phi_i^{-1} \tag{8.4}$$

as an integral operator on $(\mathbb{H}^{n+1})_0^2$. After linking the front face to the hyperbolic case, we now can reduce to the estimate to the hyperbolic case as follows.

$$\sup_{z'} \|K\|_{L_z^r(V_i)} = C \sup_{z'} \|\tilde{K}\|_{L_z^r(V_i')} \le C|t|^{-(n+1)/2},$$

where $K$ is mapped to $\tilde{K}$ on hyperbolic space and the $L^r$ norm of $\tilde{K}$ on hyperbolic space. □

**Proof of (7.3) and (7.4).** The long time behaviour results from the interpolation among

$$\|\cdot\|_{L^1 \to L^r} \le Ct^{-3/2} \quad \text{for any } r > 2, \tag{8.5}$$

$$\|\cdot\|_{L^{r'} \to L^\infty} \le Ct^{-3/2} \quad \text{for any } r > 2, \tag{8.6}$$

$$\|\cdot\|_{L^{r'} \to L^r} \le Ct^{-3/2} \quad \text{for any } r > 2, \tag{8.7}$$

provided $t > 1$.

The proofs of (8.5) and (8.6) are exactly the same with (8.1) and (8.2).

The novelty in the proof of (8.7) is a non-trivial non-Euclidean ingredient called the Kunze–Stein phenomenon, which is named after Kunze and Stein [25]. Specifically, the Kunze–Stein phenomenon on hyperbolic space $\mathbb{H}^{n+1}$ at $(2, 2)$ is expressed as

$$\|f * F\|_{L^2(\mathbb{H}^{n+1})} \le C\|f\|_{L^2(\mathbb{H}^{n+1})} \cdot \int_0^\infty |F(\rho)|(1+\rho)e^{n\rho/2} d\rho,$$

for any $f, F \in C_0(\mathbb{H}^{n+1})$, provided $F(\rho)$ is a radial function. See Cowling's work [14] for a general result on semi-simple Lie groups. There is a generalized inequality

$$\|f * F\|_{L^r(\mathbb{H}^{n+1})} \le C\|f\|_{L^{r'}(\mathbb{H}^{n+1})} \cdot \left( \int_0^\infty |F(\rho)|^{r/2}(1+\rho)e^{n\rho/2} d\rho \right)^{2/r}, \quad \text{for } r \ge 2 \tag{8.8}$$

obtained by Anker and Pierfelice [2].

According to the long time dispersive estimates (5.1)–(5.2),[7] we consider a kernel $K_t(z, z') = t^{-3/2}(1 + d(z, z'))e^{-nd(z,z')/2}$ on $X_0^2$ and decompose it as

$$K_t = K_t \cdot \chi_U + K_t \cdot (1 - \chi_U),$$

where $U$ is a small neighbourhood of the front face.

The part away the front face is proved like the short time case

$$\|K_t\|_{L^{r'} \to L^r} = \|K_t\|_{L^r(X_0^2 \setminus U)} \le t^{-3/2} \left( \int (1 + d(z, z'))^{nr/2} e^{-nrd(z,z')} dg_z dg_{z'} \right)^{1/r} \le Ct^{-3/2}.$$

[7] Note $t^{-(n+1)/2} < t^{-3/2}$ for the intermediate times $1 < |t - s| < 1 + d(z, z')$.

For the part near the front face, we link the front face to hyperbolic space as in (8.4) and then have

$$\left\| \int K * f \right\|_{L^r(V_i)} = C \left\| \int \tilde{K} * \tilde{f} \right\|_{L^r(V_i')} \leq C t^{-3/2} \|f\|_{L^{r'}(V_i)},$$

by invoking (8.8), where $K$ and $f$ are mapped to $\tilde{K}$ and $\tilde{f}$ on hyperbolic space respectively. □

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None declared.


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