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Energy identity and necklessness for a sequence of Sacks–Uhlenbeck maps to a sphere [☆]

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Abstract

Let u be a map from a Riemann surface M to a Riemannian manifold N and $\alpha > 1$, the α energy functional is defined as

$$E_\alpha(u) = \frac{1}{2}\int_M [(1 + |\nabla u|^2)^\alpha - 1]dV.$$

We call u_α a sequence of Sacks–Uhlenbeck maps if u_α are critical points of E_α and

$$\sup_{\alpha>1} E_\alpha(u_\alpha) < \infty.$$

In this paper, we show the energy identity and necklessness for a sequence of Sacks–Uhlenbeck maps during blowing up, if the target N is a sphere S^{K-1}. The energy identity can be used to give an alternative proof of Perelman's result [15] that the Ricci flow from a compact orientable prime non-aspherical 3-dimensional manifold becomes extinct in finite time (cf. [3,4]).

Keywords: Energy identity; Sacks–Uhlenbeck map; Harmonic sphere

1. Introduction

Let M be a compact Riemann surface and N be a compact Riemannian manifold without boundary. By Nash's embedding theorem, N can be embedded isometrically into an Euclidean space R^K. For $u \in H^{1,2}(M, N)$, one defines the energy of u by

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$$E(u)=\frac{1}{2}\int_M |\bigtriangledown u|^2 dV.$$

The critical points of the energy functional are called harmonic maps.

Given a metric h on N and a nontrivial homotopy class $[\beta]\in\pi_1(C^0\cap H^{1,2}(M,N))$, following the usual variational principle, we define $W(h,[\beta])$ (this is called the width by Colding–Minicozzi in [3]) by

$$W(h,[\beta])=\inf_{\gamma\in[\beta]}\sup_{s\in[0,1]} E(\gamma(s)). \tag{1.1}$$

It was proved by J. Jost [8] that there exists a sequence $\gamma^j\in[\beta]$ such that

$$W(h,[\beta])=\lim_{j\to\infty}\max_{s\in[0,1]} E(\gamma^j(s)), \tag{1.2}$$

and there is a sequence $s_j\in[0,1]$, a harmonic map u_0 from M to N and some harmonic spheres $\phi_1,\cdots,\phi_m$ in N such that $\gamma^j(s_j)\to u_0$ weakly in $H^{1,2}(M,N)$ and

$$W(h,[\beta])=E(u_0)+\sum_{i=1}^m E(\phi_i). \tag{1.3}$$

This last identity is called the energy identity.

To get harmonic maps, Sacks–Uhlenbeck [19] considered the perturbed energy functional

$$E_\alpha(u)=\frac{1}{2}\int_M [(1+|\bigtriangledown u|^2)^\alpha-1]dV,$$

for $u\in H^{1,2\alpha}(M,N)$ where $\alpha>1$. The Euler–Lagrange equation for the functional E_α is

$$\triangle u+\frac{(\alpha-1)\langle\nabla^2u(\nabla u),\nabla u\rangle}{1+|\bigtriangledown u|^2}=A(u)(du,du), \tag{1.4}$$

where A is the second fundamental form of N in R^K.

For any $\alpha>1$, the perturbed energy E_α satisfies the Palais–Smale condition ([19]), so there is a map $\gamma^\alpha:[0,1]\to C^0\cap H^{1,2\alpha}(M,N)$ with $\gamma^\alpha\in[\beta]$ such that

$$E_\alpha(\gamma^\alpha(s_\alpha))=\inf_{\gamma\in[\beta]}\sup_{s\in[0,1]} E_\alpha(\gamma(s)), \tag{1.5}$$

where $s_\alpha\in[0,1]$. Sacks–Uhlenbeck proved that there is a sequence $\alpha_n\to 1$ as $n\to\infty$ such that $u_n:=\gamma^{\alpha_n}(s_{\alpha_n})$ converges to a harmonic map u_0 outside a finite set of points Σ which is usually called blow-up set. Rescaling the maps near every point in Σ, one obtains some harmonic spheres $\phi_1,\cdots,\phi_m$, and shows the energy identity (cf. [3,4], [9]).

$$\lim_{n\to\infty}E(u_n)=E(u_0)+\sum_{i=1}^m E(\phi_i).$$

In this note, we consider a sequence of Sacks–Uhlenbeck maps u_α which are critical points of E_α with

$$\sup_{\alpha>1}E_\alpha(u_\alpha)<\infty.$$

In [13], they construct a target N and a sequence of Sacks–Uhlenbeck maps such that the energy identity fails to hold. In this paper we assume that $N=S^{K-1}$ is a sphere, and prove the energy identity for a sequence of Sacks–Uhlenbeck maps during blowing up. Furthermore, in this case we show that there is no oscillation on the neck domain, this phenomenon is called no neck or necklessness. The geometrical meaning is that the image of the base map and all the harmonic spheres during blowing up is connected.

Because the blow-up set is a finite set of points, we can choose a finite covering $\{U_i\}$ of M so that the blow up points are away from the boundaries of U_i. We see that it suffices to show the energy identity and necklessness in each U_i. Without loss of generality we assume that U_i is an unit disc B_1 in R^2 and the metric in B_1 is Euclidean. Our main results are stated in the following theorem.

Theorem 1.1. *Let $u_\alpha \in H^{1,2\alpha}(B_1, S^{K-1})$ be a critical point of E_α and*

$$\sup_{\alpha>1} E_\alpha(u_\alpha) = \Lambda < \infty.$$

If u_α tends to u weakly in $H^{1,2}(B_1, S^{K-1})$, then u is harmonic and there exist a subsequence of $\{u_\alpha\}$ (we still denote it by $\{u_\alpha\}$) and some nonnegative integer m. For any $j = 1, ..., m$, there exist a point x_α^j, a positive number r_α^j and a nonconstant harmonic sphere ψ^j (which we view as a map from $R^2 \cup \{\infty\} \to S^{K-1}$) such that

(1) $x_\alpha^j \to x^j \in B_1, r_\alpha^j \to 0$ as $\alpha \to 1$;

(2) $\lim_{\alpha\to 1}(\frac{r_\alpha^i}{r_\alpha^j} + \frac{r_\alpha^j}{r_\alpha^i} + \frac{|x_\alpha^i - x_\alpha^j|}{r_\alpha^i + r_\alpha^j}) = \infty$ for any $i \neq j$;

(3) ψ^j is a harmonic sphere in S^{K-1};

(4) ***Energy identity:***

$$\lim_{\alpha\to 1} E(u_\alpha) = E(u) + \sum_{j=1}^{m} E(\psi^j).$$

(5) ***Necklessness:***

$$\lim_{\alpha\to 1} \|u_\alpha(\cdot) - u(\cdot) - \sum_{j=1}^{m}[\psi^j(\frac{\cdot - x_\alpha^j}{r_\alpha^j}) - \psi^j(\infty)]\|_\infty = 0.$$

The energy identity of a Sacks–Uhlenbeck sequence can be used to give an alternative proof of Perelman's result that if N is a compact orientable prime non-aspherical 3-dimensional manifold with a Riemannian metric g_0, along the Ricci flow the metric $g(t)$ must become extinct in finite time ([15]). Perelman used disc-type harmonic maps and the curve shortening flow in his proof. Another proof was previously given by Colding–Minicozzi [3,4]. Other results related to the energy identity for energy E_α can be found in [9], [12], etc. The energy identity for approximation harmonic maps was studied by many authors, for example, Ding–Tian [5], Qing [17], Qing–Tian [18], Lin–Wang [14], Li–Zhu [10,11], Zhu [23], etc.

This paper is organized as below. In section 2 we recall some basic lemmas on the Lorentz spaces and the elliptic estimates. We prove the energy identity in section 3 and the necklessness in section 4.

Throughout this paper, the letter C denotes a positive constant which maybe depend on Λ, K and may vary in different cases. Furthermore we do not distinguish the sequence of u_α and its subsequences for simplicity.

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2. Basic lemmas on Lorentz spaces and elliptic equations

We recall the definitions of Lorentz spaces (cf. chapter V of [20]). The distribution of f and the non-increasing rearrangement function of f are defined as follows:

$$\lambda_f(s) = |\{x; |f(x)| > s\}|;$$
$$f^*(t) = \inf\{s \geq 0 : \lambda_f(s) \leq t\}.$$

The Lorentz space $L^{p,q}(0 < p, q < \infty)$ is defined by

$$\{f : \|f\|_{L^{p,q}} = (\frac{q}{p}\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t})^{\frac{1}{q}} < \infty\}.$$

One puts factor $\frac{q}{p}$ here so that the $L^{p,q}$-norm of $\chi_{[0,1]}$ equals to 1.

For $q=\infty$, $L^{p,\infty}$ is defined by

$$\{f : \|f\|_{L^{p,\infty}} = \sup_t t^{\frac{1}{p}} f^*(t) < \infty\}.$$

In general $\|\cdot\|_{L^{p,q}}$ may not be a norm, but when $1 \le q \le p$, it is a norm.
The following Hölder inequality on the Lorentz spaces is often used.

Lemma 2.1. *If* $1 < p_1, p_2 < \infty$, $1 \le q_1, q_2 \le \infty$ *satisfy* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1$, *then*

$$|\int f(x)g(x)dx| \le (\frac{p_1}{q_1})^{\frac{1}{q_1}} (\frac{p_2}{q_2})^{\frac{1}{q_2}} \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.$$

In this paper we also need the following interpolation theorem which is a special case of Theorem 1.4.19 in [6].

Lemma 2.2. *Suppose that* $1 \le p_1 < p_2 \le \infty$ *and* T *is a linear operator on* $L^{p_1} + L^{p_2}$. *If* T *is both bounded on* L^{p_1} *and on* L^{p_2}, *i.e.* $\|Tf\|_{p_i} \le C_i \|f\|_{p_i}$ $(i=1,2)$, *then for any* p, q *with* $p_1 < p < p_2$, $1 \le q \le \infty$, T *is bounded on* $L^{p,q}$, *i.e.*

$$\|Tf\|_{L^{p,q}} \le C\|f\|_{L^{p,q}}.$$

Now we recall some elliptic estimates.

Lemma 2.3. *Assume that* F *is supported in* B_1, *if* u *vanishes at infinity, i.e.* $u(\infty) = \lim_{|x|\to\infty} u(x) = 0$ *and solves the following equation*

$$\triangle u = div\ F,$$

then we have

$$\|\nabla u\|_2 \le \|F\|_2.$$

Proof. By the divergence theorem and Hölder inequality we obtain

$$\begin{aligned} \|\nabla u\|_2^2 &= \int |\nabla u|^2 dx = -\int u \triangle u dx = -\int u \text{div } F dx = \int F \nabla u dx \\ &\le \|\nabla u\|_2 \|F\|_2 \end{aligned}$$

which yields that

$$\|\nabla u\|_2 \le \|F\|_2.$$

The following lemma will be used in section 4.

Lemma 2.4. *Assume that* F *is supported in* B_1, *if* u *vanishes at infinity and solves the following equation*

$$\triangle u = div\ F,$$

then we have

$$\|\nabla u\|_{L^{2,1}} \le C\|F\|_{L^{2,1}}.$$

Proof. By the equation and the divergence theorem we can obtain

$$\begin{aligned} \frac{\partial u}{\partial x_i}(x) &= \int_{R^n} \frac{x_i - y_i}{|x-y|^n} \triangle u(y) dy \\ &= \int_{R^n} \frac{x_i - y_i}{|x-y|^n} \text{div } F(y) dy \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{n}\int_{R^n} \frac{\partial}{\partial y_j}\left(\frac{x_i - y_i}{|x-y|^n}\right) F^j(y)dy \\
&= \sum_{j=1}^{n}\int_{R^n} \frac{\delta^i_j |x-y|^2 - n(x_i - y_i)(x_j - y_j)}{|x-y|^{n+2}} F^j(y)dy \\
&= \sum_{j=1}^{n} T_{ij}F^j(x),
\end{aligned}
$$

where T_{ij} are classical Calderón–Zygmund singular integral operators, which are bounded in L^p space for any $1 < p < \infty$. By Lemma 2.2 we have

$$\|T_{ij}F^j\|_{L^{2,1}} \le C\|F^j\|_{L^{2,1}} \le C\|F\|_{L^{2,1}}.$$

So we obtain that

$$\|\nabla u\|_{L^{2,1}} \le \sum_{j=1}^{n} \|T_{ij}F^j\|_{L^{2,1}} \le C\|F\|_{L^{2,1}}.$$

The following result is very important in the regularity theory of elliptic equations.

Lemma 2.5. *(Wente's inequality, [21,2]) If $f, g \in H^{1,2}(R^2)$, u vanishes at infinity and*

$$\triangle u = \nabla f \nabla^{\perp} g = \partial_1 f \partial_2 g - \partial_2 f \partial_1 g,$$

then $\nabla f \nabla^{\perp} g$ belongs to the Hardy space $H^1(R^2)$ and

$$\|\nabla u\|_{L^{2,1}} \le C\|\nabla f \nabla^{\perp} g\|_{H^1} \le C\|\nabla f\|_2 \|\nabla g\|_2.$$

A similar result is proved by Bethuel.

Lemma 2.6. *([1] Lemma 2) If $\nabla f \in L^{2,\infty}(R^2)$, $\nabla g \in L^2(R^2)$, u vanishes at infinity and*

$$\triangle u = \nabla f \nabla^{\perp} g,$$

then we have

$$\|\nabla u\|_2 \le C\|\nabla f\|_{L^{2,\infty}} \|\nabla g\|_2.$$

At last we recall the following embedding theorem.

Lemma 2.7. *([7] p. 137 Theorem 3.3.4) If u vanishes at infinity and $\nabla u \in L^{2,1}(R^2)$, then $u \in C^0(R^2)$. Furthermore, one has*

$$\|u\|_{C^0} \le C\|\nabla u\|_{L^{2,1}}.$$

3. Proof of the energy identity

The following small energy regularity for Sacks–Uhlenbeck maps was proved in [19].

Lemma 3.1. *([19]) Let u_α be the critical points of E_α from a Riemann surface M to a compact manifold N without boundary ($\alpha > 1$). There exists a constant ϵ_0 such that if*

$$\sup_{\alpha > 1} E(u_\alpha, B_2) \le \epsilon_0^2,$$

we have

$$\sup_{x\in B_1} |\nabla u_\alpha(x)| \le C\|\nabla u_\alpha\|_{L^2(B_2)}.$$

As a corollary, there exists a subsequence of u_α which converges to a harmonic map u in $H^{1,2}(B_1, N)$.

Let u_α be a sequence of Sacks–Uhlenbeck maps. It follows from the small energy regularity that the blow-up set S of u_α is a finite set of points. At the blow-up point $x \in B_1$ we have, for any r with $0 < r < 1 - |x|$,

$$\sup_\alpha E(u_\alpha, B(x,r)) > \epsilon_0^2.$$

Assume that the target is a sphere S^{K-1} and u_α is a Sacks–Uhlenbeck map from M to S^{K-1}, then u_α satisfies the Euler–Lagrange equation

$$\triangle u_\alpha + \frac{(\alpha-1) < \nabla^2 u_\alpha(\nabla u_\alpha), \nabla u_\alpha >}{1+|\nabla u_\alpha|^2} = -u_\alpha|\nabla u_\alpha|^2. \tag{3.1}$$

Set $F_\alpha = (1+|\nabla u_\alpha|^2)^{\alpha-1}$, then

$$\nabla F_\alpha = (\alpha-1)(1+|\nabla u_\alpha|^2)^{\alpha-2}\nabla^2 u_\alpha(\nabla u_\alpha)$$

and the equation (3.1) can be rewritten as

$$\triangle u_\alpha + \frac{< \nabla F_\alpha, \nabla u_\alpha >}{F_\alpha} = -u_\alpha|\nabla u_\alpha|^2,$$

i.e.

$$\mathrm{div}(F_\alpha \nabla u_\alpha) = -F_\alpha u_\alpha |\nabla u_\alpha|^2. \tag{3.2}$$

The proof of the statements (1), (2) and (3) in Theorem 1.1 is standard, we omit it.

By the induction arguments [5], it suffices for us to prove the energy identity (4) and necklessness (5) in the case that there is only one bubble during the blowing up. For simplicity, we assume that the unique bubble is produced by the sequence $v_\alpha(x) = u_\alpha(r_\alpha x)$, i.e. v_α converges to a nonconstant harmonic sphere ψ strongly in $H^{1,2}_{Loc}(R^2, S^{K-1})$.

In this section we shall prove the energy identity (4), that is

$$\lim_{\alpha\to 1} E(u_\alpha) = E(u) + E(\psi).$$

Because

$$\lim_{\delta\to 0}\lim_{\alpha\to 1} E(u_\alpha, R^2 \setminus B_\delta) = E(u)$$

and

$$\lim_{R\to\infty}\lim_{\alpha\to 1} E(u_\alpha, B_{r_\alpha R}) = E(\psi),$$

it suffices to show that there is no energy loss on the neck domain, i.e.

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\to 1} E(u_\alpha, B_\delta \setminus B_{r_\alpha R}) = 0.$$

We divide the proof into some steps.

Step 1 (**The estimate of $L^{2,\infty}$ quasi-norm of ∇u_α on the neck domain**)

Given any $0 < \epsilon < \epsilon_0$, one has

$$E(\psi, R^2 \setminus B_{\frac{R}{2}}) + E(u, B_{4\delta}) < \frac{\epsilon^2}{2},$$

for R sufficiently large and δ sufficiently small. By the standard blow up argument one shows that there exists α_0 such that

$$E(u_\alpha, B_{2t} \setminus B_t) < \epsilon^2, \tag{3.3}$$

for any t with $\frac{r_\alpha R}{2} < t < 2\delta$ and any α with $1 < \alpha < \alpha_0$. Otherwise we will have another bubble which contracts to the assumption that there is only one bubble.

It follows from (3.3) that for any $x \in B_\delta \setminus B_{r_\alpha R}$, $1 < \alpha < \alpha_0$

$$|\nabla u_\alpha(x)| \le \frac{C\epsilon}{|x|}$$

which implies that

$$\|\nabla u_\alpha\|_{L^{2,\infty}(B_\delta \setminus B_{r_\alpha R})} \le C\epsilon. \tag{3.4}$$

Step 2 (**The construction of $\widetilde{u}_\alpha$ and the Hodge decomposition**)

Choose $\varphi \in C_0^\infty(B_2)$ with $\varphi = 1$ on B_1 and let $\varphi_t(x) = \varphi(\frac{x}{t})$. Set

$$\overline{u}_\alpha^1 = \frac{1}{|B_{2\delta} \setminus B_\delta|} \int_{B_{2\delta} \setminus B_\delta} u_\alpha(x)dx,$$

$$\overline{u}_\alpha^2 = \frac{1}{|B_{2r_\alpha R} \setminus B_{r_\alpha R}|} \int_{B_{2r_\alpha R} \setminus B_{r_\alpha R}} u_\alpha(x)dx$$

and

$$\widetilde{u}_\alpha(x) = \varphi_\delta(x)((1 - \varphi_{r_\alpha R}(x))(u_\alpha(x) - \overline{u}_\alpha^2) + \overline{u}_\alpha^2 - \overline{u}_\alpha^1).$$

It is clear that $\widetilde{u}_\alpha$ is supported in $B_{2\delta}$ and $\widetilde{u}_\alpha(x) = \overline{u}_\alpha^2 - \overline{u}_\alpha^1$ for $|x| < r_\alpha R$.

In the following lemma, we derive the C^0-estimate of F_α.

Lemma 3.2. *There exists a constant A independent of α such that*

$$\sup_{1<\alpha<\alpha_0} \|F_\alpha\|_{C^0} < A$$

where $\alpha_0 - 1$ is sufficiently small.

Proof. It follows from (3.4) in step 1 that, as R big enough and δ small enough,

$$|\nabla u_\alpha(x)| \le \frac{C\epsilon}{|x|}$$

for any $x \in B_\delta \setminus B_{r_\alpha R}$, $1 < \alpha < \alpha_0$ where $\alpha_0 - 1$ is small enough.

On the other hand, there exists a $\lambda > 0$ such that $E(\psi, B(y, 2\lambda)) < \epsilon_0^2$ for any $y \in B_R$. Because v_α converges to ψ strongly in $H^{1,2}_{Loc}(R^2, S^{K-1})$ and $E(\psi, B(y, 2\lambda)) < \epsilon_0^2$, by Lemma 3.1, when $\alpha - 1$ small enough, we obtain

$$|\nabla v_\alpha(y)| \le \frac{C\sqrt{E(v_\alpha, B(y, 2\lambda))}}{\lambda} \le \frac{C\sqrt{E(\psi, B(y, 2\lambda))}}{\lambda} \le C$$

which yields that for $x \in B_{r_\alpha R}$,

$$|\nabla u_\alpha(x)| \le r_\alpha^{-1} |\nabla v_\alpha(r_\alpha^{-1} x)| \le C r_\alpha^{-1}.$$

By a similar argument one shows that for $x \in B_1 \setminus B_\delta$,

$$|\nabla u_\alpha(x)| \le C\delta^{-1}.$$

So, when $\alpha - 1$ small enough, we have

$$\|\nabla u_\alpha\|_{C^0(B_1)} \le C r_\alpha^{-1}. \tag{3.5}$$

For any $R > 0$, by direct computations we obtain

$$\begin{aligned}\int_{B_R} |\nabla v_\alpha(x)|^{2\alpha} dx &= r_\alpha^{2\alpha} \int_{B_R} |\nabla u_\alpha(r_\alpha x)|^{2\alpha} dx \\ &= r_\alpha^{2\alpha-2} \int_{B_{r_\alpha R}} |\nabla u_\alpha(x)|^{2\alpha} dx \\ &\le C r_\alpha^{2\alpha-2} \int_{B_{r_\alpha R}} [(1+|\nabla u_\alpha(x)|^2)^\alpha - 1] dx \\ &\le C r_\alpha^{2\alpha-2} \Lambda.\end{aligned}$$

On the other hand, since

$$\lim_{\alpha\to 1} \|\nabla v_\alpha\|_{C^0} < \infty,$$

one has

$$\begin{aligned}\lim_{R\to\infty}\lim_{\alpha\to 1} \int_{B_R} |\nabla v_\alpha(x)|^{2\alpha} dx &= \lim_{R\to\infty} \int_{B_R} \lim_{\alpha\to 1} |\nabla v_\alpha(x)|^{2\alpha} dx \\ &= \lim_{R\to\infty} \int_{B_R} |\nabla \psi(x)|^2 dx \\ &= 2E(\psi) \\ &\ge 2\epsilon_0^2.\end{aligned}$$

So we get

$$\epsilon_0^2 \le C\Lambda \lim_{\alpha\to 1} r_\alpha^{2\alpha-2}$$

which implies that when $1 < \alpha < \alpha_0$,

$$r_\alpha^{2-2\alpha} \le \frac{C\Lambda}{\epsilon_0^2}. \tag{3.6}$$

By (3.5) and (3.6), we have

$$\|F_\alpha\|_{C^0} \le (1 + \|\nabla u_\alpha\|^2_{C^0(B_1)})^{\alpha-1} \le (1 + C r_\alpha^{-2})^{\alpha-1} \le C r_\alpha^{2-2\alpha} \le \frac{C\Lambda}{\epsilon_0^2},$$

when $1 < \alpha < \alpha_0$ with $\alpha_0 - 1$ small enough. This completes the proof of this lemma. □

By the Hodge decomposition we get

$$F_\alpha \nabla \widetilde{u}_\alpha = \nabla D_\alpha + \nabla^\perp Q_\alpha$$

where $D_\alpha,\ Q_\alpha \in H^{1,2}(B_1)$ vanish at infinity.

Step 3 (**The estimate of** $\|\nabla Q_\alpha\|_2$)

By the Hodge decomposition and direct computations we obtain

$$\begin{aligned}\triangle Q_\alpha &= \text{curl } \nabla^\perp Q_\alpha \\ &= \text{curl } (F_\alpha \nabla \widetilde{u}_\alpha - \nabla D_\alpha) \\ &= \nabla^\perp F_\alpha \nabla \widetilde{u}_\alpha \\ &= -\nabla F_\alpha \nabla^\perp \widetilde{u}_\alpha \\ &= -\nabla (F_\alpha - 1) \nabla^\perp \widetilde{u}_\alpha \\ &= -\text{div}((F_\alpha - 1)\nabla^\perp \widetilde{u}_\alpha).\end{aligned} \tag{3.7}$$

Because $\widetilde{u}_\alpha$ is supported in $B_{2\delta}$ and Q_α vanishes at infinity, by Lemma 2.3 and Lemma 3.2 we obtain

$$\begin{aligned}\|\nabla Q_\alpha\|_2 &\le \|(F_\alpha - 1)\nabla^\perp \widetilde{u}_\alpha\|_2 \\ &= \|\frac{F_\alpha - 1}{F_\alpha} F_\alpha \nabla^\perp \widetilde{u}_\alpha\|_2 \\ &\le (1 - \frac{1}{A})\|F_\alpha \nabla \widetilde{u}_\alpha\|_2. \end{aligned} \tag{3.8}$$

Therefore one gets

$$\begin{aligned}\|F_\alpha \nabla \widetilde{u}_\alpha\|_2 &\le \|\nabla D_\alpha\|_2 + \|\nabla Q_\alpha\|_2 \\ &\le \|\nabla D_\alpha\|_2 + (1 - \frac{1}{A})\|F_\alpha \nabla \widetilde{u}_\alpha\|_2\end{aligned}$$

which implies that

$$\|F_\alpha \nabla \widetilde{u}_\alpha\|_2 \le A\|\nabla D_\alpha\|_2. \tag{3.9}$$

Step 4 (**The estimate of** $\|\nabla D_\alpha\|_2$)

In this step the assumption that the target is a sphere is essential. The arguments have been used in our proof of necklessness for a sequence of approximate harmonic maps to a sphere ([11,22]).

By direct computations we obtain

$$\begin{aligned}\nabla \widetilde{u}_\alpha(x) &= \nabla(\varphi_\delta(x)((1 - \varphi_{r_\alpha R}(x))(u_\alpha(x) - \overline{u}_\alpha^2) + \overline{u}_\alpha^2 - \overline{u}_\alpha^1)) \\ &= \nabla\varphi_\delta(x)((1 - \varphi_{r_\alpha R}(x))(u_\alpha(x) - \overline{u}_\alpha^2) + \overline{u}_\alpha^2 - \overline{u}_\alpha^1) \\ &\quad + \varphi_\delta(x)(\nabla(1 - \varphi_{r_\alpha R}(x))(u_\alpha(x) - \overline{u}_\alpha^2) + (1 - \varphi_{r_\alpha R}(x))\nabla u_\alpha(x)) \\ &= \nabla\varphi_\delta(x)(u_\alpha(x) - \overline{u}_\alpha^1) - \nabla\varphi_{r_\alpha R}(x)(u_\alpha(x) - \overline{u}_\alpha^2) \\ &\quad + \varphi_\delta(x)(1 - \varphi_{r_\alpha R}(x))\nabla u_\alpha(x).\end{aligned}$$

So we get

$$\operatorname{div}(F_\alpha \nabla \widetilde{u}_\alpha) = \operatorname{div}(F_\alpha \varphi_\delta(1 - \varphi_{r_\alpha R})\nabla u_\alpha) + \operatorname{div}(F_\alpha(\nabla\varphi_\delta(u_\alpha - \overline{u}_\alpha^1) - \nabla\varphi_{r_\alpha R}(u_\alpha - \overline{u}_\alpha^2))). \tag{3.10}$$

Note that $|u_\alpha| \equiv 1$, we have

$$\sum_{j=1}^K u_\alpha^j \nabla u_\alpha^j = \frac{1}{2}\nabla \sum_{j=1}^K (u_\alpha^j)^2 = 0.$$

Now the equation (3.2) can be rewritten as

$$\begin{aligned}\operatorname{div}(F_\alpha \nabla u_\alpha^i) &= -F_\alpha \sum_{j=1}^K u_\alpha^i |\nabla u_\alpha^j|^2 \\ &= \sum_{j=1}^K F_\alpha(u_\alpha^j \nabla u_\alpha^i - u_\alpha^i \nabla u_\alpha^j)\nabla u_\alpha^j.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\operatorname{div}[F_\alpha \varphi_\delta(1 - \varphi_{r_\alpha R})\nabla u_\alpha^i] \\ &= \varphi_\delta(1 - \varphi_{r_\alpha R})\operatorname{div}(F_\alpha \nabla u_\alpha^i) + F_\alpha \nabla u_\alpha^i \nabla(\varphi_\delta(1 - \varphi_{r_\alpha R})) \\ &= \varphi_\delta(1 - \varphi_{r_\alpha R})\sum_{j=1}^K F_\alpha(u_\alpha^j \nabla u_\alpha^i - u_\alpha^i \nabla u_\alpha^j)\nabla u_\alpha^j + F_\alpha \nabla u_\alpha^i \nabla(\varphi_\delta(1 - \varphi_{r_\alpha R})) \\ &= \sum_{j=1}^K F_\alpha(u_\alpha^j \nabla u_\alpha^i - u_\alpha^i \nabla u_\alpha^j)\nabla(\varphi_\delta(1 - \varphi_{r_\alpha R})u_\alpha^j)\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{K} F_\alpha (u_\alpha^j \nabla u_\alpha^i - u_\alpha^i \nabla u_\alpha^j) u_\alpha^j \nabla(\varphi_\delta(1-\varphi_{r_\alpha R})) + F_\alpha \nabla u_\alpha^i \nabla(\varphi_\delta(1-\varphi_{r_\alpha R}))\\
&= \sum_{j=1}^{K} F_\alpha (u_\alpha^j \nabla u_\alpha^i - u_\alpha^i \nabla u_\alpha^j) \nabla(\varphi_\delta(1-\varphi_{r_\alpha R}) u_\alpha^j) - \sum_{j=1}^{K} F_\alpha (u_\alpha^j)^2 \nabla u_\alpha^i \nabla(\varphi_\delta(1-\varphi_{r_\alpha R}))\\
&\quad + F_\alpha \nabla u_\alpha^i \nabla(\varphi_\delta(1-\varphi_{r_\alpha R}))\\
&= \sum_{j=1}^{K} F_\alpha (u_\alpha^j \nabla u_\alpha^i - u_\alpha^i \nabla u_\alpha^j) \nabla(\varphi_\delta(1-\varphi_{r_\alpha R}) u_\alpha^j) - F_\alpha \nabla u_\alpha^i \nabla(\varphi_\delta(1-\varphi_{r_\alpha R}))\\
&\quad + F_\alpha \nabla u_\alpha^i \nabla(\varphi_\delta(1-\varphi_{r_\alpha R}))\\
&= \sum_{j=1}^{K} F_\alpha (u_\alpha^j \nabla u_\alpha^i - u_\alpha^i \nabla u_\alpha^j) \nabla(\varphi_\delta(1-\varphi_{r_\alpha R}) u_\alpha^j).
\end{aligned}
\tag{3.11}
$$

Because

$$
\begin{aligned}
&\mathrm{div}[F_\alpha (u_\alpha^j \nabla u_\alpha^i - u_\alpha^i \nabla u_\alpha^j)]\\
&= u_\alpha^j \mathrm{div}(F_\alpha \nabla u_\alpha^i) + F_\alpha \nabla u_\alpha^j \nabla u_\alpha^i - u_\alpha^i \mathrm{div}(F_\alpha \nabla u_\alpha^j) - F_\alpha \nabla u_\alpha^i \nabla u_\alpha^j\\
&= -u_\alpha^j u_\alpha^i |\nabla u_\alpha|^2 + u_\alpha^i u_\alpha^j |\nabla u_\alpha|^2\\
&= 0,
\end{aligned}
$$

we can find $G_{\alpha,ij} \in H^{1,2}(R^2)$ such that

$$
F_\alpha (u_\alpha^j \nabla u_\alpha^i - u_\alpha^i \nabla u_\alpha^j) = \nabla^\perp G_{\alpha,ij} \text{ on } B_1. \tag{3.12}
$$

Using the idea in the construction of $\widetilde{u}_\alpha$ in step 2, we construct $\widetilde{G}_\alpha$. Set

$$
\overline{G}_\alpha^1 = \frac{1}{|B_{4\delta} \setminus B_{2\delta}|} \int_{B_{4\delta}\setminus B_{2\delta}} G_\alpha(x)dx,
$$

$$
\overline{G}_\alpha^2 = \frac{1}{|B_{r_\alpha R} \setminus B_{\frac{r_\alpha R}{2}}|} \int_{B_{r_\alpha R}\setminus B_{\frac{r_\alpha R}{2}}} G_\alpha(x)dx
$$

and

$$
\widetilde{G}_\alpha(x) = \varphi_{2\delta}(x)((1-\varphi_{\frac{r_\alpha R}{2}}(x))(G_\alpha(x) - \overline{G}_\alpha^2) + \overline{G}_\alpha^2 - \overline{G}_\alpha^1).
$$

Note that $\varphi_\delta(1-\varphi_{r_\alpha R})u_\alpha^j$ is supported in $B_{2\delta} \setminus B_{r_\alpha R}$ and $\widetilde{G}_\alpha = G_\alpha$ on $B_{2\delta} \setminus B_{r_\alpha R}$, by (3.10), (3.11) and (3.12) we obtain

$$
\begin{aligned}
&\mathrm{div}(F_\alpha \nabla \widetilde{u}_\alpha)\\
&= \mathrm{div}(F_\alpha \varphi_\delta(1-\varphi_{r_\alpha R})\nabla u_\alpha(x)) + \mathrm{div}(F_\alpha(\nabla\varphi_\delta(u_\alpha - \overline{u}_\alpha^1) - \nabla\varphi_{r_\alpha R}(u_\alpha - \overline{u}_\alpha^2)))\\
&= \sum_{j=1}^{K} F_\alpha (u_\alpha^j \nabla u_\alpha - u_\alpha \nabla u_\alpha^j) \nabla(\varphi_\delta(1-\varphi_{r_\alpha R}) u_\alpha^j)\\
&\quad + \mathrm{div}(F_\alpha(\nabla\varphi_\delta(u_\alpha - \overline{u}_\alpha^1) - \nabla\varphi_{r_\alpha R}(u_\alpha - \overline{u}_\alpha^2)))\\
&= \nabla^\perp G_\alpha \nabla(\varphi_\delta(1-\varphi_{r_\alpha R})u_\alpha) + \mathrm{div}(F_\alpha(\nabla\varphi_\delta(u_\alpha - \overline{u}_\alpha^1) - \nabla\varphi_{r_\alpha R}(u_\alpha - \overline{u}_\alpha^2)))\\
&= \nabla^\perp \widetilde{G}_\alpha \nabla(\varphi_\delta(1-\varphi_{r_\alpha R})u_\alpha)\\
&\quad + \mathrm{div}(F_\alpha(\nabla\varphi_\delta(u_\alpha - \overline{u}_\alpha^1) - \nabla\varphi_{r_\alpha R}(u_\alpha - \overline{u}_\alpha^2))).
\end{aligned}
\tag{3.13}
$$

We estimate the $L^{2,\infty}$ quasi-norm of $\nabla\widetilde{G}_\alpha$. For any t with $\frac{r_\alpha R}{2} \le t \le 2\delta$, it is easy to see that

$$\begin{aligned}\|\nabla G_\alpha\|_{L^2(B_{2t}\setminus B_t)} &= \|\nabla^\perp G_\alpha\|_{L^2(B_{2t}\setminus B_t)}\\ &\le \sum_{i,j}\|F_\alpha(u_\alpha^j\nabla u_\alpha^i - u_\alpha^i\nabla u_\alpha^j)\|_{L^2(B_{2t}\setminus B_t)}\\ &\le C\|F_\alpha\|_{C^0}\|u_\alpha\|_{C^0}\|\nabla u_\alpha\|_{L^2(B_{2t}\setminus B_t)}\\ &\le C\|\nabla u_\alpha\|_{L^2(B_{2t}\setminus B_t)}\\ &\le C\epsilon.\end{aligned}$$

It follows from (3.4) that

$$\|\nabla G_\alpha\|_{L^{2,\infty}(B_{2\delta}\setminus B_{r_\alpha R})} \le C\|\nabla u_\alpha\|_{L^{2,\infty}(B_{2\delta}\setminus B_{r_\alpha R})} \le C\epsilon.$$

So, by the Poincaré inequality we can get

$$\begin{aligned}\|\nabla\widetilde{G}_\alpha\|_{L^{2,\infty}} &\le C(\|\nabla\widetilde{G}_\alpha\|_{L^2(R^2\setminus B_{2\delta})} + \|\nabla\widetilde{G}_\alpha\|_{L^{2,\infty}(B_{2\delta}\setminus B_{r_\alpha R})} + \|\nabla\widetilde{G}_\alpha\|_{L^2(B_{r_\alpha R})})\\ &\le C(\|\nabla(\varphi_\delta(G_\alpha - \overline{G}_\alpha^1))\|_{L^2(B_{4\delta}\setminus B_{2\delta})} + \|\nabla G_\alpha\|_{L^{2,\infty}(B_{2\delta}\setminus B_{r_\alpha R})} +\\ &\quad \|\nabla((1-\varphi_{r_\alpha R})(G_\alpha - \overline{G}_\alpha^2))\|_{L^2(B_{r_\alpha R}\setminus B_{\frac{r_\alpha R}{2}})})\\ &\le C(\|\nabla G_\alpha\|_{L^2(B_{4\delta}\setminus B_{2\delta})} + \|(G_\alpha - \overline{G}_\alpha^1)\nabla\varphi_\delta\|_{L^2(B_{4\delta}\setminus B_{2\delta})} + \epsilon\\ &\quad + \|(G_\alpha - \overline{G}_\alpha^2)\nabla\varphi_{r_\alpha R}\|_{L^2(B_{r_\alpha R}\setminus B_{\frac{r_\alpha R}{2}})} + \|\nabla G_\alpha\|_{L^2(B_{r_\alpha R}\setminus B_{\frac{r_\alpha R}{2}})})\\ &\le C(\|\nabla G_\alpha\|_{L^2(B_{4\delta}\setminus B_{2\delta})} + \epsilon + \|\nabla G_\alpha\|_{L^2(B_{r_\alpha R}\setminus B_{\frac{r_\alpha R}{2}})})\\ &\le C\epsilon. \end{aligned} \tag{3.14}$$

We solve the following equations with $\Phi_1, \Phi_2 = 0$ at infinity

$$\triangle\Phi_1 = \nabla^\perp\widetilde{G}_\alpha\nabla(\varphi_\delta(1-\varphi_{r_\alpha R})u_\alpha);$$
$$\triangle\Phi_2 = \mathrm{div}(F_\alpha(\nabla\varphi_\delta(u_\alpha - \overline{u}_\alpha^1) - \nabla\varphi_{r_\alpha R}(u_\alpha - \overline{u}_\alpha^2))).$$

By Lemma 2.6 and (3.14) we obtain

$$\begin{aligned}\|\nabla\Phi_1\|_2 &\le C\|\nabla\widetilde{G}_\alpha\|_{L^{2,\infty}}\|\nabla(\varphi_\delta(1-\varphi_{r_\alpha R})u_\alpha)\|_2\\ &\le C\epsilon(\|\nabla u_\alpha\|_2 + \|\nabla(\varphi_\delta(1-\varphi_{r_\alpha R}))\|_2)\\ &\le C\epsilon.\end{aligned} \tag{3.15}$$

By Lemma 3.1 we have

$$|u_\alpha(x) - \overline{u}_\alpha^1| \le C\epsilon, x\in B_{2\delta}\setminus B_\delta;\quad |u_\alpha(x) - \overline{u}_\alpha^2| \le C\epsilon, x\in B_{2r_\alpha R}\setminus B_{r_\alpha R}.$$

It follows from Lemma 2.3 that

$$\begin{aligned}\|\nabla\Phi_2\|_2 &\le \|F_\alpha((u_\alpha - \overline{u}_\alpha^1)\nabla\varphi_\delta - (u_\alpha - \overline{u}_\alpha^2)\nabla\varphi_{r_\alpha R})\|_2\\ &\le A(\|(u_\alpha - \overline{u}_\alpha^1)\nabla\varphi_\delta\|_2 + \|(u_\alpha - \overline{u}_\alpha^2)\nabla\varphi_{r_\alpha R}\|_2)\\ &\le C\epsilon(\|\nabla\varphi_\delta\|_2 + \|\nabla\varphi_{r_\alpha R}\|_2)\\ &\le C\epsilon.\end{aligned} \tag{3.16}$$

Since $\triangle D_\alpha = \mathrm{div}\,(F_\alpha\nabla\widetilde{u}_\alpha) = \triangle\Phi_1 + \triangle\Phi_2$ and $D_\alpha = 0$ at infinity, we have $D_\alpha = \Phi_1 + \Phi_2$. It follows from (3.15) and (3.16) that

$$\|\nabla D_\alpha\|_2 \le \|\nabla\Phi_1\|_2 + \|\nabla\Phi_2\|_2 \le C\epsilon. \tag{3.17}$$

By (3.9) and (3.17) we get

$$\begin{aligned}\|\nabla u_\alpha\|_{L^2(B_\delta\setminus B_{2r_\alpha R})} &\leq \|\nabla\widetilde{u}_\alpha\|_2\\ &\leq \|F_\alpha\nabla\widetilde{u}_\alpha\|_2\\ &\leq A\|\nabla D_\alpha\|_2\\ &\leq C\epsilon.\end{aligned}$$

This completes the proof of the energy identity.

4. Proof of the necklessness

By the induction arguments [5], it suffices for us to prove the result in the case that there is only one bubble, i.e.

$$\lim_{\alpha\to 1}\|u_\alpha(\cdot)-u(\cdot)-[\psi(\frac{\cdot-x_\alpha}{r_\alpha})-\psi(\infty)]\|_\infty=0. \tag{4.1}$$

It follows from Lemma 3.1 that $u_\alpha\to u$ in $C^\infty(R^2\setminus B_\delta)$ for any $\delta>0$ which implies that

$$\begin{aligned}&\lim_{\alpha\to 1}\|u_\alpha(\cdot)-u(\cdot)-[\psi(\frac{\cdot-x_\alpha}{r_\alpha})-\psi(\infty)]\|_{L^\infty(R^2\setminus B_\delta)}\\ \leq&\lim_{\alpha\to 1}(\|u_\alpha(\cdot)-u(\cdot)\|_{L^\infty(R^2\setminus B_\delta)}+\|\psi(\frac{\cdot-x_\alpha}{r_\alpha})-\psi(\infty)\|_{L^\infty(R^2\setminus B_\delta)})\\ =&0.\end{aligned}$$

Similarly, for any $R>0$ we obtain

$$\begin{aligned}&\lim_{\alpha\to 1}\|u_\alpha(\cdot)-u(\cdot)-[\psi(\frac{\cdot-x_\alpha}{r_\alpha})-\psi(\infty)]\|_{L^\infty(B_{r_\alpha R})}\\ \leq&\lim_{\alpha\to 1}\|u_\alpha(\cdot)-\psi(\frac{\cdot-x_\alpha}{r_\alpha})\|_{L^\infty(B_{r_\alpha R})}+|u(0)-\psi(\infty)|\\ =&0.\end{aligned}$$

On the other hand, it is easy to see that

$$\lim_{R\to\infty}\lim_{\alpha\to 1}\|\psi(\frac{\cdot-x_\alpha}{r_\alpha})-\psi(\infty)\|_{L^\infty(R^2\setminus B_{r_\alpha R})}=0.$$

To obtain (4.1), it is only left to show that

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\to 1}\|u_\alpha(\cdot)-u(\cdot)\|_{L^\infty(B_\delta\setminus B_{2r_\alpha R})}=0.$$

Since u_α and u are continuous and

$$\lim_{\delta\to 0}\lim_{\alpha\to 1}\|u_\alpha(\cdot)-u(\cdot)\|_{L^\infty(R^2\setminus B_\delta)}=0,$$

we need only to prove that

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\to 1}\sup_{x,y\in B_\delta\setminus B_{2r_\alpha R}}|u_\alpha(x)-u_\alpha(y)|=0, \tag{4.2}$$

i.e. there is no oscillation on the neck domain.

From the construction of $\widetilde{u}_\alpha$, we see that $\widetilde{u}_\alpha=u_\alpha$ on $B_\delta\setminus B_{2r_\alpha R}$. By Lemma 2.7 we get that

$$\begin{aligned}\sup_{x,y\in B_\delta\setminus B_{2r_\alpha R}}|u_\alpha(x)-u_\alpha(y)| &\leq \sup_{x,y}|\widetilde{u}_\alpha(x)-\widetilde{u}_\alpha(y)|\\ &\leq \|\widetilde{u}_\alpha\|_{C^0}\\ &\leq C\|\nabla\widetilde{u}_\alpha\|_{L^{2,1}}.\end{aligned}$$

In this section we shall prove

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\to 1}\|\nabla\widetilde{u}_\alpha\|_{L^{2,1}}=0 \tag{4.3}$$

which yields the necklessness (4.1).

As in section 3, we divide the proof into some steps.

Step 1 (**The Hodge decomposition**)

In section 3, we showed that, for any $\epsilon>0$ there exist $\delta>0$, $R>0$ and $\alpha_0>1$ such that

$$\|\nabla u_\alpha\|_{L^2(B_{2\delta}\setminus B_{\frac{r_\alpha R}{2}})}<\epsilon;\quad \|\nabla\widetilde{u}_\alpha\|_2<\epsilon \tag{4.4}$$

when $1<\alpha<\alpha_0$.

By the Hodge decomposition $F_\alpha\nabla\widetilde{u}_\alpha=\nabla D_\alpha+\nabla^\perp Q_\alpha$ we have

$$\|\nabla\widetilde{u}_\alpha\|_{L^{2,1}}\le\|F_\alpha\nabla\widetilde{u}_\alpha\|_{L^{2,1}}\le\|\nabla D_\alpha\|_{L^{2,1}}+\|\nabla Q_\alpha\|_{L^{2,1}}. \tag{4.5}$$

Step 2 (**The estimate of** $\|\nabla Q_\alpha\|_{L^{2,1}}$)

Because we have already proved the energy identity, we have the following identity ([12]). For completeness, we sketch the proof below.

Lemma 4.1. *With the same notations and assumptions in section 3, we have*

$$\lim_{\alpha\to 1}\|F_\alpha\|_{C^0}=1.$$

Proof. By an argument similar to the one used in obtaining Monotonicity inequality of stationary harmonic maps [16], we get

$$\begin{aligned}&\int_{\partial B_t}(1+|\nabla u_\alpha|^2)^{\alpha-1}(|\frac{\partial u_\alpha}{\partial r}|^2-\frac{1}{2\alpha}|\nabla u_\alpha|^2)ds_0\\&=\frac{\alpha-1}{\alpha t}\int_{B_t}(1+|\nabla u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx+O(t).\end{aligned} \tag{4.6}$$

Integrating t from $r_\alpha R$ to δ, we get

$$\begin{aligned}&\int_{B_\delta\setminus B_{r_\alpha R}}F_\alpha(|\frac{\partial u_\alpha}{\partial r}|^2-\frac{1}{2\alpha}|\nabla u_\alpha|^2)dx\\&=\int_{r_\alpha R}^{\delta}\frac{\alpha-1}{\alpha t}\int_{B_t}F_\alpha|\nabla u_\alpha|^2dxdt+\delta O(\delta).\end{aligned}$$

It is clear that

$$\begin{aligned}&\int_{r_\alpha R}^{\delta}\frac{\alpha-1}{\alpha t}\int_{B_t}F_\alpha|\nabla u_\alpha|^2dxdt\\&\ge\int_{r_\alpha R}^{\delta}\frac{\alpha-1}{\alpha t}\int_{B_{r_\alpha R}}|\nabla u_\alpha|^2dxdt\\&=\frac{2(\alpha-1)}{\alpha}\ln\frac{\delta}{r_\alpha R}E(u_\alpha,B_{r_\alpha R}).\end{aligned}$$

Letting $\alpha \to 1$, one obtains

$$\begin{aligned}
&\lim_{\alpha\to 1}\int_{r_\alpha R}^{\delta}\frac{\alpha-1}{\alpha t}\int_{B_t}F_\alpha|\nabla u_\alpha|^2dxdt\\
&\geq C\lim_{\alpha\to 1}\frac{\alpha-1}{\alpha}\ln\frac{\delta}{r_\alpha R}E(\psi)\\
&=C\lim_{\alpha\to 1}(\alpha-1)(\ln\frac{\delta}{R}-\ln r_\alpha)E(\psi)\\
&=C\lim_{\alpha\to 1}\ln r_\alpha^{1-\alpha}E(\psi).
\end{aligned}\tag{4.7}$$

On the other hand, it follows from (4.4) and Lemma 3.1 that

$$\begin{aligned}
&\int_{B_\delta\setminus B_{r_\alpha R}}F_\alpha(|\frac{\partial u_\alpha}{\partial r}|^2-\frac{1}{2\alpha}|\nabla u_\alpha|^2)dx\\
&\leq 2AE(u_\alpha, B_\delta\setminus B_{r_\alpha R})\\
&\leq C\epsilon^2.
\end{aligned}\tag{4.8}$$

By (4.7) and (4.8) we have

$$\begin{aligned}
&\lim_{\alpha\to 1}\ln r_\alpha^{1-\alpha}E(\psi)\\
&\leq C\lim_{\alpha\to 1}\int_{r_\alpha R}^{\delta}\frac{\alpha-1}{\alpha t}\int_{B_t}F_\alpha|\nabla u_\alpha|^2dxdt\\
&\leq C(\int_{B_\delta\setminus B_{r_\alpha R}}F_\alpha(|\frac{\partial u_\alpha}{\partial r}|^2-\frac{1}{2\alpha}|\nabla u_\alpha|^2)dx+\delta O(\delta))\\
&\leq C(\epsilon^2+\delta)
\end{aligned}$$

which yields that

$$\lim_{\alpha\to 1}\ln r_\alpha^{1-\alpha}=0,$$

that is

$$\lim_{\alpha\to 1}r_\alpha^{1-\alpha}=1.\tag{4.9}$$

Using the same arguments as that in the proof of Lemma 3.2, by (3.5) and (4.9), we can get

$$\begin{aligned}
1&\leq\lim_{\alpha\to 1}\|F_\alpha\|_{C^0}\\
&\leq\lim_{\alpha\to 1}(1+Cr_\alpha^{-2})^{\alpha-1}\\
&=\lim_{\alpha\to 1}r_\alpha^{2-2\alpha}\\
&=1.
\end{aligned}$$

So the lemma is proved. □

By the same computations as that in (3.7) we obtain

$$\triangle Q_\alpha=-\mathrm{div}((F_\alpha-1)\nabla^\perp\widetilde{u}_\alpha).\tag{4.10}$$

Because $\widetilde{u}_\alpha$ is supported in $B_{2\delta}$ and Q_α vanishes at infinity, by Lemma 2.4 one can get

$$\begin{aligned}\|\nabla Q_\alpha\|_{L^{2,1}} &\le C\|(F_\alpha - 1)\nabla^\perp \widetilde{u}_\alpha\|_{L^{2,1}}\\ &= C\|(1-\frac{1}{F_\alpha})F_\alpha \nabla \widetilde{u}_\alpha\|_{L^{2,1}}.\end{aligned}$$

It follows from Lemma 4.1 that

$$\|\nabla Q_\alpha\|_{L^{2,1}} \le \frac{1}{2}\|F_\alpha \nabla \widetilde{u}_\alpha\|_{L^{2,1}}$$

when $\alpha - 1$ small enough, which implies that

$$\|F_\alpha \nabla \widetilde{u}_\alpha\|_{L^{2,1}} \le 2\|\nabla D_\alpha\|_{L^{2,1}}. \tag{4.11}$$

Step 3 (**The estimate of** $\|\nabla D_\alpha\|_{L^{2,1}}$)
Here we use the same notations and the similar arguments as that in step 4 of section 3.
We first estimate $\|\nabla \widetilde{G}_\alpha\|_2$. By the definition of G_α, $\widetilde{G}_\alpha$ and the same computations as that in (3.14) we get

$$\begin{aligned}\|\nabla \widetilde{G}_\alpha\|_2 &\le C\|\nabla G_\alpha\|_{L^2(B_{2\delta}\setminus B_{\frac{r_\alpha R}{2}})}\\ &= C\|\nabla^\perp G_\alpha\|_{L^2(B_{2\delta}\setminus B_{\frac{r_\alpha R}{2}})}\\ &\le C\sum_i\sum_j \|F_\alpha(u_\alpha^j \nabla u_\alpha^i - u_\alpha^i \nabla u_\alpha^j)\|_{L^2(B_{2\delta}\setminus B_{\frac{r_\alpha R}{2}})}\\ &\le C\|\nabla u_\alpha\|_{L^2(B_{2\delta}\setminus B_{\frac{r_\alpha R}{2}})}.\end{aligned}$$

By (4.4) we have

$$\|\nabla \widetilde{G}_\alpha\|_2 \le C\epsilon. \tag{4.12}$$

In section 3, using the special structure of the sphere we have derived

$$\begin{aligned}\operatorname{div}(F_\alpha \nabla \widetilde{u}_\alpha) &= \nabla^\perp \widetilde{G}_\alpha \nabla(\varphi_\delta(1-\varphi_{r_\alpha R})u_\alpha)\\ &\quad + \operatorname{div}(F_\alpha(\nabla\varphi_\delta(u_\alpha - \overline{u}_\alpha^1) - \nabla\varphi_{r_\alpha R}(u_\alpha - \overline{u}_\alpha^2))).\end{aligned} \tag{4.13}$$

Let Φ_1, Φ_2 be the solutions of the following equations with $\Phi_1, \Phi_2 = 0$ at infinity,

$$\begin{aligned}&\triangle\Phi_1 = \nabla^\perp \widetilde{G}_\alpha \nabla(\varphi_\delta(1-\varphi_{r_\alpha R})u_\alpha);\\ &\triangle\Phi_2 = \operatorname{div}(F_\alpha(\nabla\varphi_\delta(u_\alpha - \overline{u}_\alpha^1) - \nabla\varphi_{r_\alpha R}(u_\alpha - \overline{u}_\alpha^2))).\end{aligned}$$

By Lemma 2.5 and (4.12) we get

$$\begin{aligned}\|\nabla\Phi_1\|_{L^{2,1}} &\le C\|\nabla \widetilde{G}_\alpha\|_2 \|\nabla(\varphi_\delta(1-\varphi_{r_\alpha R})u_\alpha)\|_2\\ &\le C\epsilon(\|\nabla u_\alpha\|_2 + \|\nabla(\varphi_\delta(1-\varphi_{r_\alpha R}))\|_2)\\ &\le C\epsilon.\end{aligned} \tag{4.14}$$

On the other hand, since

$$|u_\alpha(x) - \overline{u}_\alpha^1| \le C\epsilon, x \in B_{2\delta}\setminus B_\delta;\ |u_\alpha(x) - \overline{u}_\alpha^2| \le C\epsilon, x\in B_{2r_\alpha R}\setminus B_{r_\alpha R},$$

by Lemma 2.4 and Lemma 3.2 we have

$$\begin{aligned}\|\nabla\Phi_2\|_{L^{2,1}} &\le C\|F_\alpha((u_\alpha - \overline{u}_\alpha^1)\nabla\varphi_\delta - (u_\alpha - \overline{u}_\alpha^2)\nabla\varphi_{r_\alpha R})\|_{L^{2,1}}\\ &\le C(\|(u_\alpha - \overline{u}_\alpha^1)\nabla\varphi_\delta\|_{L^{2,1}} + \|(u_\alpha - \overline{u}_\alpha^2)\nabla\varphi_{r_\alpha R}\|_{L^{2,1}})\\ &\le C\epsilon(\|\nabla\varphi_\delta\|_{L^{2,1}} + \|\nabla\varphi_{r_\alpha R}\|_{L^{2,1}})\\ &\le C\epsilon.\end{aligned} \tag{4.15}$$

Note that $D_\alpha = \Phi_1 + \Phi_2$, we see that (4.14) and (4.15) imply

$$\|\nabla D_\alpha\|_{L^{2,1}} \le \|\nabla \Phi_1\|_{L^{2,1}} + \|\nabla \Phi_2\|_{L^{2,1}} \le C\epsilon. \tag{4.16}$$

It follows from (4.11) and (4.16) that

$$\begin{aligned}\|\nabla \widetilde{u}_\alpha\|_{L^{2,1}} &\le \|F_\alpha \nabla \widetilde{u}_\alpha\|_{L^{2,1}} \\ &\le 2\|\nabla D_\alpha\|_{L^{2,1}} \\ &\le C\epsilon\end{aligned}$$

which implies (4.3). This proves that there is no oscillation on the neck domain.

Conflict of interest statement

There is no any conflict of interest.

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