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Asymptotic stability of a composite wave of two viscous shock waves for the one-dimensional radiative Euler equations

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Abstract

This paper is devoted to the study of the wellposedness of the radiative Euler equations. By employing the anti-derivative method, we show the unique global-in-time existence and the asymptotic stability of the solutions of the radiative Euler equations for the composite wave of two viscous shock waves with small strength. This method developed here is also helpful to other related problems with similar analytical difficulties.

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Keywords: Radiative Euler equations; Viscous shock waves; Diffusion wave; Stability

1. Introduction

The radiative Euler equations are a fundamental system to describe the motion of the compressible gas with radiation heat transfer phenomenon. Mathematically, the radiative Euler equations are a hyperbolic–elliptic coupled system with the following form:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla p = 0, \\ \left\{\rho\left(e + \frac{|u|^2}{2}\right)\right\}_t + \mathrm{div}\left\{\rho u\left(e + \frac{|u|^2}{2}\right) + pu\right\} + \mathrm{div} q = 0, \\ -\nabla \mathrm{div} q + aq + b\nabla\theta^4 = 0, \end{cases} \tag{1.1}$$

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where ρ, u, p, e, θ, and q are respectively the mass density, velocity, pressure, internal energy, absolute temperature, and the radiative heat flux. Positive constants a and b depend only on the concerned gas itself. As the classic compressible Euler equations, the first three equations in (1.1) stand for the conservation of mass, momentum and energy respectively. The fourth equation in (1.1) is related to the radiation heat transfer phenomenon, and one can refer [32,45,53] for more details. System (1.1) can also be derived by the non-relativistic limit (speed of light tending to $+\infty$) from a hyperbolic–kinetic system, where the radiation is described by the photon density. The photon density satisfies a transport equation with the interaction kernel given by the Stefan–Boltzmann law. Details in this direction can be found in [1,9,17,30].

Let $v = \frac{1}{\rho}$ be the specific volume, then one can rewrite the one-dimensional radiative Euler equations in the Lagrangian coordinates that

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \frac{u^2}{2}\right)_t + (pu)_x + q_x = 0, \\ -\left(\frac{q_x}{v}\right)_x + avq + b\left(\theta^4\right)_x = 0. \end{cases} \tag{1.2}$$

In this paper, we will consider the polytropic gas, *i.e.*, the compressible flow satisfies the following thermal relations that

$$p = \frac{R\theta}{v}, \quad e = \frac{R}{\gamma - 1}\theta, \tag{1.3}$$

where $\gamma > 1$ is the adiabatic exponent and $R > 0$ is the specific gas constant. The solutions satisfy the following initial data and the far field behaviours that

$$\begin{cases} (v, u, \theta)(x, 0) = (v_0, u_0, \theta_0)(x), \\ (v, u, \theta, q)(\pm\infty, t) = (v_\pm, u_\pm, \theta_\pm, 0), \end{cases} \tag{1.4}$$

where the far field states are given constants and satisfy that $v_\pm > 0$, $\theta_\pm > 0$, and $u_\pm \in \mathbb{R}$. Then, in this paper, we will study the global existence and the asymptotic stability of the Cauchy problem (1.2) and (1.4) with the far field states that $(v_\pm, u_\pm, \theta_\pm, 0)$.

As far as we know, there are very few results on the wellposedness of the Cauchy problem of the radiative Euler equations (1.2) due to the complexity and nonlinearity of the governing equations. Almost all the results are related to the analysis of the global in time stability of the viscous Riemann solutions. More precisely, if the radiation effect is neglected, the Riemann solution consists of elementary waves such as shock waves, rarefaction waves and contact discontinuities, which are scaling invariant solutions of the Riemann problem (Euler system):

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \frac{u^2}{2}\right)_t + (pu)_x = 0, \end{cases} \tag{1.5}$$

with the Riemann-type initial data

$$(v, u, \theta)(x, 0) = \left(v_0^R, u_0^R, \theta_0^R\right)(x) := \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0. \end{cases} \tag{1.6}$$

The global in time existence of solutions around a constant state was shown in [19]. For the analysis of the rarefaction wave, if the initial data is a small perturbation of a given rarefaction wave, it was proved in [26] that the solutions converge to the rarefaction wave as $t \to +\infty$. Then in [13], the authors showed that when the absorption coefficient α tends to $+\infty$, the solutions converge to the rarefaction wave with the convergence rate $\alpha^{-\frac{1}{3}} |\ln \alpha|^2$, where the absorption coefficient α is defined by the relationship $a = 3\alpha^2$ and $b = 4\alpha\sigma$ for positive constants a, b and the Stefan–Boltzmann constants σ. The asymptotic stability of a single viscous contact wave was proved in [46,47]. The stability of the composite wave of rarefaction waves and a contact wave was investigated in [39,51]. However, for the analysis of the viscous shock wave, only the shock profile has been studied. The existence of the shock profile was studied in [27]. The asymptotic stability of the shock profile was studied in [28]. In [34], the authors discussed the stability for the case of general hyperbolic–elliptic systems including also the nonlinear stability for the radiative gas dynamics. Recently the formation of the Zeldovich spikes, which is the internal maximum in the profile of temperature, is given in [2,31].

Therefore, there is a natural question: How is the asymptotic stability of the composite wave consisting of two viscous shock waves for the radiative Euler equations (1.2)? We will give the positive answer on this problem in this paper. Our analysis is based on the anti-derivative method and energy estimates. As far as we know, it is the first time to use the anti-derivative method to study the asymptotic stability of the elementary waves for the radiative Euler equations. Since system (1.2) is less dissipative than other systems which were studied by the anti-derivative method, [5,12,14–16] for example, we need more subtle estimates to recover the regularity. We also construct a diffusion wave to eliminate the extra mass, such that (v, u, E) are conserved.

We remark that we are also motivated by the related investigations on the simplified model (Hamer model), which gives a good approximation to the fundamental system (1.2) in a certain physical situation (see [10]). The stability of shock waves for the simplified model has been extensively studied in [19–21,24,23,33–38,43]. For the other related results, one can refer to [3,4,6–8,22,40–42,48–50,52].

The rest of the paper is organised as follows: In Section 2, we construct the viscous shock waves and the diffusion wave, and state the main results. In Section 3, basic properties of these viscous waves and the anti-derivative method are introduced. In Section 4, we show the local existence. Finally, in Section 5, the *a priori* estimates are established by the energy method.

2. Viscous waves and main theorem

2.1. Viscous shock waves

In this subsection, we will construct the viscous shock waves of (1.2) based on [25,27]. For the shortness, we only sketch the arguments and omit the details. In this paper, we consider the situation when the Riemann solution of problem (1.5) and (1.6) consists of two shock waves, *i.e.*, there exists an intermediate state (v_m, u_m, θ_m) such that (v_-, u_-, θ_-) connects with (v_m, u_m, θ_m) by the 1-shock wave with the shock speed $s_1 < 0$, and (v_m, u_m, θ_m) connects with (v_+, u_+, θ_+) by the 3-shock wave with the shock speed $s_3 > 0$.

It is well-known that for any given (v_-, u_-, θ_-) with $v_- > 0$, such Riemann solution exists provided that (v_+, u_+, θ_+) is located on a curved surface in a small neighbourhood of (v_-, u_-, θ_-). Let Ω_- be the neighbourhood of (v_-, u_-, θ_-), *i.e.*,

$$\Omega_- := \left\{(v, u, \theta)\Big|\, |(v - v_-, u - u_-, \theta - \theta_-)| \leq \bar{\delta}\right\}, \tag{2.1}$$

where $\bar{\delta}$ is a positive constant depending only on (v_-, u_-, θ_-). In order to describe the strength of the shock waves, let

$$\delta_1 = |v_m - v_-| + |u_m - u_-| + |\theta_m - \theta_-|,$$

$$\delta_3 = |v_m - v_+| + |u_m - u_+| + |\theta_m - \theta_+|, \tag{2.2}$$

and let

$$\delta = \min\{\delta_1, \delta_3\}. \tag{2.3}$$

If $\bar{\delta}$ is small, then for $(v_+, u_+, \theta_+) \in \Omega_-$, it holds that (cf. [44]),

$$\delta \leq C\, |(v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-)|\,, \tag{2.4}$$

where C is a positive constant depending only on (v_-, u_-, θ_-). Moreover, we assume that there exists a constant $C > 0$, such that

$$\delta_1 + \delta_3 \leq C\delta. \tag{2.5}$$

Next let us consider the viscous i-shock wave of (1.2) with the form

$$Z_i = (V_i, U_i, \Theta_i, Q_i)(x - s_i t), \quad i = 1, 3.$$

If $i = 1$, Z_1 corresponds to the 1-shock wave with the shock speed $s_1 < 0$ connecting the far field states $z_- = (v_-, u_-, \theta_-, 0)$ and $z_m = (v_m, u_m, \theta_m, 0)$. Since the decay rate of q_{xx} is expected in a higher order, the viscous 1-shock wave of (1.2) is the solution with the form $(v, u, \theta, q) = (V_1, U_1, \Theta_1, Q_1)(\xi)$, where $\xi = x - s_1 t$, satisfying the following equation that

$$\begin{cases} -s_1 V_1' - U_1' = 0, \\ -s_1 U_1' + P_1' = 0, \\ -s_1 \left(\frac{R}{\gamma-1} \Theta_1 + \frac{U_1^2}{2} \right)' + (P_1 U_1)' = \left(\frac{4b}{aV_1} \Theta_1^3 \Theta_1' \right)', \\ Q_1 = -\frac{b}{aV_1} \left(\Theta_1^4 \right)' = -\frac{b}{aV_1} \Theta_1^3 \Theta_1', \\ (V_1, U_1, \Theta_1, Q_1)(-\infty) = (v_-, u_-, \theta_-, 0), \\ (V_1, U_1, \Theta_1, Q_1)(+\infty) = (v_m, u_m, \theta_m, 0), \end{cases} \tag{2.6}$$

where $P_1 = p(V_1, \Theta_1)$, $p_\pm = p(v_\pm, \theta_\pm)$, $e_\pm = e(\theta_\pm)$, $p_m = p(v_m, \theta_m)$, and $e_m = e(\theta_m)$. In addition, the shock speed s_1 and the far field states of Z_1 given in (2.6) must satisfy the Rankine–Hugoniot condition and the following entropy condition

$$\lambda_1(v_-, \theta_-) = -\sqrt{\gamma p_- / v_-} > s_1 > -\sqrt{\gamma p_m / v_m} = \lambda_1(v_m, \theta_m). \tag{2.7}$$

It is easy to see that the Rankine–Hugoniot condition and entropy condition (2.7) imply that

$$s_1^2 = \frac{\gamma p_-}{v_m} \left(1 - \frac{d_-}{1 + d_-} \right), \quad \theta_m = \theta_- \left(1 - \frac{v_- + v_m}{v_-} \frac{d_-}{1 + d_-} \right), \quad v_- > v_m, \tag{2.8}$$

where $d_- = \frac{\gamma-1}{2} \frac{v_m - v_-}{v_m} < 0$. Here we may assume that $\bar{\delta}$ is suitably small to assure that $|d_-| < 1$.

By [25,27], we can construct a solution Z_1 of (2.6), which is unique up to the shift of ξ. We omit the details, since the arguments can be found in [25,27].

Similarly, if $i = 3$, Z_3 corresponds to the 3-shock wave of system (1.5) with the shock speed $s_3 > 0$ connecting the far field states $z_m = (v_m, u_m, \theta_m, 0)$ and $z_+ = (v_+, u_+, \theta_+, 0)$. The solution satisfies the following equation that

$$\begin{cases} -s_3 V_3' - U_3' = 0, \\ -s_3 U_3' + P_3' = 0, \\ -s_3 \left(\frac{R}{\gamma-1} \Theta_3 + \frac{U_3^2}{2} \right)' + (P_3 U_3)' = \left(\frac{4b}{aV_3} \Theta_3^3 \Theta_3' \right)', \\ Q_3 = -\frac{b}{aV_3} \left(\Theta_3^4 \right)' = -\frac{b}{aV_3} \Theta_1^3 \Theta_3', \\ (V_3, U_3, \Theta_3, Q_3)(-\infty) = (v_m, u_m, \theta_m, 0), \\ (V_3, U_3, \Theta_3, Q_3)(+\infty) = (v_+, u_+, \theta_+, 0), \end{cases} \tag{2.9}$$

under the entropy condition

$$\lambda_3(v_m, \theta_m) = \sqrt{\gamma p_m / v_m} > s_3 > \sqrt{\gamma p_+ / v_+} = \lambda_3(v_+, \theta_+).$$

We omit the details, since the arguments can be found in [25,27] too.

2.2. *Diffusion wave*

In this subsection, by using the method in [11], we will construct a diffusion wave $Z^d(x, t) = \left(v^d, u^d, \theta^d, q^d \right)(x, t)$, which connects the same constant state $z_m = (v_m, u_m, \theta_m, 0)$ at the positive and negative infinity.

First, we temporarily (*the exact definition will be given in* (2.16) *and* (2.17)) expect that v^d and θ^d are of the following form

$$v^d = \frac{R}{p_m}\tilde{\Theta}, \quad \theta^d = \tilde{\Theta}$$

and the radiative heat flux q^d is

$$q^d := -\frac{b}{av^d}\left(\tilde{\Theta}^4\right)_x$$

for a function $\tilde{\Theta}$ with the far field states that $\tilde{\Theta}(\pm\infty, t) = \theta_m$. Then the first equation of (1.2) is

$$\left(\frac{R}{p_m}\tilde{\Theta}\right)_t - u^d_x = 0. \tag{2.10}$$

The second and third equations together yield that

$$\frac{R}{\gamma-1}\tilde{\Theta}_t + p_m u^d_x = \frac{4b}{a}\left(\frac{\tilde{\Theta}^3\tilde{\Theta}_x}{v^d}\right)_x. \tag{2.11}$$

Substituting (2.10) into (2.11), we have a nonlinear diffusion equation for $\tilde{\Theta}$

$$\tilde{\Theta}_t = \frac{\gamma-1}{R\gamma}\frac{4bp_m}{aR}\left(\tilde{\Theta}^2\tilde{\Theta}_x\right)_x. \tag{2.12}$$

In order to avoid the nonlinearity of the equation above, we further approximate the equation (2.12) by the following linear heat equation

$$\tilde{\Theta}_t = k\,\tilde{\Theta}_{xx}, \qquad \text{where} \quad \tilde{\Theta}(\pm\infty, t) = \theta_m, \quad k = \frac{\gamma-1}{R\gamma}\frac{4bp_m\theta_m^2}{aR}. \tag{2.13}$$

Let constant β_2 be

$$\beta_2 = \int (\tilde{\Theta}(x,t) - \theta_m)dx. \tag{2.14}$$

Thus, by employing the heat kernel formulation, the solution of (2.13) is

$$\tilde{\Theta} = \theta_m + \frac{\beta_2}{\sqrt{4\pi k(1+t)}}e^{-\frac{x^2}{4k(1+t)}}. \tag{2.15}$$

Now define θ^d to be

$$\theta^d = \tilde{\Theta} - \frac{\gamma-1}{2R}\left(u^d - u_m\right)^2, \tag{2.16}$$

and v^d and u^d to be

$$v^d = \frac{R}{p_m}\tilde{\Theta}, \quad u^d = u_m + \frac{aR}{p_m}\tilde{\Theta}_x. \tag{2.17}$$

2.3. *Composite wave of two viscous shock waves and a diffusion wave*

Based on the construction of the viscous shock waves and the diffusion wave, we can construct a composite wave, which is the asymptotic states of the solutions of the initial value problem (1.2) and (1.4) as $t \to \infty$. Set

$$E = \theta + \frac{\gamma-1}{2R}u^2 \qquad \text{and} \qquad m(x,t) = (v, u, E)^\top(x,t). \tag{2.18}$$

Following the arguments in [29], the asymptotic state is expected to be a composite wave $M(x,t) = \left(V, U, \overline{E}_{\beta_1,\beta_3}\right)^\top(x,t)$, which is given by

$$
\begin{aligned}
&V = V_1(x - s_1 t + \beta_1) + V_3(x - s_3 t + \beta_3) - v_m + (v^d(x,t) - v_m),\\
&U = U_1(x - s_1 t + \beta_1) + U_3(x - s_3 t + \beta_3) - u_m + (u^d(x,t) - u_m),\\
&\overline{E}_{\beta_1,\beta_3} = E_1(x - s_1 t + \beta_1) + E_3(x - s_3 t + \beta_3) - E_m + (E^d(x,t) - E_m),\\
&\Theta = \overline{E}_{\beta_1,\beta_3} - \frac{\gamma - 1}{2R} U^2,\\
&Q = Q_1(x - s_1 t + \beta_1) + Q_3(x - s_3 t + \beta_3) + q^d(x,t),
\end{aligned}
\tag{2.19}
$$

where $E_i = \Theta_i + \frac{\gamma-1}{2R} U_i^2 (i = 1, 3, m)$ and $E^d = \theta^d + \frac{\gamma-1}{2R}(u^d)^2$. It follows from (2.16) that

$$
E^d - E_m = \left(\tilde{\Theta} - \theta_m\right) + \frac{\gamma - 1}{R}\left(u^d - u_m\right) u_m,
$$

and

$$
\int (E^d - E_m) dx = \beta_2.
$$

Similarly as done in [5], the constant vector $(\beta_1, \beta_2, \beta_3)$ is determined by

$$
\int_{\mathbb{R}} (m(x,0) - M(x,0))\, dx = 0. \tag{2.20}
$$

We remark that (2.20) allows us to apply the anti-derivative method.

2.4. Main theorem

Before stating the main theorem, let us introduce several notations which we will use throughout this paper. $L^p(\mathbb{R})(1 \le p \le \infty)$ represents the Lebesgue space on $\mathbb{R}$ with norm $\|\cdot\|_{L^p}$. For $k \in \mathbb{Z}_+$, $H^k(\mathbb{R})$ represents the Sobolev space with the norm $\|\cdot\|_k$. It is easy to see that $\|\cdot\|_0 = \|\cdot\|_{L^2}$. To simplify the notation, we set $\|\cdot\| := \|\cdot\|_0 = \|\cdot\|_{L^2}$. Finally, we denote by $C^k(I; H^p)$ the k-times continuously differentiable functions in the interval I with the range in $H^p(\mathbb{R})$; and denote by $L^2(I; H^p)$ the space of L^2 functions in the interval I with the range in $H^p(\mathbb{R})$.

We are ready to introduce the main result of this paper. For any fixed $(v_-, u_-, \theta_-, 0)$, we assume that $(v_+, u_+, \theta_+, 0) \in \Omega_-$ and the Riemann solution of (1.5) and (1.6) with the initial data $(v_\pm, u_\pm, \theta_\pm, 0)$ consists of two shock waves. Let

$$
\begin{aligned}
&\overline{v} = V_1(x) + V_3(x) - v_m, \qquad \overline{u} = U_1(x) + U_3(x) - u_m,\\
&\overline{E} = E_1(x) + E_3(x) - E_m, \qquad \overline{\theta} = \overline{E} - \frac{\gamma - 1}{2R}\overline{u}^2, \qquad \overline{q} = Q_1(x) + Q_3(x).
\end{aligned}
\tag{2.21}
$$

Suppose the initial data $(v_0, u_0, \theta_0, q_0)$ satisfies that

$$
v_0 - \bar{v},\ u_0 - \bar{u},\ \theta_0 - \bar{\theta} \in H^2 \cap L^1, \quad q_0 - \bar{q} \in H^3 \cap L^1. \tag{2.22}
$$

Then (V, U, Θ, Q) in (2.19) is well defined and satisfies (2.20). Furthermore, let

$$
\begin{aligned}
\Phi_0(x) &= \int_{-\infty}^{x} [v_0(y) - V(y,0)]\,dy, \\
\Psi_0(x) &= \int_{-\infty}^{x} [u_0(y) - U(y,0)]\,dy, \\
\overline{W}_0(x) &= \int_{-\infty}^{x} \left[\left(\frac{R}{\gamma-1}\theta_0 + \frac{u_0^2}{2}\right)(y) - \left(\frac{R}{\gamma-1}\Theta + \frac{U^2}{2}\right)(y,0)\right]dy, \\
\overline{\Gamma}_0(x) &= \int_{-\infty}^{x} [(v_0 q_0)(y) - (VQ)(y,0)]\,dy,
\end{aligned}
\tag{2.23}
$$

and assume that

$$
(\Phi_0, \Psi_0, \overline{W}_0, \overline{\Gamma}_0) \in H^3. \tag{2.24}
$$

Let

$$
\begin{aligned}
I(v_0,u_0,\theta_0,q_0) = \ & \|(v_0 - V(\cdot,0), u_0 - U(\cdot,0), \theta_0 - \Theta(\cdot,0))\|_{H^2\cap L^1} \\
& + \|q_0 - Q(\cdot,0))\|_{H^3\cap L^1} + \|(\Phi_0, \Psi_0, \overline{W}_0, \overline{\Gamma}_0)\|_{L^2}.
\end{aligned}
$$

Then, our main result is the following theorem.

Theorem 2.1. *Assume that* (2.2), (2.22) *and* (2.24) *holds, and* $1 < \gamma < 3$. *Then there exist positive constants* δ_0 *and* ϵ_0 *such that if*

$$
\begin{aligned}
&|(v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-)| \le \delta_0, \\
&I\,(v_0,u_0,\theta_0,q_0) \le \epsilon_0,
\end{aligned}
$$

then the Cauchy problem (1.2) *and* (1.4) *admits a unique global in time solution* (v,u,θ,q) *satisfying that*

$$
\begin{aligned}
&(v - V, u - U, \theta - \Theta) \in C^0\left([0,\infty); H^2\right), q - Q \in C^0\left([0,\infty); H^3\right) \\
&(v - V, u - U, \theta - \Theta) \in L^2\left([0,\infty); H^2\right), q - Q \in L^2\left([0,\infty); H^3\right),
\end{aligned}
$$

and the asymptotic behaviour that

$$
\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |(v - V, u - U, \theta - \Theta, q - Q)\,(x,t)| = 0. \tag{2.25}
$$

3. Preliminaries and mathematical reformulation

In this section, we will introduce several properties of the viscous shock waves and the diffusion wave, and then introduce the anti-derivative method to reformulate the problem.

3.1. Preliminaries

In this subsection, we will recall several properties of viscous shock waves and the diffusion wave. First, let us introduce two notations which will be frequently used throughout this paper. $A \lesssim B$ means that there is a constant $C > 0$ which does not depend on δ such that $A \le CB$. Next, $A \approx B$ means that it holds that

$$
|A - B| \lesssim \delta^2 e^{-c\delta(|x|+t)} + \frac{|\beta_2|}{(1+t)^{\frac{3}{2}}} e^{-\frac{cx^2}{1+t}} + \delta e^{-c(|x|+t)}.
$$

For the properties of viscous shock waves, as the ones listed in Proposition 2.1 of [5], we have

Proposition 3.1. *For any fixed* $(v_-, u_-, \theta_-, 0)$, *suppose* $(v_+, u_+, \theta_+, 0) \in \Omega_-$ *and suppose the Riemann solution of* (1.5) *consists of two shock waves whose strengths satisfy* (2.4) *and* (2.5). *Then problems* (2.6) *and* (2.9) *admit smooth solutions* $(V_1, U_1, \Theta_1, Q_1)(x - s_1 t)$ *and* $(V_3, U_3, \Theta_3, Q_3)(x - s_3 t)$ *respectively which are unique up to the spatial shift and satisfy the following:*

(1)

$$U_{ix}(x - s_i t) = -s_i V_{ix}(x - s_i t) < 0,$$
$$|\Theta_{ix}(x - s_i t)| \lesssim |V_{ix}(x - s_i t)|, \quad x \in \mathbb{R},\ t \geq 0, \quad i = 1, 3; \tag{3.1}$$

(2) *There exist positive constants* c *and* C *such that for* $i = 1, 3$,

$$\begin{aligned}
&|(V_1 - v_m, U_1 - u_m, \Theta_1 - \theta_m)(x - s_1 t)| \lesssim \delta_1 e^{-c\delta_1 |x - s_1 t|},\\
&|(V_3 - v_m, U_3 - u_m, \Theta_3 - \theta_m)(x - s_3 t)| \lesssim \delta_3 e^{-c\delta_3 |x - s_3 t|},\\
&|(V_{ix}, U_{ix}, \Theta_{ix}, V_{ixx}, U_{ixx}, \Theta_{ixx}, Q_i, Q_{ix}, Q_{ixx})(x - s_i t)| \lesssim \delta_i^2 e^{-c\delta_i |x - s_i t|},\\
&\left| s_i^2 - \frac{\gamma p_m}{v_m} \right| \lesssim \delta_i.
\end{aligned} \tag{3.2}$$

The first three inequalities are respectively valid in the regions $\{x > s_1 t,\ t \geq 0\}$, $\{x < s_3 t,\ t \geq 0\}$ *and* $\{x \in \mathbb{R},\ t \geq 0\}$.

Next, for the diffusion wave obtained by (2.16) and (2.17), by the straightforward calculations, we know that vector $Z^d = \left(v^d, u^d, \theta^d, q^d\right)$ satisfies that

$$\begin{cases}
v_t^d - u_x^d = 0,\\
u_t^d + p_x^d = \left(R_1^d\right)_x,\\
\left(\frac{R}{\gamma - 1}\theta^d + \frac{(u^d)^2}{2}\right)_t + \left(p^d u^d\right)_x + q_x^d = \left(R_2^d\right)_x,\\
-\left(\frac{q_x^d}{v^d}\right)_x + a v^d q^d + b\left({\theta^d}^4\right)_x = \left(R_3^d\right)_x,\\
\left(v^d, u^d, \theta^d, q^d\right)(\pm\infty, t) = (v_m, u_m, \theta_m, 0).
\end{cases} \tag{3.3}$$

Note that

$$p^d - p_m = -\frac{\gamma - 1}{2v^d}\left(u^d - u_m\right)^2,$$

so the errors R_1^d, R_2^d and R_3^d are given by

$$\begin{aligned}
R_1^d &= \frac{Ra^2}{p_m}\tilde{\Theta}_{xx} - \frac{\gamma - 1}{2Rv^d}\left(u^d - u_m\right)^2,\\
R_2^d &= \frac{4bp_m}{aR}\left(\frac{\tilde{\Theta} - \theta^d}{\theta^d}\right)\tilde{\Theta}^2\tilde{\Theta}_x - \frac{\gamma - 1}{2v^d}\left(u^d - u_m\right)^2 u^d + \frac{Ra^2 u_m}{p_m}\tilde{\Theta}_{xx}
\end{aligned} \tag{3.4}$$

and

$$R_3^d = b\left[\left(\theta^d\right)^4 - \tilde{\Theta}^4\right] + \frac{4bp_m^2}{aR^2}\frac{\left(\tilde{\Theta}^2\tilde{\Theta}_x\right)_x}{\tilde{\Theta}}. \tag{3.5}$$

By (2.15), for $x \in \mathbb{R},\ t \geq 0$, we have that

$$\left|Z^d - z_m\right| \lesssim \frac{|\beta_2|}{(1 + t)^{\frac{1}{2}}} e^{-\frac{x^2}{4a(1+t)}}, \tag{3.6}$$

and that

$$\left|\left(\frac{\partial}{\partial x}\right)^i\left(R_1^d, R_2^d, R_3^d\right)\right| \lesssim \frac{|\beta_2|}{(1+t)^{\frac{3}{2}}}e^{-\frac{x^2}{4a(1+t)}}, \quad 0 \le i \le 3. \tag{3.7}$$

Finally, let us introduce properties of the asymptotic state $Z(x,t) = (V, U, \Theta, Q)^\top(x,t)$ defied by (2.19). Let

$$\begin{aligned}\left(\overline{R}_3\right)_x &= VQ - V_1Q_1 - V_3Q_3 - v^dq^d \\ &= Q_1\left[(V_3 - v_m) + \left(v^d - v_m\right)\right] + Q_3\left[(V_1 - v_m) + \left(v^d - v_m\right)\right] + q^d\left[(V_1 - v_m) + (V_3 - v_m)\right].\end{aligned}$$

Then one has

$$\begin{aligned}&-\left(\tfrac{Q_x}{V}\right)_x + aVQ + b\left(\Theta^4\right)_x \\ &= a\left(\overline{R}_3\right)_x + aV_1Q_1 + aV_3Q_3 + av^dq^d + b\left(\Theta^4\right)_x - \left(\tfrac{Q_x}{V}\right)_x \\ &= a\left(\overline{R}_3\right)_x + b\left[\Theta^4 - \left(\Theta_1^4 + \Theta_3^4 - \theta_m^4\right) - \left(\left(\theta^d\right)^4 - \theta_m^4\right)\right]_x \\ &\quad - \left(\tfrac{Q_x}{V} - \tfrac{Q_{1x}}{V_1} - \tfrac{Q_{3x}}{V_3} - \tfrac{q_x^d}{v^d}\right)_x.\end{aligned}$$

Therefore, it follows from (2.6), (2.9) and (3.3), that $Z(x,t)$ approximately satisfies (1.2) with errors in the following way

$$\begin{cases} V_t - U_x = 0, \\ U_t + P_x = (R_1)_x, \\ \left(\frac{R}{\gamma-1}\Theta + \frac{U^2}{2}\right)_t + (PU)_x + Q_x = (R_2)_x, \\ -\left(\frac{Q_x}{V}\right)_x + aVQ + b\left(\Theta^4\right)_x = (R_3)_x, \end{cases} \tag{3.8}$$

where $(V, U, \Theta, Q)(\pm\infty, t) = (v_\pm, u_\pm, \theta_\pm, 0)$, $P = \frac{R\Theta}{V}$, and the errors R_1, R_2 and R_3 are given by

$$\begin{aligned} R_1 &= P - \left[P_1 + P_3 - p_m + \left(p^d - p_m\right)\right] + R_1^d, \\ R_2 &= PU - \left[P_1U_1 + P_3U_3 - p_mu_m + \left(p^du^d - p_mu_m\right)\right] + R_2^d, \\ R_3 &= a\overline{R}_3 + b\left\{\Theta^4 - \left(\Theta_1^4 + \Theta_3^4 - \theta_m^4\right) - \left[\left(\theta^d\right)^4 - \theta_m^4\right]\right\} - \left(\frac{Q_x}{V} - \frac{Q_{1x}}{V_1} - \frac{Q_{3x}}{V_3} - \frac{q_x^d}{v^d}\right). \end{aligned} \tag{3.9}$$

For the errors, we can show the estimates that

$$\left(\frac{\partial}{\partial x}\right)^i\left(R_j\right) \approx 0, \quad j = 1, 2, 3, \quad 0 \le i \le 3. \tag{3.10}$$

Without loss of the generality, we only show the estimate of R_1 when $i = 0$, since the proof of the other estimates is similar. Note that

$$\begin{aligned}\Theta &= \Theta_1 + \Theta_3 - \theta_m + \left(\theta^d - \theta_m\right) \\ &\quad - \tfrac{\gamma-1}{R}\left[(U_1 - u_m)(U_3 - u_m) + (U_1 - u_m)\left(u^d - u_m\right) + (U_3 - u_m)\left(u^d - u_m\right)\right].\end{aligned}$$

Then we have the estimate that

$$|R_1| \lesssim \left[\left|u^d - u_m\right|^2 + |Z_1 - z_m||Z_3 - z_m| + \left(|Z_1 - z_m| + |Z_3 - z_m|\right)\left|z^d - z_m\right|\right] + \left|R_1^d\right|. \tag{3.11}$$

The wave interaction terms at the right hand side of (3.11) can be estimated by applying Proposition 3.1 and (3.6), since we can see that

$$|Z_1 - z_m||Z_3 - z_m| \lesssim \delta_1\delta_3\left(e^{-c\delta_1(|x|+t)+c\delta_1|\beta_1|} + e^{-c\delta_3(|x|+t)+c\delta_3|\beta_3|}\right),$$

and

$$|Z_i - z_m| \left|z^d - z_m\right| \lesssim \delta_i e^{-c\delta_i(|x|+t)+c\delta_i|\beta_i|} \frac{|\beta_2|}{(1+t)^{\frac{1}{2}}} e^{-\frac{cx^2}{1+t}} + \delta_i|\beta_2|e^{-c(|x|+t)}, (i=1,3).$$

Note that δ_1 and δ_3 are small in the same order by (2.5), so

$$\begin{aligned}
&|Z_1 - z_m||Z_3 - z_m| \lesssim \delta^2 e^{-c\delta(|x|+t)},\\
&\left|(Z_1 - z_m)z^d,\ (Z_3 - z_m)z^d\right| \lesssim \delta^2 e^{-c\delta(|x|+t)} + \frac{|\beta_2|}{(1+t)^{\frac{3}{2}}} e^{-\frac{cx^2}{1+t}} + \delta e^{-c(|x|+t)},\\
&|Z_{1x}(Z_3 - z_m),\ Z_{3x}(Z_1 - z_m)| \lesssim \delta^3 e^{-c\delta(|x|+t)},\\
&\left|Z_{ix}z^d,\ (Z_1 - z_m)z_x^d,\ (Z_3 - z_m)z_x^d\right| \lesssim \delta^2 e^{-c\delta(|x|+t)} + \frac{|\beta_2|}{(1+t)^{\frac{3}{2}}} e^{-\frac{cx^2}{1+t}} + \delta e^{-c(|x|+t)}.
\end{aligned} \tag{3.12}$$

It means that

$$|R_1| \lesssim \delta^2 e^{-c\delta(|x|+t)} + \frac{|\beta_2|}{(1+t)^{\frac{3}{2}}} e^{-\frac{cx^2}{1+t}} + \delta e^{-c(|x|+t)}. \tag{3.13}$$

By the definition of $\approx$, estimate (3.13) is equivalent to that $R_1 \approx 0$.

3.2. Anti-derivative method

In this subsection, we will introduce the anti-derivative method by reformulating the Cauchy problem (1.2) and (1.4) into an integrated system of (1.2). Set

$$\begin{aligned}
\Phi(x,t) &= \int_{-\infty}^{x} (v - V)(y,t)\,dy,\\
\Psi(x,t) &= \int_{-\infty}^{x} (u - U)(y,t)\,dy,\\
\overline{W}(x,t) &= \int_{-\infty}^{x} \left[\left(\frac{R}{\gamma-1}\theta + \frac{u^2}{2}\right) - \left(\frac{R}{\gamma-1}\Theta + \frac{U^2}{2}\right)\right](y,t)\,dy,\\
W(x,t) &= \frac{\gamma-1}{R}(\overline{W} - U\Psi)(x,t),\\
\Gamma(x,t) &= \int_{-\infty}^{x} (vq - VQ)(y,t)\,dy.
\end{aligned} \tag{3.14}$$

Obviously,

$$\begin{aligned}
&v - V = \Phi_x, \quad \theta - \Theta = W_x + \tfrac{\gamma-1}{R}\left(U_x\Psi - \tfrac{1}{2}\Psi_x^2\right),\\
&u - U = \Psi_x, \quad q - Q = \tfrac{1}{V+\Phi_x}\left(\Gamma_x - Q\Phi_x\right),
\end{aligned} \tag{3.15}$$

where we expect $(\Phi, \Psi, W, \Gamma) \in C([0,\infty); H^3)$. Then, substituting (3.15) to (1.2), subtracting (3.8), and integrating the deduced system with respect to x, we have the integrated system for (Φ, Ψ, W, Γ) that

$$\begin{cases} \Phi_t - \Psi_x = 0, \\ \Psi_t + R\left(\dfrac{\Theta + W_x + \frac{\gamma-1}{R}\left(U_x\Psi - \frac{1}{2}\Psi_x^2\right)}{V+\Phi_x} - \dfrac{\Theta}{V}\right) = -R_1, \\ \dfrac{R}{\gamma-1}W_t + U_t\Psi + R\left(\dfrac{\Theta + W_x + \frac{\gamma-1}{R}\left(U_x\Psi - \frac{1}{2}\Psi_x^2\right)}{V+\Phi_x}\right)\Psi_x \\ \qquad = -R_2 + UR_1 - \dfrac{\Gamma_x - Q\Phi_x}{V+\Phi_x}, \\ a\Gamma + \dfrac{1}{V+\Phi_x}\left(\dfrac{\Gamma_x - Q\Phi_x}{V+\Phi_x}\right)_x - \dfrac{Q_x\Phi_x}{V(V+\Phi_x)} \\ \qquad = -b\left\{\left[\Theta + W_x + \dfrac{\gamma-1}{R}\left(U_x\Psi - \dfrac{1}{2}\Psi_x^2\right)\right]^4 - \Theta^4\right\} - R_3, \end{cases} \tag{3.16}$$

with the initial data that

$$(\Phi, \Psi, W, \Gamma)(0) = (\Phi_0, \Psi_0, W_0, \Gamma_0) \in H^3. \tag{3.17}$$

Then we only need to study the Cauchy problem (3.16) and (3.17) instead of problem (1.2) and (1.4) to finish the proof of Theorem 2.1.

4. Local existence

In this section, we will consider the local existence of the Cauchy problem (3.16) and (3.17). For any interval $I \subseteq \mathbb{R}_+$, define the solution set $X(I)$ as

$$X(I) = \left\{(\Phi, \Psi, W, w) \in C(I; H^3) |\ (\Phi_x, \Psi_x, W_x, \Gamma_x) \in L^2(I; H^2), w \in L^2(I; H^3)\right\},$$

where

$$\xi = \theta - \Theta = W_x + \frac{\gamma-1}{R}\left(U_x\Psi - \frac{1}{2}\Psi_x^2\right), \quad w = q - Q = \frac{\Gamma_x - Q\Phi_x}{V+\Phi_x}.$$

We further choose a positive constant $\bar{\delta}_0$ for the given $(v_-, u_-, \theta_-, 0)$ such that if $\delta \lesssim \bar{\delta}_0$, then

$$\sup_{x\in\mathbb{R}, t\geq 0} |(V - v_-, U - u_-, \Theta - \theta_-)(x,t)| \leq \frac{1}{4}\min(v_-, \theta_-), \tag{4.1}$$

where δ is given by (2.3). By the definition of Q in (2.19), it holds that

$$\sup_{x\in\mathbb{R}, t\geq 0} |Q(x,t)| \lesssim \sup_{x\in\mathbb{R}, t\geq 0} \left(|Q_1'| + |Q_3'| + |\widetilde{\Theta}_x|\right) \lesssim \delta^2 e^{-c\delta(|x|+t)} + \frac{|\beta_2|}{(1+t)^{\frac{1}{2}}} e^{-\frac{cx^2}{1+t}} \lesssim \delta + |\beta_2|. \tag{4.2}$$

We also choose a positive constant $\bar{\epsilon}_0\,(\leq \frac{1}{2}\min(v_-, \theta_-))$ such that if $\|(\Phi, \Psi, W, \Gamma)(t)\|_2 \leq \bar{\epsilon}_0$, then

$$\sup_{x\in\mathbb{R}} \left|\left(\Phi, \Psi, W_x + \frac{\gamma-1}{R}\left(U_x\Psi - \frac{1}{2}\Psi_x^2\right), \frac{\Gamma_x - Q\Phi_x}{V+\Phi_x}\right)(x,t)\right| \leq \frac{1}{2}\min(v_-, \theta_-). \tag{4.3}$$

Then we have the following proposition about the local existence of the Cauchy problem (3.16) and (3.17).

Proposition 4.1. (Local existence) *For any fixed $(v_-, u_-, \theta_-, 0)$, there exist positive constants $\bar{\epsilon}_1\,(\leq \bar{\epsilon}_0)$ and $\bar{C}_1\,(\bar{C}_1\bar{\epsilon}_1 \leq \bar{\epsilon}_0)$ such that the following statements hold. Under the assumption that $\delta \lesssim \bar{\delta}_0$ and that $M \in (0, \bar{\epsilon}_1)$, there exists a positive constant $t_0 = t_0(M)$, such that if $\|(\Phi(\cdot,\tau), \Psi(\cdot,\tau), W(\cdot,\tau), w(\cdot,\tau))\|_{H^3} \leq M$, then the Cauchy problem (3.16) with the initial condition (3.17) via replacing $t = 0$ by $t = \tau$, admits a unique solution $(\Phi, \Psi, W, w) \in X([\tau, \tau + t_0])$ satisfying*

$$\sup_{t\in[\tau,\tau+t_0]} \|(\Phi, \Psi, W, w)(\cdot,t)\|_3 \leq \bar{C}_1 M.$$

Proof. Since the main ideas and techniques are similar as the ones in [5], we sketch the process and only list the difference here for the shortness. Without loss of the generality, we only consider the case that $\tau = 0$. First note that $\bar{C}_1 M \leq \bar{C}_1 \bar{\epsilon}_1 \leq \bar{\epsilon}_0$ is suitable small, then the Sobolev's inequality with (4.1)–(4.3) together implies the local solutions constructed in Proposition 4.1 satisfy that

$$\frac{1}{4}v_- \leq (V + \Phi_x)(x,t) \leq \frac{7}{4}v_-, \quad \frac{1}{4}\theta_- \leq (\Theta + \xi)(x,t) \leq \frac{7}{4}\theta_-. \tag{4.4}$$

Differentiate (3.16) with respect to x, and introduce the new variables (ϕ, ψ, ξ, w) that

$$\begin{aligned} &\phi = \Phi_x, \quad \xi = \theta - \Theta = W_x + \frac{\gamma - 1}{R}(U_x \Psi - \frac{1}{2}\Psi_x^2), \\ &\psi = \Psi_x, \quad w = q - Q = \frac{\Gamma_x - Q\Phi_x}{V + \Phi_x}. \end{aligned} \tag{4.5}$$

System (3.16) can be rewritten as follows

$$\begin{cases} \Phi_t = \psi, \\ \Psi_t = -\left(\frac{R\xi}{V+\phi} - \frac{P\phi}{V+\phi}\right) - R_1, \\ \frac{R}{\gamma-1}W_t + U\Psi_x = \frac{R(\Theta+\xi)}{V+\phi}\psi - w + UR_1 - R_2, \\ a\Gamma + \frac{1}{V+\phi}\left(\frac{\Gamma_x}{V+\phi}\right)_x = \frac{1}{V+\phi}\left(\frac{Q\phi}{V+\phi}\right)_x + \frac{Q_x\phi}{V(V+\phi)} - b\left[(\Theta+\xi)^4 - \Theta^4\right] - R_3, \end{cases} \tag{4.6}$$

with the initial data

$$(\Phi, \Psi, W, \Gamma)(x, 0) = (\Phi_0, \Psi_0, W_0, \Gamma_0)(x). \tag{4.7}$$

Moreover, (3.16) again can be rewritten as the following quasi-linear system of (ϕ, ψ, ξ, w)

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t - \dfrac{R(\Theta+\xi)}{(V+\phi)^2}\phi_x = F_1(\phi, \xi, \phi_x, \xi_x), \\ \dfrac{R}{\gamma - 1}\xi_t + \dfrac{R(\Theta+\xi)}{V+\phi}\psi_x = F_2(\phi, \xi, \psi_x, w_x), \\ aw - \dfrac{1}{V+\phi}\left(\dfrac{w_x}{V+\phi}\right)_x = F_3(\phi, \xi, \phi_x, \xi_x), \end{cases} \tag{4.8}$$

where

$$\begin{aligned} F_1(\phi, \xi, \phi_x, \xi_x) &:= \frac{P_x\phi}{V+\phi} - \frac{R\xi_x}{V+\phi} + \frac{R\xi - P\phi}{(V+\phi)^2}V_x - R_{1x}, \\ F_2(\phi, \xi, \psi, w) &:= -w_x - \frac{R\xi - P\phi}{V+\phi}U_x + UR_{1x} - R_{2x}, \\ F_3(\phi, \xi, \phi_x, \xi_x) &:= -\frac{4b(\Theta+\xi)^3}{(V+\phi)}\xi_x + \frac{4b(\Theta+\xi)^3}{\Theta^2(V+\phi)}(2\Theta+\xi)\Theta_x\xi \\ &\quad - \frac{1}{V+\phi}\left[\frac{Q_x\phi}{(V+\phi)V}\right]_x - \frac{a(\Theta+\xi)^5 Q}{V+\phi}\left[\frac{V+\phi}{(\Theta+\xi)^5} - \frac{V}{\Theta^5}\right]. \end{aligned} \tag{4.9}$$

The initial data is

$$\begin{aligned} (\phi, \psi, \xi, w)(x, 0) &= (\phi_0, \psi_0, \xi_0, w_0)(x) \\ &= \left(\Phi_0, \Psi_0, W_{0x} + \tfrac{\gamma-1}{R}\left(U_x\Psi_0 - \tfrac{1}{2}\Psi_{0x}^2\right)\right) \in H^2. \\ w(x, 0) &= \frac{\Gamma_{0x} - Q\Phi_{0x}}{V + \Phi_{0x}} \in H^3. \end{aligned} \tag{4.10}$$

Therefore, we only need to look for the existence of solutions of (4.8)–(4.10) for $t \in [0, t_0]$ with the estimate that

$$M_0 = \sup_{t\in[0,t_0]} \left(\|(\phi, \psi, \xi)(t)\|_2^2 + \|w(t)\|_3^2 \right) \leq \bar{\epsilon}_0.$$

Note that the left hand side of (4.8) for (ϕ, ψ, ξ) is strictly hyperbolic, and the right hand side of (4.8) can be regarded as the lower order terms. Therefore, applying the method in [18,46], for any given function w (or Γ) with $M_0 \leq \bar{\epsilon}_0$, we can show the local existence and uniqueness of solution $(\phi, \psi, \xi) \in C([0, t_0], H^2)$ for a suitably small $t_0 = t_0(M_0) > 0$, provided that $\|(\Phi, \Psi, W, w)(t)\|_3^2 \leq M_0 \leq \bar{\epsilon}_1$ small. The remaining equation, the last equation of (4.8) is a second order differential equation, so plugging the obtained solution (ϕ, ψ, ξ) into $(4.8)_4$, we can easily have a solution $w \in C([0, t_0], H^3)$. Therefore, by applying the fixed point theorem and a straightforward argument, we can obtain a unique solution $(\phi, \psi, \xi) \in C([0, t_0], H^2)$ and $w \in C([0, t_0], H^3)$ for a suitably small time $t_0 = t_0(M_0) > 0$.

Next, we can put (ϕ, ψ, ξ, w) into (4.6) and obtain the unique existence of solution $(\Phi, \Psi, W, w) \in C([0, t_0], H^3)$ of (4.6) and (4.7). So we complete the proof of this proposition. □

5. The a priori estimates and the global existence

Based on the local existence, Proposition 4.1, we should next establish an *a priori* estimate for the solution of (3.16) and (3.17) to show the global in time existence.

Let

$$N(T) = \sup_{t\in[0,T]} \left(\|(\Phi, \Psi, W, \Gamma)(t)\|^2 + \|(\phi, \psi, \xi)(t)\|_2^2 + \|w(t)\|_3^2 \right).$$

In this section, we will show the *a priori* estimate as follows.

Proposition 5.1. (*a priori* estimates) *Under the same assumptions in Theorem 2.1, there exist positive constants $\delta_0\,(\leq \bar{\delta}_0)$ and $\epsilon_0\,(\leq \bar{\epsilon}_1)$, such that if $(\Phi, \Psi, W, \Gamma) \in X([0, T])$ is the solution of (3.16) and (3.17) for some $T > 0$, $|\beta_2| + \delta \leq \delta_0$, and $N(T) \leq \epsilon_0$, then for $t \in [0, T]$, it holds that,*

$$\begin{aligned} &N(t) + \int_0^t \left(\left\| (|U_{1x}| + |U_{3x}|)^{\frac{1}{2}} (\Psi, W)(\tau) \right\|^2 + \|(\Gamma, \Gamma_x, W_x)(\tau)\|^2 \right) d\tau \\ &+ \int_0^t \left(\|(\phi, \psi, \xi)(\tau)\|_2^2 + \|w(\tau)\|_3^2 \right) d\tau \lesssim \left(N(0) + \delta^{\frac{1}{2}} + |\beta_2| \right). \end{aligned} \tag{5.1}$$

First, we remark that based on Proposition 5.1, one can complete the proof of Theorem 2.1.

Proof of Theorem 2.1. Choosing δ_0, and $N(T) \leq \epsilon_0$ suitably small, we can construct the global solution for $(\Phi, \Psi, W, \Gamma) \in X([0, +\infty])$ by combining Propositions 4.1 and 5.1, and can show the estimate (5.1) holds for all $t \in [0, +\infty)$. Furthermore, it holds

$$\int_0^\infty \|(\Phi_x, \Psi_x, \Gamma_x, W_x)(t)\|^2 dt + \frac{d}{dt} \int_0^\infty \|(\Phi_x, \Psi_x, \Gamma_x, W_x)(t)\|^2 dt < +\infty, \tag{5.2}$$

which together with the Sobolev inequality yields the asymptotic behaviour of the solution that

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |(\Phi, \Psi, W, \Gamma, \Phi_x, \Psi_x, W_x, \Gamma_x)(x, t)| = 0. \tag{5.3}$$

Therefore, we obtain the global in time solution as in Theorem 2.1 with the asymptotic behaviour (2.25). □

Therefore, the remaining task is to show the *a prior* estimate, Proposition 5.1.

Let $(\Phi, \Psi, W, \Gamma) \in X([0,T])$ be a solution of (3.16) for some $T>0$, $\delta \le \delta_0 (\le \bar{\delta}_0)$, and $N(T) \le \epsilon_0 (\le \bar{\epsilon}_1)$, where ϵ_0 and δ_0 are suitably small and will be chosen later. By the Sobolev's inequality with (4.1)–(4.3) together, V, $V+\Phi_x$, Θ, and $\Theta+\xi$ are uniformly positive on $[0,T]$, and satisfy that

$$\inf_{x,t} V \ge \frac{3v_-}{4}, \quad \inf_{x,t}(V+\Phi_x) \ge \frac{v_-}{4}, \quad \inf_{x,t} \Theta \ge \frac{3\theta_-}{4}, \quad \inf_{x,t}(\Theta+\xi) \ge \frac{\theta_-}{4}.$$

In order to derive the *a prior* estimate (5.1), we rewrite system (3.16) as

$$\begin{cases} \Phi_t - \Psi_x = 0, \\ \Psi_t - \frac{P}{V}\Phi_x + \frac{R}{V}W_x + \frac{\gamma-1}{V}U_x\Psi = N_1 - R_1, \\ \frac{R}{\gamma-1}W_t + P\Psi_x + U_t\Psi + \frac{\Gamma_x}{V} - \frac{Q\Phi_x}{V} = N_2 - R_2 + UR_1, \\ a\Gamma - \left(\frac{\Gamma_x - Q\Phi_x}{v}\right)_x + 4b\Theta^3(W_x + \frac{\gamma-1}{R}U_x\Psi) = N_3 - R_3, \end{cases} \tag{5.4}$$

where

$$N_1 = \frac{\gamma-1}{2V}\Psi_x^2 - \left(p - P + \frac{P}{V}\Phi_x - \frac{R}{V}(\theta-\Theta)\right),$$
$$N_2 = (P-p)\Psi_x - \frac{\Phi_x}{vV}(\Gamma_x - Q\Phi_x),$$
$$N_3 = -\frac{Q_x\Phi_x}{vV} - 2b\Theta^3\frac{\gamma-1}{R}\Psi_x^2 + o(\xi^2),$$
$$p = \frac{R\theta}{v}, \quad v = V+\Phi_x, \quad \xi = \theta - \Theta = W_x + \frac{\gamma-1}{R}\left(U_x\Psi - \frac{1}{2}\Psi_x^2\right).$$

The initial data is

$$(\Phi, \Psi, W, \Gamma)(x,0) = (\Phi_0, \Psi_0, W_0, \Gamma_0)(x). \tag{5.5}$$

Notice that if $\|(\Phi, \Psi, W)\|_3$ is small, we have that

$$N_1 = O(1)\left(\Phi_x^2 + \Psi_x^2 + W_x^2 + |U_x|\Psi^2\right),$$
$$N_2 = O(1)\left(\Phi_x^2 + \Psi_x^2 + W_x^2 + |U_x|\Psi^2 + |\Phi_x||\Gamma_x|\right),$$
$$N_3 = O(1)\left(|Q_x||\Phi_x| + \Psi_x^2 + W_x^2 + |U_x|\Psi^2\right).$$

Then the proof of Proposition 5.1 is divided into the proof of the following six lemmas.

Lemma 5.1. *Under the conditions listed in Proposition 5.1, if δ_0 and ϵ_0 are suitably small, it holds that*

$$\|(\Phi, \Psi, W)(t)\|^2 + \int_0^t \int_{\mathbb{R}} \left\{(|U_{1x}| + |U_{3x}|)\left(\Psi^2 + W^2\right) + \Gamma^2 + \Gamma_x^2\right\} dx d\tau$$
$$\lesssim \left(\|(\Phi_0, \Psi_0, W_0)\|^2 + \delta^{\frac{1}{2}} + |\beta_2|\right) + (\epsilon_0 + \delta_0)\int_0^t \|(\Phi_x, \Psi_x, W_x)(\tau)\|^2 d\tau. \tag{5.6}$$

Proof. Multiplying equation $(5.4)_1$ by Φ and $(5.4)_2$ by $\frac{V}{P}\Psi$ respectively, and adding together, we have that

$$\left\{\frac{\Phi^2}{2} + \frac{V}{2P}\Psi^2\right\}_t + \left[-\left(\frac{V}{2P}\right)_t + \frac{\gamma-1}{P}U_x\right]\Psi^2 + (-\Phi\Psi)_x + \frac{R}{P}W_x\Psi = \frac{V}{P}\Psi(N_1 - R_1). \tag{5.7}$$

Multiplying equation $(5.4)_3$ by $\frac{R}{P^2}W$ and $(5.4)_4$ by $\frac{R}{4bVP^2\Theta^3}\Gamma$ respectively, and adding all the resulted equations together, we obtain

$$\left\{\frac{R^2}{2(\gamma-1)P^2}W^2\right\}_t+\frac{R^2P_t}{(\gamma-1)P^3}W^2+\frac{R}{P}W\Psi_x+\frac{R}{P^2}U_tW\Psi+\frac{R}{VP^2}QW\Phi_x$$
$$+\frac{aV}{4bR\Theta^5}\Gamma^2+\frac{\Gamma_x-Q\Phi_x}{v}\left(\frac{V}{4bR\Theta^5}\Gamma\right)_x-\left(\frac{\Gamma_x-Q\Phi_x}{v}\frac{V}{4bR\Theta^5}\Gamma\right)_x+\frac{R}{VP^2}(\Gamma W)_x$$
$$+\frac{\gamma-1}{VP^2}U_x\Psi\Gamma=\frac{V}{4bR\Theta^5}\Gamma(N_3-R_3)+\frac{R}{P^2}W(N_2-R_2+UR_1). \tag{5.8}$$

Combining (5.7) and (5.8) together, we get

$$\left\{\frac{\Phi^2}{2}+\frac{V}{2P}\Psi^2+\frac{R^2}{2(\gamma-1)P^2}W^2\right\}_t+A\Psi^2++\frac{R^2P_t}{(\gamma-1)P^3}W^2+\frac{aV}{4bR\Theta^5}\Gamma^2$$
$$+\frac{V}{4bRv\Theta^5}\Gamma_x^2+\left\{-\Phi\Psi-\frac{\Gamma_x-Q\Phi_x}{v}\frac{V}{4bR\Theta^5}\Gamma+\frac{R}{VP^2}\Gamma W+\frac{R}{P}W\Psi\right\}_x+J_1$$
$$=\frac{V}{P}\Psi(N_1-R_1)+\frac{R}{P^2}W(N_2-R_2+UR_1)+\frac{V}{4bR\Theta^5}\Gamma(N_3-R_3), \tag{5.9}$$

where

$$A=-\left(\frac{V}{2P}\right)_t+\frac{\gamma-1}{P}U_x,$$
$$\begin{aligned}J_1&=\frac{RP_x}{P^2}W\Psi+\frac{R}{P^2}U_tW\Psi+\frac{R}{VP^2}QW\Phi_x+\left(\frac{R}{VP^2}\right)_x\Gamma W+\frac{\gamma-1}{VP^2}U_x\Psi\Gamma\\&\quad-\frac{V}{4bR\Theta^5}\Gamma_x\frac{Q\Phi_x}{v}+\frac{\Gamma_x-Q\Phi_x}{v}\left(\frac{V}{4bR\Theta^5}\right)_x\Gamma\\&=\frac{RR_{1x}}{P^2}W\Psi+\frac{R}{VP^2}QW\Phi_x+\left(\frac{R}{VP^2}\right)_x\Gamma W+\frac{\gamma-1}{VP^2}U_x\Psi\Gamma\\&\quad-\frac{V}{4bR\Theta^5}\Gamma_x\frac{Q\Phi_x}{v}+\frac{\Gamma_x-Q\Phi_x}{v}\left(\frac{V}{4bR\Theta^5}\right)_x\Gamma.\end{aligned}$$

We estimate the terms A and J_1 one by one. By (3.7), (3.9) and (3.10), it holds that

$$P_t\approx P_{1t}+P_{3t},\quad P_x\approx P_{1x}+P_{3x},\quad U_x\approx U_{1x}+U_{3x}.$$

Therefore

$$A\approx\left[-\left(\frac{V_1}{2P_1}\right)_t+\frac{\gamma-1}{P_1}U_{1x}\right]+\left[-\left(\frac{V_3}{2P_3}\right)_t+\frac{\gamma-1}{P_3}U_{3x}\right]=:A_1+A_3.$$

By (3.8) and (3.10), we know that

$$P_{it}=-s_iP_{ix}\approx s_iU_{it}=s_i^2(-U_{ix})>0.$$

Then it follows from Proposition 3.1 and (3.8) that

$$\begin{aligned}A_i&=-\frac{V_{it}}{2P_i}+\frac{V_iP_{it}}{2P_i^2}+\frac{\gamma-1}{P_i}U_{ix}\\&\approx\frac{-U_{ix}}{2P_i^2}\left[\left(s_i^2V_i-\gamma P_i\right)+(3-\gamma)P_i\right]\\&\geq c|U_{ix}|\left[(3-\gamma)p_m-C\delta_i\right].\end{aligned}$$

Since $\gamma\in(1,3)$, choosing δ_0 suitably small, we have that there exists a positive constant c, such that

$$A\Psi^2\geq c(|U_{1x}|+|U_{3x}|)\Psi^2-\tilde{R}\Psi^2, \tag{5.10}$$

where and also in the followings $\tilde{R}$ stands for some error function which satisfies that $\tilde{R}\approx 0$.

Next, for J_1, note that

$$
\begin{aligned}
Q &= Q_{1x} + Q_{3x} + Q^c_x \\
&= -\frac{4b\Theta^3}{aV_1}\Theta_{1x} - \frac{4b\Theta^3}{aV_3}\Theta_{3x} - \frac{4b(\theta^c)^3}{av^c}\theta^c_x \\
&\approx \Theta_{1x} + \Theta_{3x},
\end{aligned} \tag{5.11}
$$

where we use the fact that $(\theta^d_x)^2 \approx 0$ and $R_{1,x} \approx 0$. Then by the fact that $|Q| \lesssim \delta_0$, we have

$$
\begin{aligned}
J_1 &\geq -|QW\Phi_x| - |U_x\Psi\Gamma| - |\Gamma_x Q\Phi_x| - O(1)(V_x, \Theta_x)|\Gamma\Gamma_x + \Gamma W + Q\Gamma\Phi_x| \\
&\geq -\delta_0\left(\Gamma^2 + \Gamma_x^2 + \Phi_x^2\right) - C\delta_0\left(|U_{1x}| + |U_{3x}|\right)\left(\Psi^2 + W^2\right) - \tilde{R}\left(\Psi^2 + W^2\right).
\end{aligned} \tag{5.12}
$$

Next let us consider ΨN_1, WN_2 and ΓN_3, which are the terms at the right hand side of (5.8) and (5.9). From the condition $N(T) \leq \epsilon_0$ and the Sovolev's inequality, it easily follows that

$$
\begin{aligned}
&|\Psi N_1| \lesssim \epsilon_0\left[\Phi_x^2 + \Psi_x^2 + W_x^2 + (|U_{1x}| + |U_{3x}|)\,\Psi^2\right] + \tilde{R}\Psi^2, \\
&|WN_2| \lesssim \epsilon_0(\Phi_x^2 + \Psi_x^2 + W_x^2 + \xi^2 + (|U_{1x}| + |U_{3x}|)\Psi^2) + \tilde{R}\Psi^2. \\
&|\Gamma N_3| \lesssim \epsilon_0(\Phi_x^2 + \xi_x^2 + \Phi_x^2).
\end{aligned} \tag{5.13}
$$

Finally, for all the error terms like $\tilde{R}\Psi W_x$, $\tilde{R}\Psi$ and $\tilde{R}W$ at the right hand side of (5.7) and $(\Psi^2 + W^2)\tilde{R}$ at the right hand side of (5.8)–(5.12), recalling the fact that $R_1 \approx 0$ and $R_2 \approx 0$ by (3.10) and the definition of "$\approx$", we know that all the integrations of such terms on $\mathbb{R} \times (0, t)$ are estimated by

$$
\delta_0 \int_0^t \|(\Psi_x, W_x)(\tau)\|^2 \, d\tau + \delta^{\frac{1}{2}} + |\beta_2|. \tag{5.14}
$$

For example, for the terms $\tilde{R}\Psi W_x$ and $\tilde{R}\Psi$, we have that

$$
\int_0^t \int_{\mathbb{R}} |\tilde{R}|(|\Psi W_x| + |\Psi|)\, dx d\tau \lesssim \delta_0 \int_0^t \|W_x\|^2 \, d\tau + \int_0^t \int_{\mathbb{R}} |\tilde{R}||\Psi| \, dx d\tau, \tag{5.15}
$$

and

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}} |\tilde{R}||\Psi| \, dx d\tau \lesssim \int_0^t \int_{\mathbb{R}} \delta^2 \mathrm{e}^{-c\delta(|x|+\tau)}|\Psi| \, dx d\tau \\
&\quad + \int_0^t \int_{\mathbb{R}} \frac{|\beta_2|}{(1+\tau)^{\frac{3}{2}}} \mathrm{e}^{-\frac{cx^2}{1+\tau}}|\Psi| \, dx d\tau + C\int_0^t \int_{\mathbb{R}} \delta \mathrm{e}^{-c(|x|+\tau)}|\Psi| \, dx d\tau \\
&\lesssim \int_0^t \delta^{\frac{3}{2}} e^{-c\delta\tau}\|\Psi\| d\tau + \int_0^t \frac{|\beta_2|}{(1+\tau)^{\frac{5}{4}}}\|\Psi\| d\tau + \delta \\
&\lesssim \delta^{\frac{1}{2}} + |\beta_2|.
\end{aligned} \tag{5.16}
$$

The other error terms can be similarly estimated. We omit the details for the shortness.

Combining all the estimates (5.8)–(5.14), integrating (5.7) on $\mathbb{R} \times (0, t)$ and choosing ϵ_0 and δ_0 suitably small, we have the desired estimate (5.6). This completes the proof of Lemma 5.1. □

We remark that in order to show (5.10), we need the assumption that $\gamma \in (1, 3)$.

In order to derive the estimates of higher order derivatives, we recall that from the perturbation equation (3.16), the equations of $(\phi, \psi, \xi, w)(x, t)$ are

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t - \left(\frac{P}{v}\phi\right)_x + \left(\frac{R}{v}\xi\right)_x = -R_{1x}, \\ \frac{R}{\gamma-1}\xi_t + p\psi_x + \left(\frac{R\xi}{v} - \frac{P\phi}{v}\right)U_x + w_x = UR_{1x} - R_{2x}, \\ \left(\frac{\xi}{\theta}\right)_x - \frac{\xi\Theta_x}{\theta\Theta} - \frac{\Theta}{4b\theta^5}\left(\frac{w_x}{v} - \frac{Q_x\phi}{vV}\right)_x + \frac{a\Theta}{4b}\left(\frac{vq}{\theta^5} - \frac{VQ}{\Theta^5}\right) = 0, \end{cases} \tag{5.17}$$

with the conditions that

$$(\phi, \psi, \xi, w)(x, 0) = (\phi_0, \psi_0, \xi_0, w_0)(x),$$

$$(\phi, \psi, \xi, w)(\pm\infty, t) = (0, 0, 0, 0).$$

Then we have the following lemma.

Lemma 5.2. *Under the conditions listed in the Proposition 5.1, if δ_0 and ϵ_0 are suitably small, it holds that*

$$\begin{aligned} &\|(\phi, \psi, \xi)(t)\|_1^2 + \int_0^t \|(w, w_x, w_{xx})(\tau)\|^2 \, d\tau \\ &\lesssim \|(\phi_0, \psi_0, \xi_0)\|_1^2 + \delta^{\frac{1}{2}} + |\beta_2| + (\epsilon_0 + \delta_0) \int_0^t \|(\phi, \psi, \xi)(\tau)\|_1^2 d\tau. \end{aligned} \tag{5.18}$$

Proof. Multiplying $(5.17)_1$ by $\frac{P}{v}\phi$, $(5.17)_2$ by ψ, $(5.17)_3$ by $\frac{\xi}{\theta}$, and $(5.17)_4$ by w respectively, and combining the results together, we get

$$\begin{aligned} &\left\{\frac{P}{2v}\phi^2 + \frac{\psi^2}{2} + \frac{R\xi^2}{2(\gamma-1)\theta}\right\}_t + \frac{\Theta}{4b\theta^5 v}w_x^2 + \frac{a\Theta v}{4b\theta^5}w^2 \\ &\quad + \left\{\frac{R}{v}\xi\psi - \frac{P}{v}\phi\psi + \frac{\xi}{\theta}w - \frac{\Theta w}{4b\theta^5}\left(\frac{w_x}{v} - \frac{Q_x\phi}{vV}\right)\right\}_x + J_2 \\ &= -R_{1x}\psi + \frac{\xi}{\theta}\left(UR_{1x} - R_{2x}\right), \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} J_2 &= \left(\frac{Pv_t}{2v^2} - \frac{P_t}{2v}\right)\phi^2 + \frac{\xi}{\theta}\left(\frac{R\xi}{v} - \frac{P\phi}{v}\right)U_x + \left(\frac{\Theta}{4b\theta^5}\right)_x w\left(\frac{w_x}{v} - \frac{Q_x\phi}{vV}\right) \\ &\quad - \frac{\Theta w_x}{4b\theta^5}\frac{Q_x\phi}{vV} - \frac{\Theta_x\xi w}{\theta\Theta} + \frac{a\Theta Q w\Theta}{b}\left(\frac{v}{4\theta^5} - \frac{V}{4\Theta^5}\right) + \frac{R\theta_t}{2(\gamma-1)\theta^2}\xi^2 \\ &\geq -C\left(\epsilon_0 + \delta_0\right)\left(\phi^2 + \xi^2 + w^2 + w_x^2\right). \end{aligned}$$

Note that the terms at the right hand side of (5.19) can be dealt with in the same way as (5.16), such that

$$\int_0^t \int_{\mathbb{R}} \left\{-R_{1x}\psi + \frac{\xi}{\theta}\left(UR_{1x} - R_{2x}\right)\right\} dx d\tau \lesssim \delta^{\frac{1}{2}} + |\beta_2|.$$

Integrating (5.19) on $\mathbb{R} \times (0, t)$, and choosing ϵ_0 and δ_0 suitably small, we have the desired estimate that

$$\|(\phi,\psi,\xi)(t)\|^2+\int_0^t\|(w,w_x)(\tau)\|^2\,d\tau$$
$$\lesssim\|(\phi_0,\psi_0,\xi_0)\|^2+\delta^{\frac12}+|\beta_2|+(\epsilon_0+\delta_0)\int_0^t\|(\phi,\psi,\xi)(\tau)\|^2d\tau.$$

Similarly, multiplying (5.17)$_{1x}$ by $\frac{P}{v}\phi_x$, (5.17)$_{2x}$ by ψ_x, (5.17)$_{3x}$ by $\frac{\xi_x}{\theta}$, and (5.17)$_{4x}$ by w_x respectively, combining the results together, we get that

$$\begin{aligned}&\left\{\frac{P}{2v}\phi_x^2+\frac{\psi_x^2}{2}+\frac{R^2}{2(\gamma-1)\theta}\xi_x^2\right\}_t+\frac{\Theta}{4b\theta^5v}w_{xx}^2+\frac{a\Theta v}{4b\theta^5}w_x^2+J_3\\&+\left\{\left(\frac{\xi}{\theta}\right)_xw_x-\frac{\Theta w_x}{4b\theta^5}\left(\frac{w_x}{v}-\frac{Q_x\phi}{vV}\right)_x+\left(\frac{R}{v}\xi-\frac{P}{v}\phi\right)_x\psi_x\right\}_x\\&=-R_{1xx}\psi_x+\frac{\xi_x}{\theta}(UR_{1x}-R_{2x})_x\,,\end{aligned}\tag{5.20}$$

where

$$\begin{aligned}J_3=&\frac{\xi}{\theta^2}\theta_xw_{xx}+\left(\frac{\Theta}{4b\theta^5}\right)_xw_x\left(\frac{w_x}{v}-\frac{Q_x\phi}{vV}\right)_x-\frac{\Theta w_{xx}}{4b\theta^5}\left[\frac{v_xw_x}{v^2}+\left(\frac{Q_x\phi}{vV}\right)_x\right]\\&+\left\{-\frac{\Theta_x\xi}{\theta\Theta}+\frac{a\Theta Q}{4b}\left(\frac{v}{\theta^5}-\frac{V}{\Theta^5}\right)\right\}_xw_x+\left(\frac{a\Theta v}{4b\theta^5}\right)ww_x-\left(\frac{R}{v}\right)_x\xi\psi_{xx}+p_x\psi_x\frac{\xi_x}{\theta}\\&\geq-C\,(\epsilon_0+\delta_0)\left(\phi^2+\phi_x^2+w_x^2+w_{xx}^2+\xi^2+\xi_x^2+\psi_x^2\right).\end{aligned}$$

Integrating (5.20) on $\mathbb{R}\times(0,t)$ and choosing ϵ_0 and δ_0 suitably small, we have the estimate that

$$\begin{aligned}&\|(\phi_x,\psi_x,\xi_x)(t)\|^2+\int_0^t\|(w_x,w_{xx})(\tau)\|^2\,d\tau\\&\lesssim\|(\phi_0,\psi_0,\xi_0)\|_1^2+\delta^{\frac12}+|\beta_2|+(\epsilon_0+\delta_0)\int_0^t\|(\phi,\psi,\xi)(\tau)\|_1^2d\tau.\end{aligned}\tag{5.21}$$

Combining all the results together, we have the desired estimate (5.18). This completes the proof of Lemma 5.2. □

For the estimate of second order derivatives, it holds that

Lemma 5.3. *Under the conditions listed in the Proposition 5.1, if δ_0 and ϵ_0 are suitably small, it holds that*

$$\begin{aligned}&\|(\phi_{xx},\psi_{xx},\xi_{xx})(t)\|^2+\int_0^t\|w_{xxx}(\tau)\|^2\,d\tau\\&\lesssim\|(\phi_0,\psi_0,\xi_0)\|_2^2+\delta^{\frac12}+|\beta_2|+(\epsilon_0+\delta_0)\int_0^t\|(\phi,\psi,\xi)(\tau)\|_2^2d\tau.\end{aligned}\tag{5.22}$$

Proof. Multiplying (5.17)$_{1xx}$ by $(\frac{P}{v}\phi_x)_x$, (5.17)$_{2xx}$ by ψ_{xx}, (5.17)$_{3xx}$ by $\frac{\xi_{xx}}{\theta}$, and (5.17)$_{4xx}$ by w_{xx} respectively, we get

$$\left\{\frac{P}{2v}\phi_{xx}^2+\frac{\psi_{xx}^2}{2}+\frac{R\xi_{xx}^2}{2(\gamma-1)\theta}\right\}_t+\frac{\Theta}{4b\theta^5 v}w_{xxx}^2+J_4+J_5$$
$$+\left\{\left(\frac{\xi}{\theta}\right)_{xx}w_{xx}-\frac{\Theta w_{xx}}{4b\theta^5}\left(\frac{w_x}{v}-\frac{Q_x\phi}{vV}\right)_{xx}-\left(\frac{P}{v}\phi-\frac{R}{v}\xi\right)_{xx}\psi_{xx}\right\}_x$$
$$=-R_{1xxx}\psi_{xx}+(UR_{1x}-R_{2x})_{xx}\frac{\xi_{xx}}{\theta}, \tag{5.23}$$

where

$$J_4:=-\left(\frac{P}{2v}\right)_t\phi_{xx}^2+(2p_x\psi_{xx}+p_{xx}\psi_x)\frac{\xi_{xx}}{\theta}+\left[\frac{a\Theta}{4b}\left(\frac{vq}{\theta^5}-\frac{VQ}{\Theta^5}\right)-\frac{\xi\Theta_x}{\theta\Theta}\right]_{xx}w_{xx}$$
$$+\left[\left(\frac{R\xi}{v}-\frac{P\phi}{v}\right)U_x\right]_{xx}\frac{\xi_{xx}}{\theta}+\frac{\Theta w_{xxx}}{4b\theta^5}\left[2w_{xx}\left(\frac{1}{v}\right)_x+w_x\left(\frac{1}{v}\right)_{xx}-\left(\frac{Q_x\phi}{vV}\right)_{xx}\right]$$
$$\geq-\frac{\Theta}{16b\theta^5 v}w_{xxx}^2-Cw_{xx}^2$$
$$-C(\epsilon_0+\delta_0)\left[|(\phi,\psi,\xi,w)|^2+|(\phi_x,\psi_x,\xi_x,w_x)|^2+|(\phi_{xx},\psi_{xx},\xi_{xx},w_{xx})|^2\right],$$

and

$$J_5:=\left(\frac{P}{v}\right)_{xx}\phi\psi_{xxx}+2\left(\frac{P}{v}\right)_x\phi_x\psi_{xxx}+\frac{R\xi_x v_x}{v^2}\psi_{xxx}-R\xi\left(\frac{1}{v}\right)_{xx}\psi_{xxx}$$
$$=\left(\frac{P_{xx}}{v}-2\frac{P_xv_x}{v^2}+P\frac{2v_x^2}{v^3}\right)\phi\psi_{xxx}+2\left(\frac{P}{v}\right)_x\phi_x\psi_{xxx}+\frac{R\xi_xv_x}{v^2}\psi_{xxx}$$
$$-R\xi\frac{2v_x^2}{v^3}\psi_{xxx}+\frac{R\xi-P\phi}{v^2}V_{xx}\psi_{xxx}+\frac{R\xi-P\phi}{v^2}\phi_{xx}\psi_{xxx}$$
$$=\sum_{i=1}^{6}J_5^i, \tag{5.24}$$

where

$$\sum_{i=1}^{5}J_5^i=\left\{\left[\left(\frac{P_{xx}}{v}-2\frac{P_xv_x}{v^2}+P\frac{2v_x^2}{v^3}\right)\phi+2\left(\frac{P}{v}\right)_x\phi_x+\frac{R\xi_xv_x}{v^2}-R\xi\frac{2v_x^2}{v^3}+\frac{R\xi-P\phi}{v^2}V_{xx}\right]\psi_{xx}\right\}_x$$
$$-\left\{\left(\frac{P_{xx}}{v}-2\frac{P_xv_x}{v^2}+P\frac{2v_x^2}{v^3}\right)\phi+2\left(\frac{P}{v}\right)_x\phi_x+\frac{R\xi_xv_x}{v^2}-R\xi\frac{2v_x^2}{v^3}+\frac{R\xi-P\phi}{v^2}V_{xx}\right\}_x\psi_{xx},$$
$$J_5^6=\frac{R\xi-P\phi}{v^2}\phi_{xx}\phi_{txx}$$
$$=\left(\frac{R\xi-P\phi}{2v^2}\phi_{xx}^2\right)_t-\left(\frac{R\xi-P\phi}{2v^2}\right)_t\phi_{xx}^2.$$

Here we have used the equation that $\phi_t=\psi_x$. Note that system (1.2) is less dissipative, so we need more careful estimates on the terms contain ψ_{xxx} in J_5. Integrating J_4, J_5 on $\mathbb{R}\times(0,t)$ and choosing ϵ_0 and δ_0 suitably small, we have the estimate

$$\int_0^t\int_{\mathbb{R}}(J_4+J_5)dxd\tau\geq-\frac{\Theta}{16b\theta^5 v}\int_0^t\|w_{xxx}(\tau)\|^2d\tau-C\int_0^t\|w_{xx}(\tau)\|^2d\tau$$
$$-C(\epsilon_0+\delta_0)\|\phi_{xx}(t)\|^2-C(\epsilon_0+\delta_0)\int_0^t\|(\phi,\psi,\xi,w)(\tau)\|_2^2d\tau. \tag{5.25}$$

Therefore, integrating (5.23) on $\mathbb{R}\times(0,t)$ and choosing ϵ_0 and δ_0, the terms on the right hand side of (5.23) can also be dealt with in the same way as (5.16) to get (5.22). This completes the proof of Lemma 5.3. □

Next, we should deal with the term $\int_0^t \|(\phi,\psi)(\tau)\|_2^2 d\tau$.

Lemma 5.4. *Under the conditions listed in the Proposition 5.1, if δ_0 and ϵ_0 are suitably small, it holds that*

$$\int_0^t \|(\phi,\psi)(\tau)\|_2^2 d\tau \lesssim N(0) + \delta^{\frac{1}{2}} + |\beta_2| + \int_0^t \|\xi(\tau)\|_2^2 d\tau. \tag{5.26}$$

Proof. Multiplying the equation $(5.4)_2$ by $-\frac{P}{2}\Phi_x$, $(5.4)_3$ by Ψ_x, and $(5.4)_4$ by $\frac{R^2}{V4b\Theta^3}W_x$ respectively, and adding the resulted equations, we have

$$\begin{aligned}
&\left\{\frac{R}{(\gamma-1)}W\Psi_x - \frac{P}{2}\Phi_x\Psi\right\}_t + \left\{\frac{P}{2}\Phi_t\Psi - \frac{R}{(\gamma-1)}W\Psi_t\right\}_x \\
&+\frac{P^2}{2V}\Phi_x^2 - \frac{RP}{2V}\Phi_x W_x + \frac{R^2}{V}W_x^2 + \frac{P}{2}\Psi_x^2 + J_6 \\
&= \left[\frac{P}{2}\Phi_x + \frac{R}{(\gamma-1)}W_x\right](N_1 - R_1) + \Psi_x(N_2 - R_2) + \frac{R^2}{V4b\Theta^3}W_x(N_3 - R_3),
\end{aligned} \tag{5.27}$$

where

$$\begin{aligned}
J_6 &= -\frac{P_x}{2}\Psi\Psi_x + \frac{P_t}{2}\Psi\Phi_x - \frac{\gamma-1}{2V}PU_x\Psi\Phi_x + U_t\Psi\Psi_x + \frac{\Gamma_x\Psi_x}{V} - \frac{Q\Phi_x\Psi_x}{vV} \\
&\quad + \left(a\Gamma - w_x + 4b\Theta^3\frac{\gamma-1}{R}U_x\Psi\right)\frac{R^2}{V4b\Theta^3}W_x \\
&\geq -\eta\left(\Phi_x^2 + \Psi_x^2 + W_x^2\right) - c\left(|U_{1x}| + |U_{3x}|\right)\Psi^2 + \Gamma^2 + w_x^2.
\end{aligned}$$

By the positivity of P and V, it holds that

$$\frac{P^2}{2V}\Phi_x^2 - \frac{RP}{2V}\Phi_x W_x + \frac{R^2}{V}W_x^2 \geq c(\Phi_x^2 + W_x^2).$$

Then integrating (5.27) and choosing δ_0, η suitably small, we have

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}} \left(\Phi_x^2 + \Psi_x^2 + W_x^2\right) dx d\tau &\lesssim \|(\Psi, W, \Phi_x, \Psi_x)(t)\|^2 + \|(\Psi_0, W_0, \Phi_{0x}, \Psi_{0x})\|^2 \\
&\quad + \int_0^t\int_{\mathbb{R}} \left[\left(|U_{1,x}| + |U_{3,x}|\right)\Psi^2 + \Gamma^2 + w_x^2\right] dx d\tau \\
&\quad + \int_0^t\int_{\mathbb{R}} [(|\Phi_x| + |W_x|)|N_1 - R_1| + |\Psi_x||N_2 - R_2| \\
&\quad + |W_x||N_3 - R_3| + |\tilde{R}|\Psi^2\Big] dx d\tau.
\end{aligned} \tag{5.28}$$

Following the proof of Lemma 5.1 and also using (5.16) to control the last term of (5.28), (5.28) can be further estimated such that:

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}} \left(\Phi_x^2 + \Psi_x^2 + W_x^2\right) dx d\tau &\lesssim \|(\Phi_0, \Psi_0, W_0)\|^2 + \|(\phi_0, \psi_0, \xi_0)\|_1^2 + \delta^{\frac{1}{2}} + |\beta_2| \\
&\quad + (\epsilon_0 + \delta_0)\int_0^t \|(\phi, \psi, \xi)(\tau)\|_1^2 d\tau.
\end{aligned} \tag{5.29}$$

Similarly, multiplying $(5.17)_2$ by $-\frac{P}{2}\phi_x$, $(5.17)_3$ by ψ_x, $(5.17)_{2,x}$ by $-\frac{P}{2}\phi_{xx}$, and $(5.17)_{3,x}$ by ψ_{xx} respectively, adding them all and integrating the resulted formula, we can also get

$$\int_0^t \int_{\mathbb{R}} \left(\phi_x^2 + \psi_x^2 + \phi_{xx}^2 + \psi_{xx}^2\right) dx d\tau \lesssim N(0) + \delta^{\frac{1}{2}} + |\beta_2| + (\epsilon_0 + \delta_0) \int_0^t \|(\phi, \psi, \xi)(\tau)\|_2^2 d\tau.$$

Putting the results together and taking δ_0 and ϵ_0 suitably small, we have the desired estimate (5.26) immediately. This completes the proof of Lemma 5.4. □

Now combining all the three lemmas above, we have the estimate that

$$\begin{aligned}
&\|(\Phi, \Psi, W)(t)\|^2 + \|(\phi, \psi, \xi)(t)\|_2^2 + \int_0^t \left(\|(\phi, \psi)(\tau)\|_2^2 + \|w(\tau)\|_3^2\right) d\tau \\
&+ \int_0^t \left[\left\|(|U_{1x}| + |U_{3x}|)^{\frac{1}{2}} (\Psi, W)(\tau)\right\|^2 + \|(\Gamma, \Gamma_x, W_x)(\tau)\|^2\right] d\tau \\
\lesssim\ & N(0) + \delta^{\frac{1}{2}} + +|\beta_2| + (\epsilon_0 + \delta_0) \int_0^t \|\xi(\tau)\|_2^2 d\tau.
\end{aligned} \tag{5.30}$$

Then we need to deal with the term $\int_0^t \|\xi(\tau)\|_2^2 d\tau$.

Lemma 5.5. *Under the conditions listed in the Proposition 5.1, if δ_0 and ϵ_0 are suitably small, it holds that*

$$\int_0^t \|\xi(\tau)\|_2^2 d\tau \lesssim N(0) + \delta^{\frac{1}{2}} + |\beta_2|. \tag{5.31}$$

Proof. Since

$$\xi^2 = \left[W_x + \frac{\gamma - 1}{R}\left(U_x \Psi - \frac{1}{2}\Psi_x^2\right)\right]^2 \le C\left(W_x^2 + U_x^2 \Psi^2 + \Psi_x^4\right),$$

and by (5.30), we have that

$$\begin{aligned}
\int_0^t \|\xi(\tau)\|^2 d\tau &\lesssim \int_0^t \int_{\mathbb{R}} \left[\left(|U_{1,x}| + |U_{3,x}|\right) \Psi^2 + W_x^2\right] dx d\tau + \epsilon_0 \int_0^t \|\Psi_x(\tau)\|^2 d\tau \\
&\lesssim N(0) + \delta^{\frac{1}{2}} + +|\beta_2| + (\epsilon_0 + \delta_0) \int_0^t \|\xi(\tau)\|_2^2 d\tau.
\end{aligned} \tag{5.32}$$

On the other hand, by $(5.17)_4$, we get

$$\xi_x = \frac{\xi \xi_x}{\theta} + \frac{\xi^2 \Theta_x}{\theta \Theta} + \frac{\Theta}{4b\theta^4}\left(\frac{w_x}{v} - \frac{Q_x \phi}{vV}\right)_x - \frac{a\Theta v}{b\theta^4} w - \frac{a\Theta Q}{4b\theta}\left(\frac{v}{\theta^5} - \frac{V}{\Theta^5}\right). \tag{5.33}$$

Multiplying equation (5.33) above by ξ_x, we get

$$\begin{aligned}
\xi_x^2 &= \frac{\xi \xi_x^2}{\theta} + \frac{\xi^2 \xi_x \Theta_x}{\theta \Theta} + \frac{\Theta}{4b\theta^4}\left(\frac{w_x}{v} - \frac{Q_x \phi}{vV}\right)_x \xi_x - \frac{a\Theta v}{b\theta^4} w \xi_x - \frac{a\Theta Q}{4b\theta}\left(\frac{v}{\theta^5} - \frac{V}{\Theta^5}\right)\xi_x \\
&\lesssim \frac{1}{4}\xi_x^2 + (\epsilon_0 + \delta_0)\left(\xi^2 + \phi^2 + \phi_x^2\right) + \left(w^2 + w_x^2 + w_{xx}^2\right).
\end{aligned} \tag{5.34}$$

Integrating (5.34), we have that

$$\begin{aligned}\int_0^t \|\xi_x(\tau)\|^2 d\tau &\lesssim \int_0^t \|w(\tau)\|_2^2 d\tau + (\epsilon_0+\delta_0)\int_0^t \|(\xi,\phi,\phi_x)(\tau)\|^2 d\tau \\ &\lesssim N(0)+\delta^{\frac{1}{2}}+|\beta_2|+(\epsilon_0+\delta_0)\int_0^t \|\xi(\tau)\|_2^2 d\tau. \end{aligned} \tag{5.35}$$

Similarly, taking derivative ∂_x on (5.33) and multiplying the resulted equation by ξ_{xx}, we have

$$\int_0^t \|\xi_{xx}(\tau)\|^2 d\tau \lesssim N(0)+\delta^{\frac{1}{2}}+|\beta_2|+(\epsilon_0+\delta_0)\int_0^t \|\xi(\tau)\|_2^2 d\tau. \tag{5.36}$$

Putting (5.32), (5.35) and (5.36) together and taking δ_0 and ϵ_0 suitably small, we have the desired estimate (5.31) immediately. This completes the proof of Lemma 5.5. □

Finally, we will give the estimate of $\|w(t)\|_3^2$.

Lemma 5.6. *If δ_0 and ϵ_0 are suitably small, it holds that*

$$\|\Gamma(t)\|_1^2 + \|w(t)\|_3^2 \lesssim N(0)+\delta^{\frac{1}{2}}+|\beta_2|. \tag{5.37}$$

Proof. Multiplying the equation $(5.4)_4$ by Γ, it yields that

$$a\Gamma^2 - \left(\frac{\Gamma_x - Q\Phi_x}{v}\Gamma\right)_x + \frac{\Gamma_x^2}{v} = \frac{Q\Phi_x\Gamma_x}{v} - 4b\Theta^3\xi\Gamma + (N_3-R_3)\Gamma. \tag{5.38}$$

Integrating the equation above by x, we get

$$\begin{aligned}\int_{\mathbb{R}} a\Gamma^2 dx + \int_{\mathbb{R}} \frac{\Gamma_x^2}{v} dx &= \int_{\mathbb{R}} \left(\frac{Q\Phi_x\Gamma_x}{v} - 4b\Theta^3\xi\Gamma + (N_3-R_3)\Gamma\right) dx \\ &\le \left(\frac{1}{4}+\epsilon_0\right)\left\|\left(\Gamma, \frac{\Gamma_x}{\sqrt{v}}\right)\right\|^2 + \|\xi_x\|^2 + C(\epsilon_0+\delta)\,\|(\phi,\psi,\xi)\|_1^2. \end{aligned} \tag{5.39}$$

By (5.30) and (5.31), it holds that

$$\|\Gamma(t)\|_1^2 \lesssim N(0)+\delta^{\frac{1}{2}}+|\beta_2|. \tag{5.40}$$

On the other hand, $(5.17)_4$ can be rewritten as

$$avw - \left(\frac{w_x}{v}\right)_x = -\left(\frac{Q_x\phi}{vV}\right)_x - aQ\phi - 4b\theta^3\xi_x - 4b\left(\theta^3-\Theta^3\right)\Theta_x w - R_{3x}. \tag{5.41}$$

Multiplying the equation above by w and integrating it, we have

$$\begin{aligned}\int_{\mathbb{R}} avw^2 dx + \int_{\mathbb{R}} \frac{w_x^2}{v} dx &= \int_{\mathbb{R}} \left[-\left(\frac{Q_x\phi}{vV}\right)_x - aQ\phi - 4b\theta^3\xi_x - 4b\left(\theta^3-\Theta^3\right)\Theta_x w - R_{3x}\right] w dx \\ &\le \left(\frac{1}{4}+\epsilon_0\right)\|(w,w_x)\|^2 + \left(\|Q\|_1^2 + \|(\phi,\xi)\|_2^2\right) + (\epsilon_0+\delta)\,\|(\phi,\psi,\xi)\|_2^2. \end{aligned} \tag{5.42}$$

By (5.30) and (5.31), it holds that

$$\|w\|_1^2 \lesssim N(0)+\delta^{\frac{1}{2}}+|\beta_2|. \tag{5.43}$$

Differentiating (5.41) with respect to x, one has

$$(avw)_x - \left(\frac{w_x}{v}\right)_{xx} = \left[-\left(\frac{Q_x\phi}{vV}\right)_x - aQ\phi - 4b\theta^3\xi_x - 4b\left(\theta^3-\Theta^3\right)\Theta_x w - R_{3x}\right]_x. \tag{5.44}$$

Multiplying equation (5.44) by $-w_{xxx}$ and integrating it, we have

$$\begin{aligned}
&\int_{\mathbb{R}} a v w_{xx}^2 dx + \int_{\mathbb{R}} \frac{w_{xxx}^2}{v} dx \\
&\le \int_{\mathbb{R}} \left[2a v_x w_x w_{xx} + a v_{xx} w w_{xx} + 2\left(\frac{1}{v}\right)_x w_{xx} w_{xxx} + \left(\frac{1}{v}\right)_{xx} w_x w_{xxx} \right] dx \\
&\quad + \int_{\mathbb{R}} \left[\left(\frac{Q_x \phi}{v V}\right)_x + a Q\phi + 4b\theta^3 \xi_x + 4b\left(\theta^3 - \Theta^3\right)\Theta_x w + R_{3x} \right]_x w_{xxx} dx \\
&\lesssim \left(\frac{1}{8} + \epsilon_0\right) \|(w_{xx}, w_{xxx})\|^2 + \left(\|Q\|_1^2 + \|(w, \phi, \xi)\|_2^2\right) + (\epsilon_0 + \delta)\, \|(\phi, \psi, \xi)\|_2^2.
\end{aligned} \tag{5.45}$$

Therefore it follows that

$$\|w_{xx}\|_1^2 \lesssim N(0) + \delta^{\frac{1}{2}} + |\beta_2|. \tag{5.46}$$

Combining (5.40), (5.43) and (5.46) together, we have (5.37). This completes the proof of Lemma 5.6. □

Now combining Lemma 5.1–Lemma 5.6 together, we have (5.1) and then finish the proof of the Proposition 5.1.

Conflict of interest statement

No conflict of interest.

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References

[1] C. Buet, B. Despres, Asymptotic analysis of fluid models for the coupling of radiation and hydrodynamics, J. Quant. Spectrosc. Radiat. Transf. 85 (2004) 385–418.
[2] J. Coulombel, T. Goudon, P. Lafitte, C. Lin, Analysis of large amplitude shock profiles for non-equilibrium radiative hydrodynamics: formation of Zeldovich spikes, Shock Waves 22 (2012) 181–197.
[3] R. Duan, K. Fellner, C. Zhu, Energy method for multi-dimensional balance laws with non-local dissipation, J. Math. Pures Appl. 93 (2010) 572–598.
[4] R. Duan, L. Ruan, C. Zhu, Optimal decay rates to conservation laws with diffusion-type terms of regularity-gain and regularity-loss, Math. Models Methods Appl. Sci. 22 (2012) 1250012, 39 pp..
[5] L. Fan, A. Matsumura, Asymptotic stability of a composite wave of two viscous shock waves for a one-dimensional system of non-viscous and heat-conductive ideal gas, J. Differ. Equ. 258 (2015) 1129–1157.
[6] M. Francesco, Initial value problem and relaxation limits of the Hamer model for radiating gases in several space variables, Nonlinear Differ. Equ. Appl. 13 (2007) 531–562.
[7] W. Gao, C. Zhu, Asymptotic decay toward the planar rarefaction waves for a model system of the radiating gas in two dimensions, Math. Models Methods Appl. Sci. 18 (2008) 511–541.
[8] W. Gao, L. Ruan, C. Zhu, Decay rates to the planar rarefaction waves for a model system of the radiating gas in n-dimensions, J. Differ. Equ. 244 (2008) 2614–2640.
[9] P. Godillon-Lafitte, T. Goudon, A coupled model for radiative transfer: Doppler effects, equilibrium and non equilibrium diffusion asymptotics, SIAM J. Multiscale Model. Simul. 4 (2005) 1245–1279.

[10] K. Hamer, Nonlinear effects on the propagation of sound waves in a radiating gas, Q. J. Mech. Appl. Math. 24 (1971) 155–168.
[11] L. Hsiao, T.-P. Liu, Convergence to diffusion waves for solutions of a system of hyperbolic conservation laws with damping, Comment. Phys.-Math. 143 (1992) 599–605.
[12] F. Huang, J. Li, A. Matsumura, Asymptotic stability of combination of viscous contact wave with rarefaction waves for one-dimensional compressible Navier–Stokes system, Arch. Ration. Mech. Anal. 197 (2010) 89–116.
[13] F. Huang, X. Li, Convergence to the rarefaction wave for a model of radiating gas in one-dimension, Acta Math. Appl. Sin. Engl. Ser. 32 (2016) 239–256.
[14] F. Huang, A. Matsumura, Stability of a composite wave of two viscous shock waves for the full compressible Navier–Stokes equation, Comment. Phys.-Math. 289 (2009) 841–861.
[15] F. Huang, A. Matsumura, Z. Xin, Stability of contact discontinuities for the 1-D compressible Navier–Stokes equations, Arch. Ration. Mech. Anal. 179 (2006) 55–77.
[16] F. Huang, H. Zhao, On the global stability of contact discontinuity for compressible Navier–Stokes equations, Rend. Semin. Mat. Univ. 109 (2003) 283–305.
[17] S. Jiang, F. Li, F. Xie, Nonrelativistic limit of the compressible Navier–Stokes–Fourier-P1 approximation model arising in radiation hydrodynamics, SIAM J. Math. Anal. 47 (2015) 3726–3746.
[18] S. Kawashima, Systems of a Hyperbolic–Parabolic Composite Type, with Applications to the Equations of Magneto-Hydrodynamics, doctoral thesis in Kyoto University, 1984, http://hdl.handle.net/2433/97887.
[19] S. Kawashima, Y. Nikkuni, S. Nishibata, The initial value problem for hyperbolic–elliptic coupled systems and applications to radiation hydrodynamics, in: Analysis of Systems of Conservation Laws, Aachen, 1997, in: Chapman Hall/CRC Monogr. Surv. Pure. Appl. Math., vol. 99, Chapman Hall/CRC, Boca Raton, FL, 1999, pp. 87–127.
[20] S. Kawashima, S. Nishibata, Shock waves for a model system of a radiating gas, SIAM J. Math. Anal. 30 (1999) 95–117.
[21] S. Kawashima, S. Nishibata, Cauchy problem for a model system of the radiating gas: weak solutions with a jump and classical solutions, Math. Models Methods Appl. Sci. 9 (1999) 69–91.
[22] C. Lattanzio, P. Marcati, Global well-posedness and relaxation limits of a model for radiating gas, J. Differ. Equ. 190 (2003) 439–465.
[23] C. Lattanzio, C. Mascia, T. Nguyen, R. Plaza, K. Zumbrun, Stability of scalar radiative shock profiles, SIAM J. Math. Anal. 41 (2009) 2165–2206.
[24] C. Lattanzio, C. Mascia, D. Serre, Shock waves for radiative hyperbolic–elliptic systems, Indiana Univ. Math. J. 56 (2007) 2601–2640.
[25] C. Lattanzio, C. Mascia, D. Serre, Nonlinear hyperbolic–elliptic coupled systems arising in radiation dynamics. Hyperbolic problems: theory, numerics, applications, in: S. Benzoni-Gavage, D. Serre (Eds.), Proceedings of the XIth International Conference, Ecole Normale Supérieure, Lyon, July 17–21, 2006, Springer-Verlag, Berlin, Heidelberg, 2008, pp. 661–669.
[26] C. Lin, Asymptotic stability of rarefaction waves in radiative hydrodynamics, Commun. Math. Sci. 9 (2011) 207–223.
[27] C. Lin, J. Coulombel, T. Goudon, Shock profiles for non-equilibrium radiating gas, Physica D 218 (2006) 83–94.
[28] C. Lin, J. Coulombel, T. Goudon, Asymptotic stability of shock profiles in radiative hydrodynamics, C. R. Math. Acad. Sci. Paris 345 (2007) 625–628.
[29] T. Liu, Nonlinear stability of shock waves for viscous conservation laws, Mem. Am. Math. Soc. 56 (1985) 1–108.
[30] R. Lowrie, J. Morel, J. Hittinger, The coupling of radiation and hydrodynamics, Astrophys. J. 521 (1999) 432–450.
[31] C. Mascia, Small, medium and large shock waves for radiative Euler equations, Physica D 245 (2013) 46–56.
[32] D. Mihalas, B. Mihalas, Foundations of Radiation Hydrodynamics, Oxford University Press, New York, 1984.
[33] T. Nguyen, Multi-dimensional stability of planar Lax shocks in hyperbolic–elliptic coupled systems, J. Differ. Equ. 252 (2012) 382–411.
[34] T. Nguyen, R. Plaza, K. Zumbrun, Stability of radiative shock profiles for hyperbolic–elliptic coupled systems, Physica D 239 (2010) 428–453.
[35] M. Nishikawa, S. Nishibata, Convergence rates toward the travelling waves for a model system of the radiating gas, Math. Methods Appl. Sci. 30 (2007) 649–663.
[36] M. Ohnawa, Convergence rates towards the traveling waves for a model system of radiating gas with discontinuities, Kinet. Relat. Models 5 (2012) 857–872.
[37] M. Ohnawa, L^∞-stability of continuous shock waves in a radiating gas model, SIAM J. Math. Anal. 46 (2014) 2136–2159.
[38] M. Ohnawa, L^∞-stability of discontinuous traveling waves in a hyperbolic–elliptic coupled system, SIAM J. Math. Anal. 48 (2016) 3820–3839.
[39] C. Rohde, W. Wang, F. Xie, Hyperbolic–hyperbolic relaxation limit for a 1D compressible radiation hydrodynamics model: superposition of rarefaction and contact waves, Commun. Pure Appl. Anal. 12 (2013) 2145–2171.
[40] L. Ruan, J. Zhang, Asymptotic stability of rarefaction wave for hyperbolic–elliptic coupled system in radiating gas, Acta Math. Sci. Ser. B Engl. Ed. 27 (2007) 347–360.
[41] L. Ruan, C. Zhu, Asymptotic decay toward rarefaction wave for a hyperbolic–elliptic coupled system on half space, J. Partial Differ. Equ. 21 (2008) 173–192.
[42] L. Ruan, C. Zhu, Asymptotic behavior of solutions to a hyperbolic–elliptic coupled system in multi-dimensional radiating gas, J. Differ. Equ. 249 (2010) 2076–2110.
[43] D. Serre, L^1-stability of constants in a model for radiating gases, Commun. Math. Sci. 1 (2003) 197–205.
[44] J. Smoller, Shock Waves and Reaction–Diffusion Equations, Springer-Verlag, New York–Berlin, 1983.
[45] W. Vincenti, C. Kruger, Introduction to Physical Gas Dynamics, Wiley, New York, 1965.
[46] J. Wang, F. Xie, Singular limit to strong contact discontinuity for a 1D compressible radiation hydrodynamics model, SIAM J. Math. Anal. 43 (2011) 1189–1204.
[47] J. Wang, F. Xie, Asymptotic stability of viscous contact wave for the 1D radiation hydrodynamics system, J. Differ. Equ. 251 (2011) 1030–1055.

[48] W. Wang, W. Wang, Blow up and global existence of solutions for a model system of the radiating gas, Nonlinear Anal. 81 (2013) 12–30.
[49] W. Wang, W. Wang, The pointwise estimates of solutions for a model system of the radiating gas in multi-dimensions, Nonlinear Anal. 71 (2009) 1180–1195.
[50] W. Wang, W. Wang, Z. Wu, Decay of a model system of radiating gas, Math. Methods Appl. Sci. 37 (2014) 2331–2340.
[51] F. Xie, Nonlinear stability of combination of viscous contact wave with rarefaction waves for a 1D radiation hydrodynamics model, Discrete Contin. Dyn. Syst., Ser. B 17 (2012) 1075–1100.
[52] T. Yang, H. Zhao, BV estimates on Lax–Friedrichs' scheme for a model of radiating gas, Appl. Anal. 83 (2004) 533–539.
[53] Y. Zeldovich, Y. Raizer, Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, Academic Press, New York, 1966.