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Robust local Hölder rigidity of circle maps with breaks

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Abstract

We prove that, for every $\varepsilon \in (0, 1)$, every two $C^{2+\alpha}$-smooth ($\alpha > 0$) circle diffeomorphisms with a break point, i.e. circle diffeomorphisms with a single singular point where the derivative has a jump discontinuity, with the same irrational rotation number $\rho \in (0, 1)$ and the same size of the break $c \in \mathbb{R}_+\setminus\{1\}$, are conjugate to each other via a conjugacy which is $(1-\varepsilon)$-Hölder continuous at the break points. An analogous result does not hold for circle diffeomorphisms even when they are analytic.

Keywords: Rigidity; Conjugacy; Circle maps with breaks

1. Introduction

The rigidity theory of circle diffeomorphisms is a classic topic in dynamical systems, which started with the work of Arnol'd [2] and was largely developed by Herman [8] and Yoccoz [23] (see also [9] and [15]). It concerns an implied regularity (often smoothness) of conjugacies between maps that belong to the same topological conjugacy class. Over the last twenty-five years a major focus has been put on understanding the rigidity properties of circle diffeomorphisms with a single singular point where the derivative has a jump discontinuity (*circle maps with a break*) or vanishes (*critical circle maps*). This paper presents the first rigidity result for circle maps with a break that holds for *all* irrational rotation numbers. It concerns a phenomenon not previously seen and establishes a result whose analog for circle diffeomorphisms does not hold.

The first result on the rigidity of circle diffeomorphisms concerns the smoothness of conjugations for analytic diffeomorphisms of a circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, close to a rotation $R_\rho : x \mapsto x + \rho \mod 1$, with $\rho \in (0, 1)\setminus\mathbb{Q}$. Arnol'd [2] proved, using methods of Kolmogorov–Arnol'd–Moser theory, that any analytic circle diffeomorphism with a Diophantine rotation number ρ, sufficiently close to the rotation R_ρ, is analytically conjugate to R_ρ. He also made a conjecture, proved almost two decades later by Herman [8], that the closeness to the rotation is not necessary for this claim to hold true. In fact, Herman proved that any C^∞-smooth (C^ω-smooth) circle diffeomorphism with a Diophantine rotation number ρ is C^∞-smoothly (C^ω-smoothly) conjugate to the rotation R_ρ. The required smoothness of the

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maps was further weakened by Yoccoz [23], establishing generic $C^{1+\epsilon}$-rigidity, with $\epsilon > 0$, of C^r smooth ($r \geq 3$) circle diffeomorphisms. A natural approach to Herman's theory is based on renormalization. Renormalizations of circle diffeomorphisms converge to linear maps with slope 1. A recent result [15] shows that $C^{2+\alpha}$-smooth circle diffeomorphisms with a Diophantine rotation number ρ of class $D(\delta)$, with $0 \leq \delta < \alpha < 1$, are $C^{1+\alpha-\delta}$-smoothly conjugate to R_ρ. A number ρ is Diophantine of class $D(\delta)$, for some $\delta \geq 0$, if there exists $\mathcal{C} > 0$ such that $|\rho - p/q| > \mathcal{C}/q^{2+\delta}$, for every $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. On the other hand, *robust rigidity*, i.e., rigidity for all irrational rotation numbers, does not hold even for analytic circle diffeomorphisms. In fact, Arnol'd constructed examples of analytic circle diffeomorphisms with the same Liouville (non-Diophantine) irrational rotation number for which the conjugacy is essentially singular.

We recently proved a sequence of results on the rigidity of circle maps with breaks that can be considered an extension of Herman's theory of the linearization of circle diffeomorphisms. In [12,13], we proved that, for almost all irrational $\rho \in (0, 1)$, any two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break, with the same rotation number ρ and the same size of the break $c \in \mathbb{R}_+\backslash\{1\}$ (i.e., the same square root of the ratio of the left and right derivatives at the break point), are C^1-smoothly conjugate to each other. This generic C^1-rigidity result follows from the exponential convergence of renormalizations of these maps that we proved in [12]. In fact, in [12], we proved that, for *all* irrational ρ, renormalizations f_n and $\widetilde{f}_n$ of any two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break T and $\widetilde{T}$, with the same irrational rotation number ρ and the same size of the break c approach each other exponentially fast (in the C^2-topology), i.e., there exist $\lambda \in (0, 1)$ and $C > 0$ such that

$$\| f_n - \widetilde{f}_n \|_{C^2[-1,0]} \leq C\lambda^n. \tag{1.1}$$

The exponential rate of convergence λ is universal and depends only on the size of the break c and α (for $\alpha < 1$, $\lambda = \mu^\alpha$, with $\mu \in (0, 1)$ independent of α). Partial results concerning the convergence of renormalizations restricted to sets of rotation numbers of zero Lebesgue measure, were previously obtained in [10,16]. A set S_{rig} of rotation numbers ρ for which C^1-rigidity holds [12,13] can be characterized, using the continued fraction expansion $\rho = [k_1, k_2, \dots]$, as follows. S_{rig} is the set of all ρ for which there exists a constant $C_1 > 0$ and $\lambda_1 \in (\lambda, 1)$ such that $k_{n+1} \leq C_1\lambda_1^{-n}$ for all $n \in 2\mathbb{N}$, if $c < 1$, or for all $n \in 2\mathbb{N} - 1$, if $c > 1$. The difference between n odd and n even comes from the difference in the behavior of the corresponding subsequences of renormalizations. We also proved [11] that, although generic, C^1-rigidity does not hold for all irrational rotation numbers. These results are analogous to those for circle diffeomorphisms. A recent result of one of us [19] shows that, for almost all irrational rotation numbers, $C^{1+\epsilon}$-rigidity of circle maps with breaks does not hold for any $\epsilon > 0$, contrary to the case of circle diffeomorphisms. The set S_{non} of rotation numbers for which $C^{1+\epsilon}$-rigidity does not hold includes all irrational numbers $\rho \in (0, 1)$, for which there is subsequence of k_{n+1}, with $n \in 2\mathbb{N}$, if $c < 1$, or with $n \in 2\mathbb{N} - 1$, if $c > 1$, which grows faster than linearly in n.

The smaller set of rotation numbers for which $C^{1+\epsilon}$-rigidity holds, for some $\epsilon > 0$, for circle maps with breaks, in comparison to circle diffeomorphisms, is the consequence of the *strongly unbounded geometry* of these maps. While, in the case of circle diffeomorphisms, the ratio of lengths of neighboring elements of dynamical partitions $\mathcal{P}_n$ is at most of the order of the partial quotient k_{n+1}, in the case of circle maps with a break, this ratio can be exponentially large in k_{n+1}. This can also be compared with analytic critical circle maps whose *bounded geometry*, i.e., the property that this ratio is bounded, is ultimately responsible for their robust C^1-rigidity. Namely, Khanin and Teplinsky proved [14] that any two analytic critical circle maps with the same irrational rotation number and the same order of the critical point are C^1-smoothly conjugate to each other. A critical point x_c is said to be of order $\beta > 1$ if the derivative of the map for x near x_c behaves as $|x - x_c|^{\beta-1}$. The result is based on the exponential convergence of renormalizations that was proved by de Faria and de Melo [7] for bounded type rotation numbers and extended to all irrational rotation numbers by Yampolsky [22]. In fact, de Faria and de Melo proved that a stronger, $C^{1+\epsilon}$-rigidity, of analytic critical circle maps holds for generic irrational rotation numbers [7]. They also proved that such a result cannot be extended to all irrational rotation numbers in the C^∞-class of maps [6]. A local result of Khmelev and Yampolsky [18] suggested that the analytic case might be different. Nevertheless, for any $\epsilon > 0$, Avila [3] constructed examples of analytic critical circle maps, with the same irrational rotation number and the same order of the critical point, that are not $C^{1+\epsilon}$-smoothly conjugate to each other. All positive rigidity results for critical circle maps with non-analytic critical points are, at the moment, conditional, due to the lack of proof of the convergence of renormalization in this case.

Contrary to the case of critical circle maps, as already mentioned above, robust C^1-rigidity does not hold even for analytic circle maps with a break. In [11], we even constructed pairs of analytic circle maps with a break, with the same irrational rotation number and the same size of the break, for which no conjugacy between them is Lipschitz continuous. The rotation numbers ρ of these maps have a rapidly growing (faster than some exponential function) subsequence of odd-indexed digits in the continued fraction expansion k_{n+1} of ρ, if $c < 1$, or even-indexed digits, if $c > 1$. In [11], we also proved that the conjugacy that maps the break point of one map into the break point of the other can be arbitrarily bad. More precisely, for any modulus of continuity, we constructed examples of analytic circle maps with a break, with the same irrational rotation number and the same size of the break, such that the conjugacy that maps the break point of one map into the break point of the other is not uniformly continuous with that modulus of continuity.

The main result of this paper is given by the following theorem.

Definition 1.1. Let $x_0 \in \mathbb{T}^1$. A function $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$ is locally β-Hölder continuous or β-Hölder continuous at x_0 or $\varphi(x_0)$ if there exists $\mathcal{C} > 0$ such that, for all $x \in \mathbb{T}^1$,

$$\mathcal{C}^{-1}|x - x_0|^{\frac{1}{\beta}} \le |\varphi(x) - \varphi(x_0)| \le \mathcal{C}|x - x_0|^{\beta}. \tag{1.2}$$

The conjugacy is β-Hölder continuous if it is β-Hölder continuous at each $x \in \mathbb{T}^1$.

Theorem 1.2. *Let $\varepsilon \in (0,1)$, $c \in \mathbb{R}_+\backslash\{1\}$, $\alpha \in (0,1)$ and let ρ be any irrational number in $(0,1)$. Then, for any two $C^{2+\alpha}$-smooth circle diffeomorphisms T and $\widetilde{T}$ with break points at x_c and $\widetilde{x}_c$, respectively, with the same rotation number ρ and the same size of the break c, there is a point $x_0 \in \mathbb{T}^1$ such that the conjugacy $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$ that satisfies $\varphi \circ T \circ \varphi^{-1} = \widetilde{T}$ and $\varphi(x_0) = \widetilde{x}_c$ is $(1-\varepsilon)$-Hölder continuous at the break points.*

Remark 1. For any $\varepsilon > 0$, this result establishes robust local $(1-\varepsilon)$-Hölder rigidity of $C^{2+\alpha}$-smooth circle diffeomorphisms with a break. This is the first rigidity result for such maps that holds for *all* irrational rotation numbers. An analogous result does not hold for circle diffeomorphisms, even when they are analytic.

Remark 2. We emphasize that the construction of the $(1-\varepsilon)$-Hölder continuous conjugacy in general requires a non-trivial "shift" of the preimage of the break point, i.e., for some irrational rotation numbers, $x_0 = \varphi^{-1}(\widetilde{x}_c) \neq x_c$. No previous rigidity result for circle maps with a break required such a "shift". In fact, this is the first rigidity result for circle diffeomorphisms with a singular point in general which involves conjugacy which does not map the singular point of one map into the singular point of the other.

In addition to being part of the rigidity theory of circle homeomorphisms, rigidity results for circle maps with breaks are also important for understanding properties of the generalized interval exchange transformations. Although quite natural, these transformations were introduced only recently by Marmi, Moussa and Yoccoz [20]. They are obtained by replacing linear branches with slope 1 of an interval exchange transformation by smooth diffeomorphisms. Just as an interval exchange transformation of two intervals can be seen as a rigid rotation on a circle, a generalized interval exchange transformation of two intervals is a circle map with two break points. As these two points lie on the same orbit of the map, the map can be piecewise-smoothly conjugated to a circle map with one point of break. Marmi, Moussa and Yoccoz considered the linearizable case of an arbitrary number of intervals [20], when there are no break points. The special case of cyclic permutations, which corresponds to circle maps with more points of break, but with product of the sizes of breaks equal to 1, was considered by Cunha and Smania [4,5]. In this case, renormalizations approach piecewise linear maps. In the case of circle maps with breaks with the product of the sizes of breaks along some orbit not equal to 1, the renormalizations are essentially non-linear and approach piecewise fractional linear transformations.

This paper is organized as follows. In Section 2, we review basic facts about dynamical partitions and renormalizations of circle homeomorphisms – the main technical tools that we use in this paper. In Section 3, we give a criterion of (local) Hölder continuity of a conjugacy between two circle homeomorphisms. In Section 4, we obtain some general estimates on the geometry of dynamical partitions. In particular, we show that the lengths of the corresponding fundamental intervals are asymptotically the same on the logarithmic scale. In Section 5, we prove that, after an appropriate shift of indexes, the renormalized intervals of the next level partition inside the fundamental intervals of dynamical partitions are, in some sense, comparable. Finally, in Section 6, we construct a particular conjugacy and prove Theorem 1.2.

2. Preliminaries

For every orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$ of the circle $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$, there exists a (unique up to an additive integer constant) continuous and strictly increasing function $\mathcal{T} : \mathbb{R} \to \mathbb{R}$, called a lift of T, that satisfies $\mathcal{T}(x+1) = \mathcal{T}(x) + 1$, for every $x \in \mathbb{R}$. Poincaré showed that, for every such $T : \mathbb{T}^1 \to \mathbb{T}^1$, there is a unique rotation number ρ, given by the limit $\rho := \lim_{n\to\infty} \mathcal{T}^n(x)/n \mod 1$, where $\mathcal{T}$ is any lift of T. Renormalizations of an orientation-preserving homeomorphism of a circle T, with a rotation number $\rho \in (0,1)$ are defined using the continued fraction expansion

$$\rho = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \dots}}}, \tag{2.1}$$

that we also write as $\rho = [k_1, k_2, k_3, \dots]$. The sequence of integers $(k_n)_{n\in\mathbb{N}}$, called *partial quotients*, is infinite if and only if ρ is irrational. Every irrational ρ defines uniquely the sequence of partial quotients. Conversely, every infinite sequence of partial quotients defines uniquely an irrational number ρ as the limit of the sequence of rational convergents $p_n/q_n = [k_1, k_2, \dots, k_n]$. It is well-known that this sequence forms a sequence of best rational approximations of an irrational ρ, i.e., there are no rational numbers with denominators smaller or equal to q_n, that are closer to ρ than p_n/q_n. The rational convergents can also be defined recursively by $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$, starting with $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$.

To define renormalizations of an orientation-preserving homeomorphism of a circle T, with an irrational rotation number ρ, we start with a *marked point* $x_0 \in \mathbb{T}^1$, and consider the marked trajectory $x_i = T^i x_0$, with $i \in \mathbb{N}$. The subsequence $(x_{q_n})_{n\in\mathbb{N}}$ indexed by the denominators q_n of the sequence of rational convergents of the rotation number ρ, will be called the sequence of *dynamical convergents*. It follows from the simple arithmetic properties of the rational convergents that the sequence of dynamical convergents $(x_{q_n})_{n\in\mathbb{N}}$ for the rigid rotation R_ρ has the property that its subsequence with n odd approaches x_0 from the left monotonically and the subsequence with n even approaches x_0 from the right monotonically. Since all circle homeomorphisms with the same irrational rotation number are combinatorially equivalent, the order of the dynamical convergents of T is the same.

The interval $[x_{q_n}, x_0]$, for n odd, and $[x_0, x_{q_n}]$, for n even, will be denoted by $\Delta_0^{(n)}$ and called the n-th renormalization segment associated to x_0. The n-th renormalization segment associated to x_i will be denoted by $\Delta_i^{(n)}$. It follows from the properties of the continued fractions that the only points of the orbit $\{x_i : 0 < i \le q_{n+1}\}$ that belong to $\Delta_0^{(n-1)}$ are $\{x_{q_{n-1}+iq_n} : 0 \le i \le k_{n+1}\}$.

A certain number of images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$, under the iterates of the map T, cover the whole circle without overlapping beyond the end points and form the n-th *dynamical partition* of the circle

$$\mathcal{P}_n := \{T^i \Delta_0^{(n-1)} : 0 \le i < q_n\} \cup \{T^i \Delta_0^{(n)} : 0 \le i < q_{n-1}\}. \tag{2.2}$$

The intervals $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ will be called the *fundamental intervals* of $\mathcal{P}_n$. We also define $\bar{\Delta}_0^{(n-1)} := \Delta_0^{(n-1)} \cup \Delta_0^{(n)}$ and the renormalization parameter $a_n := \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|}$, characterizing the geometry of dynamical partitions.

The n-th *renormalization* of an orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$, with a rotation number ρ, with respect to the marked point $x_0 \in \mathbb{T}^1$, is a function $f_n : [-1, 0] \to \mathbb{R}$ obtained from the restriction of T^{q_n} to $\Delta_0^{(n-1)}$, by rescaling the coordinates, in the following way. If τ_n is the affine change of coordinates that maps $x_{q_{n-1}}$ into -1 and x_0 into 0, then

$$f_n := \tau_n \circ T^{q_n} \circ \tau_n^{-1}. \tag{2.3}$$

Definition (2.3) is valid for all $n \in \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, if and only if ρ is irrational; otherwise, n is less than or equal to the length of the continued fraction expansion of ρ. If we identify x_0 with zero, then τ_n is exactly the multiplication by $(-1)^n/|\Delta_0^{(n-1)}|$. Here, and in what follows, $|I|$ denotes the length of an interval I on the circle $\mathbb{T}^1$. Notice that $f_n(0) = a_n$.

When necessary to state explicitly which marked point x_0 the quantities $\Delta_i^{(n)}$, $\bar{\Delta}_0^{(n-1)}$, a_n, $\mathcal{P}_n$, f_n and τ_n are associated to, they are denoted by $\Delta_i^{(n)}(x_0)$, $\bar{\Delta}_0^{(n-1)}(x_0)$, $a_n(x_0)$, $\mathcal{P}_{n,x_0}$, f_{n,x_0} and τ_{n,x_0}, respectively.

This paper concerns circle diffeomorphisms (maps) with a break, i.e., homeomorphisms of a circle for which there exists a point $x_c \in \mathbb{T}^1$, such that

(i) $T \in C^r(\mathbb{T}^1 \backslash \{x_c\})$,
(ii) $T'(x)$ is bounded from below by a positive constant on $\mathbb{T}^1 \backslash \{x_c\}$, and
(iii) the one-sided derivatives $T'_-(x_c)$ and $T'_+(x_c)$ at x_c are such that the *size of the break*

$$c := \sqrt{\frac{T'_-(x_c)}{T'_+(x_c)}} \neq 1.$$

In this paper, we will reserve the notation $\Delta_i^{(n)}$, $\bar{\Delta}_0^{(n-1)}$, a_n, $\mathcal{P}_n$, f_n and τ_n for the quantities associated to the marked point $x_0 = x_c$. The corresponding quantities, associated to the map $\widetilde{T}$ will be denoted by $\widetilde{\Delta}_i^{(n)}$, $\widetilde{\bar{\Delta}}_0^{(n-1)}$, $\widetilde{a}_n$, $\widetilde{\mathcal{P}}_n$, $\widetilde{f}_n$ and $\widetilde{\tau}_n$.

Since for circle maps with a break $V := \mathrm{Var}_{\mathbb{T}^1} \ln T' < \infty$, we have $|\ln(T^{q_n})'(x)| \leq V$, for all $x \in \mathbb{T}^1$, by Denjoy's lemma [21]. Therefore, we have the uniform bound

$$|\ln f_n'(x)| \leq V, \tag{2.4}$$

for all $x \in [-1, 0]$.

It was proved in [17] that the renormalizations of circle maps with a break approach a particular family of fractional linear transformations. Namely (see also [12]), for every $c \in \mathbb{R}_+ \backslash \{1\}$ and $\alpha \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that the renormalizations f_n, $n \in \mathbb{N}_0$, of a $C^{2+\alpha}$-smooth circle map T, with a break of size c, satisfy

$$\| f_n - F_n \|_{C^2[-1,0]} \leq C\lambda^n, \tag{2.5}$$

for some $C > 0$, where $F_n \equiv F_{a_n, b_n, M_n, c_n} : [-1, 0] \to \mathbb{R}$,

$$F_n(z) := \frac{a_n + (a_n + b_n M_n)z}{1 - (M_n - 1)z}, \tag{2.6}$$

with

$$a_n := \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|}, \quad b_n := \frac{|\Delta_0^{(n-1)}| - |\Delta_{q_{n-1}}^{(n)}|}{|\Delta_0^{(n-1)}|}, \quad M_n := \exp\left(\sum_{i=0}^{q_n - 1} \int_{x_{q_{n-1}+i}}^{x_i} \frac{T''(x)}{2T'(x)}\, dx \right). \tag{2.7}$$

Further information on closeness of renormalizations for maps with the derivative in a Zygmund class (under suitable arithmetic conditions) have been obtained in [1].

We end this section with a few more comments about the notation. For functions $f, g : \mathcal{D} \to \mathbb{R}$, with a domain $\mathcal{D}$, we write $f(x) = \mathcal{O}(g(x))$ if there is a constant $K > 0$, independent of $x \in \mathcal{D}$, such that $|f(x)| \leq K|g(x)|$. We write $f(x) = \Theta(g(x))$ if there is a constant $K > 0$, independent of $x \in \mathcal{D}$, such that $K^{-1}g(x) \leq f(x) \leq Kg(x)$.

3. A criterion of Hölder continuity of the conjugacy

In this section, we state and prove a criterion of Hölder regularity of the conjugacy.

Proposition 3.1 *(Criterion of local Hölder regularity). Let $\gamma \in (0, 1)$ and $x \in \mathbb{T}^1$. Let $\widetilde{T}, T : \mathbb{T}^1 \to \mathbb{T}^1$ be two orientation-preserving circle homeomorphisms and $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$ a homeomorphism satisfying*

$$\varphi \circ T \circ \varphi^{-1} = \widetilde{T}. \tag{3.1}$$

If there exist $\sigma > 0$ and $\delta > 0$ such that, for all $y \in \mathbb{T}^1$ satisfying $|x - y| < \delta$, there exist $J \in \mathbb{N}$ and a finite sequence of intervals $\Delta_j \subset [x, y]$, $j = 1, \ldots, J$, such that

(i) $\sum_{j=1}^{J} |\varphi(\Delta_j)| \geq \sigma |\varphi(x) - \varphi(y)|$,

(ii) $\sum_{j=1}^{J} |\Delta_j| \ge \sigma |x-y|$,
(iii) $(\forall j : 1 \le j \le J)$ $|\Delta_j| \ge \sigma |x-y|^2$,
(iv) $(\forall j : 1 \le j \le J)$ $|\varphi(\Delta_j)| \ge \sigma |\varphi(x)-\varphi(y)|^2$,
(v) $(\forall j : 1 \le j \le J)$

$$\gamma < \frac{\ln|\varphi(\Delta_j)|}{\ln|\Delta_j|} < 2-\gamma, \tag{3.2}$$

then the conjugacy φ and its inverse φ^{-1} are $2\gamma-1$-Hölder continuous at x and $\varphi(x)$, respectively.

Remark 3. In this paper, $[x, y]$ denotes the shortest arc on $\mathbb{T}^1$ with end points at x and y. $|x-y|$ denotes the shortest arc distance on $\mathbb{T}^1$, i.e., the length of $[x, y]$.

Proof. It follows from (3.2) that, for all $x \in \mathbb{T}^1$ and all $\Delta_j \subset [x, y]$ we have $|\varphi(\Delta_j)| \le |\Delta_j|^\gamma$ and $|\Delta_j| \le |\varphi(\Delta_j)|^\gamma$. Using (i) and (iii), we have

$$|\varphi(x)-\varphi(y)| \le \sigma^{-1}\sum_{j=1}^{J}|\varphi(\Delta_j)| \le \sigma^{-1}\sum_{j=1}^{J}|\Delta_j|^\gamma \le \sigma^{\gamma-2}|x-y|^{2\gamma-1}. \tag{3.3}$$

This proves that φ is $2\gamma-1$-Hölder continuous at x. The $2\gamma-1$-Hölder continuity of φ^{-1} at $\varphi(x)$ is established similarly, using (ii) and (iv),

$$|x-y| \le \sigma^{-1}\sum_{j=1}^{J}|\Delta_j| \le \sigma^{-1}\sum_{j=1}^{J}|\varphi(\Delta_j)|^{\frac{1}{2-\gamma}} \le \sigma^{\frac{1}{2-\gamma}-2}|x-y|^{\frac{2}{2-\gamma}-1}. \tag{3.4}$$

and the fact that $\frac{1}{2-\gamma} > \gamma$, for $\gamma \in (0, 1)$. □

It was shown in [11] that, for every $c \in \mathbb{R}_+ \backslash \{1\}$, there are irrational numbers $\rho \in (0, 1)$ and pairs of circle diffeomorphisms T and $\widetilde{T}$ with breaks at x_c and $\widetilde{x}_c$, respectively, with the same rotation number ρ and the same size of the break c, such that the conjugacy φ that satisfies (3.1) and $\varphi(x_c) = \widetilde{x}_c$ is not Hölder continuous at x_c. The main goal of this paper is to determine a point x_0, for any such pairs of maps, such that the assumptions of Proposition 3.1 are satisfied, with the intervals Δ_j chosen from among the intervals of dynamical partitions $\mathcal{P}_{n,x_0}$.

4. Estimates on the renormalization parameters

In the following, let T and $\widetilde{T}$ be two circle diffeomorphisms with breaks at x_c and $\widetilde{x}_c$, respectively, with the same irrational rotation number $\rho \in (0, 1)$ and the same size of the break $c \in \mathbb{R}\backslash\{1\}$. In this section, we obtain some general estimates on the renormalization parameters a_n and $\widetilde{a}_n$ and show that the logarithms of the lengths of the corresponding fundamental intervals of T and $\widetilde{T}$ are asymptotically the same.

Proposition 4.1. *Let $\lambda_1 \in (\lambda, 1)$ and $\lambda_2 \in (\sqrt{\lambda/\lambda_1}, 1)$. There exists $C_2 > 0$ such that, if $c_n > 1$ or if $c_n < 1$ and $k_{n+1} \le C_1\lambda_1^{-n}$, then*

$$\left|\frac{\widetilde{a}_n}{a_n} - 1\right| \le C_2\lambda_2^n. \tag{4.1}$$

Remark 4. If $c_n > 1$, (4.1) can actually be strengthened by replacing λ_2 with λ.

Proof. Let $\lambda_3 \in (\lambda/\lambda_2, \lambda_1\lambda_2)$. If $c_n < 1$ and $a_n \ge C_3\lambda_3^n$, for some $C_3 > 0$, the claim follows directly from the exponential closeness of renormalizations (1.1), since $\lambda_2 > \lambda/\lambda_3$ and

$$|\widetilde{a}_n - a_n| = |\widetilde{f}_n(0) - f_n(0)| \le C\lambda^n. \tag{4.2}$$

If $c_n > 1$, the claim follows from the same estimate since, in that case, a_n is bounded from below by a positive constant (see Proposition 3.3 of [12]).

Now, assume that $c_n < 1$ and $a_n < C_3\lambda_3^n$. We assume that n is sufficiently large such that the renormalizations are concave downwards (see Proposition 3.6 of [12]). If $\widetilde{a}_n/a_n > 1 + C_2\lambda_2^n$, then there is a constant $C_4 > 0$ such that

$$\frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})|}{|\tau_n(\Delta^{(n)}_{q_{n-1}})|} > 1 + C_4\lambda_2^n. \tag{4.3}$$

This follows from the fact that $|\tau_n(\Delta^{(n)}_{q_{n-1}})| = f'_{n-1}(\zeta)|\tau_n(\Delta^{(n)}_0)| = f'_{n-1}(\zeta)a_n$, where $\zeta \in \tau_{n-1}(\Delta^{(n)}_0)$, and $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})| = \widetilde{f}'_{n-1}(\widetilde{\zeta})|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_0)| = \widetilde{f}'_{n-1}(\widetilde{\zeta})\widetilde{a}_n$, where $\widetilde{\zeta} \in \widetilde{\tau}_{n-1}(\widetilde{\Delta}^{(n)}_0)$, using again (1.1) and the Denjoy estimate (2.4). Namely,

$$\frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})|}{|\tau_n(\Delta^{(n)}_{q_{n-1}})|} = \frac{|\widetilde{f}'_{n-1}(\widetilde{\zeta})|}{|f'_{n-1}(\zeta)|}\frac{\widetilde{a}_n}{a_n} > (1 + \mathcal{O}(\lambda^n + a_n))(1 + C_2\lambda_2^n) > 1 + C_4\lambda_2^n. \tag{4.4}$$

Here, we have also used that $|\zeta - \widetilde{\zeta}| \le C_5 a_n < C_3C_5\lambda_3^n$, for some $C_5 > 0$.

Furthermore, there is a constant $C_6 > 0$ such that

$$\frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n+1)}_0)|}{|\tau_n(\Delta^{(n+1)}_0)|} = \frac{\widetilde{a}_{n+1}}{a_{n+1}}\frac{\widetilde{a}_n}{a_n} > (1 + \mathcal{O}(\lambda^n))(1 + C_2\lambda_2^n) > 1 + C_6\lambda_2^n. \tag{4.5}$$

Therefore, there is a constant $C_7 > 0$ such that

$$\begin{aligned}\frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n+1}})|}{|\tau_n(\Delta^{(n)}_{q_{n+1}})|} &= \frac{\widetilde{a}_n(1 + \widetilde{a}_{n+1}(1 - \widetilde{f}'_n(\widetilde{\zeta}')))}{a_n(1 + a_{n+1}(1 - f'_n(\zeta')))} \\ &= \frac{\widetilde{a}_n}{a_n}\left(1 + \frac{(\widetilde{a}_{n+1} - a_n)(1 - \widetilde{f}'_n(\widetilde{\zeta}')) + a_n(f'_n(\zeta') - \widetilde{f}'_n(\widetilde{\zeta}'))}{1 + a_{n+1}(1 - f'_n(\zeta'))}\right) \\ &> (1 + \mathcal{O}(\lambda^n) + a_n\mathcal{O}(\lambda^n + a_n))(1 + C_2\lambda_2^n) > 1 + C_7\lambda_2^n,\end{aligned} \tag{4.6}$$

where $\zeta' \in \tau_n(\Delta^{(n+1)}_0)$ and $\widetilde{\zeta}' \in \widetilde{\tau}_n(\widetilde{\Delta}^{(n+1)}_0)$. Here, we have used that $|\zeta' - \widetilde{\zeta}'| \le C_8 a_n \le C_3C_8\lambda_3^n$, for some $C_8 > 0$, in addition to using $|\tau_n(\Delta^{(n+1)}_{q_n})| = f'_n(\zeta')|\tau_n(\Delta^{(n+1)}_0)| = f'_n(\zeta')a_{n+1}a_n$ and $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n+1)}_{q_n})| = \widetilde{f}'_n(\widetilde{\zeta}')|\widetilde{\tau}_n(\widetilde{\Delta}^{(n+1)}_0)| = \widetilde{f}'_n(\widetilde{\zeta}')\widetilde{a}_{n+1}\widetilde{a}_n$. We have also used (1.1) and the Denjoy estimate (2.4).

Since

$$\begin{aligned}\frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})|}{|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|} &= \frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})|}{|\tau_n(\Delta^{(n)}_{q_{n-1}})|}\prod_{j=0}^{i-1}\frac{\widetilde{f}'_n(\widetilde{\zeta}_j)}{f'_n(\zeta_j)}, \\ \frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})|}{|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|} &= \frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n+1}})|}{|\tau_n(\Delta^{(n)}_{q_{n+1}})|}\prod_{j=i}^{k_{n+1}-1}\left(\frac{\widetilde{f}'_n(\widetilde{\zeta}_j)}{f'_n(\zeta_j)}\right)^{-1},\end{aligned} \tag{4.7}$$

where $\zeta_j \in \tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$ and $\widetilde{\zeta}_j \in \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$, we can obtain that, for some $C_9 > 0$,

$$\frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})|}{|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|} > 1 + C_9\lambda_2^n, \tag{4.8}$$

for all $0 \le i \le k_{n+1}$ such that the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n}) \subset [-1, -1 + \lambda_3^n] \cup [-\lambda_3^n, 0]$. All but at most order n of the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})$ satisfy this condition. Starting with estimate (4.3) and using the first of the identities (4.7), we obtain

$$\frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})|}{|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|} > (1 + C_4\lambda_2^n)(1 + \mathcal{O}(\lambda_3^n))^{-C_1\lambda_1^{-n}}, \tag{4.9}$$

and, thus, (4.8) follows for i such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n}) \subset [-1, -1+\lambda_3^n]$. Here, we have used that $|\widetilde{f}_n'(\widetilde{\zeta}_j) - f_n'(\zeta_j)| \le C_{10}\lambda_3^n$, where $C_{10} > 0$, and $\lambda < \lambda_3 < \lambda_1\lambda_2$. Similarly, starting with estimate (4.6) and using the second of the identities (4.7), we obtain (4.8) for i such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n}) \subset [-\lambda_3^n, 0]$.

Let ξ_i and ξ_{i+1} be the left and right endpoints of the interval $\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})$. Let, similarly, $\widetilde{\xi}_i$ and $\widetilde{\xi}_{i+1}$ be the left and right endpoints of the interval $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})$. Let $r_i = \widetilde{\xi}_i - \xi_i$. Estimates (4.8) imply that for i such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n}) \subset [-1, -1+\lambda_3^n]$, $r_i \ge C_{11}\lambda_2^n|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})|$, for some $C_{11} > 0$, and n large enough. Here, we have also used that, for all such i, $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})|$ is of the same order as $\sum_{j=0}^{i}|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})|$. This follows from the fact that for such i, $f_n'(\zeta_i) - c_n^{-1} = \mathcal{O}(\lambda_3^n)$ and $\widetilde{f}_n'(\widetilde{\zeta}_i) - c_n^{-1} = \mathcal{O}(\lambda_3^n)$ and, therefore, the length of the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})$ increases exponentially with i. Similarly, for i such that the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n}) \subset [-\lambda_3^n, 0]$, we have $r_i \le -C_{12}\lambda_2^n|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})|$, for some $C_{12} > 0$, and n large enough. Let $i_{\min}$ be the index i of the longest of the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n}) \subset [-1, -1+\lambda_3^n]$. If such $i_{\min}$ does not exist we set $i_{\min} := 0$. Similarly, let $i_{\max}$ be the index i of the longest of the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n}) \subset [-\lambda_3^n, 0]$. If such $i_{\max}$ does not exist, we set $i_{\max} := k_{n+1}$. Since $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+i_{\min}q_n})|$ and $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+i_{\max}q_n})|$ are at least of the order of λ_3^n, we obtain that $r_{i_{\min}} \ge C_{13}\lambda_2^n\lambda_3^n$ and $r_{i_{\max}} \le -C_{13}\lambda_2^n\lambda_3^n$, for some $C_{13} > 0$, and all n large enough. We can now extend these estimates using the following relation

$$r_{i+1} = \widetilde{f}_n'(\widetilde{\zeta}_i')r_i + \mathcal{O}(\lambda^n), \tag{4.10}$$

where $\widetilde{\zeta}_i' \in (\xi_i, \widetilde{\xi}_i)$. By iterating this relation, we obtain

$$r_i \ge r_{i_{\min}} \prod_{j=i_{\min}}^{i-1} \widetilde{f}_n'(\widetilde{\zeta}_j') - C_{14}\lambda^n \sum_{k=0}^{i-i_{\min}-1} \prod_{j=i-k}^{i-1} \widetilde{f}_n'(\widetilde{\zeta}_j'), \tag{4.11}$$

where $C_{14} > 0$. For any $\kappa > 0$, there exists $\varkappa > 0$, such that if $\widetilde{\zeta}_i' \in [-1, -1+\varkappa]$, then $|\widetilde{f}_n'(\widetilde{\zeta}_i') - c_n^{-1}| < \kappa$ and if $\widetilde{\zeta}_i' \in [-\varkappa, 0]$, then $|\widetilde{f}_n'(\widetilde{\zeta}_i') - c_n| < \kappa$. Therefore, if κ is small enough, and i is such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n}) \subset [-1, -1+\varkappa]$, the derivatives in (4.11) are larger than and bounded away from 1. Consequently, the sum of the products in (4.11) is of the order of the maximal product. Therefore,

$$r_i \ge C_{13}\lambda_2^n\lambda_3^n - C_{15}\lambda^n \ge C_{16}\lambda_2^n\lambda_3^n, \tag{4.12}$$

for some $C_{15}, C_{16} > 0$ and n large enough. Similarly, for i such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n}) \subset [-\varkappa, 0]$, we obtain that

$$r_i \le -C_{17}\lambda_2^n\lambda_3^n, \tag{4.13}$$

for some $C_{17} > 0$ and all n large enough. Using (4.10), each of the estimates (4.12) and (4.13) can be extended to i such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n}) \cap (-1+\varkappa, -\varkappa) \ne \emptyset$. This leads to a contradiction. The claim follows. □

Proposition 4.2. *There exist $C_{18}, C_{19} > 0$ such that, if $c_n < 1$ and $k_{n+1} > C_{18}$, then*

$$\left| \frac{\ln a_n}{\frac{1}{2}k_{n+1}\ln c_n} - 1 \right| \le C_{19} \max\left\{ \frac{\ln k_{n+1}}{k_{n+1}}, \lambda^n \right\}. \tag{4.14}$$

Proof. Let us consider two subintervals of $[-1, 0]$, $L_1 := [-1, -1+1/k_{n+1}]$ and $L_2 := [f_n^{k_{n+1}}(-1) - 1/k_{n+1}, f_n^{k_{n+1}}(-1)]$, and the set of points $\mathcal{S} := \{f_n^j(-1) : j = 1, \dots, k_{n+1}\}$. Let m_1 and m_2 be the cardinalities of the sets $\mathcal{S} \cap L_1$ and $\mathcal{S} \cap L_2$, respectively. Then, there is $C_{20} > 0$ such that

$$k_{n+1} - (m_1 + m_2) \le C_{20} \ln k_{n+1}, \tag{4.15}$$

since the cardinality of the set $\mathcal{S} \setminus (L_1 \cup L_2)$ is of the order of $\ln k_{n+1}$. This follows from the fact that, for $c_n < 1$ and sufficiently large n, the second derivative of the renormalizations f_n'' is bounded away from zero and negative (see Proposition 3.6 of [12]).

If $b_{n,1} = (f_n)'_+(-1)$ and $b_{n,2} = (f_n)'_-(0)$, and $M \geq \max_{z\in[-1,0]} |f_n''(z)|$, then

$$\begin{aligned} \frac{C_{21}^{-1} b_{n,1}^{-m_1}}{k_{n+1}} &\leq |f_n(-1)+1| \leq \frac{C_{21} b_{n,1}^{-m_1}}{k_{n+1}} \left(1 - \frac{M}{b_{n,1}k_{n+1}}\right)^{-m_1}, \\ \frac{C_{21}^{-1} b_{n,2}^{m_2}}{k_{n+1}} &\leq |f_n^{k_{n+1}}(-1) - f_n^{k_{n+1}-1}(-1)| \leq \frac{C_{21} b_{n,2}^{m_2}}{k_{n+1}} \left(1 + \frac{2M}{b_{n,2}k_{n+1}}\right)^{m_2}, \end{aligned} \tag{4.16}$$

for some $C_{21} > 0$. The last inequality is obtained under the assumption $|f_n^{k_{n+1}}(-1)| < 1/k_{n+1}$. It follows from the Denjoy estimate (2.4) that

$$e^{-3V} |f_n^{k_{n+1}}(-1) - f_n^{k_{n+1}-1}(-1)| \leq |f_n(-1)+1| \leq e^{3V} |f_n^{k_{n+1}}(-1) - f_n^{k_{n+1}-1}(-1)|. \tag{4.17}$$

Since both $m_1, m_2 \leq k_{n+1}$, this implies that, for some $C_{22} > 0$,

$$C_{22}^{-1} b_{n,2}^{m_2} \leq b_{n,1}^{-m_1} \leq C_{22} b_{n,2}^{m_2}. \tag{4.18}$$

Using (4.15), for some $C_{23} > 0$, we have

$$\begin{aligned} \left| m_1 - \frac{\ln b_{n,2}}{\ln b_{n,1}^{-1} + \ln b_{n,2}} k_{n+1} \right| &\leq C_{23} \ln k_{n+1}, \\ \left| m_2 - \frac{\ln b_{n,1}^{-1}}{\ln b_{n,1}^{-1} + \ln b_{n,2}} k_{n+1} \right| &\leq C_{23} \ln k_{n+1}. \end{aligned} \tag{4.19}$$

It follows that $|f_n^{k_{n+1}}(-1)| < C_{24} b_{n,1}^{-m_1}/k_{n+1} < 1/k_{n+1}$, for some $C_{24} > 0$, if k_{n+1} is large enough. Since, by (2.5), $|b_{n,1} - F_n'(-1)| \leq C\lambda^n$ and $F_n'(-1) = c_n^{-1} + \mathcal{O}(a_n)$ (due to Proposition 3.2 of [12]), the claim follows. $\square$

Corollary 4.3. *Let $\lambda_4 \in (\lambda^{1/3}, 1)$. There exist $C_{25} > 0$ and $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$ such that $c_n < 1$, we have*

$$\left| \frac{\ln \widetilde{a}_n}{\ln a_n} - 1 \right| \leq C_{25} \lambda_4^n. \tag{4.20}$$

Proof. Let $\lambda_1 = \lambda^{1/3}$. If $k_{n+1} \leq C_1 \lambda_1^{-n}$, the claim follows from Proposition 4.1. If $k_{n+1} > C_1 \lambda_1^{-n}$, the claim follows from Proposition 4.2. We have also used the fact that, if $c_n < 1$, then, for $n \geq N_1$ and $N_1 \in \mathbb{N}$ large enough, $a_n < c_n < 1$ (see Proposition 3.3 in [12]). $\square$

Proposition 4.4.

$$\lim_{n\to\infty} \frac{\ln |\widetilde{\Delta}_0^{(n)}|}{\ln |\Delta_0^{(n)}|} = 1. \tag{4.21}$$

Proof. Let $\epsilon > 0$. Since $\Delta_0^{(n)} = \prod_{k=1}^n a_k$, we have $\ln |\Delta_0^{(n)}| = \ln \prod_{k=1\,:\,c_k>1}^{n} a_k + \ln \prod_{k=1\,:\,c_k<1}^{n} a_k$. If $N_2 \in \mathbb{N}$ and $N_2 \geq N_1$, using Proposition 4.1 and Corollary 4.3, we obtain

$$\begin{aligned} \frac{\ln |\widetilde{\Delta}_0^{(n)}|}{\ln |\Delta_0^{(n)}|} &= 1 + \frac{\ln \prod_{k=1\,:\,c_k>1}^{n} (1+\mathcal{O}(\lambda_2^k)) + \sum_{k=1\,:\,c_k<1}^{N_2-1} \ln a_k \mathcal{O}(\lambda_4^k)}{\ln |\Delta_0^{(n)}|} + \frac{\sum_{k=N_2\,:\,c_k<1}^{n} \ln a_k \mathcal{O}(\lambda_4^k)}{\ln |\Delta_0^{(n)}|} \\ &= 1 + \frac{\mathcal{O}(1) + \Psi_1(N_2)}{\ln |\Delta_0^{(n)}|} + \mathcal{O}(\lambda_4^{N_2}) \frac{\ln \prod_{k=N_2\,:\,c_k<1}^{n} a_k}{\ln |\Delta_0^{(n)}|}, \end{aligned} \tag{4.22}$$

where $\Psi_1(N_2)$ is a constant that depends on N_2, but does not depend on n. Since $|\Delta_0^{(n)}|$ decrease at least exponentially with n and since, for sufficiently large k and $c_k > 1$, a_k are bounded both from above and from below by positive constants (see Proposition 3.3 of [12]), we have

$$\frac{\ln \prod_{k=N_2\,:\,c_k<1}^{n} a_k}{\ln|\Delta_0^{(n)}|} = 1 - \frac{\ln \prod_{k=1\,:\,c_k<1}^{N_2-1} a_k}{\ln|\Delta_0^{(n)}|} - \frac{\ln \prod_{k=1\,:\,c_k>1}^{n} a_k}{\ln|\Delta_0^{(n)}|} = \mathcal{O}(1) - \frac{\Psi_2(N_2)}{\ln|\Delta_0^{(n)}|}, \tag{4.23}$$

where $\Psi_2(N_2)$ is a constant that depends on N_2 only. It follows from (4.22) that if N_2 has been chosen large enough, there exists $N_3 \geq N_2$ such that, for all $n \geq N_3$, we have

$$\left| \frac{\ln|\widetilde{\Delta}_0^{(n)}|}{\ln|\Delta_0^{(n)}|} - 1 \right| < \epsilon. \tag{4.24}$$

The claim follows. $\square$

5. Estimates on the renormalized intervals of the next level partition and the shift of indexes

The following proposition was proved in [13].

Proposition 5.1. ([13]) *Let $\lambda_5 = \max\{\lambda_2, \lambda^{\frac{(1+\alpha)\alpha}{8(2+\alpha)}}\}$. There exists $C_{26} > 0$ such that, for all $n \in \mathbb{N}$ such that either $c_n > 1$ or $c_n < 1$ and $k_{n+1} \leq C_1\lambda_1^{-n}$, we have*

$$\left| \frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})|}{|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|} - 1 \right| \leq C_{26}\lambda_5^n, \tag{5.1}$$

for all i such that $0 \leq i < k_{n+1}$.

Let $x_i := T^i(x_c)$.

Proposition 5.2. *For every $\alpha \in (0,1)$, $\rho \in (0,1)\backslash\mathbb{Q}$ and $c \in \mathbb{R}_+\backslash\{1\}$, there exists $\lambda \in (0,1)$ and, for every $C^{2+\alpha}$-smooth circle map T with a break of size c and rotation number ρ, there exists $C_{27} > 0$, such that, if $c_n < 1$ then, for every $i = 1, \ldots, q_n$, we have*

$$\| f_{n,x_{i-q_n}} - F_n^{(0)} \|_{C^2[-1,0]} \leq C_{27}(\lambda^n + a_n), \tag{5.2}$$

where

$$F_n^{(0)}(z) = \frac{c_n z}{1 + (1-c_n)z}. \tag{5.3}$$

Proof. The proof of the claim is similar to the proof of (2.5), using Proposition 3.2 of [12] and the fact that $\frac{|\Delta_0^{(n)}(x_{i-q_n})|}{|\Delta_0^{(n-1)}(x_{i-q_n})|} = \Theta(a_n)$, for $i = 1, \ldots, q_n - 1$, due to the bounded distortion of the ratio $\frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|}$ under the action of T^{-i}. $\square$

Let $\mathcal{S}_{n,x_i} := \{f^j_{n,x_i}(-1) : j = 1, \ldots, k_{n+1}\}$.

Proposition 5.3. *Let $\epsilon_1 > 0$ and let $n_1 = n_1(n,i)$ be the cardinality of $\mathcal{S}_{n,x_i} \cap M_1$, where $M_1 := [-1, -1+\epsilon_1]$. There exists $C_{28} > 0$ such that, if $c_n < 1$ and $k_{n+1} > C_{28}n$, then, for $i = 0, \ldots, q_n - 1$,*

$$n_1 = \frac{1}{2}k_{n+1} + \mathcal{O}(\lambda^n k_{n+1} + \ln k_{n+1}). \tag{5.4}$$

Proof. Since the distortion of the ratio $\frac{|\Delta_{i-q_n}^{(n)}|}{|\Delta_{i-q_n}^{(n-1)}|}$ under T^{q_n-i} is bounded, $|\Delta_i^{(n-1)}| = |\Delta_{i-q_n}^{(n-1)}|(1+\mathcal{O}(a_n))$, for $i = 1, \ldots, q_n - 1$. It follows that, for sufficiently large n, the cardinality of the set $\mathcal{S}_{n,x_{i-q_n}} \cap M_1$, that we will denote by $\bar{n}_1$, can differ from n_1 by at most 2. Here, we have used Proposition 5.2 and, therefore, that the distance between successive points $f_{n,x_{i-q_n}}^j(-1)$ grows exponentially with j. Using Proposition 5.2 again, in particular that the second derivative of $f_{n,x_{i-q_n}}$ is bounded both from below and above by negative constants and that the derivatives $f'_{n,x_{i-q_n}}(-1)$ and $f'_{n,x_{i-q_n}}(0)$ can be made arbitrary close to c_n^{-1} and c_n, respectively, by choosing n and k_{n+1} sufficiently large, one can prove, completely analogously to the proof of the first inequality in (4.19) (see the proof of Proposition 4.2), that

$$\bar{n}_1 = \frac{\ln b_{n,x_{i-q_n},2}}{\ln b_{n,x_{i-q_n},1}^{-1} + \ln b_{n,x_{i-q_n},2}} k_{n+1} + \mathcal{O}(\ln k_{n+1}), \tag{5.5}$$

where $b_{n,x_{i-q_n},1} = (f_{n,x_{i-q_n}})'_+(-1)$ and $b_{n,x_{i-q_n},2} = (f_{n,x_{i-q_n}})'_-(0)$. Here, we have also used the fact that the cardinality of the set $\mathcal{S}_{n,x_{i-q_n}} \cap (M_1 \backslash L_1)$ (see the proof of Proposition 4.2) is of the order of $\ln k_{n+1}$.

Since it follows from Proposition 4.2 and Proposition 5.2 that $(f_{n,x_{i-q_n}})'_+(-1) - c_n^{-1} = \mathcal{O}(\lambda^n)$ and $(f_{n,x_{i-q_n}})'_+(0) - c_n = \mathcal{O}(\lambda^n)$, for $k_{n+1} > C_{28}n$ and $C_{28} > 0$ sufficiently large, the claim follows from (5.5). □

Let $\widetilde{x}_i := \widetilde{T}^i(\widetilde{x}_c)$ and $\widetilde{\mathcal{S}}_{n,\widetilde{x}_i} := \{\widetilde{f}_{n,\widetilde{x}_i}^j(-1) : j = 1, \ldots, k_{n+1}\}$. An immediate corollary of Proposition 5.3 is the following.

Corollary 5.4. *Let $\lambda_6 \in (\lambda_1, 1)$. Let $\epsilon_1 > 0$ and let n_1 and $\widetilde{n}_1$ be the cardinalities of $\mathcal{S}_{n,x_i} \cap M_1$ and $\widetilde{\mathcal{S}}_{n,\widetilde{x}_i} \cap M_1$, where $M_1 := [-1, -1+\epsilon_1]$. There exists $K_1 > 0$, depending on T and $\widetilde{T}$ only, such that, if $c_n < 1$ and $k_{n+1} > C_1\lambda_1^{-n}$, then, for $i = 0, \ldots, q_n - 1$,*

$$|n_1 - \widetilde{n}_1| \le K_1 \epsilon(n) k_{n+1}, \tag{5.6}$$

where $\epsilon(n) = \lambda^n + \frac{\ln k_{n+1}}{k_{n+1}} \le \Theta(\lambda_6^n)$.

The following proposition shows that, after a proper shift of indexes, i_n, the lengths of the intervals $\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+iq_n})$ and $\tau_n(\Delta_{q_{n-1}+(i+i_n)q_n})$ are of the same order.

To simplify the notation, let $J_i := \tau_n(\Delta_{q_{n-1}+iq_n}^{(n)})$ and $\widetilde{J}_i := \widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+iq_n}^{(n)})$. It follows from Proposition 5.2 that, for $c_n < 1$, $k_{n+1} \ge C_1\lambda_1^{-n}$ and n large enough, the renormalizations f_n and $\widetilde{f}_n$ are uniformly concave downwards with derivatives at -1 and 0 close to c_n^{-1} and c_n, respectively. Therefore, there are unique points z_n^* and $\widetilde{z}_n^*$ such that $f'_n(z_n^*) = 1$ and $\widetilde{f}_n(\widetilde{z}_n^*) = 1$. Let $i^{(n)}$ and $\widetilde{i}^{(n)}$ be the indexes of two intervals $J_{i^{(n)}}$ and $\widetilde{J}_{\widetilde{i}^{(n)}}$ such that $z_n^* \in J_{i^{(n)}}$ and $\widetilde{z}_n^* \in \widetilde{J}_{\widetilde{i}^{(n)}}$. We define

$$i_n := i^{(n)} - \widetilde{i}^{(n)}. \tag{5.7}$$

If $i^{(n)}$ or $\widetilde{i}^{(n)}$ is not defined uniquely, we choose i_n to take the value that maximizes $|i_n|$.

It follows from Corollary 5.4 that $|i_n| < C_{29}\epsilon(n)k_{n+1}$, for some $C_{29} > 0$.

Proposition 5.5. *For sufficiently small $\epsilon_2 > 0$, there exists $C_{30} > 0$, such that if $c_n < 1$ and $k_{n+1} \ge C_1\lambda_1^{-n}$ then, for every i satisfying $0 \le i \le k_{n+1}$ and $|i - \widetilde{i}^{(n)}| \le \epsilon_2\lambda^{-n}$, we have*

$$\left| \ln \frac{|\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+iq_n}^{(n)})|}{|\tau_n(\Delta_{q_{n-1}+(i+i_n)q_n}^{(n)})|} \right| \le C_{30}. \tag{5.8}$$

Proof. It is easy to see that the lengths of the intervals $J_{i^{(n)}}$ and $\widetilde{J}_{\widetilde{i}^{(n)}}$ are of order 1. It follows that, for every $\varkappa > 0$, there exists $C_{31} > 0$, such that for all i such that $(J_{i+i_n} \cup \widetilde{J}_i) \cap M_0 \ne \emptyset$, where $M_0 = (-1+\varkappa, -\varkappa)$, we have

$$C_{31}^{-1} \le \frac{|\widetilde{J}_i|}{|J_{i+i_n}|} \le C_{31}. \tag{5.9}$$

We will now extend this estimate for i such that $0 \le i \le k_{n+1}$ and $|i - \widetilde{i}_n| \le \epsilon_2 k_{n+1}$, using the recursion relation

$$\frac{|\widetilde{J}_{i+1}|}{|J_{i+i_n+1}|} = \frac{|\widetilde{J}_i|}{|J_{i+i_n}|} \frac{\widetilde{f}_n'(\widetilde{\zeta}_i)}{f_n'(\zeta_{i+i_n})}, \tag{5.10}$$

where $\zeta_i \in J_i$ and $\widetilde{\zeta}_i \in \widetilde{J}_i$. If $i_{\min}^{(n)}$ and $i_{\max}^{(n)}$ are the smallest and largest values of i such that $(J_{i+i_n} \cup \widetilde{J}_i) \cap M_0 \neq \emptyset$, we have

$$\frac{|\widetilde{J}_i|}{|J_{i+i_n}|} = \frac{|\widetilde{J}_{i_{\min}^{(n)}}|}{|J_{i_{\min}^{(n)}+i_n}|} \prod_{j=i}^{i_{\min}^{(n)}-1} \left(\frac{\widetilde{f}_n'(\widetilde{\zeta}_j)}{f_n'(\zeta_{j+i_n})} \right)^{-1}, \tag{5.11}$$

for $i < i_{\min}^{(n)}$, and

$$\frac{|\widetilde{J}_i|}{|J_{i+i_n}|} = \frac{|\widetilde{J}_{i_{\max}^{(n)}}|}{|J_{i_{\max}^{(n)}+i_n}|} \prod_{j=i_{\max}^{(n)}}^{i-1} \frac{\widetilde{f}_n'(\widetilde{\zeta}_j)}{f_n'(\zeta_{j+i_n})}, \tag{5.12}$$

for $i > i_{\max}^{(n)}$, as long as $0 \le i < k_{n+1}$ and $0 \le i + i_n < k_{n+1}$. It follows from Proposition 5.2 that, for any $\kappa > 0$, there exists $\varkappa > 0$, such that if $\zeta_i, \widetilde{\zeta}_i \in M_1$, where $M_1 = [-1, -1 + \varkappa]$, then $|f_n'(\zeta_i) - c_n^{-1}| < \kappa$ and $|\widetilde{f}_n'(\widetilde{\zeta}_i) - c_n^{-1}| < \kappa$ and if $\zeta_i, \widetilde{\zeta}_i \in M_2$, where $M_2 = [-\varkappa, 0]$, then $|f_n'(\zeta_i) - c_n| < \kappa$ and $|\widetilde{f}_n'(\widetilde{\zeta}_i) - c_n| < \kappa$.

Since the second derivatives f_n'' and $\widetilde{f}_n''$ are bounded, it follows from (5.11) that

$$\begin{aligned}
\frac{|\widetilde{J}_i|}{|J_{i+i_n}|} &= \frac{|\widetilde{J}_{i_{\min}^{(n)}}|}{|J_{i_{\min}^{(n)}+i_n}|} \prod_{j=i}^{i_{\min}^{(n)}-1} (1 + \mathcal{O}(\max\{|J_{j+i_n}|, |\widetilde{J}_j|\}) + \lambda^n)) \\
&= \frac{|\widetilde{J}_{i_{\min}^{(n)}}|}{|J_{i_{\min}^{(n)}+i_n}|} \left(1 + \sum_{j=i}^{i_{\min}^{(n)}-1} \mathcal{O}(\max\{|J_{j+i_n}|, |\widetilde{J}_j|\}) \right) \Theta\left((1+\lambda^n)^{i_{\min}-i} \right) \\
&= \frac{|\widetilde{J}_{i_{\min}^{(n)}}|}{|J_{i_{\min}^{(n)}+i_n}|} (1 + \mathcal{O}(\varkappa))\Theta(1).
\end{aligned} \tag{5.13}$$

In the last step, we have used that $i_{\min}^{(n)} - i \le \widetilde{i}^{(n)} - i \le \epsilon_2 \lambda_1^{-n}$, for $\varkappa$ small enough. We have also used the fact that, for all j satisfying $i \le j < i_{\min}^{(n)}$, $J_{j+i_n}, \widetilde{J}_j \subset M_1$. This follows from the fact that $|\widetilde{i}^{(n)} - i_n| > C_{32} \lambda_1^{-n}$, for some $C_{32} > 0$, and, therefore, $|\widetilde{i}^{(n)} - i| \le \epsilon_2 \lambda_1^{-n} < |\widetilde{i}^{(n)} - i_n|$, for $\epsilon_2 > 0$ small enough. This proves the claim for $i < i_{\min}^{(n)}$.

Using (5.12), one can similarly obtain

$$\frac{|\widetilde{J}_i|}{|J_{i+i_n}|} = \frac{|\widetilde{J}_{i_{\max}^{(n)}}|}{|J_{i_{\max}^{(n)}+i_n}|} (1 + \mathcal{O}(\varkappa))\Theta(1), \tag{5.14}$$

for $i > i_{\max}^{(n)}$ satisfying $i - \widetilde{i}^{(n)} \le \epsilon_2 \lambda_1^{-n}$, and $\epsilon_2 > 0$ small enough. The claim follows. □

An immediate corollary of the previous proposition is the following.

Corollary 5.6. *Under the assumptions of Proposition 5.5, we have*

$$\left| \frac{\ln |\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+iq_n}^{(n)})|}{\ln |\tau_n(\Delta_{q_{n-1}+(i+i_n)q_n}^{(n)})|} - 1 \right| \le \frac{C_{30}}{|\ln |\tau_n(\Delta_{q_{n-1}+(i+i_n)q_n}^{(n)})||}. \tag{5.15}$$

Proposition 5.7. *Let $\lambda_6 \in (\lambda_1, 1)$ and $\epsilon_2 > 0$. There exists $C_{33} > 0$ such that, if $c_n < 1$ and $k_{n+1} \ge C_1 \lambda_1^{-n}$, for all i such that $0 \le i < k_{n+1}$ and $|i - \widetilde{i}^{(n)}| > \epsilon_2 \lambda_1^{-n}$, we have*

$$\left|\frac{\ln|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})|}{\ln|\tau_n(\Delta^{(n)}_{q_{n-1}+(i+i_n)q_n})|}-1\right|\le C_{33}\lambda_6^n. \tag{5.16}$$

Proof. Let $i_l^{(n)}$ and $i_r^{(n)}$ be the smallest and largest values of i for which (5.8) holds. Since $\widetilde{i}^{(n)}-i_l^{(n)}$ and $i_r^{(n)}-\widetilde{i}^{(n)}$ are at least of the order of λ_1^{-n}, it follows from Proposition 5.2 that there is $C_{34}>0$ such that $|\ln|\widetilde{J}_{i_l^{(n)}}||, |\ln|\widetilde{J}_{i_r^{(n)}}||\ge C_{34}\lambda_1^{-n}$. Corollary 5.6 then implies that there exists $C_{35}>0$ such that

$$\left|\frac{\ln|\widetilde{J}_{i_l^{(n)}}|}{\ln|J_{i_l^{(n)}+i_n}|}-1\right|,\left|\frac{\ln|\widetilde{J}_{i_r^{(n)}}|}{\ln|J_{i_r^{(n)}+i_n}|}-1\right|\le C_{35}\lambda_1^n. \tag{5.17}$$

We will now extend this estimate for $0\le i<i_l^{(n)}$ and $i_r^{(n)}<i<k_{n+1}$, using the following relation

$$\frac{\ln|\widetilde{J}_{j+1}|}{\ln|J_{j+1+i_n}|}=\frac{\ln|\widetilde{J}_j|+\ln(\widetilde{T}^{q_n})'(\widetilde{\mathfrak{z}}_j)}{\ln|J_{j+i_n}|+\ln(T^{q_n})'(\mathfrak{z}_{j+i_n})}, \tag{5.18}$$

where $\mathfrak{z}_j\in\Delta^{(n)}_{q_{n-1}+jq_n}$ and $\widetilde{\mathfrak{z}}_j\in\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}$.

We will first extend the estimate (5.17) to $i_r^{(n)}<i<k_{n+1}$; for $0\le i<i_l^{(n)}$, the analysis is similar. By iterating (5.18), we obtain

$$\frac{\ln|\widetilde{J}_i|}{\ln|J_{i+i_n}|}=\frac{\ln|\widetilde{J}_{i_r^{(n)}}|+\sum_{j=i_r^{(n)}}^{i-1}\ln(\widetilde{T}^{q_n})'(\widetilde{\mathfrak{z}}_j)}{\ln|J_{i_r^{(n)}+i_n}|+\sum_{j=i_r^{(n)}}^{i-1}\ln(T^{q_n})'(\mathfrak{z}_{j+i_n})}. \tag{5.19}$$

For $i_r^{(n)}<j<i_{\max}:=\min\{k_{n+1},k_{n+1}-i_n\}$, the derivatives satisfy

$$\left|\frac{(\widetilde{T}^{q_n})'(\widetilde{\mathfrak{z}}_j)}{(T^{q_n})'(\mathfrak{z}_{j+i_n})}-1\right|\le C_{36}\lambda^n, \tag{5.20}$$

for some $C_{36}>0$, since $(T^{q_n})'(\mathfrak{z}_j)=f_n'(\zeta_j)$ and $(\widetilde{T}^{q_n})'(\widetilde{\mathfrak{z}}_j)=\widetilde{f}_n'(\widetilde{\zeta}_j)$ and, for sufficiently large n, all the points $\zeta_{j+i_n},\widetilde{\zeta}_j$ belong to an interval $L_2:=[-d,0]$, where $0<d\le c_n^{C_{37}\lambda_1^{-n}}$, for some $C_{37}>0$. Here, we have also used that, by Proposition 5.2, for $i_r^{(n)}<j<i_{\max}$, $|f_n'(\zeta_{j+i_n})-c_n|=\mathcal{O}(\lambda^n)$ and $|\widetilde{f}_n'(\widetilde{\zeta}_j)-c_n|=\mathcal{O}(\lambda^n)$. Therefore, for $i_r^{(n)}<i<i_{\max}$, we obtain

$$\frac{\ln|\widetilde{J}_i|}{\ln|J_{i+i_n}|}-1=\frac{(i-i_r^{(n)})\mathcal{O}(\lambda^n)}{\ln|J_{i+i_n}|}+\mathcal{O}(\lambda_1^n)=\frac{(i-i_r^{(n)})\mathcal{O}(\lambda^n)}{\Theta(\lambda_1^{-n})+\Theta(i-i_r^{(n)})}+\mathcal{O}(\lambda_1^n)=\mathcal{O}(\lambda_1^n). \tag{5.21}$$

If $i_n>0$, then $i_{\max}<k_{n+1}$. To extend estimate (5.17) to i satisfying $i_{\max}<i<k_{n+1}$, we use the following estimate, similar to (5.19), which was also obtained from (5.18),

$$\frac{\ln|\widetilde{J}_i|}{\ln|J_{i+i_n}|}=\frac{\ln|\widetilde{J}_{i_{\max}-1}|+\sum_{j=i_{\max}-1}^{i-1}\ln(\widetilde{T}^{q_n})'(\widetilde{\mathfrak{z}}_j)}{\ln|J_{i_{\max}+i_n-1}|+\sum_{j=i_{\max}-1}^{i-1}\ln(T^{q_n})'(\mathfrak{z}_{j+i_n})}. \tag{5.22}$$

For $i_{\max}\le j<k_{n+1}$, however, the derivatives $(\widetilde{T}^{q_n})'(\widetilde{\mathfrak{z}}_j)$ and $(T^{q_n})'(\mathfrak{z}_{j+i_n})$ can differ by (at most) a constant, as follows from Proposition 5.2. The number of these terms, however, is bounded by i_n and is, therefore, of the order of $\epsilon(n)k_{n+1}$, which is small in comparison to k_{n+1}. For $i_{\max}\le i<k_{n+1}$, we, therefore, obtain

$$\frac{\ln|\widetilde{J}_i|}{\ln|J_{i+i_n}|}-1=\frac{k_{n+1}\mathcal{O}(\lambda_1^n)+k_{n+1}\mathcal{O}(\epsilon(n))}{\ln|J_{i+i_n}|}=\mathcal{O}(\lambda_1^n)+\mathcal{O}\left(\frac{\ln k_{n+1}}{k_{n+1}}\right), \tag{5.23}$$

taking into account that $|\ln|J_{i_{\max}}||=\Theta(k_{n+1})$. The claim follows. □

6. Choice of the conjugacy and proof of the main result

In the previous section, we considered intervals of dynamical partitions $\mathcal{P}_n$ and $\widetilde{\mathcal{P}}_n$ of circle diffeomorphisms with a break T and $\widetilde{T}$, constructed with the corresponding marked points x_c and $\widetilde{x}_c$, respectively. For the map T, we will now consider intervals of dynamical partitions $\mathcal{P}_{n,x_0}$, constructed with a marked point x_0 that will be defined below.

We will assume that the rotation number $\rho \in (0,1)\backslash\mathbb{Q}$ of T and $\widetilde{T}$ is such that there is an infinite increasing sequence of positive integers $(\ell_i)_{i\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$ for which $c_n < 1$, we have:

(i) $k_{n+1} > C_1\lambda_1^{-n}$, if $n = \ell_i$, for some $i \in \mathbb{N}$;
(ii) $k_{n+1} \le C_1\lambda_1^{-n}$, if $n \neq \ell_i$, for any $i \in \mathbb{N}$.

If this is not the case, i.e., if the sequence $(\ell_i)_{i\in\mathbb{N}}$ is finite or empty, the claim of Theorem 1.2 follows directly from the fact that T and $\widetilde{T}$ are conjugate to each other via a C^1-smooth conjugacy φ that satisfies $\varphi(x_c) = \widetilde{x}_c$ [12,13].

For all $n \in \mathbb{N}$ such that $n = \ell_i$, for some $i \in \mathbb{N}$, let $\mathrm{i}_n := i_n$, where i_n is the integer defined by (5.7). For all $n \in \mathbb{N}$ such that $n \neq \ell_i$, for any $i \in \mathbb{N}$, we define $\mathrm{i}_n := 0$.

Let $x_0^{(n)} := T^{\sum_{m=1}^{n} \mathrm{i}_m q_m} x_c$, for $n \in \mathbb{N}$, and $x_0^{(0)} := x_c$.

Notice that $|x_0^{(\ell_i)} - x_0^{(\ell_{-1})}|$ is of the order of the length of i_{ℓ_i} consecutive "long" intervals of partition $\mathcal{P}_{\ell_i+1}$, nearest to the point $x_0^{(\ell_i-1)}$. Since the number of such intervals is small compared to k_{ℓ_i+1}, they all belong either to $\Delta_0^{(\ell_i-1)}(x_0^{(\ell_i-1)})$ or to $\Delta_{-q_{\ell_i-1}}^{(\ell_i-1)}(x_0^{(\ell_i-1)})$. The following proposition gives an estimate on this distance.

Proposition 6.1. *Let $\epsilon_3 > 0$. There exist $N_4 \in \mathbb{N}$ and $C_{38} > 0$ such that, for all $n \ge N_4$, we have*

$$a_{\ell_i}(x_0^{(\ell_i-1)}) \le C_{38} c_{\ell_i}^{(\frac{1}{2}-\epsilon_3)k_{\ell_i+1}} \tag{6.1}$$

and

$$|x_0^{(\ell_i)} - x_0^{(\ell_i-1)}| \le C_{38} c_{\ell_i}^{(\frac{1}{2}-\epsilon_3)k_{\ell_i+1}} |\Delta_0^{(\ell_i-1)}(x_0^{(\ell_i-1)})|. \tag{6.2}$$

Proof. Since $k_{\ell_i+1} > C_1\lambda_1^{-\ell_i}$, Proposition 5.3 implies that $a_{\ell_i}(x_0^{(\ell_i-1)}) = c_{\ell_i}^{\frac{1}{2}k_{\ell_i+1}+\mathcal{O}(\lambda_6^n)k_{\ell_i+1}}$. Since

$$|x_0^{(\ell_i)} - x_0^{(\ell_i-1)}| = \mathcal{O}\left(c_{\ell_i}^{\mathcal{O}(\epsilon(n)k_{\ell_i+1})}\right) a_{\ell_i}(x_0^{(\ell_i-1)})|\Delta_0^{(\ell_i-1)}(x_0^{(\ell_i-1)})|, \tag{6.3}$$

the claim follows. □

Let $\ell_0 := 0$. Let $s_n := \max\{i \in \mathbb{N}_0 : \ell_i \le n\}$.

Proposition 6.2.

$$x_0 := \lim_{n\to\infty} x_0^{(n)} \in \mathbb{T}^1. \tag{6.4}$$

Proof. Let $n > m$. It follows from Proposition 6.1 that

$$|x_0^{(n)} - x_0^{(m)}| = |x_0^{(\ell_{s_n})} - x_0^{(\ell_{s_m})}| \le \sum_{i=s_m+1}^{s_n} |x_0^{(\ell_i)} - x_0^{(\ell_{i-1})}| \le C_{39} \sum_{i=s_m+1}^{s_n} \lambda_1^{\ell_i} \le C_{40}\lambda_1^{\ell_{s_m+1}}, \tag{6.5}$$

where $C_{39}, C_{40} > 0$, and, therefore, $(x_0^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence on $\mathbb{T}^1$ and, thus, convergent. □

Lemma 6.3. *There exists $C_{41} > 0$ such that the following holds for $0 \le j < k_{n+1}$. For all $n \in \mathbb{N}$ such that $n \neq \ell_i$, for all $i \in \mathbb{N}$, we have*

$$\left| \ln \frac{|\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+jq_n}^{(n)})|}{|\tau_{n,x_0}(\Delta_{q_{n-1}+jq_n}^{(n)}(x_0))|} \right| \le C_{41}. \tag{6.6}$$

If $n = \ell_i$, for some $i \in \mathbb{N}$, we have

$$\left| \ln \frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|}{|\tau_{n,x_0}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0))|} \right| \leq C_{41} \max\{1, \lambda_6^n |\ln|\tau_{n,x_0}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0))||\}. \tag{6.7}$$

Proof. Consider first the case $n \neq \ell_i$, for any $i \in \mathbb{N}$. We would like to estimate the ratio

$$\frac{|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})|}{|\tau_{n,x_0}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0))|} = \frac{|\tau_{n,x_0^{(\ell_{s_n})}}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0^{(\ell_{s_n})}))|}{|\tau_{n,x_0}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0))|} \prod_{i=1}^{s_n} \frac{|\tau_{n,x_0^{(\ell_{i-1})}}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0^{(\ell_{i-1})}))|}{|\tau_{n,x_0^{(\ell_i)}}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0^{(\ell_i)}))|}. \tag{6.8}$$

Notice that $x_0^{(\ell_i)} = T^{\mathfrak{i}_{\ell_i} q_{\ell_i}} x_0^{(\ell_{i-1})}$. The ratio in the product is the reciprocal of the distortion of the ratio $|\tau_{n,x_0^{(\ell_{i-1})}}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0^{(\ell_{i-1})}))|$ under the action of $T^{\mathfrak{i}_{\ell_i} q_{\ell_i}}$ and can be estimated as

$$\frac{|\tau_{n,x_0^{(\ell_{i-1})}}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0^{(\ell_{i-1})}))|}{|\tau_{n,x_0^{(\ell_i)}}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0^{(\ell_i)}))|} = 1 + \mathcal{O}\left(\sum_{j=0}^{\mathfrak{i}_{\ell_i} q_{\ell_i} - 1} |\Delta_j^{(n-1)}(x_0^{(\ell_{i-1})})| \right) = 1 + \mathcal{O}(c_{\ell_i}^{(\frac{1}{2}-\epsilon_3) C_1 \lambda_1^{-\ell_i}}), \tag{6.9}$$

since $n > \ell_{s_n}$.

To estimate the ratio in front of product in (6.8), notice that Proposition 6.1 implies

$$|x_0 - x_0^{(\ell_{s_n})}| = \mathcal{O}(c_{\ell_{s_n+1}}^{(\frac{1}{2}-\epsilon_3) C_1 \lambda_1^{-\ell_{s_n+1}}}) |\Delta_0^{(\ell_{s_n+1}-1)}(x_0^{(\ell_{s_n})})|. \tag{6.10}$$

Due to the Denjoy estimate (2.4), the distances $|T^{q_{n-1}}(x_0) - T^{q_{n-1}}(x_0^{(\ell_{s_n})})|$ and $|T^{q_{n-1}+q_n}(x_0) - T^{q_{n-1}+q_n}(x_0^{(\ell_{s_n})})|$ are of the same order. Since, $\ell_{s_n+1} > n$, we have that $|\Delta_0^{(\ell_{s_n+1}-1)}(x_0^{(\ell_{s_n})})| \leq e^V |\Delta^{(n)}_{q_{n-1}}(x_0^{(\ell_{s_n})}))|$ and, therefore, using (6.10), we obtain

$$\frac{|\Delta^{(n)}_{q_{n-1}}(x_0^{(\ell_{s_n})}))| - |\Delta^{(n)}_{q_{n-1}}(x_0))|}{|\Delta^{(n)}_{q_{n-1}}(x_0^{(\ell_{s_n})}))|} = \mathcal{O}(c_{\ell_{s_n+1}}^{(\frac{1}{2}-\epsilon_3) C_1 \lambda_1^{-\ell_{s_n+1}}}) \tag{6.11}$$

and

$$\frac{|\Delta_0^{(n-1)}(x_0^{(\ell_{s_n})}))| - |\Delta_0^{(n-1)}(x_0))|}{|\Delta_0^{(n-1)}(x_0^{(\ell_{s_n})}))|} = \mathcal{O}(c_{\ell_{s_n+1}}^{(\frac{1}{2}-\epsilon_3) C_1 \lambda_1^{-\ell_{s_n+1}}}). \tag{6.12}$$

Let $\xi_{j,x_0} := T^{q_{n-1}+jq_n} x_0$ and $\xi_{j,x_0^{(\ell_{s_n})}} := T^{q_{n-1}+jq_n} x_0^{(\ell_{s_n})}$. Let $r_j := |\xi_{j,x_0} - \xi_{j,x_0^{(\ell_{s_n})}}|$. Since the distortion of the ratio $r_0 / |\Delta^{(n)}_{q_{n-1}}(x_0^{(\ell_{s_n})}))|$ under the action of T^{jq_n}, for $j = 1, \ldots, k_{n+1}$, is bounded, we obtain that the ratio in front of the product in (6.8) can be estimated as

$$\frac{|\tau_{n,x_0^{(\ell_{s_n})}}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0^{(\ell_{s_n})}))|}{|\tau_{n,x_0}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0))|} = 1 + \mathcal{O}(c_{\ell_{s_n+1}}^{(\frac{1}{2}-\epsilon_3) C_1 \lambda_1^{-\ell_{s_n+1}}}). \tag{6.13}$$

Therefore, the ratio in (6.8) can be estimated as

$$\frac{|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})|}{|\tau_{n,x_0}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0))|} = \prod_{i=1}^{s_n+1} \left(1 + \mathcal{O}(c_{\ell_i}^{(\frac{1}{2}-\epsilon_3) C_1 \lambda_1^{-\ell_i}}) \right) = 1 + \mathcal{O}(c_{\ell_1}^{(\frac{1}{2}-\epsilon_3) C_1 \lambda_1^{-\ell_1}}). \tag{6.14}$$

The first claim, (6.6), follows from this estimate and Proposition 5.1. To prove the second claim, (6.7) (for $n = \ell_i$, for some $i \in \mathbb{N}$), we similarly have

$$\frac{|\tau_n(\Delta^{(n)}_{q_{n-1}+(j+i_n)q_n})|}{|\tau_{n,x_0}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0))|}=\frac{|\tau_{n,x_0^{(\ell_{s_n})}}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0^{(\ell_{s_n})}))|}{|\tau_{n,x_0}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0))|}\frac{|\tau_{n,x_0^{(\ell_{s_n-1})}}(\Delta^{(n)}_{q_{n-1}+(j+i_n)q_n}(x_0^{(\ell_{s_n-1})}))|}{|\tau_{n,x_0^{(\ell_{s_n})}}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0^{(\ell_{s_n})}))|}\cdot\prod_{i=1}^{s_n-1}\frac{|\tau_{n,x_0^{(\ell_{i-1})}}(\Delta^{(n)}_{q_{n-1}+(j+i_n)q_n}(x_0^{(\ell_{i-1})}))|}{|\tau_{n,x_0^{(\ell_i)}}(\Delta^{(n)}_{q_{n-1}+(j+i_n)q_n}(x_0^{(\ell_i)}))|}. \tag{6.15}$$

Using the same arguments as above, we can estimate the first ratio and the product of the ratios. To estimate the second ratio, notice that $n=\ell_{s_n}$ and $\Delta^{(n)}_{q_{n-1}+(j+i_n)q_n}(x_0^{(\ell_{s_n-1})})=\Delta^{(n)}_{q_{n-1}+jq_n}(x_0^{(\ell_{s_n})})$. We, therefore, obtain

$$\frac{|\tau_n(\Delta^{(n)}_{q_{n-1}+(j+i_n)q_n})|}{|\tau_{n,x_0}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0))|}=\prod_{i=1}^{s_n+1}\left(1+\mathcal{O}(c_{\ell_i}^{(\frac12-\epsilon_3)C_1\lambda_1^{-\ell_i}})\right)=1+\mathcal{O}(c_{\ell_1}^{(\frac12-\epsilon_3)C_1\lambda_1^{-\ell_1}}). \tag{6.16}$$

The claim (6.7) follows from identity (6.16), Proposition 5.5 and Proposition 5.7. □

Proposition 6.4.

$$\lim_{n\to\infty}\frac{\ln|\widetilde{\Delta}_0^{(n)}|}{\ln|\Delta_0^{(n)}(x_0)|}=1. \tag{6.17}$$

Proof. Let $\epsilon_4>0$. We will estimate first

$$\ln\frac{|\Delta_0^{(n-1)}(x_0)|}{|\Delta_0^{(n-1)}|}=\ln\frac{|\Delta_0^{(n-1)}(x_0)|}{|\Delta_0^{(n-1)}(x_0^{(\ell_{s_n})})|}+\sum_{i=1}^{s_n}\ln\frac{|\Delta_0^{(n-1)}(x_0^{(\ell_i)})|}{|\Delta_0^{(n-1)}(x_0^{(\ell_{i-1})})|}. \tag{6.18}$$

We use the same notation as in the proof of Lemma 6.3. Since $x_0^{(\ell_i)}=T^{\mathrm{i}_{\ell_i}q_{\ell_i}}x_0^{(\ell_{i-1})}$, using the Denjoy estimate (2.4) and (6.10), we have

$$\left|\ln\frac{|\Delta_0^{(n-1)}(x_0)|}{|\Delta_0^{(n-1)}|}\right|\le C_{42}c_{\ell_{s_n+1}}^{(\frac12-\epsilon_3)C_1\lambda_1^{-\ell_{s_n+1}}}+V\sum_{i=1}^{s_n}|i_{\ell_i}|, \tag{6.19}$$

where $C_{42}>0$. Therefore,

$$\begin{aligned}\frac{\left|\ln\frac{|\Delta_0^{(n-1)}(x_0)|}{|\Delta_0^{(n-1)}|}\right|}{|\ln|\Delta_0^{(n-1)}||}&\le\frac{C_{42}c_{\ell_{s_n+1}}^{(\frac12-\epsilon_3)C_1\lambda_1^{-\ell_{s_n+1}}}+V\sum_{i=1}^{s_n}|i_{\ell_i}|}{\sum_{i=1}^{n-1}|\ln a_i|}\\&\le\frac{C_{42}c_{\ell_{s_n+1}}^{(\frac12-\epsilon_3)C_1\lambda_1^{-\ell_{s_n+1}}}+VC_{29}\sum_{i=1}^{s_n}\epsilon(\ell_i)k_{\ell_i+1}}{C_{43}\sum_{i=1}^{s_n}k_{\ell_i+1}},\end{aligned} \tag{6.20}$$

for some $C_{43}>0$. The last quantity can be made arbitrarily small for $n\ge N_5$, by choosing $N_5\in\mathbb{N}$ and C_1 large enough (such that ℓ_1 is sufficiently large).

The claim now follows from

$$\frac{\ln|\widetilde{\Delta}_0^{(n-1)}|}{\ln|\Delta_0^{(n-1)}(x_0)|}=\frac{\ln|\widetilde{\Delta}_0^{(n-1)}|}{\ln|\Delta_0^{(n-1)}|+\ln\frac{|\Delta_0^{(n-1)}(x_0)|}{|\Delta_0^{(n-1)}|}} \tag{6.21}$$

and Proposition 4.4 since, for $n\ge N_5$,

$$\left|\frac{\ln|\widetilde{\Delta}_0^{(n-1)}|}{\ln|\Delta_0^{(n-1)}(x_0)|}-1\right|<\epsilon_4. \qquad \square \tag{6.22}$$

Let φ be the conjugacy between T and $\widetilde{T}$ that satisfies (3.1) and $\varphi(x_0)=\widetilde{x}_c$.

Lemma 6.5.

$$\lim_{n\to\infty}\max_{0\le j<k_{n+1}}\frac{\ln|\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}|}{\ln|\Delta^{(n)}_{q_{n-1}+jq_n}(x_0)|}=1,\quad \lim_{n\to\infty}\min_{0\le j<k_{n+1}}\frac{\ln|\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}|}{\ln|\Delta^{(n)}_{q_{n-1}+jq_n}(x_0)|}=1. \tag{6.23}$$

Proof. The claim follows from Lemma 6.3 and Proposition 6.4, taking into account that

$$\left|\frac{\ln|\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}|}{\ln|\Delta^{(n)}_{q_{n-1}+jq_n}(x_0)|}-1\right|=\left|\frac{\ln\frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|}{|\tau_{n,x_0}(\Delta^{(n)}_{q_{n-1}+jq_n}(x_0))|}+\ln\frac{|\widetilde{\Delta}^{(n-1)}_0|}{|\Delta^{(n-1)}_0(x_0)|}}{\ln|\Delta^{(n)}_{q_{n-1}+jq_n}(x_0)|}\right|, \tag{6.24}$$

and that $\max_{0\le j<k_{n+1}}|\Delta_{q_{n-1}+jq_n}(x_0)|$ decreases at least exponentially with n. $\square$

Proposition 6.6.

$$\lim_{n\to\infty}\max_{0<j\le k_{n+2}}\frac{\ln|\widetilde{\Delta}^{(n+1)}_{jq_{n+1}}|}{\ln|\Delta^{(n+1)}_{jq_{n+1}}(x_0)|}=1,\quad \lim_{n\to\infty}\min_{0<j\le k_{n+2}}\frac{\ln|\widetilde{\Delta}^{(n+1)}_{jq_{n+1}}|}{\ln|\Delta^{(n+1)}_{jq_{n+1}}(x_0)|}=1. \tag{6.25}$$

Proof. Notice that $\Delta^{(n+1)}_{jq_{n+1}}(x_0)\subset\Delta^{(n)}_{q_{n+1}-q_n}(x_0)$, for $0<j\le k_{n+2}$. Since

$$\frac{|\widetilde{\Delta}^{(n+1)}_{jq_{n+1}}|}{|\Delta^{(n+1)}_{jq_{n+1}}(x_0)|}=\frac{|\widetilde{\Delta}^{(n+1)}_{q_n+jq_{n+1}}|}{|\Delta^{(n+1)}_{q_n+jq_{n+1}}(x_0)|}\frac{(T^{q_n})'(\mathfrak{z})}{(\widetilde{T}^{q_n})'(\widetilde{\mathfrak{z}})}, \tag{6.26}$$

where $\mathfrak{z}\in\Delta^{(n+1)}_{jq_{n+1}}(x_0)$ and $\widetilde{\mathfrak{z}}\in\widetilde{\Delta}^{(n+1)}_{jq_{n+1}}$, we have

$$\left|\frac{\ln|\widetilde{\Delta}^{(n+1)}_{jq_{n+1}}|}{\ln|\Delta^{(n+1)}_{jq_{n+1}}(x_0)|}-1\right|=\left|\frac{\left(\frac{\ln|\widetilde{\Delta}^{(n+1)}_{q_n+jq_{n+1}}|}{\ln|\Delta^{(n+1)}_{q_n+jq_{n+1}}(x_0)|}-1\right)\ln|\Delta^{(n+1)}_{q_n+jq_{n+1}}(x_0)|+\ln\frac{(T^{q_n})'(\mathfrak{z})}{(\widetilde{T}^{q_n})'(\widetilde{\mathfrak{z}})}}{\ln|\Delta^{(n+1)}_{jq_{n+1}}(x_0)|}\right|. \tag{6.27}$$

The claim follows from the latter identity by using Denjoy bound (2.4) and Lemma 6.5 since $|\Delta^{(n+1)}_{jq_{n+1}}(x_0)|\le e^V|\Delta^{(n+1)}_{q_n+jq_{n+1}}(x_0))|$. $\square$

Proposition 6.7. *If $n\ne\ell_i$, for any $i\in\mathbb{N}$, then*

$$a_n(x_0)=\Theta(a_n). \tag{6.28}$$

Proof. Similar to (6.8), we have

$$\frac{a_n}{a_n(x_0)}=\frac{|\tau_n(\Delta^{(n)}_0)|}{|\tau_{n,x_0}(\Delta^{(n)}_0(x_0))|}=\frac{|\tau_{n,x_0^{(\ell_{s_n})}}(\Delta^{(n)}_0(x_0^{(\ell_{s_n})}))|}{|\tau_{n,x_0}(\Delta^{(n)}_0(x_0))|}\prod_{i=1}^{s_n}\frac{|\tau_{n,x_0^{(\ell_i-1)}}(\Delta^{(n)}_0(x_0^{(\ell_i-1)}))|}{|\tau_{n,x_0^{(\ell_i)}}(\Delta^{(n)}_0(x_0^{(\ell_i)}))|}. \tag{6.29}$$

The ratio in the product is the reciprocal of the distortion of the ratio $|\tau_{n,x_0^{(\ell_i-1)}}(\Delta^{(n)}_0(x_0^{(\ell_i-1)}))|$ under the action of $T^{\mathfrak{i}_{\ell_i}q_{\ell_i}}$ and, since $n>\ell_{s_n}$, it can be estimated, similar to (6.9), as

$$\frac{|\tau_{n,x_0^{(\ell_i-1)}}(\Delta^{(n)}_0(x_0^{(\ell_i-1)}))|}{|\tau_{n,x_0^{(\ell_i)}}(\Delta^{(n)}_0(x_0^{(\ell_i)}))|}=1+\mathcal{O}\left(\sum_{j=0}^{\mathfrak{i}_{\ell_i}q_{\ell_i}-1}|\Delta^{(n-1)}_j(x_0^{(\ell_i-1)})|\right)=1+\mathcal{O}(c_{\ell_i}^{(\frac12-\epsilon_3)C_1\lambda_1^{-\ell_i}}). \tag{6.30}$$

To estimate the ratio in front of product in (6.29), notice that, due to (6.10) and Denjoy estimate (2.4), we obtain

$$\frac{|\Delta_0^{(n)}(x_0))|}{|\Delta_0^{(n)}(x_0^{(\ell_{s_n})}))|} = 1 + \mathcal{O}(c_{\ell_{s_n+1}}^{(\frac{1}{2}-\epsilon_3)C_1\lambda_1^{-\ell_{s_n+1}}}) \tag{6.31}$$

and

$$\frac{|\Delta_0^{(n-1)}(x_0))|}{|\Delta_0^{(n-1)}(x_0^{(\ell_{s_n})}))|} = 1 + \mathcal{O}(c_{\ell_{s_n+1}}^{(\frac{1}{2}-\epsilon_3)C_1\lambda_1^{-\ell_{s_n+1}}}). \tag{6.32}$$

Therefore, the ratio in (6.29) can be estimated as

$$\frac{a_n}{a_n(x_0)} = \prod_{i=1}^{s_n+1}\left(1 + \mathcal{O}(c_{\ell_i}^{(\frac{1}{2}-\epsilon_3)C_1\lambda_1^{-\ell_i}})\right) = 1 + \mathcal{O}(c_{\ell_1}^{(\frac{1}{2}-\epsilon_3)C_1\lambda_1^{-\ell_1}}). \tag{6.33}$$

The claim follows. □

Proof of Theorem 1.1. To prove the claim we will verify that the assumptions of Proposition 3.1 are satisfied with $x = x_0$ and the intervals Δ_j chosen among the intervals of partitions $\mathcal{P}_{n,x_0}$, for $n \in \mathbb{N}$. Proposition 6.4, Lemma 6.5 and Proposition 6.6 give us that, for every $\varepsilon > 0$, there exists $N_6 \in \mathbb{N}$ such that, for all $n \geq N_6$, $0 \leq \bar{j} < k_{n+1}$ and $0 < \hat{j} \leq k_{n+2}$,

$$1 - \frac{\varepsilon}{2} < \frac{\ln|\widetilde{\Delta}^{(n)}_{q_{n-1}+\bar{j}q_n}|}{\ln|\Delta^{(n)}_{q_{n-1}+\bar{j}q_n}(x_0)|}, \frac{\ln|\widetilde{\Delta}_0^{(n-1)}|}{\ln|\Delta_0^{(n-1)}(x_0)|}, \frac{\ln|\widetilde{\Delta}^{(n+1)}_{\hat{j}q_{n+1}}|}{\ln|\Delta^{(n+1)}_{\hat{j}q_{n+1}}(x_0)|} < 1 + \frac{\varepsilon}{2}. \tag{6.34}$$

Let us choose $\delta > 0$ small enough such that the interval $[x_0 - \delta, x_0 + \delta]$ is contained inside the interval $\bar{\Delta}_0^{(N_6)}$. For every $y \in (-\delta, \delta)$, there exists $n > N_6$, such that the interval $[x_0, y] \subset \Delta_0^{(n-1)}(x_0)$ and $[x_0, y] \not\subset \Delta_0^{(n+1)}(x_0)$. Consider the following partitions of $\Delta_0^{(n-1)}(x_0)$: $\mathcal{Q}_{n+1,x_0} := \{\Delta^{(n)}_{q_{n-1}+\bar{j}q_n}(x_0) : 0 \leq \bar{j} < k_{n+1}\} \cup \{\Delta_0^{(n+1)}(x_0)\}$ and

$$\mathcal{G}_{n+1,x_0} := \mathcal{Q}_{n+1,x_0} \backslash \{\Delta^{(n)}_{q_{n+1}-q_n}(x_0)\} \cup \{\Delta^{(n+2)}_{q_{n+1}-q_n}(x_0)\} \cup \{\Delta^{(n+1)}_{\hat{j}q_{n+1}}(x_0) : 0 < \hat{j} \leq k_{n+2}\}. \tag{6.35}$$

Denote the corresponding partitions of $\widetilde{\Delta}_0^{(n-1)}$ by $\widetilde{\mathcal{Q}}_{n+1}$ and $\widetilde{\mathcal{G}}_{n+1}$, respectively.

Recall that if $c_n > 1$, a_n and $\widetilde{a}_n$ are bounded from below by a positive constant (see Proposition 3.3 in [12]). Due to Proposition 6.7, $a_n(x_0)$ is also bounded from below by a positive constant.

Consider first the case $c_n < 1$. It follows from the discussion above and the Denjoy estimate (2.4) that the lengths of the intervals $\Delta^{(n)}_{q_{n+1}-q_n}(x_0)$, $\Delta_0^{(n)}(x_0)$ and $\Delta_0^{(n+1)}(x_0)$ are of the same order. Due to the Denjoy estimate (2.4), for every $C_{44} > 0$, there exists $\epsilon_5 > 0$, such that if $k_{n+1} \leq C_{44}$, then $a_n(x_0) > \epsilon_5$. For every $\epsilon_6 > 0$, there exist $\varkappa_1 > 0$, $N_7 \geq N_6$, and $C_{44} > 0$ such that if $n \geq N_7$ and $k_{n+1} > C_{44}$, then $|f'_{n,x_0}(z) - c_n| \leq \epsilon_6$, for $z \in [-\varkappa_1, \tau_{n,x_0}(T^{-q_n}x_0)]$. Therefore, the length of the intervals $\Delta^{(n)}_{q_{n-1}+\bar{j}q_n}(x_0) \subset \tau^{-1}_{n,x_0}([-\varkappa_1, 0])$ decreases exponentially with $\bar{j}$. Consequently, if $y \in \Delta^{(n)}_{q_{n-1}+\bar{j}q_n}(x_0)$, for some $\bar{j}$, there is an interval of partition $\mathcal{Q}_{n+1,x_0}$ whose length is of the same order as $|x_0 - y|$: if $\bar{j} < k_{n+1} - 1$, then there is j such that $\bar{j} < j < k_{n+1}$ and $|\Delta^{(n)}_{q_{n-1}+jq_n}(x_0)| = \Theta(|x_0 - y|)$; if $\bar{j} = k_{n+1} - 1$, then $|\Delta_0^{(n+1)}(x_0)| = \Theta(|x_0 - y|)$. Similarly, if $\bar{j} < k_{n+1} - 1$, then $|\widetilde{\Delta}^{(n)}_{q_{n-1}+(\bar{j}+1)q_n}| = \Theta(|\varphi(x_0) - \varphi(y)|)$; if $\bar{j} = k_{n+1} - 1$, then $|\widetilde{\Delta}_0^{(n+1)}| = \Theta(|\varphi(x_0) - \varphi(y)|)$. This interval satisfies conditions (i)–(iv) of Proposition 3.1. By (6.34), condition (v) of Proposition 3.1 is also satisfied with $\gamma = 1 - \frac{\varepsilon}{2}$.

If $c_n > 1$, $|\Delta_0^{(n+1)}(x_0)|$ can actually be much smaller than $|\Delta^{(n)}_{q_{n+1}-q_n}(x_0)|$, if k_{n+2} is very large. In this case, we need to consider the extended partition $\mathcal{G}_{n+1,x_0}$ of $\Delta_0^{(n-1)}(x_0)$. Since the lengths of the intervals $\Delta^{(n)}_{q_{n+1}-q_n}(x_0)$, $\Delta_0^{(n)}(x_0)$ and $\Delta_0^{(n-1)}(x_0)$ are of the same order, if $y \in \Delta^{(n)}_{q_{n-1}+\bar{j}q_n}(x_0)$ and $\bar{j} < k_{n+1} - 1$, then $|\Delta^{(n)}_{q_{n+1}-q_n}(x_0)| = \Theta(|x_0 - y|)$ and $|\widetilde{\Delta}^{(n)}_{q_{n+1}-q_n}(x_0)| = \Theta(|\varphi(x_0) - \varphi(y)|)$. If $y \in \Delta^{(n)}_{q_{n+1}-q_n}(x_0)$, then either $y \in \Delta^{(n+2)}_{q_{n+1}-q_n}(x_0)$ or $y \in \Delta^{(n+1)}_{\hat{j}q_{n+1}}(x_0)$ for some $\hat{j}$ satisfying $0 < \hat{j} \le k_{n+2}$. Since $c_{n+1} < 1$, for every $\epsilon_7 > 0$, there exist $\varkappa_2 > 0$, $N_8 \ge N_6$ and $C_{45} > 0$ such that $|f'_{n+1,x_{q_{n+1}-q_n}}(z) - c_n^{-1}| \le \epsilon_7$, for $z \in [-1+\varkappa_2, \tau_{n+1,x_{q_{n+1}-q_n}}(T^{2q_{n+1}}x_0)]$, for $n \ge N_8$ and $k_{n+2} > C_{45}$. Similar analysis as before gives us that if $y \in \Delta^{(n+2)}_{q_{n+1}-q_n}(x_0)$ or $y \in \Delta^{(n+1)}_{\hat{j}q_{n+1}}(x_0)$ for some $\hat{j} < k_{n+2}$, there is j satisfying $\hat{j} < j \le k_{n+2}$, $|\Delta^{(n+1)}_{jq_{n+1}}(x_0)| = \Theta(|x_0 - y|)$ and $|\widetilde{\Delta}^{(n+1)}_{jq_{n+1}}| = \Theta(|\varphi(x_0) - \varphi(y)|)$; if $y \in \Delta^{(n+1)}_{q_{n+1}}(x_0)$, then $|\Delta^{(n+1)}_0(x_0)| = \Theta(|x_0 - y|)$ and $|\widetilde{\Delta}^{(n+1)}_0| = \Theta(|\varphi(x_0) - \varphi(y)|)$. Therefore, conditions (i)–(iv) of Proposition 3.1 are satisfied. By (6.34), condition (v) of Proposition 3.1 is also satisfied with $\gamma = 1 - \frac{\varepsilon}{2}$.

Proposition 3.1 shows that φ and φ^{-1} are $(1-\varepsilon)$-Hölder continuous at x_0 and $\widetilde{x}_c$, respectively. By exchanging the roles of T and $\widetilde{T}$, due to the symmetry in the definition (5.7), we can easily see that φ^{-1} and φ are $(1-\varepsilon)$-Hölder continuous at $\varphi(x_c)$ and x_c, respectively. The claim follows. □

Conflict of interest statement

There is no conflict of interest.

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