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Local well-posedness for quasi-linear NLS with large Cauchy data on the circle ☆

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Abstract

We prove local in time well-posedness for a large class of quasilinear Hamiltonian, or parity preserving, Schrödinger equations on the circle. After a paralinearization of the equation, we perform several paradifferential changes of coordinates in order to transform the system into a paradifferential one with symbols which, at the positive order, are constant and purely imaginary. This allows to obtain a priori energy estimates on the Sobolev norms of the solutions.

Keywords: NLS; Quasi-linear PDEs; Para-differential calculus; Local wellposedness; Dispersive equations; Energy method

1. Introduction

1.1. Main results

In this paper we study the initial value problem (IVP)

$$\begin{cases} \mathrm{i}\partial_t u + \partial_{xx} u + P * u + f(u, u_x, u_{xx}) = 0, \quad u = u(t,x), \quad x \in \mathbb{T}, \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, the nonlinearity f is in $C^\infty(\mathbb{C}^3;\mathbb{C})$ in the *real sense* (i.e. $f(z_1,z_2,z_3)$ is C^∞ as function of $\mathrm{Re}(z_i)$ and $\mathrm{Im}(z_i)$ for $i = 1,2,3$) vanishing at order 2 at the origin, the potential $P(x) = \sum_{j\in\mathbb{Z}} \hat{p}(j)\frac{e^{\mathrm{i}jx}}{\sqrt{2\pi}}$ is a function in $C^1(\mathbb{T};\mathbb{C})$ with real Fourier coefficients $\hat{p}(j) \in \mathbb{R}$ for any $j \in \mathbb{Z}$ and $P * u$ denotes the convolution between P and $u = \sum_{j\in\mathbb{Z}} \hat{u}(j)\frac{e^{\mathrm{i}jx}}{\sqrt{2\pi}}$

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$$P * u(x) := \int_{\mathbb{T}} P(x-y)u(y)dy = \sum_{j\in\mathbb{Z}} \hat{p}(j)\hat{u}(j)e^{\mathrm{i}jx}. \tag{1.2}$$

Our aim is to prove the local existence, uniqueness and regularity of the classical solution of (1.1) on Sobolev spaces

$$H^s := H^s(\mathbb{T};\mathbb{C}) : \left\{ u(x) = \sum_{k\in\mathbb{Z}} \hat{u}(k)\frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}} \ : \ \|u\|^2_{H^s} := \sum_{j\in\mathbb{Z}} \langle j\rangle^{2s}|\hat{u}(j)|^2 < \infty \right\}, \tag{1.3}$$

where $\langle j\rangle := \sqrt{1+|j|^2}$ for $j\in\mathbb{Z}$, for s large enough.

Similar problems have been studied in the case $x\in\mathbb{R}^n$, $n\geq 1$. For $x\in\mathbb{R}$, in the paper [23], it was considered the fully nonlinear Schrödinger type equation $\mathrm{i}\partial_t u = F(t,x,u,u_x,u_{xx})$; it has been shown that the IVP associated to this equation is locally in time well posed in $H^\infty(\mathbb{R};\mathbb{C})$ (where $H^\infty(\mathbb{R};\mathbb{C})$ denotes the intersection of all Sobolev spaces $H^s(\mathbb{R};\mathbb{C})$, $s\in\mathbb{R}$) if the function F satisfies some suitable ellipticity hypotheses.

Concerning the n-dimensional case the IVP for quasi-linear Schrödinger equations has been studied in [20] in the Sobolev spaces $H^s(\mathbb{R}^n;\mathbb{C})$ with s sufficiently large. Here the key ingredient used to prove energy estimates is a Doi's type lemma which involves pseudo-differential calculus for symbols defined on the Euclidean space $\mathbb{R}^n$.

Coming back to the case $x\in\mathbb{T}$ we mention [8]. In this paper it is shown that if s is big enough and if the size of the initial datum u_0 is sufficiently small, then (1.1) is well posed in the Sobolev space $H^s(\mathbb{T})$ if $P=0$ and f is *Hamiltonian* (in the sense of Hypothesis 1.1). The proof is based on a Nash–Moser–Hörmander implicit function theorem and the required energy estimates are obtained by means of a procedure of reduction to constant coefficients of the equation (as done in [16], [17]).

We remark that, even for the short time behavior of the solutions, there are deep differences between the problem (1.1) with periodic boundary conditions ($x\in\mathbb{T}$) and (1.1) with $x\in\mathbb{R}$. Indeed Christ proved in [13] that the following family of problems

$$\begin{cases} \partial_t u + \mathrm{i}u_{xx} + u^{p-1}u_x = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{1.4}$$

is ill-posed in all Sobolev spaces $H^s(\mathbb{T})$ for any integer $p\geq 2$ and it is well-posed in $H^s(\mathbb{R})$ for $p\geq 3$ and s sufficiently large. The ill-posedness of (1.4) is very strong, in [13] it has been shown that its solutions have the following norm inflation phenomenon: for any $\varepsilon>0$ there exists a solution u of (1.4) and a time $t_\varepsilon\in(0,\varepsilon)$ such that

$$\|u_0\|_{H^s}\leq\varepsilon \quad \text{and} \quad \|u(t_\varepsilon)\|_{H^s} > \varepsilon^{-1}.$$

The examples exhibited in [13] suggest that some assumptions on the nonlinearity f in (1.1) are needed. In this paper we prove local well-posedness for (1.1) in two cases. The first one is the *Hamiltonian* case. We assume that equation (1.1) can be written in the complex Hamiltonian form

$$\partial_t u = \mathrm{i}\nabla_{\bar{u}}\mathcal{H}(u), \tag{1.5}$$

with Hamiltonian function

$$\mathcal{H}(u) = \int_{\mathbb{T}} -|u_x|^2 + (P*u)\bar{u} + F(u,u_x)dx, \tag{1.6}$$

for some real valued function $F\in\mathbb{C}^\infty(\mathbb{C}^2;\mathbb{R})$ and where $\nabla_{\bar{u}} := (\nabla_{\mathrm{Re}(u)} + \mathrm{i}\nabla_{\mathrm{Im}(u)})/2$ and ∇ denotes the $L^2(\mathbb{T};\mathbb{R})$ gradient. Note that the assumption $\hat{p}(j)\in\mathbb{R}$ implies that the Hamiltonian $\int_{\mathbb{T}}(P*u)\bar{u}dx$ is real valued. We denote by $\partial_{z_i} := (\partial_{\mathrm{Re}(z_i)} - \mathrm{i}\partial_{\mathrm{Im}(z_i)})/2$ and $\partial_{\bar{z}_i} := (\partial_{\mathrm{Re}(z_i)} + \mathrm{i}\partial_{\mathrm{Im}(z_i)})/2$ for $i=1,2$ the Wirtinger derivatives. We assume the following.

Hypothesis 1.1 *(***Hamiltonian structure***)*. We assume that the nonlinearity f in equation (1.1) has the form

$$\begin{aligned} f(z_1,z_2,z_3) = (\partial_{\bar{z}_1}F)(z_1,z_2) - \Big(&(\partial_{z_1\bar{z}_2}F)(z_1,z_2)z_2 + \\ &(\partial_{\bar{z}_1\bar{z}_2}F)(z_1,z_2)\bar{z}_2 + (\partial_{z_2\bar{z}_2}F)(z_1,z_2)z_3 + (\partial_{\bar{z}_2\bar{z}_2}F)(z_1,z_2)\bar{z}_3\Big), \end{aligned} \tag{1.7}$$

where F is a real valued C^∞ function (in the real sense) defined on $\mathbb{C}^2$ vanishing at 0 at order 3.

Under the hypothesis above equation (1.1) is *quasi-linear* in the sense that the non linearity depends linearly on the variable z_3. We remark that Hypothesis 1.1 implies that the nonlinearity f in (1.1) has the Hamiltonian form

$$f(u,u_x,u_{xx}) = (\partial_{\bar{z}_1}F)(u,u_x) - \frac{d}{dx}[(\partial_{\bar{z}_2}F)(u,u_x)].$$

The second case is the *parity preserving* case.

Hypothesis 1.2 *(***Parity preserving structure***).* Consider the equation (1.1). Assume that f is a C^∞ function in the real sense defined on $\mathbb{C}^3$ and that it vanishes at order 2 at the origin. Assume P has real Fourier coefficients. Assume moreover that f and P satisfy the following

1. $f(z_1,z_2,z_3) = f(z_1,-z_2,z_3)$;
2. $(\partial_{z_3} f)(z_1,z_2,z_3) \in \mathbb{R}$;
3. $P(x) = \sum_{j\in\mathbb{Z}} \hat{p}(j)e^{\mathrm{i}jx}$ is such that $\hat{p}(j) = \hat{p}(-j) \in \mathbb{R}$ (this means that $P(x) = P(-x)$).

Note that item 1 in Hypothesis 1.2 implies that if $u(x)$ is even in x then $f(u,u_x,u_{xx})$ is even in x; item 3 implies that if $u(x)$ is even in x so is $P * u$. Therefore the space of functions even in x is invariant for (1.1). We assume item 2 to avoid parabolic terms in the non linearity, so that (1.1) is a *Schrödinger-type* equation; note that in this case the equation may be *fully-nonlinear*, i.e. the dependence on the variable z_3 is not necessary linear.

In order to treat initial data with big size we shall assume also the following *ellipticity condition*.

Hypothesis 1.3 *(***Global ellipticity***).* We assume that there exist constants $\mathtt{c}_1, \mathtt{c}_2 > 0$ such that the following holds. If f in (1.1) satisfies Hypothesis 1.1 (i.e. has the form (1.7)) then

$$\begin{aligned} &1 - \partial_{z_2}\partial_{\bar{z}_2}F(z_1,z_2) \geq \mathtt{c}_1, \\ &\big((1-\partial_{z_2}\partial_{\bar{z}_2}F)^2 - |\partial_{\bar{z}_2}\partial_{\bar{z}_2}F|^2\big)(z_1,z_2) \geq \mathtt{c}_2 \end{aligned} \tag{1.8}$$

for any (z_1,z_2) in $\mathbb{C}^2$. If f in (1.1) satisfies Hypothesis 1.2 then

$$\begin{aligned} &1 + \partial_{z_3} f(z_1,z_2,z_3) \geq \mathtt{c}_1, \\ &\big((1+\partial_{z_3} f)^2 - |\partial_{\bar{z}_3} f|^2\big)(z_1,z_2,z_3) \geq \mathtt{c}_2 \end{aligned} \tag{1.9}$$

for any (z_1,z_2,z_3) in $\mathbb{C}^3$.

The main result of the paper is the following.

Theorem 1.1 *(***Local existence***). Consider equation* (1.1)*, assume Hypothesis 1.1 (respectively Hypothesis 1.2) and Hypothesis 1.3. Then there exists $s_0 > 0$ such that for any $s \geq s_0$ and for any u_0 in $H^s(\mathbb{T};\mathbb{C})$ (respectively any u_0 even in x in the case of Hypothesis 1.2) there exists $T > 0$, depending only on $\|u_0\|_{H^s}$, such that the equation* (1.1) *with initial datum u_0 has a unique classical solution $u(t,x)$ (resp. $u(t,x)$ even in x) such that*

$$u(t,x) \in C^0\Big([0,T); H^s(\mathbb{T})\Big) \bigcap C^1\Big([0,T); H^{s-2}(\mathbb{T})\Big).$$

Moreover there is a constant $C > 0$ depending on $\|u_0\|_{H^{s_0}}$ and on $\|P\|_{C^1}$ such that

$$\sup_{t\in[0,T)} \|u(t,\cdot)\|_{H^s} \leq C\|u_0\|_{H^s}.$$

We make some comments about Hypotheses 1.1, 1.2 and 1.3. We remark that the class of Hamiltonian equations satisfying Hypothesis 1.1 is different from the parity preserving one satisfying Hypothesis 1.2. For instance the equation

$$\partial_t u = \mathrm{i}\Big[(1+|u|^2)u_{xx} + u_x^2\bar{u} + (u-\bar{u})u_x\Big] \tag{1.10}$$

has the form (1.5) with Hamiltonian function

$$\mathcal{H} = \int_{\mathbb{T}} -|u_x|^2(1+|u|^2) + |u|^2(u_x + \bar{u}_x)dx,$$

but does not have the parity preserving structure (in the sense of Hypothesis 1.2). On the other hand the equation

$$\partial_t u = \mathrm{i}(1+|u|^2)u_{xx} \tag{1.11}$$

has the parity preserving structure but is not Hamiltonian with respect to the symplectic form $(u,v) \mapsto \mathrm{Re}\int_{\mathbb{T}} \mathrm{i}u\bar{v}dx$. To check this fact one can reason as done in the appendix of [25]. Both the examples (1.10) and (1.11) satisfy the ellipticity Hypothesis 1.3. Furthermore there are examples of equations that satisfy Hypothesis 1.1 or Hypothesis 1.2 but do not satisfy Hypothesis 1.3, for instance

$$\partial_t u = \mathrm{i}(1-|u|^2)u_{xx}. \tag{1.12}$$

The equation (1.12) has the parity preserving structure and it has the form (1.1) with $P \equiv 0$ and $f(u,u_x,u_{xx}) = -|u|^2 u_{xx}$, therefore such an f violates (1.9) for $|u| \geq 1$. Nevertheless we are able to prove local existence for equations with this kind of non-linearity if the size of the initial datum is sufficiently small; indeed, since f in (1.1) is a C^∞ function vanishing at the origin, conditions (1.9) in the case of Hypothesis 1.2 and (1.8) in the case of Hypothesis 1.1 are always locally fulfilled for $|u|$ small enough. More precisely we have the following theorem.

Theorem 1.2 *(***Local existence for small data***). Consider equation* (1.1) *and assume only Hypothesis 1.1 (respectively Hypothesis 1.2). Then there exists $s_0 > 0$ such that for any $s \geq s_0$ there exists $r_0 > 0$ such that, for any $0 \leq r \leq r_0$, the thesis of Theorem 1.1 holds for any initial datum u_0 in the ball of radius r of $H^s(\mathbb{T};\mathbb{C})$ centered at the origin.*

Our method requires a high regularity of the initial datum. In the rest of the paper we have not been sharp in quantifying the minimal value of s_0 in Theorems 1.1 and 1.2. The reason for which we need regularity is to perform suitable changes of coordinates and having a symbolic calculus at a sufficient order, which requires smoothness of the functions of the phase space.

The convolution potential P in equation (1.1) is motivated by possible future applications. For instance the potential P can be used, as external parameter, in order to modulate the linear frequencies with the aim of studying the long time stability of the small amplitude solutions of (1.1) by means of Birkhoff Normal Forms techniques. For semilinear NLS-type equation this has been done in [9]. As far as we know there are no results regarding quasi-linear NLS-type equations. For quasi-linear equations we quote [14], [15] for the Klein–Gordon and [10] for the capillary Water Waves.

1.2. Functional setting and ideas of the proof

Here we introduce the phase space of functions and we give some ideas of the proof. It is useful for our purposes to work on the product space $H^s \times H^s := H^s(\mathbb{T};\mathbb{C}) \times H^s(\mathbb{T};\mathbb{C})$, in particular we will often use its subspace

$$\begin{aligned} &\mathbf{H}^s := \mathbf{H}^s(\mathbb{T},\mathbb{C}^2) := \big(H^s \times H^s\big) \cap \mathcal{U}, \\ &\mathcal{U} := \{(u^+,u^-) \in L^2(\mathbb{T};\mathbb{C}) \times L^2(\mathbb{T};\mathbb{C}) \,:\, u^+ = \overline{u^-}\}, \end{aligned} \tag{1.13}$$

endowed with the product topology. On $\mathbf{H}^0$ we define the scalar product

$$(U,V)_{\mathbf{H}^0} := \int_{\mathbb{T}} U \cdot \overline{V} dx. \tag{1.14}$$

We introduce also the following subspaces of H^s and of $\mathbf{H}^s$ made of even functions in $x \in \mathbb{T}$

$$H^s_e := \{u \in H^s \,:\, u(x) = u(-x)\}, \qquad \mathbf{H}^s_e := (H^s_e \times H^s_e) \cap \mathbf{H}^0. \tag{1.15}$$

We define the operators $\lambda[\cdot]$ and $\bar{\lambda}[\cdot]$ by linearity as

$$\begin{aligned} &\lambda[e^{\mathrm{i}jx}] := \lambda_j e^{\mathrm{i}jx}, \qquad \lambda_j := (\mathrm{i}j)^2 + \hat{p}(j), \qquad j \in \mathbb{Z}, \\ &\bar{\lambda}[e^{\mathrm{i}jx}] := \lambda_{-j} e^{\mathrm{i}jx}, \end{aligned} \tag{1.16}$$

where $\hat{p}(j)$ are the Fourier coefficients of the potential P in (1.2). Let us introduce the following matrices

$$E := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbb{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.17}$$

and set

$$\Lambda U := \begin{pmatrix} \lambda[u] \\ \overline{\lambda}\,[\bar{u}] \end{pmatrix}, \qquad \forall\ U = (u, \bar{u}) \in \mathbf{H}^s. \tag{1.18}$$

We denote by $\mathfrak{P}$ the linear operator on $\mathbf{H}^s$ defined by

$$\mathfrak{P}[U] := \begin{pmatrix} P * u \\ \bar{P} * \bar{u} \end{pmatrix}, \quad U = (u, \bar{u}) \in \mathbf{H}^s, \tag{1.19}$$

where $P * u$ is defined in (1.2). With this formalism we have that the operator Λ in (1.18) and (1.16) can be written as

$$\Lambda := \begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx} \end{pmatrix} + \mathfrak{P}. \tag{1.20}$$

It is useful to rewrite the equation (1.1) as the equivalent system

$$\partial_t U = \mathrm{i} E \Lambda U + \mathtt{F}(U), \qquad \mathtt{F}(U) := \begin{pmatrix} f(u, u_x, u_{xx}) \\ \overline{f(u, u_x, u_{xx})} \end{pmatrix}, \tag{1.21}$$

where $U = (u, \bar{u})$. The first step is to rewrite (1.21) as a paradifferential system by using the paralinearization formula of Bony (see for instance [21], [24]). In order to do that, we will introduce rigorously classes of symbols in Section 3, here we follow the approach used in [10]. Roughly speaking we shall deal with functions $\mathbb{T} \times \mathbb{R} \ni (x, \xi) \to a(x, \xi)$ with limited smoothness in x satisfying, for some $m \in \mathbb{R}$, the following estimate

$$|\partial_\xi^\beta a(x, \xi)| \le C_\beta \langle \xi \rangle^{m-\beta}, \ \ \forall \beta \in \mathbb{N}, \tag{1.22}$$

where $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$. These functions will have limited smoothness in x because they will depend on x through the dynamical variable U which is in $\mathbf{H}^s(\mathbb{T})$ for some s. From the symbol $a(x, \xi)$ one can define the *paradifferential* operator $\mathrm{Op}^{\mathcal{B}}(a(x, \xi))[\cdot]$, acting on periodic functions of the form $u(x) = \sum_{j \in \mathbb{Z}} \hat{u}(j) \frac{e^{\mathrm{i} j x}}{\sqrt{2\pi}}$, in the following way:

$$\mathrm{Op}^{\mathcal{B}}(a(x, \xi))[u] := \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{\mathrm{i} k x} \left(\sum_{j \in \mathbb{Z}} \chi \left(\frac{k - j}{\langle j \rangle} \right) \hat{a}(k - j, j) \hat{u}(j) \right), \tag{1.23}$$

where $\hat{a}(k, j)$ is the k^{th}-Fourier coefficient of the 2π-periodic in x function $a(x, \xi)$, and where $\chi(\eta)$ is a C_0^∞ function supported in a sufficiently small neighborhood of the origin. With this formalism (1.21) is equivalent to the paradifferential system

$$\partial_t U = \mathrm{i} E \mathcal{G}(U)[U] + \mathcal{R}(U), \tag{1.24}$$

where $\mathcal{G}(U)[\cdot]$ is

$$\begin{aligned} \mathcal{G}(U)[\cdot] &:= \begin{pmatrix} \mathrm{Op}^{\mathcal{B}}((\mathrm{i}\xi)^2 + a(x, \xi))[\cdot] & \mathrm{Op}^{\mathcal{B}}(b(x, \xi))[\cdot] \\ \mathrm{Op}^{\mathcal{B}}(\overline{b(x, -\xi)})[\cdot] & \mathrm{Op}^{\mathcal{B}}((\mathrm{i}\xi)^2 + \overline{a(x, -\xi)})[\cdot] \end{pmatrix}, \\ a(x, \xi) &:= a(U; x, \xi) = \partial_{u_{xx}} f (\mathrm{i}\xi)^2 + \partial_{u_x} f (\mathrm{i}\xi) + \partial_u f, \\ b(x, \xi) &:= b(U; x, \xi) = \partial_{\bar{u}_{xx}} f (\mathrm{i}\xi)^2 + \partial_{\bar{u}_x} f (\mathrm{i}\xi) + \partial_{\bar{u}} f, \end{aligned} \tag{1.25}$$

and where $\mathcal{R}(U)$ is a smoothing operator

$$\mathcal{R}(\cdot) : \mathbf{H}^s \to \mathbf{H}^{s+\rho},$$

for any $s > 0$ large enough and $\rho \sim s$. Note that the symbols in (1.25) are of order 2, i.e. they satisfy (1.22) with $m = 2$. One of the most important property of being a paradifferential operator is the following: if U is sufficiently regular, namely $U \in \mathbf{H}^{s_0}$ with s_0 large enough, then $\mathcal{G}(U)[\cdot]$ extends to a bounded linear operator from $\mathbf{H}^s$ to $\mathbf{H}^{s-2}$ for any s in $\mathbb{R}$. This paralinearization procedure will be discussed in detail in Section 4, in particular in Lemma 4.1 and Proposition 4.1. Since equation (1.1) is quasi-linear the proofs of Theorems 1.1, 1.2 do not rely on direct fixed point arguments; these arguments are used to study the local theory for the semi-linear equations (i.e. when the nonlinearity f in (1.1) depends only on u). The local theory for the semi-linear Schrödinger type equations is, nowadays, well understood; for a complete overview we refer to [12]. Our approach is based on the following quasi-linear iterative scheme (a similar one is used for instance in [2]). We consider the sequence of linear problems

$$\mathcal{A}_0 := \begin{cases} \partial_t U_0 - \mathrm{i}E\partial_{xx}U_0 = 0, \\ U_0(0) = U^{(0)}, \end{cases} \tag{1.26}$$

and for $n \geq 1$

$$\mathcal{A}_n := \begin{cases} \partial_t U_n - \mathrm{i}E\mathcal{G}(U_{n-1})[U_n] - \mathcal{R}(U_{n-1}) = 0, \\ U_n(0) = U^{(0)}, \end{cases} \tag{1.27}$$

where $U^{(0)}(x) = (u_0(x), \overline{u_0}(x))$ with $u_0(x)$ given in (1.1). The goal is to show that there exists $s_0 > 0$ such that for any $s \geq s_0$ the following facts hold:

1. the iterative scheme is well-defined, i.e. there is $T > 0$ such that for any $n \geq 0$ there exists a unique solution U_n of the problem $\mathcal{A}_n$ which belongs to the space $C^0([0, T)]; \mathbf{H}^s)$;
2. the sequence $\{U_n\}_{n\geq 0}$ is bounded in $C^0([0, T)]; \mathbf{H}^s)$;
3. $\{U_n\}_{n\geq 0}$ is a Cauchy sequence in $C^0([0, T)]; \mathbf{H}^{s-2})$.

From these properties the limit function U belongs to the space $L^\infty([0, T); \mathbf{H}^s)$. In the final part of Section 6 we show that actually U is a *classical* solution of (1.1), namely U solves (1.21) and it belongs to $C^0([0, T); \mathbf{H}^s)$.

Therefore the key point is to obtain energy estimates for the linear problem in V

$$\begin{cases} \partial_t V - \mathrm{i}E\mathcal{G}(U)[V] - \mathcal{R}(U) = 0, \\ V(0) = U^{(0)}, \end{cases} \tag{1.28}$$

where $\mathcal{G}$ is given in (1.25) and $U = U(t, x)$ is a fixed function defined for $t \in [0, T], T > 0$, regular enough and $\mathcal{R}(U)$ is regarded as a non homogeneous forcing term. Note that the regularity in time and space of the coefficients of operators $\mathcal{G}, \mathcal{R}$ depends on the regularity of the function U. Our strategy is to perform a paradifferential change of coordinates $W := \Phi(U)[V]$ such that the system (1.28) in the new coordinates reads

$$\begin{cases} \partial_t W - \mathrm{i}E\widetilde{\mathcal{G}}(U)[W] - \widetilde{\mathcal{R}}(U) = 0, \\ W(0) = \Phi(U^{(0)})[U^{(0)}], \end{cases} \tag{1.29}$$

where the operator $\widetilde{\mathcal{G}}(U)[\cdot]$ is self-adjoint with constant coefficients in $x \in \mathbb{T}$ and $\widetilde{\mathcal{R}}(U)$ is a bounded term. More precisely we show that the operator $\widetilde{\mathcal{G}}(U)[\cdot]$ has the form

$$\begin{gathered} \widetilde{\mathcal{G}}(U)[\cdot] := \begin{pmatrix} \mathrm{Op}^{\mathcal{B}}((\mathrm{i}\xi)^2 + m(U;\xi))[\cdot] & 0 \\ 0 & \mathrm{Op}^{\mathcal{B}}((\mathrm{i}\xi)^2 + m(U;\xi))[\cdot] \end{pmatrix}, \\ m(U;\xi) := m_2(U)(\mathrm{i}\xi)^2 + m_1(U)(\mathrm{i}\xi) \in \mathbb{R}, \end{gathered} \tag{1.30}$$

with $m(U;\xi)$ real valued and independent of $x \in \mathbb{T}$. Since the symbol $m(U;\xi)$ is real valued the linear operator $\mathrm{i}E\widetilde{\mathcal{G}}(U)$ generates a well defined flow on $L^2 \times L^2$, since it has also constant coefficients in x it generates a flow on $H^s \times H^s$ for $s \geq 0$. This idea of conjugation to constant coefficients up to bounded remainder has been developed in order to study the linearized equation associated to quasi-linear system in the context of Nash–Moser iterative scheme. For instance we quote the papers [6], [7] on the KdV equation, [17], [18] on the NLS equation and [19], [11], [5], [1] on the water waves equation, in which such techniques are used in studying the existence of periodic and quasi-periodic solutions. Here, dealing with the paralinearized equation (1.24), we adapt the changes of coordinates, for instance performed in [17], to the paradifferential context following the strategy introduced in [10] for the water waves equation.

Comments on Hypotheses 1.1, 1.2 and 1.3.

Consider the following linear system

$$\partial_t V - \mathrm{i}E\mathfrak{L}(x)\partial_{xx}V = 0, \tag{1.31}$$

where $\mathfrak{L}(x)$ is the non constant coefficient matrix

$$\mathfrak{L}(x) := \begin{pmatrix} 1+a_2(x) & b_2(x) \\ \overline{b_2(x)} & 1+a_2(x) \end{pmatrix}, \quad a_2 \in C^\infty(\mathbb{T};\mathbb{R}),\ \ b_2 \in C^\infty(\mathbb{T};\mathbb{C}). \tag{1.32}$$

Here we explain how to diagonalize and conjugate to constant coefficients the system (1.31) at the highest order, we also discuss the role of the Hypotheses 1.1, 1.2 and 1.3. The analogous analysis for the paradifferential system (1.28) is performed in Section 5.

First step: diagonalization at the highest order. We want to transform (1.31) into the system

$$\partial_t V_1 = \mathrm{i}E\left(A_2^{(1)}(x)\partial_{xx}V_1 + A_1^{(1)}(x)\partial_x V_1 + A_0^{(1)}(x)V_1\right), \tag{1.33}$$

where $A_1^{(1)}(x)$, $A_0^{(1)}(x)$ are 2×2 matrices of functions, and $A_2^{(1)}(x)$ is the diagonal matrix of functions

$$A_2^{(1)}(x) = \begin{pmatrix} 1+a_2^{(1)}(x) & 0 \\ 0 & 1+a_2^{(1)}(x) \end{pmatrix},$$

for some real valued function $a_2^{(1)}(x) \in C^\infty(\mathbb{T};\mathbb{R})$. See Section 5.1 for the paradifferential linear system (1.28). The matrix $E\mathfrak{L}(x)$ can be diagonalized through a regular transformation if the determinant of $E\mathfrak{L}(x)$ is strictly positive, i.e. there exists $c>0$ such that

$$\det\Big(E\mathfrak{L}(x)\Big) = (1+a_2(x))^2 - b_2(x)^2 \geq c, \tag{1.34}$$

for any $x\in\mathbb{T}$. Note that the eigenvalues of $E\mathfrak{L}(x)$ are $\lambda_{1,2}(x) = \pm\sqrt{\det E\mathfrak{L}(x)}$. Let $\Phi_1(x)$ be the matrix of functions such that

$$\Phi_1(x)\big(E\mathfrak{L}(x)\big)\Phi_1^{-1}(x) = EA_2^{(1)}(x),$$

where $(1+a_2^{(1)}(x))$ is the positive eigenvalue of $E\mathfrak{L}(x)$. One obtains the system (1.33) by setting $V_1 := \Phi_1(x)V$.

Note that condition (1.34) is the transposition at the linear level of the second inequality in (1.8) or (1.9). Note also that if $\|a_2\|_{L^\infty}, \|b_2\|_{L^\infty} \leq r$ then condition (1.34) is automatically fulfilled for r small enough.

Second step: reduction to constant coefficients at the highest order. In order to understand the role of the first bound in conditions (1.8) and (1.9) we perform a further step in which we reduce the system (1.33) to

$$\partial_t V_2 = \mathrm{i}E\left(A_2^{(2)}\partial_{xx}V_2 + A_1^{(2)}(x)\partial_x V_2 + A_0^{(2)}(x)V_2\right), \tag{1.35}$$

where $A_1^{(2)}(x)$, $A_0^{(2)}(x)$ are 2×2 matrices of functions, and

$$A_2^{(2)} = \begin{pmatrix} m_2 & 0 \\ 0 & m_2 \end{pmatrix},$$

for some constant $m_2\in\mathbb{R}$, $m_2>0$. See Section 5.3 for the reduction of the paradifferential linear system (1.28). In order to do this we use the torus diffeomorphism $x\to x+\beta(x)$ for some periodic function $\beta(x)$ with inverse given by $y\to y+\gamma(y)$ with $\gamma(y)$ periodic in y. We define the following linear operator $(Au)(x) = u(x+\beta(x))$, such operator is invertible with inverse given by $(A^{-1}v)(y) = v(y+\gamma(y))$. This change of coordinates transforms (1.33) into (1.35) where

$$A_2^{(2)}(x) = \begin{pmatrix} A[(1+a_2^{(1)}(y))(1+\gamma_y(y))^2] & 0 \\ 0 & A[(1+a_2^{(1)}(y))(1+\gamma_y(y))^2] \end{pmatrix}. \tag{1.36}$$

Then the highest order coefficient does not depend on $y\in\mathbb{T}$ if

$$(1+a_2^{(1)}(y))(1+\gamma_y)^2 = m_2,$$

with $m_2 \in \mathbb{R}$ independent of y. This equation can be solved by setting

$$\begin{aligned} m_2 &:= \left[2\pi \left(\int_{\mathbb{T}} \frac{1}{\sqrt{1+a_2^{(1)}(x)}} dx \right)^{-1} \right]^2, \\ \gamma(y) &:= \partial_y^{-1} \left(\sqrt{\frac{m_2}{1+a_2^{(1)}(y)}} - 1 \right), \end{aligned} \tag{1.37}$$

where ∂_y^{-1} is the Fourier multiplier with symbol $1/(\mathrm{i}\xi)$, hence it is defined only on zero mean functions. This justifies the choice of m_2. Note that m_2, γ in (1.37) are well-defined if $(1+a_2^{(1)}(x))$ is real and strictly positive for any $x \in \mathbb{T}$. This is the first condition in (1.8) and (1.9).

Third step: reduction at lower orders. One can show that it is always possible to conjugate system (1.35) to a system of the form

$$\partial_t V_3 = \mathrm{i}E \left(A_2^{(3)} \partial_{xx} V_2 + A_1^{(3)}(x) \partial_x V_2 + A_0^{(3)}(x) V_2 \right), \tag{1.38}$$

where $A_2^{(3)} \equiv A_2^{(2)}$ and

$$A_1^{(3)} := \begin{pmatrix} m_1 & 0 \\ 0 & \overline{m_1} \end{pmatrix},$$

with $m_1 \in \mathbb{C}$ and $A_0^{(3)}(x)$ is a matrix of functions up to bounded operators. See Sections 5.2, 5.4 for the analogous reduction for paradifferential linear system (1.28).

It turns out that no extra hypotheses are needed to perform this third step. We obtained that the unbounded term in the r.h.s. of (1.38) is pseudo-differential constant coefficients operator with symbol $m(\xi) := m_2(\mathrm{i}\xi)^2 + m_1(\mathrm{i}\xi)$. This is not enough to get energy estimates because the operator $A_2^{(3)}\partial_{xx} + A_1^{(3)}\partial_x$ is not self-adjoint since the symbol $m(\xi)$ is not *a-priori* real valued.

This example gives the idea that the global ellipticity hypothesis Hypothesis 1.3 (or the smallness of the initial datum), are needed to conjugate the highest order term of $\mathcal{G}$ in (1.28) to a diagonal and constant coefficient operator. Of course there are no *a-priori* reasons to conclude that $\widetilde{\mathcal{G}}$ is self-adjoint. This operator is self-adjoint if and only if its symbol $m(U;\xi)$ in (1.30) is real valued for any $\xi \in \mathbb{R}$. The Hamiltonian Hypothesis 1.1 implies that $m_1(U)$ in (1.30) is purely imaginary, while the parity preserving Assumption 1.2 guarantees that $m_1(U) \equiv 0$. Indeed it is shown Lemma 4.2 that if f is Hamiltonian (i.e. satisfies Hypothesis 1.1) then the operator $\mathcal{G}(U)[\cdot]$ is formally self-adjoint w.r.t. the scalar product of $L^2 \times L^2$. In our reduction procedure we use transformations which preserve this structure. On the other hand in the case that f is parity preserving (i.e. satisfies Hypothesis 1.2) then, in Lemma 4.3 it is shown that the operator $\mathcal{G}(U)[\cdot]$ maps even functions in even functions if U is even in $x \in \mathbb{T}$. In this case we apply only transformations which preserve the parity of the functions. An operator of the form $\widetilde{\mathcal{G}}$ as in (1.30) preserves the subspace of even function only if $m_1(U) = 0$.

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2. Linear operators

We define some special classes of linear operators on spaces of functions.

Definition 2.1. Let $A : H^s \to H^{s'}$, for some $s, s' \in \mathbb{R}$, be a linear operator. We define the operator $\overline{A}$ as

$$\overline{A}[h] := \overline{A[\bar{h}]}, \qquad h \in H^s. \tag{2.1}$$

Definition 2.2 *(***Reality preserving***)*. Let $A, B : H^s \to H^{s'}$, for some $s, s' \in \mathbb{R}$, be linear operators. We say that a matrix of linear operators $\mathfrak{F}$ is *reality* preserving if has the form

$$\mathfrak{F} := \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}, \tag{2.2}$$

for A and B linear operators.

Remark 2.1. Given $s, s' \in \mathbb{R}$, one can easily check that a *reality* preserving linear operator $\mathfrak{F}$ of the form (2.2) is such that

$$\mathfrak{F} : \mathbf{H}^s \to \mathbf{H}^{s'}. \tag{2.3}$$

Given an operator $\mathfrak{F}$ of the form (2.2) we denote by $\mathfrak{F}^*$ its adjoint with respect to the scalar product (1.14)

$$(\mathfrak{F}U, V)_{\mathbf{H}^0} = (U, \mathfrak{F}^* V)_{\mathbf{H}^0}, \quad \forall\, U,\ V \in \mathbf{H}^s.$$

One can check that

$$\mathfrak{F}^* := \begin{pmatrix} A^* & \overline{B}^* \\ B^* & \overline{A}^* \end{pmatrix}, \tag{2.4}$$

where A^* and B^* are respectively the adjoints of the operators A and B with respect to the complex scalar product on $L^2(\mathbb{T}; \mathbb{C})$

$$(u, v)_{L^2} := \int_{\mathbb{T}} u \cdot \bar{v} dx, \quad u, v \in L^2(\mathbb{T}; \mathbb{C}).$$

Definition 2.3 *(***Self-adjointness***)*. Let $\mathfrak{F}$ be a reality preserving linear operator of the form (2.2). We say that $\mathfrak{F}$ is *self-adjoint* if $A, A^*, B, B^* : H^s \to H^{s'}$, for some $s, s' \in \mathbb{R}$ and

$$A^* = A, \quad \overline{B} = B^*. \tag{2.5}$$

We have the following definition.

Definition 2.4 *(***Parity preserving***)*. Let $A : H^s \to H^{s'}$, for some $s, s' \in \mathbb{R}$ be a linear operator. We say that A is *parity* preserving if

$$A : H^s_e \to H^{s'}_e, \tag{2.6}$$

i.e. maps even functions in even functions of $x \in \mathbb{T}$. Let $\mathfrak{F} : \mathbf{H}^s \to \mathbf{H}^{s'}$ be a reality preserving operator of the form (2.2). We say that $\mathfrak{F}$ is parity preserving if the operators A, B are parity preserving operators.

Remark 2.2. Given $s, s' \in \mathbb{R}$, and let $\mathfrak{F} : \mathbf{H}^s \to \mathbf{H}^{s'}$ be a reality and parity preserving operator of the form (2.2). One can check that

$$\mathfrak{F} : \mathbf{H}^s_e \to \mathbf{H}^{s'}_e. \tag{2.7}$$

We note that Λ in (1.18) has the following properties:

- the operator Λ is reality preserving (according to Definition 2.2).
- the operator Λ is self-adjoint according to Definition 2.3 since the coefficients $\hat{p}(j)$ for $j \in \mathbb{Z}$ are real;
- under the parity preserving assumption Hypothesis 1.2 the operator Λ is parity preserving according to Definition 2.4, since $\hat{p}(j) = \hat{p}(-j)$ for $j \in \mathbb{Z}$.

Hamiltonian and parity preserving vector fields Let $\mathfrak{F}$ be a reality preserving, self-adjoint (or parity preserving respectively) operator as in (2.2) and consider the linear system

$$\partial_t U = \mathrm{i}E\mathfrak{F}U, \tag{2.8}$$

on $\mathbf{H}^s$ where E is given in (1.17). We want to analyze how the properties of the system (2.8) change under the conjugation through maps

$$\Phi : \mathbf{H}^s \to \mathbf{H}^s,$$

which are reality preserving. We have the following lemma.

Lemma 2.1. *Let $\mathcal{X} : \mathbf{H}^s \to \mathbf{H}^{s-m}$, for some $m \in \mathbb{R}$ and $s > 0$ be a reality preserving, self-adjoint operator according to Definitions 2.2, 2.3 and assume that its flow*

$$\partial_\tau \Phi^\tau = \mathrm{i}E\mathcal{X}\Phi^\tau, \qquad \Phi^0 = \mathbb{1}, \tag{2.9}$$

satisfies the following. The map Φ^τ is a continuous function in $\tau \in [0,1]$ with values in the space of bounded linear operators from $\mathbf{H}^s$ to $\mathbf{H}^s$ and $\partial_\tau\Phi^\tau$ is continuous as well in $\tau \in [0,1]$ with values in the space of bounded linear operators from $\mathbf{H}^s$ to $\mathbf{H}^{s-m}$.

Then the map Φ^τ satisfies the condition

$$(\Phi^\tau)^*(-\mathrm{i}E)\Phi^\tau = -\mathrm{i}E. \tag{2.10}$$

Proof. First we note that the adjoint operator $(\Phi^\tau)^*$ satisfies the equation $\partial_\tau(\Phi^\tau)^* = (\Phi^\tau)^*\mathcal{X}(-\mathrm{i}E)$. Therefore one can note that

$$\partial_\tau\Big[(\Phi^\tau)^*(-\mathrm{i}E)\Phi^\tau\Big] = 0,$$

which implies $(\Phi^\tau)^*(-\mathrm{i}E)\Phi^\tau = (\Phi^0)^*(-\mathrm{i}E)\Phi^0 = -\mathrm{i}E$. □

Lemma 2.2. *Consider a reality preserving, self-adjoint linear operator $\mathfrak{F}$ (i.e. which satisfies (2.2) and (2.5)) and a reality preserving map Φ. Assume that Φ satisfies condition (2.10) and consider the system*

$$\partial_t W = \mathrm{i}E\mathfrak{F}W, \qquad W \in \mathbf{H}^s. \tag{2.11}$$

By setting $V = \Phi W$ one has that the system (2.11) reads

$$\partial_t V = \mathrm{i}E\mathcal{Y}V, \tag{2.12}$$

$$\mathcal{Y} := -\mathrm{i}E\Phi(\mathrm{i}E)\mathfrak{F}\Phi^{-1} - \mathrm{i}E(\partial_t\Phi)\Phi^{-1}, \tag{2.13}$$

and $\mathcal{Y}$ is self-adjoint, i.e. it satisfies conditions (2.5).

Proof. One applies the changes of coordinates and one gets the form in (2.13). We prove that separately each term of $\mathcal{Y}$ is self-adjoint. Note that by (2.10) one has that $(-\mathrm{i}E)\Phi = (\Phi^*)^{-1}(-\mathrm{i}E)$, hence $-\mathrm{i}E\Phi(\mathrm{i}E)\mathfrak{F}\Phi^{-1} = (\Phi^*)^{-1}\mathfrak{F}\Phi^{-1}$. Then

$$\Big((\Phi^*)^{-1}\mathfrak{F}\Phi^{-1}\Big)^* = (\Phi^{-1})^*\mathfrak{F}[(\Phi^*)^{-1}]^*, \tag{2.14}$$

since $\mathfrak{F}$ is self-adjoint. Moreover we have that $(\Phi^{-1})^* = (\Phi^*)^{-1}$. Indeed again by (2.10) one has that

$$\Phi^{-1} = (\mathrm{i}E)\Phi^*(-\mathrm{i}E), \qquad (\Phi^{-1})^* = (\mathrm{i}E)\Phi(-\mathrm{i}E), \quad \Phi^* = (-\mathrm{i}E)\Phi^{-1}(\mathrm{i}E)$$

Hence one has

$$(\Phi^{-1})^*\Phi^* = (\mathrm{i}E)\Phi(-\mathrm{i}E)(-\mathrm{i}E)\Phi^{-1}(\mathrm{i}E) = -(\mathrm{i}E)(\mathrm{i}E) = \mathbb{1}. \tag{2.15}$$

Then by (2.14) we conclude that $(-\mathrm{i}E)\Phi\mathrm{i}E\Phi^{-1}$ is self-adjoint. Let us study the second term of (2.13). First note that

$$\partial_t[\Phi^*] = -(\Phi^*)(-\mathrm{i}E)(\partial_t\Phi)\Phi^{-1}(\mathrm{i}E), \qquad (\partial_t\Phi)^* = \Phi^*(\mathrm{i}E)(\partial_t(\Phi^*))^*\Phi^{-1}(\mathrm{i}E) \tag{2.16}$$

then

$$\Big((-\mathrm{i}E)(\partial_t\Phi)(\Phi^{-1})\Big)^* = (\Phi^{-1})^*(\partial_t\Phi)^*(\mathrm{i}E) = (-\mathrm{i}E)(\partial_t(\Phi^*))^*\Phi^{-1}. \tag{2.17}$$

By (2.16) we have $\partial_t(\Phi^*) = (\partial_t\Phi)^*$, hence we get the result. □

Lemma 2.3. *Consider a reality and parity preserving linear operator $\mathfrak{F}$ (i.e. (2.2) and (2.7) hold) and a map Φ as in (2.2) which is parity preserving (see Definition 2.4). Consider the system*

$$\partial_t W = \mathrm{i}E\mathfrak{F}W, \qquad W \in \mathbf{H}^s. \tag{2.18}$$

By setting $V = \Phi W$ one has that the system (2.11) reads

$$\partial_t V = \mathrm{i}E\mathcal{Y}V, \tag{2.19}$$

$$\mathcal{Y} := -\mathrm{i}E\Phi(\mathrm{i}E)\mathfrak{F}\Phi^{-1} - \mathrm{i}E(\partial_t\Phi)\Phi^{-1}, \tag{2.20}$$

and $\mathcal{Y}$ is reality preserving and parity preserving, i.e. satisfies condition (2.2) and (2.7).

Proof. It follows straightforward by the Definitions 2.4 and 2.2. □

3. Paradifferential calculus

3.1. Classes of symbols

We introduce some notation. If $K \in \mathbb{N}$, I is an interval of $\mathbb{R}$ containing the origin, $s \in \mathbb{R}^+$ we denote by $C^K_{*\mathbb{R}}(I, \mathbf{H}^s(\mathbb{T}, \mathbb{C}^2))$, sometimes by $C^K_{*\mathbb{R}}(I, \mathbf{H}^s)$, the space of continuous functions U of $t \in I$ with values in $\mathbf{H}^s(\mathbb{T}, \mathbb{C}^2)$, which are K-times differentiable and such that the k-th derivative is continuous with values in $\mathbf{H}^{s-2k}(\mathbb{T}, \mathbb{C}^2)$ for any $0 \le k \le K$. We endow the space $C^K_{*\mathbb{R}}(I, \mathbf{H}^s)$ with the norm

$$\sup_{t\in I} \|U(t,\cdot)\|_{K,s}, \quad \text{where} \quad \|U(t,\cdot)\|_{K,s} := \sum_{k=0}^{K} \left\|\partial_t^k U(t,\cdot)\right\|_{\mathbf{H}^{s-2k}}. \tag{3.1}$$

Moreover if $r \in \mathbb{R}^+$ we set

$$B^K_s(I, r) := \left\{ U \in C^K_{*\mathbb{R}}(I, \mathbf{H}^s) : \sup_{t\in I} \|U(t,\cdot)\|_{K,s} < r \right\}. \tag{3.2}$$

Definition 3.1 *(Symbols)*. Let $m \in \mathbb{R}$, $K' \le K$ in $\mathbb{N}$, $r > 0$. We denote by $\Gamma^m_{K,K'}[r]$ the space of functions $(U; t, x, \xi) \mapsto a(U; t, x, \xi)$, defined for $U \in B^K_{\sigma_0}(I, r)$, for some large enough σ_0, with complex values such that for any $0 \le k \le K - K'$, any $\sigma \ge \sigma_0$, there are $C > 0$, $0 < r(\sigma) < r$ and for any $U \in B^K_{\sigma_0}(I, r(\sigma)) \cap C^{k+K'}_{*\mathbb{R}}(I, \mathbf{H}^\sigma)$ and any $\alpha, \beta \in \mathbb{N}$, with $\alpha \le \sigma - \sigma_0$

$$\left|\partial_t^k \partial_x^\alpha \partial_\xi^\beta a(U; t, x, \xi)\right| \le C \|U\|_{k+K',\sigma} \langle\xi\rangle^{m-\beta}, \tag{3.3}$$

for some constant $C = C(\sigma, \|U\|_{k+K',\sigma_0})$ depending only on σ and $\|U\|_{k+K',\sigma_0}$.

Remark 3.1. In the rest of the paper the time t will be treated as a parameter. In order to simplify the notation we shall write $a(U; x, \xi)$ instead of $a(U; t, x, \xi)$. On the other hand we will emphasize the x-dependence of a symbol a. We shall denote by $a(U; \xi)$ only those symbols which are independent of the variable $x \in \mathbb{T}$.

Remark 3.2. If one compares the latter definition of class of symbols with the one given in Section 2 in [10] one note that they have been more precise on the expression of the constant C in the r.h.s. of (3.3). First of all we do not need such precision since we only want to study local theory. Secondly their classes are modeled in order to work in a small neighborhood of the origin.

Lemma 3.1. *Let $a \in \Gamma^m_{K,K'}[r]$ and $U \in B^K_{\sigma_0}(I,r)$ for some σ_0. One has that*

$$\sup_{\xi\in\mathbb{R}} \langle\xi\rangle^{-m} \|a(U;\cdot,\xi)\|_{K-K',s} \leq C\,\|U\|_{K,s+\sigma_0+1}\,. \tag{3.4}$$

for $s \geq 0$.

Proof. Assume that $s \in \mathbb{N}$. We have

$$\begin{aligned}\|a(U;x,\xi)\|_{K-K',s} &\leq C_1 \sum_{k=0}^{K-K'} \sum_{j=0}^{s-2k} \|\partial_t^k \partial_x^j a(U;\cdot,\xi)\|_{L^\infty} \\ &\leq C_2\langle\xi\rangle^m \sum_{k=0}^{K-K'} \|U\|_{k+K',s+\sigma_0},\end{aligned} \tag{3.5}$$

with $C_1, C_2 > 0$ depend only on s, K and $\|U\|_{k+K',\sigma_0}$, and where we used formula (3.3) with $\sigma = s + \sigma_0$. Equation (3.5) implies (3.4) for $s \in \mathbb{N}$. The general case $s \in \mathbb{R}_+$, follows by using the log-convexity of the Sobolev norm by writing $s = [s]\tau + (1-\tau)(1+[s])$ where $[s]$ is the integer part of s and $\tau \in [0,1]$. □

We define the following special subspace of $\Gamma^0_{K,K'}[r]$.

Definition 3.2 *(***Functions***).* Let $K' \leq K$ in $\mathbb{N}$, $r > 0$. We denote by $\mathcal{F}_{K,K'}[r]$ the subspace of $\Gamma^0_{K,K'}[r]$ made of those symbols which are independent of ξ.

3.2. Quantization of symbols

Given a smooth symbol $(x,\xi) \to a(x,\xi)$, we define, for any $\sigma \in [0,1]$, the quantization of the symbol a as the operator acting on functions u as

$$\mathrm{Op}_\sigma(a(x,\xi))u = \frac{1}{2\pi} \int_{\mathbb{R}\times\mathbb{R}} e^{\mathrm{i}(x-y)\xi} a(\sigma x + (1-\sigma)y, \xi) u(y) dy d\xi. \tag{3.6}$$

This definition is meaningful in particular if $u \in C^\infty(\mathbb{T})$ (identifying u to a 2π-periodic function). By decomposing u in Fourier series as $u = \sum_{j\in\mathbb{Z}} \hat{u}(j)\frac{e^{ijx}}{\sqrt{2\pi}}$, we may calculate the oscillatory integral in (3.6) obtaining

$$\mathrm{Op}_\sigma(a)u := \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} \hat{a}\big(k-j, (1-\sigma)k + \sigma j\big)\hat{u}(j) \right) e^{\mathrm{i}kx}, \quad \forall\ \sigma \in [0,1], \tag{3.7}$$

where $\hat{a}(k,\xi)$ is the k^{th}-Fourier coefficient of the 2π-periodic function $x \mapsto a(x,\xi)$. In the paper we shall use two particular quantizations:

Standard quantization We define the standard quantization by specifying formula (3.7) for $\sigma = 1$:

$$\mathrm{Op}(a)u := \mathrm{Op}_1(a)u = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} \hat{a}\big(k-j, j\big)\hat{u}(j) \right) e^{\mathrm{i}kx}; \tag{3.8}$$

Weyl quantization We define the Weyl quantization by specifying formula (3.7) for $\sigma = \frac{1}{2}$:

$$\mathrm{Op}^W(a)u := \mathrm{Op}_{\frac{1}{2}}(a)u = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} \hat{a}\big(k-j, \frac{k+j}{2}\big)\hat{u}(j) \right) e^{\mathrm{i}kx}. \tag{3.9}$$

Moreover one can transform the symbols between different quantization by using the formulas

$$\mathrm{Op}(a) = \mathrm{Op}^W(b), \qquad \text{where} \quad \hat{b}(j,\xi) = \hat{a}(j, \xi - \frac{j}{2}). \tag{3.10}$$

In order to define operators starting from the classes of symbols introduced before, we reason as follows. Let $n \in \mathbb{Z}$, we define the projector on n-th Fourier mode as

$$(\Pi_n u)(x) := \hat{u}(n)\frac{e^{\mathrm{i}nx}}{\sqrt{2\pi}}; \quad u(x) = \sum_{n\in\mathbb{Z}} \hat{u}(n)\frac{e^{\mathrm{i}jx}}{\sqrt{2\pi}}. \tag{3.11}$$

For $U \in B^K_s(I,r)$ (as in Definition 3.1), a symbol a in $\Gamma^m_{K,K'}[r]$, and $v \in C^\infty(\mathbb{T},\mathbb{C})$ we define

$$\mathrm{Op}(a(U;j))[v] := \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} \Pi_{k-j} a(U;j) \Pi_j v \right). \tag{3.12}$$

Equivalently one can define $\mathrm{Op}^W(a)$ according to (3.9).

We want to define a *paradifferential* quantization. First we give the following definition.

Definition 3.3 *(***Admissible cut-off functions***).* We say that a function $\chi \in C^\infty(\mathbb{R}\times\mathbb{R};\mathbb{R})$ is an admissible cut-off function if it is even with respect to each of its arguments and there exists $\delta > 0$ such that

$$\operatorname{supp}\chi \subset \left\{(\xi',\xi) \in \mathbb{R}\times\mathbb{R}; |\xi'| \le \delta\langle\xi\rangle\right\}, \qquad \xi \equiv 1 \;\; \text{for} \;\; |\xi'| \le \frac{\delta}{2}\langle\xi\rangle.$$

We assume moreover that for any derivation indices α and β

$$|\partial^\alpha_\xi \partial^\beta_{\xi'} \chi(\xi',\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-\beta}, \;\; \forall\alpha, \; \beta \in \mathbb{N}.$$

An example of function satisfying the condition above, and that will be extensively used in the rest of the paper, is $\chi(\xi',\xi) := \widetilde{\chi}(\xi'/\langle\xi\rangle)$, where $\widetilde{\chi}$ is a function in $C^\infty_0(\mathbb{R};\mathbb{R})$ having a small enough support and equal to one in a neighborhood of zero. For any $a \in C^\infty(\mathbb{T})$ we shall use the following notation

$$(\chi(D)a)(x) = \sum_{j\in\mathbb{Z}} \chi(j)\Pi_j a. \tag{3.13}$$

Proposition 3.1 *(***Regularized symbols***). Fix $m \in \mathbb{R}$, $p, K, K' \in \mathbb{N}$, $K' \le K$ and $r > 0$. Consider $a \in \Gamma^m_{K,K'}[r]$ and χ an admissible cut-off function according to Definition 3.3. Then the function*

$$a_\chi(U;x,\xi) := \sum_{n\in\mathbb{Z}} \chi(n,\xi)\,\Pi_n a(U;x,\xi) \tag{3.14}$$

belongs to $\Gamma^m_{K,K'}[r]$.

For the proof we refer the reader to the remark after Definition 2.2.2 in [10].

We define the Bony quantization in the following way. Consider an admissible cut-off function χ and a symbol a belonging to the class $\Gamma^m_{K,K'}[r]$, we set

$$\mathrm{Op}^{\mathcal{B}}(a(U;x,j))[v] := \mathrm{Op}(a_\chi(U;x,j))[v], \tag{3.15}$$

where a_χ is defined in (3.14). Analogously we define the Bony–Weyl quantization

$$\mathrm{Op}^{\mathcal{B}W}(b(U;x,j))[v] := \mathrm{Op}^W(b_\chi(U;x,j))[v]. \tag{3.16}$$

The definition of the operators $\mathrm{Op}^{\mathcal{B}}(b)$ and $\mathrm{Op}^{\mathcal{B}W}(b)$ is independent of the choice of the cut-off function χ modulo smoothing operators that we define now.

Definition 3.4 *(***Smoothing remainders***).* Let $K' \leq K \in \mathbb{N}$, $\rho \geq 0$ and $r > 0$. We define the class of remainders $\mathcal{R}^{-\rho}_{K,K'}[r]$ as the space of maps $(V, u) \mapsto R(V)u$ defined on $B^K_{s_0}(I, r) \times C^K_{*\mathbb{R}}(I, H^{s_0}(\mathbb{T}, \mathbb{C}))$ which are linear in the variable u and such that the following holds true. For any $s \geq s_0$ there exists a constant $C > 0$ and $r(s) \in]0, r[$ such that for any $V \in B^K_{s_0}(I, r) \cap C^K_{*\mathbb{R}}(I, H^s(\mathbb{T}, \mathbb{C}^2))$, any $u \in C^K_{*\mathbb{R}}(I, H^s(\mathbb{T}, \mathbb{C}))$, any $0 \leq k \leq K - K'$ and any $t \in I$ the following estimate holds true

$$\left\| \partial_t^k \left(R(V)u\right)(t, \cdot)\right\|_{H^{s-2k+\rho}} \leq \sum_{k'+k''=k} C\Big[\|u\|_{k'',s} \|V\|_{k'+K',s_0} + \|u\|_{k'',s_0} \|V\|_{k'+K',s} \Big], \tag{3.17}$$

where $C = C(s, \|V\|_{k'+K',s_0})$ is a constant depending only on s and $\|V\|_{k'+K',s_0}$.

Lemma 3.2. *Consider χ_1 and χ_2 admissible cut-off functions. Fix $m \in \mathbb{R}$, $r > 0$, $K' \leq K \in \mathbb{N}$. Then for $a \in \Gamma^m_{K,K'}[r]$, we have* $\mathrm{Op}(a_{\chi_1} - a_{\chi_2}) \in \mathcal{R}^{-\rho}_{K,K'}[r]$ *for any $\rho \in \mathbb{N}$.*

For the proof we refer the reader to the remark after the proof of Proposition 2.2.4 in [10].

Now we state a proposition describing the action of paradifferential operators defined in (3.15) and in (3.16).

Proposition 3.2 *(***Action of paradifferential operators***). Let $r > 0$, $m \in \mathbb{R}$, $K' \leq K \in \mathbb{N}$ and consider a symbol $a \in \Gamma^m_{K,K'}[r]$. There exists $s_0 > 0$ such that for any $U \in B^K_{s_0}(I, r)$, the operator $\mathrm{Op}^{\mathcal{B}W}(a(U; x, \xi))$ extends, for any $s \in \mathbb{R}$, as a bounded operator from the space $C^{K-K'}_{*\mathbb{R}}(I, H^s(\mathbb{T}, \mathbb{C}))$ to $C^{K-K'}_{*\mathbb{R}}(I, H^{s-m}(\mathbb{T}, \mathbb{C}))$. Moreover there is a constant $C > 0$ depending on s and on the constant in (3.3) such that*

$$\|\mathrm{Op}^{\mathcal{B}W}(\partial_t^k a(U; x, \cdot))\|_{\mathcal{L}(H^s, H^{s-m})} \leq C\|U\|_{k+K',s_0}, \tag{3.18}$$

for $k \leq K - K'$, so that

$$\left\|\mathrm{Op}^{\mathcal{B}W}(a(U; x, \xi))(v)\right\|_{K-K',s-m} \leq C\, \|U\|_{K,s_0} \|v\|_{K-K',s}, \tag{3.19}$$

for any $v \in C^{K-K'}_{*\mathbb{R}}(I, H^s(\mathbb{T}, \mathbb{C}))$.

For the proof we refer to Proposition 2.2.4 in [10].

Remark 3.3. Actually the estimates (3.18) and (3.19) follow by

$$\left\|\mathrm{Op}^{\mathcal{B}W}(a(U; x, \xi))(v)\right\|_{K,s-m} \leq C_1 \sup_{\xi\in\mathbb{R}} \langle\xi\rangle^{-m} \|a(U; \cdot, \xi)\|_{K-K',s_0} \|v\|_{K-K',s},$$

where $C_1 > 0$ is some constant depending only on s, s_0 and Remark 3.1.

Remark 3.4. We remark that Proposition 3.2 (whose proof is given in [10]) applies if a satisfies (3.3) with $|\alpha| \leq 2$ and $\beta = 0$. Moreover, by following the same proof, one can show that

$$\|\mathrm{Op}^{W}(\partial_t^k a_\chi(U; x, \cdot))\|_{\mathcal{L}(H^s, H^{s-m})} \leq C\|U\|_{k+K',s_0}, \tag{3.20}$$

if $\chi(\eta, \xi)$ is supported for $|\eta| \leq \delta\langle\xi\rangle$ for $\delta > 0$ small. Note that this is slightly different from the Definition 3.3 of admissible cut-off function since we are not requiring that $\chi \equiv 1$ for $|\eta| \leq \frac{\delta}{2}\langle\xi\rangle$.

Remark 3.5. Note that, if $m < 0$, and $a \in \Gamma^m_{K,K'}[r]$, then estimate (3.18) implies that the operator $\mathrm{Op}^{\mathcal{B}W}(a(U; x, \xi))$ belongs to the class of smoothing operators $\mathcal{R}^m_{K,K'}[r]$.

We consider paradifferential operators of the form:

$$\begin{aligned}\mathrm{Op}^{\mathcal{B}W}(A(U; x, \xi)) &:= \mathrm{Op}^{\mathcal{B}W}\begin{pmatrix} a(U; x, \xi) & b(U; x, \xi) \\ \overline{b(U; x, -\xi)} & \overline{a(U; x, -\xi)}\end{pmatrix} \\ &:= \begin{pmatrix} \mathrm{Op}^{\mathcal{B}W}(a(U; x, \xi)) & \mathrm{Op}^{\mathcal{B}W}(b(U; x, \xi)) \\ \mathrm{Op}^{\mathcal{B}W}(\overline{b(U; x, -\xi)}) & \mathrm{Op}^{\mathcal{B}W}(\overline{a(U; x, -\xi)})\end{pmatrix},\end{aligned} \tag{3.21}$$

where a and b are symbols in $\Gamma^m_{K,K'}[r]$ and U is a function belonging to $B^K_{s_0}(I,r)$ for some s_0 large enough. Note that the matrix of operators in (3.21) is of the form (2.2). Moreover it is self-adjoint (see (2.5)) if and only if

$$a(U;x,\xi)=\overline{a(U;x,\xi)}, \quad b(U;x,-\xi)=b(U;x,\xi), \tag{3.22}$$

indeed conditions (2.5) on these operators read

$$\Big(\mathrm{Op}^{\mathcal{B}W}(a(U;x,\xi))\Big)^*=\mathrm{Op}^{\mathcal{B}W}\Big(\overline{a(U;x,\xi)}\Big), \quad \overline{\mathrm{Op}^{\mathcal{B}W}(b(U;x,\xi))}=\mathrm{Op}^{\mathcal{B}W}\Big(\overline{b(U;x,-\xi)}\Big). \tag{3.23}$$

Analogously, given R_1 and R_2 in $\mathcal{R}^{-\rho}_{K,K'}[r]$, one can define a reality preserving smoothing operator on $\mathbf{H}^s(\mathbb{T},\mathbb{C}^2)$ as follows

$$R(U)[\cdot]:=\begin{pmatrix} R_1(U)[\cdot] & R_2(U)[\cdot] \\ \overline{R_2}(U)[\cdot] & \overline{R_1}(U)[\cdot]\end{pmatrix}. \tag{3.24}$$

We use the following notation for matrix of operators.

Definition 3.5 *(***Matrices***).* We denote by $\Gamma^m_{K,K'}[r]\otimes\mathcal{M}_2(\mathbb{C})$ the matrices $A(U;x,\xi)$ of the form (3.21) whose components are symbols in the class $\Gamma^m_{K,K'}[r]$. In the same way we denote by $\mathcal{R}^{-\rho}_{K,K'}[r]\otimes\mathcal{M}_2(\mathbb{C})$ the operators $R(U)$ of the form (3.24) whose components are smoothing operators in the class $\mathcal{R}^{-\rho}_{K,K'}[r]$.

Remark 3.6. An important class of *parity preserving* maps according to Definition 2.4 is the following. Consider a matrix of symbols $C(U;x,\xi)$, with U even in x, in $\Gamma^m_{K,K'}[r]\otimes\mathcal{M}_2(\mathbb{C})$ with $m\in\mathbb{N}$, if

$$C(U;x,\xi)=C(U;-x,-\xi) \tag{3.25}$$

then one can check that $\mathrm{Op}^{\mathcal{B}W}(C(U;x,\xi))$ preserves the subspace of even functions.

Moreover consider the system

$$\begin{cases}\partial_\tau\Phi^\tau(U)[\cdot]=\mathrm{Op}^{\mathcal{B}W}(C(U;x;\xi))\Phi^\tau(U)[\cdot],\\ \Phi^0(U)=\mathbb{1}.\end{cases}$$

If the flow Φ^τ is well defined for $\tau\in[0,1]$, then it defines a family of *parity preserving* maps according to Definition 2.4.

3.3. Symbolic calculus

We define the following differential operator

$$\sigma(D_x,D_\xi,D_y,D_\eta)=D_\xi D_y-D_xD_\eta, \tag{3.26}$$

where $D_x:=\frac{1}{\mathrm{i}}\partial_x$ and D_ξ,D_y,D_η are similarly defined. If a is a symbol in $\Gamma^m_{K,K'}[r]$ and b in $\Gamma^{m'}_{K,K'}[r]$, if $U\in B^K_{s_0}(I,r)$ with s_0 large enough, we define

$$(a\sharp b)_\rho(U;x,\xi):=\sum_{\ell=0}^{\rho-1}\frac{1}{\ell!}\left(\frac{i}{2}\sigma(D_x,D_\xi,D_y,D_\eta)\right)^\ell[a(U;x,\xi)b(U;y,\eta)]_{|_{x=y;\,y=\eta}}, \tag{3.27}$$

modulo symbols in $\Gamma^{m+m'-\rho}_{K,K'}[r]$. Assume also that the x-Fourier transforms $\hat{a}(\eta,\xi)$, $\hat{b}(\eta,\xi)$ are supported for $|\eta|\le\delta\langle\xi\rangle$ for small enough $\delta>0$. Then we define

$$(a\sharp b)(x,\xi):=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}e^{\mathrm{i}x(\xi^*+\eta^*)}\hat{a}(\eta^*,\xi+\frac{\xi^*}{2})\hat{b}(\xi^*,\xi-\frac{\eta^*}{2})d\xi^*d\eta^*. \tag{3.28}$$

Thanks to the hypothesis on the support of the x-Fourier transform of a and b, this integral is well defined as a distribution in (ξ^*,η^*) acting on the C^∞-function $(\xi^*,\eta^*)\mapsto e^{\mathrm{i}x(\xi^*+\eta^*)}$. Lemma 2.3.4 in [10] guarantees that according to the notation above one has

$$\mathrm{Op}^{\mathcal{B}W}(a)\circ \mathrm{Op}^{\mathcal{B}W}(b)=\mathrm{Op}^{W}(c), \quad c(x,\xi):=(a_\chi\sharp b_\chi)(x,\xi), \tag{3.29}$$

where a_χ and b_χ are defined in (3.14). We state here a Proposition asserting that the symbol $(a\sharp b)_\rho$ is the symbol of the composition up to smoothing operators.

Proposition 3.3 *(***Composition of Bony–Weyl operators***). Let a be a symbol in $\Gamma^m_{K,K'}[r]$ and b a symbol in $\Gamma^{m'}_{K,K'}[r]$, if $U\in B^K_{s_0}(I,r)$ with s_0 large enough then*

$$\mathrm{Op}^{\mathcal{B}W}(a(U;x,\xi))\circ \mathrm{Op}^{\mathcal{B}W}(b(U;x,\xi))-\mathrm{Op}^{\mathcal{B}W}((a\sharp b)_\rho(U;x,\xi)) \tag{3.30}$$

belongs to the class $\mathcal{R}^{-\rho+m+m'}_{K,K'}[r]$.

For the proof we refer to Proposition 2.3.2 in [10]. In the following we will need to compose smoothing operators and paradifferential ones, the next Proposition asserts that the outcome is another smoothing operator.

Proposition 3.4. *Let a be a symbol in $\Gamma^m_{K,K'}[r]$ with $m\geq 0$ and R be a smoothing operator in $\mathcal{R}^{-\rho}_{K,K'}[r]$. If U belongs to $B^K_{s_0}[I,r]$ with s_0 large enough, then the composition operators*

$$\mathrm{Op}^{\mathcal{B}W}(a(U;x,\xi))\circ R(U)[\cdot], \quad R(U)\circ \mathrm{Op}^{\mathcal{B}W}(a(U;x,\xi))[\cdot]$$

belong to the class $\mathcal{R}^{-\rho+m}_{K,K'}[r]$.

For the proof we refer to Proposition 2.4.2 in [10]. We can compose smoothing operators with smoothing operators as well.

Proposition 3.5. *Let R_1 be a smoothing operator in $\mathcal{R}^{-\rho_1}_{K,K'}[r]$ and R_2 in $\mathcal{R}^{-\rho_2}_{K,K'}[r]$. If U belongs to $B^K_{s_0}[I,r]$ with s_0 large enough, then the operator $R_1(U)\circ R_2(U)[\cdot]$ belongs to the class $\mathcal{R}^{-\rho}_{K,K'}[r]$, where $\rho=\min(\rho_1,\rho_2)$.*

We need also the following.

Lemma 3.3. *Fix $K,K'\in\mathbb{N}$, $K'\leq K$ and $r>0$. Let $\{c_i\}_{i\in\mathbb{N}}$ a sequence in $\mathcal{F}_{K,K'}[r]$ such that for any $i\in\mathbb{N}$*

$$\left|\partial_t^k\partial_x^\alpha c_i(U;x)\right|\leq M_i\,\|U\|_{k+K',s_0}\,, \tag{3.31}$$

for any $0\leq k\leq K-K'$ and $|\alpha|\leq 2$ and for some $s_0>0$ big enough. Then for any $s\geq s_0$ and any $0\leq k\leq K-K'$ there exists a constant $C>0$ (independent of n) such that for any $n\in\mathbb{N}$

$$\left\|\partial_t^k\left[\mathrm{Op}^{\mathcal{B}W}\Big(\prod_{i=1}^n c_i(U;x)\Big)h\right]\right\|_{H^{s-2k}}\leq C^n\prod_{i=1}^n M_i\sum_{k_1+k_2=k}\|U\|^n_{k_1+K',s_0}\|h\|_{k_2,s}\,, \tag{3.32}$$

for any $h\in C^{K-K'}_{*\mathbb{R}}(I,H^s(\mathbb{T};\mathbb{C}))$. Moreover there exists $\widetilde{C}$ such that

$$\|\mathrm{Op}^{\mathcal{B}W}\Big(\prod_{i=1}^n c_i\Big)h\|_{K-K',s}\leq \widetilde{C}^n\prod_{i=1}^n M_i\|U\|^n_{K,s_0}\|h\|_{K-K',s}, \tag{3.33}$$

for any $h\in C^{K-K'}_{*\mathbb{R}}(I,H^s(\mathbb{T};\mathbb{C}))$.

Proof. Let χ an admissible cut-off function and set $b(U;x,\xi):=(\prod_{i=1}^n c_i(U;x))_\chi$. By Liebniz rule and interpolation one can prove that

$$|\partial_t^k\partial_x^\alpha\partial_\xi^\beta b(U;x,\xi)|\leq C^n\,\|U\|^n_{k+K',s_0}\prod_{i=1}^n M_i \tag{3.34}$$

for any $0 \le k \le K - K'$, $\alpha \le 2$, any $\xi \in \mathbb{R}$ and where the constant C is independent of n. Denoting by $\widehat{b}(U;\ell,\xi) = \widehat{b}(\ell,\xi)$ the ℓ^{th} Fourier coefficient of the function $b(U;x,\xi)$, from (3.34) with $\alpha = 2$ one deduces the following decay estimate

$$|\partial_t^k \widehat{b}(\ell,\xi)| \le C^n \, \|U\|^n_{k+K',s_0} \prod_{i=1}^n M_i \langle \ell \rangle^{-2}. \tag{3.35}$$

With this setting one has

$$\begin{aligned}\mathrm{Op}^{\mathcal{B}W}\Big(\prod_{i=1}^n c_i(U;x)\Big)h &= \mathrm{Op}^W(b(U;x,\xi))h \\ &= \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}}\left(\sum_{n'\in\mathbb{Z}} \widehat{b}\Big(\ell-n', \frac{\ell+n'}{2}\Big)\widehat{h}(n')\right)e^{\mathrm{i}\ell x},\end{aligned}$$

where the sum is restricted to the set of indices such that $|\ell - n'| \le \delta \frac{|\ell+n'|}{2}$ with $0<\delta<1$ (which implies that $\ell \sim n'$). Let $0 \le k \le K-K'$, one has

$$\begin{aligned}&\left\| \partial_t^k \left[\mathrm{Op}^{\mathcal{B}W}\Big(\prod_{i=1}^n c_i(U;x)\Big)h\right]\right\|^2_{H^{s-2k}} \\ &\le C^n \sum_{k_1+k_2=k}\sum_{\ell\in\mathbb{Z}} \langle \ell\rangle^{2(s-2k)} \left| \sum_{n'\in\mathbb{Z}} \partial_t^{k_1}\left(\widehat{b}\Big(\ell-n',\frac{\ell+n'}{2}\Big)\right)\partial_t^{k_2}\Big(\widehat{h}(n')\Big)\right|^2 \\ &\le C^n \prod_{i=1}^n M_i^2 \sum_{k_1+k_2=k} \|U\|^{2n}_{k_1+K',s_0}\sum_{\ell\in\mathbb{Z}}\left(\sum_{n'\in\mathbb{Z}}\langle \ell-n'\rangle^{-2}\langle n'\rangle^{s-2k}\left|\partial_t^{k_2}\widehat{h}(n')\right|\right)^2,\end{aligned}$$

where in the last passage we have used (3.35) and that $\ell \sim n'$. By using Young inequality for sequences one can continue the chain of inequalities above and finally obtain the (3.32). The estimate (3.33) follows summing over $0 \le k \le K - K'$. □

Proposition 3.6. *Fix $K, K' \in \mathbb{N}$, $K' \le K$ and $r > 0$. Let $\{c_i\}_{i\in\mathbb{N}}$ a sequence in $\mathcal{F}_{K,K'}[r]$ satisfying the hypotheses of Lemma 3.3. Then the operator*

$$Q^{(n)}_{c_1,\ldots,c_n} := \mathrm{Op}^{\mathcal{B}W}(c_1)\circ\cdots\circ\mathrm{Op}^{\mathcal{B}W}(c_n) - \mathrm{Op}^{\mathcal{B}W}(c_1\cdots c_n) \tag{3.36}$$

belongs to the class $\mathcal{R}^{-\rho}_{K,K'}[r]$ for any $\rho \ge 0$. More precisely there exists $s_0 > 0$ such that for any $s \ge s_0$ the following holds. For any $0 \le k \le K - K'$ and any $\rho \ge 0$ there exists a constant $\mathtt{C} > 0$ (depending on $\|U\|_{K,s_0}$, s, s_0, ρ, k and independent of n) such that

$$\left\|\partial_t^k\Big(Q^{(n)}_{c_1,\ldots,c_n}[h]\Big)\right\|_{s+\rho-2k} \le \mathtt{C}^n \mathtt{M} \sum_{k_1+k_2=k}\Big(\|U\|^n_{K'+k_1,s_0}\|h\|_{k_2,s} + \|U\|^{n-1}_{K'+k_1,s_0}\|h\|_{k_2,s_0}\|U\|_{K'+k_1,s}\Big), \tag{3.37}$$

for any $n \ge 1$, any h in $C^K_{*\mathbb{R}}(I, H^s(\mathbb{T},\mathbb{C}))$, any $U \in C^K_{*\mathbb{R}}(I,\mathbf{H}^s)\cap B^K_s(I,r)$ and where $\mathtt{M} = M_1\cdots M_n$ (see (3.31)).

Proof. We proceed by induction. For $n = 1$ is trivial. Let us study the case $n = 2$. Since c_1, c_2 belong to $\mathcal{F}_{K,K'}[r]$, then $c_1 \cdot c_2 = (c_1 \sharp c_2)_\rho$ for any $\rho > 0$. Then by (3.29) there exists an admissible cut-off function χ such that

$$\begin{aligned}\mathrm{Op}^{\mathcal{B}W}(c_1)\circ\mathrm{Op}^{\mathcal{B}W}(c_2) - \mathrm{Op}^{\mathcal{B}W}(c_1\cdot c_2) &= \mathrm{Op}^{\mathcal{B}W}(c_1)\circ\mathrm{Op}^{\mathcal{B}W}(c_2) - \mathrm{Op}^{\mathcal{B}W}((c_1\sharp c_2)_\rho) \\ &= \mathrm{Op}^W((c_1)_\chi\sharp(c_2)_\chi) - \mathrm{Op}^W((c_1\sharp c_2)_{\rho,\chi}) = \mathrm{Op}^W(r_1) + \mathrm{Op}^W(r_2),\end{aligned} \tag{3.38}$$

where

$$\begin{aligned} r_1(x,\xi) &= (c_1)_\chi\sharp(c_2)_\chi - ((c_1)_\chi\sharp(c_2)_\chi)_\rho, \\ r_2(x,\xi) &= ((c_1)_\chi\sharp(c_2)_\chi)_\rho - (c_1\sharp c_2)_{\rho,\chi}.\end{aligned} \tag{3.39}$$

Then, by Lemma 2.3.3 in [10] and (3.31), one has that r_1 satisfies the bound

$$|\partial_t^k \partial_x^\ell r_1(U;x,\xi)| \leq \widetilde{C} M_1 M_2 \langle \xi \rangle^{-\rho+\ell} \|U\|^2_{k+K',s_0} \tag{3.40}$$

for any $|\ell| \leq 2$ and some universal constant $\widetilde{C} > 0$ depending only on s, s_0, ρ. Therefore Proposition 3.2 and Remark 3.4 imply that

$$\left\| \mathrm{Op}^W(\partial_t^k r_1(U;x,\cdot)) \right\|_{\mathcal{L}(H^s, H^{s+\rho-2})} \leq \widetilde{C} M_1 M_2 \|U\|^2_{k+K',s_0}, \tag{3.41}$$

for $\widetilde{C} > 0$ possibly larger than the one in (3.40), but still depending only on k, s, s_0, ρ. From the bound (3.41) one deduces the estimate (3.37) for some $\mathtt{C} \geq 2\widetilde{C}$. One can argue in the same way to estimate the term $\mathrm{Op}^W(r_2)$ in (3.38).

Assume now that (3.37) holds for $j \leq n-1$ for $n \geq 3$. We have that

$$\mathrm{Op}^{\mathcal{B}W}(c_1) \circ \cdots \circ \mathrm{Op}^{\mathcal{B}W}(c_n) = \big(\mathrm{Op}^{\mathcal{B}W}(c_1 \cdots c_{n-1}) + Q_{n-1}\big) \circ \mathrm{Op}^{\mathcal{B}W}(c_n), \tag{3.42}$$

where Q_{n-1} satisfies condition (3.37). For the term $\mathrm{Op}^{\mathcal{B}W}(c_1 \cdots c_{n-1}) \circ \mathrm{Op}^{\mathcal{B}W}(c_n)$ one has to argue as done in the case $n = 2$.

Consider the term $Q_{n-1} \circ \mathrm{Op}^{\mathcal{B}W}(c_n)$ and let $C > 0$ be the universal constant given by Lemma 3.3.

Using the inductive hypothesis on Q_{n-1} and estimate (3.32) in Lemma 3.3 (in the case $n = 1$) we have

$$\begin{aligned}
\|\partial_t^k \big(Q_{n-1} \circ \mathrm{Op}^{\mathcal{B}W}(c_n) h\big)\|_{s+\rho-2k} &\leq \mathtt{K}\mathtt{C}^{n-1} M_1 \cdots M_{n-1} \sum_{k_1+k_2=k} \sum_{j_1+j_2=k_2} C M_n \|U\|^{n-1}_{K'+k_1,s_0} \|U\|_{K'+j_1,s_0} \|h\|_{j_2,s} \\
&\quad + \mathtt{K}\mathtt{C}^{n-1} M_1 \cdots M_{n-1} \sum_{k_1+k_2=k} \sum_{j_1+j_2=k_2} C M_n \|U\|^{n-2}_{K'+k_1,s_0} \|U\|_{K'+k_1,s} \|U\|_{K'+j_1,s_0} \|h\|_{j_2,s_0} \\
&\leq \mathtt{K}\mathtt{M}\mathtt{C}^{n-1} C \sum_{k_1=0}^{k} \sum_{j_1=0}^{k-k_1} \|U\|^n_{K'+k_1+j_1,s_0} \|h\|_{k-k_1-j_1,s} \\
&\quad + \mathtt{K}\mathtt{M}\mathtt{C}^{n-1} C \sum_{k_1=0}^{k} \sum_{j_1=0}^{k-k_1} \|U\|^{n-1}_{K'+k_1+j_1,s_0} \|U\|_{K'+k_1+j_1,s} \|h\|_{k-k_1-j_1,s_0} \\
&\leq \mathtt{K}\mathtt{M}\mathtt{C}^{n-1} C \sum_{m=0}^{k} (\|U\|^n_{K'+m,s_0} \|h\|_{k-m,s} + \|U\|^{n-1}_{K'+m,s_0} \|U\|_{K'+m,s} \|h\|_{k-m,s_0})(m+1),
\end{aligned}$$

for constant $\mathtt{K}$ depending only on k. This implies (3.37) by choosing $\mathtt{C} > (k+1)C\mathtt{K}$. □

Corollary 3.1. *Fix $K, K' \in \mathbb{N}$, $K' \leq K$ and $r > 0$. Let $s(U;x)$ and $z(U;x)$ be symbols in the class $\mathcal{F}_{K,K'}[r]$. Consider the following two matrices*

$$S(U;x) := \begin{pmatrix} s(U;x) & 0 \\ 0 & \overline{s(U;x)} \end{pmatrix}, \quad Z(U;x) := \begin{pmatrix} 0 & z(U;x) \\ \overline{z(U;x)} & 0 \end{pmatrix} \in \mathcal{F}_{K,K'}[r] \otimes \mathcal{M}_2(\mathbb{C}). \tag{3.43}$$

Then one has the following

$$\begin{aligned}
&\exp\left\{\mathrm{Op}^{\mathcal{B}W}(S(U;x))\right\} - \mathrm{Op}^{\mathcal{B}W}(\{\exp S(U;x)\}) \in \mathcal{R}^{-\rho}_{K,K'}[r] \otimes \mathcal{M}_2(\mathbb{C}), \\
&\exp\left\{\mathrm{Op}^{\mathcal{B}W}(Z(U;x))\right\} - \mathrm{Op}^{\mathcal{B}W}(\{\exp Z(U;x)\}) \in \mathcal{R}^{-\rho}_{K,K'}[r] \otimes \mathcal{M}_2(\mathbb{C}),
\end{aligned}$$

for any $\rho \geq 0$.

Proof. Let us prove the result for the matrix $S(U;x)$.

Since $s(U;x)$ belongs to $\mathcal{F}_{K,K'}[r]$ then there exists $s_0 > 0$ such that if $U \in B^K_{s_0}(I,r)$, then there is a constant $\mathtt{N} > 0$ such that

$$\left|\partial_t^k \partial_x^\alpha s(U;x)\right| \leq \mathtt{N} \|U\|_{k+K',s_0},$$

for any $0 \leq k \leq K - K'$ and $|\alpha| \leq 2$. By definition one has

$$\exp\Big(\mathrm{Op}^{\mathcal{B}W}(S(U;x))\Big) = \sum_{n=0}^{\infty} \frac{\big(\mathrm{Op}^{\mathcal{B}W}(S(U;x))\big)^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} \big(\mathrm{Op}^{\mathcal{B}W}(s(U;x))\big)^n & 0 \\ 0 & \big(\mathrm{Op}^{\mathcal{B}W}(\overline{s(U;x)})\big)^n \end{pmatrix},$$

on the other hand

$$\mathrm{Op}^{\mathcal{B}W}\Big(\exp\big(S(U;x)\big)\Big) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{Op}^{\mathcal{B}W} \begin{pmatrix} \big[s(U;x)\big]^n & 0 \\ 0 & \big[\overline{s(U;x)}\big]^n \end{pmatrix}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} \mathrm{Op}^{\mathcal{B}W}\big(\big[s(U;x)\big]^n\big) & 0 \\ 0 & \mathrm{Op}^{\mathcal{B}W}\big(\big[\overline{s(U;x)}\big]^n\big) \end{pmatrix}.$$

We argue componentwise. Let h be a function in $C^K_{*\mathbb{R}}(I, H^s(\mathbb{T},\mathbb{C}))$, then using Proposition 3.6, one has

$$\left\| \sum_{n=0}^{\infty} \frac{1}{n!} \partial_t^k \Big(\big[\mathrm{Op}^{\mathcal{B}W}(s(U;x))\big]^n [h] - \mathrm{Op}^{\mathcal{B}W}\big(s(U;x)^n\big)[h] \Big) \right\|_{s+\rho-2k} \leq$$
$$\sum_{n=1}^{\infty} \frac{\mathrm{C}^n \mathrm{N}^n}{n!} \sum_{k_1+k_2=k} \Big(\|U\|^n_{K'+k_1,s_0} \|h\|_{k_2,s} + \|U\|^{n-1}_{K'+k_1,s_0} \|h\|_{k_2,s_0} \|U\|_{K'+k_1,s} \Big) \leq$$
$$\sum_{k_1+k_2=k} \big(\|U\|_{K'+k_1,s_0} \|h\|_{k_2,s} + \|U\|_{K'+k_1,s} \|h\|_{k_2,s_0} \big) \sum_{n=1}^{\infty} \frac{\mathrm{C}^n \mathrm{N}^n}{n!} \|U\|^{n-1}_{K'+k_1,s_0}.$$

Therefore we have proved the (3.17) with constant

$$C = \sum_{n=1}^{\infty} \frac{\mathrm{C}^n \mathrm{N}^n}{n!} \|U\|^{n-1}_{K'+k_1,s_0} = \frac{\exp(\mathrm{CN}\|U\|_{K'+k_1,s_0}) - 1}{\|U\|_{K'+k_1,s_0}}.$$

For the other non zero component of the matrix the argument is the same.

In order to simplify the notation, set $z(U;x) = z$ and $\overline{z(U;x)} = \bar{z}$, therefore for the matrix $Z(U;x)$, by definition, one has

$$\mathrm{Op}^{\mathcal{B}W}(\exp(Z(U;x))) = \mathrm{Op}^{\mathcal{B}W}\left(\sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} |z|^{2n} & |z|^{2n+1} z \\ |z|^{2n+1}\bar{z} & |z|^{2n} \end{pmatrix} \right).$$

On the other hand, setting $A^n_{z,\bar{z}} = \big(\mathrm{Op}^{\mathcal{B}W}(z) \circ \mathrm{Op}^{\mathcal{B}W}(\bar{z})\big)^n$ and $B^n_{z,\bar{z}} = A^n_{z,\bar{z}} \circ \mathrm{Op}^{\mathcal{B}W}(z)$, one has

$$\exp\Big(\mathrm{Op}^{\mathcal{B}W}(Z(U;x))\Big) = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} A^n_{z,\bar{z}} & B^n_{z,\bar{z}} \\ \overline{B^n_{z,\bar{z}}} & \overline{A^n_{z,\bar{z}}} \end{pmatrix}.$$

Therefore one can study each component of the matrix $\exp\Big(\mathrm{Op}^{\mathcal{B}W}(Z(U;x))\Big) - \mathrm{Op}^{\mathcal{B}W}(\exp Z(U;x))$ in the same way as done in the case of the matrix $S(U,x)$. □

4. Paralinearization of the equation

In this section we give a paradifferential formulation of the equation (1.1). In order to paralinearize the equation (1.1) we need to "double" the variables. We consider a system of equations for the variables (u^+, u^-) in $H^s \times H^s$ which is equivalent to (1.1) if $u^+ = \bar{u}^-$. More precisely we give the following definition.

Definition 4.1. Let f be the $C^\infty(\mathbb{C}^3;\mathbb{C})$ function in the equation (1.1). We define the "vector" NLS as

$$\begin{aligned}&\partial_t U = \mathrm{i}E\left[\Lambda U + \mathtt{F}(U,U_x,U_{xx})\right], \quad U\in H^s\times H^s,\\ &\mathtt{F}(U,U_x,U_{xx}) := \begin{pmatrix} f_1(U,U_x,U_{xx})\\ f_2(U,U_x,U_{xx})\end{pmatrix},\end{aligned} \tag{4.1}$$

where

$$\mathtt{F}(Z_1,Z_2,Z_3) = \begin{pmatrix} f_1(z_1^+,z_1^-,z_2^+,z_2^-,z_3^+,z_3^-)\\ f_2(z_1^+,z_1^-,z_2^+,z_2^-,z_3^+,z_3^-)\end{pmatrix}, \quad Z_i = \begin{pmatrix} z_i^+\\ z_i^-\end{pmatrix}, \quad i=1,2,3,$$

extends $(f,\overline{f})$ in the following sense. The functions f_i for $i=1,2$ are C^∞ on $\mathbb{C}^6$ (in the real sense). Moreover one has the following:

$$\begin{pmatrix} f_1(z_1,\bar z_1,z_2,\bar z_2,z_3,\bar z_3)\\ f_2(z_1,\bar z_1,z_2,\bar z_2,z_3,\bar z_3)\end{pmatrix} = \begin{pmatrix} f(z_1,z_2,z_3)\\ \overline{f(z_1,z_2,z_3)}\end{pmatrix}, \tag{4.2}$$

and

$$\begin{aligned}&\partial_{z_3^+} f_1 = \partial_{z_3^-} f_2, \quad \partial_{z_i^+} f_1 = \overline{\partial_{z_i^-} f_2}, \ \ i=1,2, \quad \partial_{z_i^-} f_1 = \overline{\partial_{z_i^+} f_2}, \quad i=1,2,3\\ &\partial_{\overline{z_i^+}} f_1 = \partial_{\overline{z_i^+}} f_2 = \partial_{\overline{z_i^-}} f_1 = \partial_{\overline{z_i^-}} f_2 = 0\end{aligned} \tag{4.3}$$

where $\partial_{\overline{z_j^\sigma}} = \partial_{\mathrm{Re}\, z_j^\sigma} + \mathrm{i}\partial_{\mathrm{Im}\, z_j^\sigma}$, $\sigma=\pm$.

Remark 4.1. In the case that f has the form

$$f(z_1,z_2,z_3) = C z_1^{\alpha_1}\bar z_1^{\beta_1} z_2^{\alpha_2}\bar z_2^{\beta_2}$$

for some $C\in\mathbb{C}$, $\alpha_i,\beta_i\in\mathbb{N}$ for $i=1,2$, a possible extension is the following:

$$\begin{aligned}&f_1(z_1^+,z_1^-,z_2^+,z_2^-) = C(z_1^+)^{\alpha_1}(z_1^-)^{\beta_1}(z_2^+)^{\alpha_2}(z_2^-)^{\beta_2},\\ &f_2(z_1^+,z_1^-,z_2^+,z_2^-) = \overline{C}(z_1^-)^{\alpha_1}(z_1^+)^{\beta_1}(z_2^-)^{\alpha_2}(z_2^+)^{\beta_2}.\end{aligned}$$

Remark 4.2. Using (4.2) one deduces the following relations between the derivatives of f and f_j with $j=1,2$:

$$\begin{aligned}&\partial_{z_i} f(z_1,z_2,z_3) = (\partial_{z_i^+} f_1)(z_1,\bar z_1,z_2,\bar z_2,z_3,\bar z_3)\\ &\partial_{\bar z_i} f(z_1,z_2,z_3) = (\partial_{z_i^-} f_1)(z_1,\bar z_1,z_2,\bar z_2,z_3,\bar z_3)\\ &\overline{\partial_{\bar z_i} f(z_1,z_2,z_3)} = (\partial_{z_i^+} f_2)(z_1,\bar z_1,z_2,\bar z_2,z_3,\bar z_3)\\ &\overline{\partial_{z_i} f(z_1,z_2,z_3)} = (\partial_{z_i^-} f_2)(z_1,\bar z_1,z_2,\bar z_2,z_3,\bar z_3).\end{aligned} \tag{4.4}$$

In the rest of the paper we shall use the following notation. Given a function $g(z_1^+,z_1^-,z_2^+,z_2^-,z_3^+,z_3^-)$ defined on $\mathbb{C}^6$ which is differentiable in the real sense, we shall write

$$\begin{aligned}&(\partial_{\partial_x^i u} g)(u,\bar u,u_x,\bar u_x,u_{xx},\bar u_{xx}) := (\partial_{z_{i+1}^+} g)(u,\bar u,u_x,\bar u_x,u_{xx},\bar u_{xx}),\\ &(\partial_{\overline{\partial_x^i u}} g)(u,\bar u,u_x,\bar u_x,u_{xx},\bar u_{xx}) := (\partial_{z_{i+1}^-} g)(u,\bar u,u_x,\bar u_x,u_{xx},\bar u_{xx}), \quad i=0,1,2.\end{aligned} \tag{4.5}$$

By Definition 4.1 one has that equation (1.1) is equivalent to the system (4.1) on the subspace $\mathbf{H}^s$.

We state the Bony paralinearization lemma, which is adapted to our case from Lemma 2.4.5 of [10].

Lemma 4.1 *(***Bony paralinearization of the composition operator***). Let f be a complex-valued function of class C^∞ in the real sense defined in a ball centered at 0 of radius $r>0$, in $\mathbb{C}^6$, vanishing at 0 at order 2. There exists a 1×2 matrix of symbols $q\in\Gamma^2_{K,0}[r]$ and a 1×2 matrix of smoothing operators $Q(U)\in\mathcal{R}^{-\rho}_{K,0}[r]$, for any ρ, such that*

$$f(U,U_x,U_{xx}) = \mathrm{Op}^{\mathcal{B}W}(q(U,U_x,U_{xx};x,\xi))[U] + Q(U)U. \tag{4.6}$$

Moreover the symbol $q(U;x,\xi)$ has the form

$$q(U;x,\xi) := d_2(U;x)(\mathrm{i}\xi)^2 + d_1(U;x)(\mathrm{i}\xi) + d_0(U;x), \tag{4.7}$$

where $d_j(U;x)$ are 1×2 matrices of symbols in $\mathcal{F}_{K,0}[r]$, for $j=0,1,2$.

Proof. By the paralinearization formula of Bony, we know that

$$f(U,U_x,U_{xx}) = T_{D_U f}U + T_{D_{U_x} f}U_x + T_{D_{U_{xx}} f}U_{xx} + R_0(U)U, \tag{4.8}$$

where $R_0(U)$ satisfies estimates (3.17) and where

$$T_{D_U f}U = \frac{1}{2\pi}\int e^{\mathrm{i}(x-y)\xi}\chi(\langle\xi\rangle^{-1}D)[c_U(U;x,\xi)]U(y)dyd\xi,$$
$$T_{D_{U_x} f}U_x = \frac{1}{2\pi}\int e^{\mathrm{i}(x-y)\xi}\chi(\langle\xi\rangle^{-1}D)[c_{U_x}(U;x,\xi)]U(y)dyd\xi,$$
$$T_{D_{U_{xx}} f}U_{xx} = \frac{1}{2\pi}\int e^{\mathrm{i}(x-y)\xi}\chi(\langle\xi\rangle^{-1}D)[c_{U_{xx}}(U;x,\xi)]U(y)dyd\xi,$$

with

$$\begin{aligned} c_U(U;x,\xi) &= D_U f, \\ c_{U_x}(U;x,\xi) &= D_{U_x} f(\mathrm{i}\xi), \\ c_{U_{xx}}(U;x,\xi) &= D_{U_{xx}} f(\mathrm{i}\xi)^2, \end{aligned} \tag{4.9}$$

for some $\chi \in C_0^\infty(\mathbb{R})$ with small enough support and equal to 1 close to 0. Using (3.10) we define the x-periodic function $b_i(U;x,\xi)$, for $i=0,1,2$, through its Fourier coefficients

$$\hat{b}_i(U;n,\xi) := \hat{c}_{U_i}(U;n,\xi-n/2) \tag{4.10}$$

where $U_i := \partial_x^i U$. In the same way we define the function $d_i(U;x,\xi)$, for $i=0,1,2$, as

$$\hat{d}_i(U;n,\xi) := \chi\left(n\langle\xi-n/2\rangle^{-1}\right)\hat{c}_{U_i}(U;n,\xi-n/2). \tag{4.11}$$

We have that $T_{D_U f}U = \mathrm{Op}^W(d_0(U,\xi))U$. We observe the following

$$\hat{d}_0(U;n,\xi) = \chi\left(n\langle\xi\rangle^{-1}\right)\widehat{D_U f}(n) + \left(\chi\left(n\langle\xi-n/2\rangle^{-1}\right) - n\langle\xi\rangle^{-1}\right)\widehat{D_U f}(n) \tag{4.12}$$

therefore if the support of χ is small enough, thanks to Lemma 3.2, we obtained

$$T_{D_U f}U = \mathrm{Op}^{\mathcal{B}W}(b_0(U;x,\xi))U + R_1(U)U, \tag{4.13}$$

for some smoothing reminder $R_1(U)$. Reasoning in the same way we get

$$\begin{aligned} T_{D_{U_x} f}U_x &= \mathrm{Op}^{\mathcal{B}W}\big(b_1(U;\xi)\big)U + R_2(U)U \\ T_{D_{U_{xx}} f}U_{xx} &= \mathrm{Op}^{\mathcal{B}W}\big(b_2(U;\xi))\big)U + R_3(U)U. \end{aligned} \tag{4.14}$$

The theorem is proved defining $Q(U) = \sum_{k=0}^3 R_k(U)$ and $q(U;x,\xi) = b_2(U;\xi) + b_1(U;\xi) + b_0(U;\xi)$. Note that the symbol q satisfies conditions (4.7) by (4.9) and formula (3.10). $\square$

We have the following Proposition.

Proposition 4.1 (**Paralinearization of the system**). *There are a matrix $A(U;x,\xi)$ in $\Gamma^2_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$ and a smoothing operator R in $\mathcal{R}^{-\rho}_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$, for any $K,r>0$ and $\rho\geq 0$ such that the system (4.1) is equivalent to*

$$\partial_t U := \mathrm{i}E\Big[\Lambda U + \mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))[U] + R(U)[U]\Big], \tag{4.15}$$

on the subspace $\mathcal{U}$ (see (1.13) and Definition 4.1) and where Λ is defined in (1.18) and (1.16). Moreover the operator $R(U)[\cdot]$ has the form (3.24), the matrix A has the form (3.21), i.e.

$$A(U;x,\xi) := \begin{pmatrix} a(U;x,\xi) & b(U;x,\xi) \\ \overline{b(U;x,-\xi)} & \overline{a(U;x,-\xi)} \end{pmatrix} \in \Gamma^2_{K,0}[r] \otimes \mathcal{M}_2(\mathbb{C}) \tag{4.16}$$

with a, b in $\Gamma^2_{K,0}[r]$. In particular we have that

$$A(U;x,\xi) = A_2(U;x)(\mathrm{i}\xi)^2 + A_1(U;x)(\mathrm{i}\xi) + A_0(U;x), \quad A_i \in \mathcal{F}_{K,0}[r] \otimes \mathcal{M}_2(\mathbb{C}), \ i = 0,1,2. \tag{4.17}$$

Proof. The functions f_1, f_2 in (4.1) satisfy the hypotheses of Lemma 4.1 for any $r > 0$. Hence the result follows by setting $q(U;x,\xi) =: (a(U;x,\xi), b(U;x,\xi))$. $\square$

In the following we study some properties of the system in (4.15).

We first prove some lemmata which translate the Hamiltonian Hypothesis 1.1, parity-preserving Hypothesis 1.2 and global ellipticity Hypothesis 1.3 in the paradifferential setting.

Lemma 4.2 (**Hamiltonian structure**). *Assume that f in (1.1) satisfies Hypothesis 1.1. Consider the matrix $A(U;x,\xi)$ in (4.16) given by Proposition 4.1. Then the term*

$$A_2(U;x)(\mathrm{i}\xi)^2 + A_1(U;x)(\mathrm{i}\xi)$$

in (4.17) satisfies conditions (3.22). More explicitly one has

$$A_2(U;x) := \begin{pmatrix} a_2(U;x) & b_2(U;x) \\ \overline{b_2(U;x)} & a_2(U;x) \end{pmatrix}, \quad A_1(U;x) := \begin{pmatrix} a_1(U;x) & 0 \\ 0 & \overline{a_1(U;x)} \end{pmatrix}, \tag{4.18}$$

with $a_2, a_1, b_2 \in \mathcal{F}_{K,0}[r]$ and $a_2 \in \mathbb{R}$.

Proof. Recalling the notation introduced in (4.5) we shall write

$$\partial_{\partial_x^i u} f := \partial_{z^+_{i+1}} f_1, \quad \partial_{\overline{\partial_x^i u}} f := \partial_{z^-_{i+1}} f_1, \quad i = 0,1,2, \tag{4.19}$$

when restricted to the real subspace $\mathcal{U}$ (see (1.13)). Using conditions (4.2), (4.3) and (4.4) one has that

$$\begin{aligned} \begin{pmatrix} f(u,u_x,u_{xx}) \\ \overline{f(u,u_x,u_{xx})} \end{pmatrix} &= \begin{pmatrix} f_1(U,U_x,U_{xx}) \\ f_2(U,U_x,U_{xx}) \end{pmatrix} \\ &= \mathrm{Op}^{\mathcal{B}}\left[\begin{pmatrix} \partial_{u_{xx}} f & \partial_{\bar{u}_{xx}} f \\ \overline{\partial_{\bar{u}_{xx}} f} & \overline{\partial_{u_{xx}} f} \end{pmatrix}(\mathrm{i}\xi)^2\right] U + \mathrm{Op}^{\mathcal{B}}\left[\begin{pmatrix} \partial_{u_x} f & \partial_{\bar{u}_x} f \\ \overline{\partial_{\bar{u}_x} f} & \overline{\partial_{u_x} f} \end{pmatrix}(\mathrm{i}\xi)\right] U + R(U)[U] \end{aligned} \tag{4.20}$$

where $R(U)$ belongs to $\mathcal{R}^0_{K,0}[r]$. By Hypothesis 1.1 we have that

$$\begin{aligned} \partial_{u_{xx}} f &= -\partial_{u_x \bar{u}_x} F, \\ \partial_{\bar{u}_{xx}} f &= -\partial_{\bar{u}_x \bar{u}_x} F, \\ \partial_{u_x} f &= -\frac{\mathrm{d}}{\mathrm{d}x}\left[\partial_{u_x \bar{u}_x} F\right] - \partial_{u \bar{u}_x} F + \partial_{u_x \bar{u}} F, \\ \partial_{\bar{u}_x} f &= -\frac{\mathrm{d}}{\mathrm{d}x}\left[\partial_{\bar{u}_x \bar{u}_x} F\right]. \end{aligned} \tag{4.21}$$

We now pass to the Weyl quantization in the following way. Set

$$c(x,\xi) = \partial_{u_{xx}} f(x)(\mathrm{i}\xi)^2 + \partial_{u_x} f(x)(\mathrm{i}\xi).$$

Passing to the Fourier side we have that

$$\widehat{c}(j,\xi - \frac{j}{2}) = (\widehat{\partial_{u_{xx}} f})(j)(\mathrm{i}\xi)^2 + \Big[(\widehat{\partial_{u_x} f})(j) - (\mathrm{i}j)(\widehat{\partial_{u_{xx}} f})(j)\Big](\mathrm{i}\xi) + \Big[\frac{(\mathrm{i}j)^2}{4}(\widehat{\partial_{u_{xx}} f})(j) - \frac{(\mathrm{i}j)}{2}(\widehat{\partial_{u_x} f})(j)\Big],$$

therefore by using formula (3.10) we have that $\mathrm{Op}^{\mathcal{B}}(c(x,\xi)) = \mathrm{Op}^{\mathcal{B}W}(a(x,\xi))$, where

$$a(x,\xi) = \partial_{u_{xx}} f(x)(\mathrm{i}\xi)^2 + [\partial_{u_x} f(x) - \frac{\mathrm{d}}{\mathrm{d}x}(\partial_{u_{xx}} f)](\mathrm{i}\xi) + \frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\partial_{u_{xx}} f) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(\partial_{u_x} f).$$

Using the relations in (4.21) we obtain a matrix A as in (4.18), and in particular we have

$$a_2(U;x) = -\partial_{u_x\bar{u}_x} F, \quad a_1(U;x) = -\partial_{u\bar{u}_x} F + \partial_{u_x\bar{u}} F, \quad b_2(U;x) = -\partial_{\bar{u}_x\bar{u}_x} F. \tag{4.22}$$

Since F is real then a_2 is real, while a_1 is purely imaginary. This implies conditions (3.22). □

Lemma 4.3 (**Parity preserving structure**). *Assume that f in (1.1) satisfies Hypothesis 1.2. Consider the matrix $A(U;x,\xi)$ in (4.16) given by Proposition 4.1. One has that $A(U;x,\xi)$ has the form (4.17) where*

$$\begin{aligned} A_2(U;x) &:= \begin{pmatrix} a_2(U;x) & b_2(U;x) \\ \overline{b_2(U;x)} & \overline{a_2(U;x)} \end{pmatrix}, \quad A_1(U;x) := \begin{pmatrix} a_1(U;x) & b_1(U;x) \\ \overline{b_1(U;x)} & \overline{a_1(U;x)} \end{pmatrix}, \\ A_0(U;x) &:= \begin{pmatrix} a_0(U;x) & b_0(U;x) \\ \overline{b_0(U;x)} & \overline{a_0(U;x)} \end{pmatrix}, \end{aligned} \tag{4.23}$$

with $a_2, b_2, a_1, b_1, a_0, b_0 \in \mathcal{F}_{K,0}[r]$ such that, for U even in x, the following holds:

$$a_2(U;x) = a_2(U;-x), \quad b_2(U;x) = b_2(U;-x), \tag{4.24a}$$

$$a_1(U;x) = -a_1(U;-x), \quad b_1(U;x) = -b_1(U;-x), \tag{4.24b}$$

$$a_0(U;x) = a_0(U;-x), \quad b_0(U;x) = b_0(U;-x), \quad U \in \mathbf{H}^s_e, \tag{4.24c}$$

and

$$a_2(U;x) \in \mathbb{R}. \tag{4.25}$$

The matrix $R(U)$ in (4.15) is parity preserving according to Definition 2.4.

Proof. Using the same notation introduced in the proof of Lemma 4.2 (recall (4.4)) we have that formula (4.20) holds. Under the Hypothesis 1.2 one has that the functions $\partial_u f, \partial_{\bar{u}} f, \partial_{u_{xx}} f, \partial_{\bar{u}_{xx}} f$ are *even* in x while $\partial_{u_x} f, \partial_{\bar{u}_x} f$ are *odd* in x. Passing to the Weyl quantization by formula (3.10) we get

$$\begin{aligned} a_2(U;x) &= \partial_{u_{xx}} f, & b_2(U;x) &= \partial_{\bar{u}_{xx}} f, \\ a_1(U;x) &= \partial_{u_x} f - \partial_x(\partial_{u_{xx}} f), & b_1(U;x) &= \partial_{\bar{u}_x} f - \partial_x(\partial_{\bar{u}_{xx}} f), \\ a_0(U;x) &= \partial_u f + \frac{1}{4}\partial_x^2(\partial_{u_{xx}} f) - \frac{1}{2}\partial_x(\partial_{u_x} f), & b_0(U;x) &= \partial_{\bar{u}} f + \frac{1}{4}\partial_x^2(\partial_{\bar{u}_{xx}} f) - \frac{1}{2}\partial_x(\partial_{\bar{u}_x} f) \end{aligned} \tag{4.26}$$

which imply conditions (4.24), while (4.25) is implied by item 2 of Hypothesis 1.2. The term R is parity preserving by difference. □

Lemma 4.4 (**Global ellipticity**). *Assume that f in (1.1) satisfies Hypothesis 1.1 (respectively Hypothesis 1.2). If f satisfies also Hypothesis 1.3 then the matrix $A_2(U;x)$ in (4.18) (resp. in (4.23)) is such that*

$$\begin{aligned} &1 + a_2(U;x) \geq \mathtt{c}_1 \\ &(1 + a_2(U;x))^2 - |b_2(U;x)|^2 \geq \mathtt{c}_2 > 0, \end{aligned} \tag{4.27}$$

where $\mathtt{c}_1$ and $\mathtt{c}_2$ are the constants given in (1.8) and (1.9).

Proof. It follows from (4.22) in the case of Hypothesis 1.1 and from (4.26) in the case of Hypothesis 1.2. □

Lemma 4.5 (**Lipschitz estimates**). *Fix $r > 0$, $K > 0$ and consider the matrices A and R given in Proposition 4.1. Then there exists $s_0 > 0$ such that for any $s \geq s_0$ the following holds true. For any $U, V \in C^K_{*\mathbb{R}}(I;\mathbf{H}^s) \cap B^K_{s_0}(I,r)$ there are constants $C_1 > 0$ and $C_2 > 0$, depending on s, $\|U\|_{K,s_0}$ and $\|V\|_{K,s_0}$, such that for any $H \in C^K_{*\mathbb{R}}(I;\mathbf{H}^s)$ one has*

$$\|\mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))[H]-\mathrm{Op}^{\mathcal{B}W}(A(V;x,\xi))[H]\|_{K,s-2}\leq C_1\|H\|_{K,s}\|U-V\|_{K,s_0} \tag{4.28}$$

$$\|R(U)[U]-R(V)[V]\|_{K,s+\rho}\leq C_2(\|U\|_{K,s}+\|V\|_{K,s})\|U-V\|_{K,s}, \tag{4.29}$$

for any $\rho\geq 0$.

Proof. We prove bound (4.28) on each component of the matrix A in (4.16) in the case that f satisfies Hypothesis 1.2. The Hamiltonian case of Hypothesis 1.1 follows by using the same arguments. From the proof of Lemma 4.3 we know that the symbol $a(U;x,\xi)$ of the matrix in (4.16) is such that $a(U;x,\xi)=a_2(U;x)(\mathrm{i}\xi)^2+a_1(U;x)(\mathrm{i}\xi)+a_0(U;x)$ where $a_i(U;x)$ for $i=0,1,2$ are given in (4.26).

By Remark 3.3 there exists $s_0>0$ such that for any $s\geq s_0$ one has

$$\|\mathrm{Op}^{\mathcal{B}W}\big((a_2(U;x)-a_2(V;x))(\mathrm{i}\xi)^2\big)h\|_{K,s-2}\leq C\sup_{\xi}\langle\xi\rangle^{-2}\|(a_2(U;x)-a_2(V;x))(\mathrm{i}\xi)^2\big)\|_{K,s_0}\|h\|_{K,s}. \tag{4.30}$$

with C depending on s,s_0. Let $U,V\in C^K_{*\mathbb{R}}(I;\mathbf{H}^s)\cap B^K_{s_0+2}(I,r)$, by Lagrange theorem, recalling the relations in (4.4), (4.5) and (4.19), one has that

$$\begin{aligned}\big(a_2(U;x)-a_2(V;x)\big)(\mathrm{i}\xi)^2&=\big((\partial_{u_{xx}}f_1)(U,U_x,U_{xx})-(\partial_{u_{xx}}f_1)(V,V_x,V_{xx})\big)(\mathrm{i}\xi)^2\\&=(\partial_U\partial_{u_{xx}}f_1)(W^{(0)},U_x,U_{xx})(U-V)(\mathrm{i}\xi)^2+\\&+(\partial_{U_x}\partial_{u_{xx}}f_1)(V,W^{(1)},U_{xx})(U_x-V_x)(\mathrm{i}\xi)^2+\\&+(\partial_{U_{xx}}\partial_{u_{xx}}f_1)(V,V_x,W^{(2)})(U_{xx}-V_{xx})(\mathrm{i}\xi)^2\end{aligned} \tag{4.31}$$

where $W^{(j)}=\partial_x^jV+t_j(\partial_x^jU-\partial_x^jV)$, for some $t_j\in[0,1]$ and $j=0,1,2$. Hence, for instance, the first summand of (4.31) can be estimated as follows

$$\begin{aligned}&\sup_{\xi}\langle\xi\rangle^{-2}\|(\partial_U\partial_{u_{xx}}f_1)(W^{(0)},U_x,U_{xx})(U-V)(\mathrm{i}\xi)^2\|_{K,s_0}\\&\quad\leq C_1\|U-V\|_{K,s_0}\sup_{U,V\in B^K_{s_0+2}(I,r)}\|(\partial_U\partial_{u_{xx}}f_1)(W^{(0)},U_x,U_{xx})\|_{K,s_0}\\&\quad\leq C_2\|U-V\|_{K,s_0},\end{aligned} \tag{4.32}$$

where C_1 depends on s_0 and C_2 depends only on s_0 and $\|U\|_{K,s_0+2}$, $\|V\|_{K,s_0+2}$ and where we have used a Moser type estimates on composition operators on H^s since f_1 belongs to $C^\infty(\mathbb{C}^6;\mathbb{C})$. We refer the reader to Lemma $A.50$ of [17] for a complete statement (see also [4], [22]). The other terms in the r.h.s. of (4.31) can be treated in the same way. Hence from (4.30) and the discussion above we have obtained

$$\|\mathrm{Op}^{\mathcal{B}W}\big((a_2(U;x)-a_2(V;x))(\mathrm{i}\xi)^2\big)h\|_{K,s-2}\leq C\|U-V\|_{K,s_0+2}\|h\|_{K,s}, \tag{4.33}$$

with C depending on s and $\|U\|_{K,s_0+2}$, $\|V\|_{K,s_0+2}$. One has to argue exactly as done above for the lower order terms $a_1(U;x)(\mathrm{i}\xi)$ and $a_0(U;x)$ of $a(U;x,\xi)$. In the same way one is able to prove the estimate

$$\|\mathrm{Op}^{\mathcal{B}W}\big((b(U;x,\xi)-b(V;x,\xi))\big)\bar{h}\|_{K,s-2}\leq C\|U-V\|_{K,s_0+2}\|\bar{h}\|_{K,s}. \tag{4.34}$$

Thus the (4.28) is proved renaming s_0 as s_0+2.

In order to prove (4.29) we show that the operator $\mathrm{d}_U(R(U)U)[\cdot]$ belongs to the class $\mathcal{R}^{-\rho}_{K,K'}[r]\otimes\mathcal{M}_2(\mathbb{C})$ for any $\rho\geq 0$ (where $d_U(R(U)U)[\cdot]$ denotes the differential of $R(U)[U]$ w.r.t. the variable U). We recall that the operator R in (4.15) is of the form

$$R(U)[\cdot]:=\begin{pmatrix}Q(U)[\cdot]\\ \overline{Q(U)[\cdot]}\end{pmatrix},$$

where $Q(U)[\cdot]$ is the 1×2 matrix of smoothing operators in (4.6) with f given in (1.1). We claim that $\mathrm{d}_U(Q(U)U)[\cdot]$ is 1×2 matrix of smoothing operators in $\mathcal{R}^{-\rho}_{K,0}[r]$. By Lemma 4.1 we know that $Q(U)[\cdot]=R_0(U)+\sum_{j=1}^{3}R_j(U)$, where R_0 is 1×2 matrix of smoothing operators coming from the Bony paralinearization formula (see (4.8)), while R_j, for $j=1,2,3$, are the 1×2 matrices of smoothing operators in (4.13) and (4.14).

One can prove the claim for the terms R_j, $j=1,2,3$, by arguing as done in the proof of (4.28). Indeed we know the explicit paradifferential structure of these remainders. For instance, by (4.10), (4.11), (4.12) and (4.13) we have that

$$R_1(U)[\cdot] := \mathrm{op}\Big(k(x,\xi)\Big)[\cdot], \tag{4.35}$$

where $k(x,\xi)=\sum_{j\in\mathbb{Z}}\hat{k}(j,\xi)e^{\mathrm{i}jx}$ and

$$k(j,\xi)=\Big(\chi\left(n\langle\xi-n/2\rangle^{-1}\right)-\chi(n\langle\xi\rangle^{-1})\Big)\widehat{D_U f}(n)$$

(see formula (4.12)). The remainders R_2, R_3 have similar expressions. We reduced to prove the claim for the term R_0. Recalling (4.9) we set

$$c(U;x,\xi):=c_U(U;x,\xi)+c_{U_x}(U;x,\xi)+c_{U_{xx}}(U;x,\xi).$$

Using this notation, formula (4.8) reads

$$f(u,u_x,u_{xx})=f_1(U,U_x,U_{xx})=\mathrm{Op}^{\mathcal{B}}(c(U;x,\xi))U+R_0(U)U. \tag{4.36}$$

Differentiating (4.36) we get

$$\mathrm{d}_U(f_1(U,U_x,U_{xx}))[H]=\mathrm{Op}^{\mathcal{B}}(c(U;x,\xi))[H]+\mathrm{Op}^{\mathcal{B}}(\partial_U c(U;x,\xi)\cdot H)[U]+\mathrm{d}_U(R_0(U)[U])[H]. \tag{4.37}$$

The l.h.s. of (4.37) is nothing but

$$\partial_U f_1(U,U_x,U_{xx})\cdot H+\partial_{U_x}f_1(U,U_x,U_{xx})\cdot H_x+\partial_{U_{xx}}f_1(U,U_x,U_{xx})\cdot H_{xx}=:G(U,H).$$

By applying the Bony paralinearization formula to $G(U,H)$ (as a function of the six variables U,U_x,U_{xx}, H,H_x,H_{xx}) we get

$$\begin{aligned}G(U,H)=&\mathrm{Op}^{\mathcal{B}}(\partial_U G(U,H))[U]+\mathrm{Op}^{\mathcal{B}}(\partial_{U_x}G(U,H))[U_x]+\mathrm{Op}^{\mathcal{B}}(\partial_{U_{xx}}G(U,H))[U_{xx}]\\&+\mathrm{Op}^{\mathcal{B}}(\partial_H G(U,H))[H]+\mathrm{Op}^{\mathcal{B}}(\partial_{H_x}G(U,H))[H_x]+\mathrm{Op}^{\mathcal{B}}(\partial_{H_{xx}}G(U,H))[H_{xx}]+R_4(U)[H],\end{aligned} \tag{4.38}$$

where $R_4(U)[\cdot]$ satisfies estimates (3.17) for any $\rho\geq 0$. By (4.9) and (4.38) we have that (4.37) reads

$$\mathrm{d}_U(R_0(U)U)[H]=R_4(U)[H]. \tag{4.39}$$

Therefore $\mathrm{d}_U(R_0(U)U)[\cdot]$ is a 1×2 matrix of operators in the class $R^{-\rho}_{K,0}[r]$ for any $\rho\geq 0$. □

5. Regularization

We consider the system

$$\begin{aligned}\partial_t V=\mathrm{i}E\Big[\Lambda V+\mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))[V]+R_1^{(0)}(U)[V]+R_2^{(0)}(U)[U]\Big],\\ U\in B^K_{s_0}(I,r)\cap C^K_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T},\mathbb{C}^2)),\end{aligned} \tag{5.1}$$

for some s_0 large, $s\geq s_0$ and where Λ is defined in (1.16). The operators $R_1^{(0)}(U)$ and $R_2^{(0)}(U)$ are in the class $\mathcal{R}^{-\rho}_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$ for some $\rho\geq 0$ and they have the reality preserving form (3.24). The matrix $A(U;x,\xi)$ satisfies the following.

Constraint 5.1. *The matrix $A(U;x,\xi)$ belongs to $\Gamma^2_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$ and has the following properties:*

- *$A(U;x,\xi)$ is* reality preserving, *i.e. has the form (3.21);*

- *the components of $A(U;x,\xi)$ have the form*

$$\begin{aligned} a(U;x,\xi) &= a_2(U;x)(\mathrm{i}\xi)^2 + a_1(U;x)(\mathrm{i}\xi),\\ b(U;x,\xi) &= b_2(U;x)(\mathrm{i}\xi)^2 + b_1(U;x)(\mathrm{i}\xi), \end{aligned} \tag{5.2}$$

  for some $a_i(U;x), b_i(U;x)$ belonging to $\mathcal{F}_{K,0}[r]$ for $i=1,2$.

In addition to Constraint 5.1 we assume that the matrix A satisfies one the following two Hypotheses:

Hypothesis 5.1 *(***Self-adjoint***).* The operator $\mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))$ is self-adjoint according to Definition 2.3, i.e. the matrix $A(U;x,\xi)$ satisfies conditions (3.22).

Hypothesis 5.2 *(***Parity preserving***).* The operator $\mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))$ is parity preserving according to Definition 2.4, i.e. the matrix $A(U;x,\xi)$ satisfies the conditions

$$A(U;x,\xi) = A(U;-x,-\xi), \qquad a_2(U;x)\in\mathbb{R}. \tag{5.3}$$

The function P in (1.2) is such that $\hat{p}(j)=\hat{p}(-j)$ for $j\in\mathbb{Z}$.

Finally we need the following *ellipticity condition*.

Hypothesis 5.3 *(***Ellipticity***).* There exist $\mathtt{c}_1,\mathtt{c}_2>0$ such that components of the matrix $A(U;x,\xi)$ satisfy the condition

$$\begin{aligned} &1+a_2(U;x)\geq \mathtt{c}_1,\\ &(1+a_2(U;x))^2 - |b_2(U;x)|^2 \geq \mathtt{c}_2 > 0, \end{aligned} \tag{5.4}$$

for any $U\in B^K_{s_0}(I,r)\cap C^K_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T},\mathbb{C}^2))$.

The goal of this section is to transform the linear paradifferential system (5.1) into a constant coefficient one up to bounded remainder.

The following result is the core of our analysis.

Theorem 5.1 *(***Regularization***). Fix $K\in\mathbb{N}$ with $K\geq 4$, $r>0$. Consider the system (5.1). There exists $s_0>0$ such that for any $s\geq s_0$ the following holds. Fix U in $B^K_{s_0}(I,r)\cap C^K_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T},\mathbb{C}^2))$ (resp. $U\in B^K_{s_0}(I,r)\cap C^K_{*\mathbb{R}}(I,\mathbf{H}^s_e(\mathbb{T},\mathbb{C}^2))$) and assume that the system (5.1) has the following structure:*

- *the operators $R_1^{(0)}$, $R_2^{(0)}$ belong to the class $\mathcal{R}^{-\rho}_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$;*
- *the matrix $A(U;x,\xi)$ satisfies Constraint 5.1,*
- *the matrix $A(U;x,\xi)$ satisfies Hypothesis 5.1 (resp. together with P satisfy Hypothesis 5.2)*
- *the matrix $A(U;x,\xi)$ satisfies Hypothesis 5.3.*

Then there exists an invertible map (resp. an invertible and parity preserving map)

$$\Phi=\Phi(U): C^{K-4}_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T},\mathbb{C}^2)) \to C^{K-4}_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T},\mathbb{C}^2)),$$

with

$$\|(\Phi(U))^{\pm 1}V\|_{K-4,s} \leq \|V\|_{K-4,s}(1+C\|U\|_{K,s_0}), \tag{5.5}$$

for a constant $C>0$ depending on s, $\|U\|_{K,s_0}$ and $\|P\|_{C^1}$ such that the following holds. There exist operators $R_1(U)$, $R_2(U)$ in $\mathcal{R}^0_{K,4}[r]\otimes\mathcal{M}_2(\mathbb{C})$, and a diagonal matrix $L(U)$ in $\Gamma^2_{K,4}[r]\otimes\mathcal{M}_2(\mathbb{C})$ of the form (3.21) satisfying condition (3.22) and independent of $x\in\mathbb{T}$, such that by setting $W=\Phi(U)V$ the system (5.1) reads

$$\partial_t W = \mathrm{i}E\Big[\Lambda W + \mathrm{Op}^{\mathcal{B}W}(L(U;\xi))[W] + R_1(U)[W] + R_2(U)[U]\Big]. \tag{5.6}$$

Remark 5.1. Note that, under the Hypothesis 5.2, if the term $R_1^{(0)}(U)[V] + R_2^{(0)}(U)[U]$ in (5.1) is *parity preserving*, according to Definition 2.4, then the flow of the system (5.1) preserves the subspace of even functions. Since the map $\Phi(U)$ in Theorem 5.1 is *parity preserving*, then Lemma 2.3 implies that also the flow of the system (5.6) preserves the same subspace.

The proof of Theorem 5.1 is divided into four steps which are performed in the remaining part of the section. We first explain our strategy and set some notation. We consider the system (5.1)

$$V_t = \mathcal{L}^{(0)}(U)[V] := \mathrm{i}E\Big[\Lambda V + \mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))[V] + R_1^{(0)}(U)[V] + R_2^{(0)}(U)[U]\Big]. \tag{5.7}$$

The idea is to construct several maps

$$\Phi_i[\cdot] := \Phi_i(U)[\cdot] : C_{*\mathbb{R}}^{K-(i-1)}(I, \mathbf{H}^s(\mathbb{T})) \to C_{*\mathbb{R}}^{K-(i-1)}(I, \mathbf{H}^s(\mathbb{T})),$$

for $i = 1, \dots, 4$ which conjugate the system $\mathcal{L}^{(i)}(U)$ to $\mathcal{L}^{(i+1)}(U)$, with $\mathcal{L}^{(0)}(U)$ in (5.7) and

$$\mathcal{L}^{(i)}(U)[\cdot] := \mathrm{i}E\Big[\Lambda + \mathrm{Op}^{\mathcal{B}W}(L^{(i)}(U;\xi))[\cdot] + \mathrm{Op}^{\mathcal{B}W}(A^{(i)}(U;x,\xi))[\cdot] + R_1^{(i)}[\cdot] + R_2^{(i)}(U)[U]\Big], \tag{5.8}$$

where $R_1^{(i)}$ and $R_2^{(i)}$ belong to $\mathcal{R}^0_{K,i}[r] \otimes \mathcal{M}_2(\mathbb{C})$, $L^{(i)}$ belong to $\Gamma^2_{K,i}[r] \otimes \mathcal{M}_2(\mathbb{C})$ and moreover they are diagonal, self-adjoint and independent of $x \in \mathbb{T}$ and finally $A^{(i)}$ are in $\Gamma^2_{K,i}[r] \otimes \mathcal{M}_2(\mathbb{C})$. As we will see, the idea is to look for Φ_i in such a way $A^{(i+1)}$ is actually a matrix with symbols of order less or equal than the order of $A^{(i)}$.

We now prove a lemma in which we study the conjugate of the convolution operator.

Lemma 5.1. *Let Q_1, Q_2 operators in the class $\mathcal{R}^0_{K,K'}[r] \otimes \mathcal{M}_2(\mathbb{C})$ and $P : \mathbb{T} \to \mathbb{R}$ a continuous function. Consider the operator $\mathfrak{P}$ defined in (1.19). Then there exists R belonging to $\mathcal{R}^0_{K,K'}[r] \otimes \mathcal{M}_2(\mathbb{C})$ such that*

$$(\mathbb{1} + Q_1(U)) \circ \mathfrak{P} \circ (\mathbb{1} + Q_2(U))[\cdot] = \mathfrak{P}[\cdot] + R(U)[\cdot]. \tag{5.9}$$

Moreover if P is even in x and the operators $Q_1(U)$ and $Q_2(U)$ are parity-preserving then the operator $R(U)$ is parity preserving according to Definition 2.4.

Proof. By linearity it is enough to show that the terms

$$Q_1(U) \circ \mathfrak{P} \circ (\mathbb{1} + Q_2(U))[h], \quad (\mathbb{1} + Q_1(U)) \circ \mathfrak{P} \circ Q_2(U)[h], \quad Q_1(U) \circ \mathfrak{P} \circ Q_2(U)[h]$$

belong to $\mathcal{R}^0_{K,K'}[r] \otimes \mathcal{M}_2(\mathbb{C})$. Note that, for any $0 \le k \le K - K'$,

$$\|\partial_t^k (P * h)\|_{H^{s-2k}} \le C \|\partial_t^k h\|_{H^{s-2k}}, \tag{5.10}$$

for some $C > 0$ depending only on $\|P\|_{L^\infty}$. The (5.10) and the estimate (3.17) on Q_1 and Q_2 imply the thesis. If P is even in x then the convolution operator with kernel P is a parity preserving operator according to Definition 2.4. Therefore if in addiction $Q_1(U)$ and $Q_2(U)$ are parity preserving so is $R(U)$. $\square$

5.1. Diagonalization of the second order operator

Consider the system (5.1) and assume the Hypothesis of Theorem 5.1. The matrix $A(U;x,\xi)$ satisfies conditions (5.2), therefore it can be written as

$$A(U;x,\xi) := A_2(U;x)(\mathrm{i}\xi)^2 + A_1(U;x)(\mathrm{i}\xi), \tag{5.11}$$

with $A_i(U;x)$ belonging to $\mathcal{F}_{K,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ and satisfying either Hypothesis 5.1 or Hypothesis 5.2. In this Section, by exploiting the structure of the matrix $A_2(U;x)$, we show that it is possible to diagonalize the matrix $E(\mathbb{1} + A_2)$ through a change of coordinates which is a multiplication operator. We have the following lemma.

Lemma 5.2. *Under the Hypotheses of Theorem 5.1 there exists $s_0 > 0$ such that for any $s \ge s_0$ there exists an invertible map (resp. an invertible and parity preserving map)*

$$\Phi_1 = \Phi_1(U) : C^K_{*\mathbb{R}}(I, \mathbf{H}^s) \to C^K_{*\mathbb{R}}(I, \mathbf{H}^s),$$

with

$$\|(\Phi_1(U))^{\pm 1} V\|_{K,s} \le \|V\|_{K,s}(1 + C\|U\|_{K,s_0}) \tag{5.12}$$

where C depends only on s and $\|U\|_{K,s_0}$ such that the following holds. There exists a matrix $A^{(1)}(U; x, \xi)$ satisfying Constraint 5.1 and Hypothesis 5.1 (resp. Hypothesis 5.2) of the form

$$\begin{aligned} A^{(1)}(U; x, \xi) &:= A_2^{(1)}(U; x)(\mathrm{i}\xi)^2 + A_1^{(1)}(U; x)(\mathrm{i}\xi), \\ A_2^{(1)}(U; x) &:= \begin{pmatrix} a_2^{(1)}(U; x) & 0 \\ 0 & a_2^{(1)}(U; x) \end{pmatrix} \in \mathcal{F}_{K,1}[r] \otimes \mathcal{M}_2(\mathbb{C}), \\ A_1^{(1)}(U; x) &:= \begin{pmatrix} a_1^{(1)}(U; x) & b_1^{(1)}(U; x) \\ \overline{b_1^{(1)}(U; x,)} & \overline{a_1^{(1)}(U; x)} \end{pmatrix} \in \mathcal{F}_{K,1}[r] \otimes \mathcal{M}_2(\mathbb{C}) \end{aligned} \tag{5.13}$$

and operators $R_1^{(1)}(U)$, $R_2^{(1)}(U)$ in $\mathcal{R}^0_{K,1}[r] \otimes \mathcal{M}_2(\mathbb{C})$ such that by setting $V_1 = \Phi(U)V$ the system (5.1) reads

$$\partial_t V_1 = \mathrm{i}E\Big[\Lambda V_1 + \mathrm{Op}^{\mathcal{B}W}(A^{(1)}(U; x, \xi))[V_1] + R_1^{(1)}(U)[V_1] + R_2^{(1)}(U)[U]\Big]. \tag{5.14}$$

Moreover there exists a constant $\mathtt{k} > 0$ such that

$$1 + a_2^{(1)}(U; x) \ge \mathtt{k}. \tag{5.15}$$

Proof. Let us consider a symbol $z(U; x)$ in the class $\mathcal{F}_{K,0}[r]$ and set

$$Z(U; x) := \begin{pmatrix} 0 & z(U; x) \\ \overline{z(U; x)} & 0 \end{pmatrix} \in \mathcal{F}_{K,0}[r] \otimes \mathcal{M}_2(\mathbb{C}). \tag{5.16}$$

Let $\Phi_1^\tau(U)[\cdot]$ the solution at time $\tau \in [0, 1]$ of the system

$$\begin{cases} \partial_\tau \Phi_1^\tau(U)[\cdot] = \mathrm{Op}^{\mathcal{B}W}(Z(U; x))\Phi_1^\tau(U)[\cdot], \\ \Phi_1^0(U)[\cdot] = \mathbb{1}[\cdot]. \end{cases} \tag{5.17}$$

Since $\mathrm{Op}^{\mathcal{B}W}(Z(U; x))$ is a bounded operator on $\mathbf{H}^s$, by standard theory of Banach space ODE we have that the flow Φ_1^τ is well defined, moreover by Proposition 3.2 one gets

$$\begin{aligned} \partial_\tau \|\Phi_1^\tau(U)V\|^2_{\mathbf{H}^s} &\le \|\Phi_1^\tau(U)V\|_{\mathbf{H}^s} \|\mathrm{Op}^{\mathcal{B}W}(Z(U; x))\Phi_1^\tau(U)V\|_{\mathbf{H}^s} \\ &\le \|\Phi_1^\tau(U)V\|^2_{\mathbf{H}^s} C\|U\|_{\mathbf{H}^{s_0}}, \end{aligned} \tag{5.18}$$

hence one obtains

$$\|\Phi_1^\tau(U)[V]\|_{\mathbf{H}^s} \le \|V\|_{\mathbf{H}^s}(1 + C\|U\|_{\mathbf{H}^{s_0}}), \tag{5.19}$$

where $C > 0$ depends only on $\|U\|_{\mathbf{H}^{s_0}}$. The latter estimate implies (5.12) for $K = 0$. By differentiating in t the equation (5.17) we note that

$$\partial_\tau \partial_t \Phi_1^\tau(U)[\cdot] = \mathrm{Op}^{\mathcal{B}W}(Z(U; x))\partial_t \Phi_1^\tau(U)[\cdot] + \mathrm{Op}^{\mathcal{B}W}(\partial_t Z(U; x))\Phi_1^\tau(U)[\cdot]. \tag{5.20}$$

Now note that, since Z belongs to the class $\mathcal{F}_{K,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$, one has that $\partial_t Z$ is in $\mathcal{F}_{K,1}[r] \otimes \mathcal{M}_2(\mathbb{C})$. By performing an energy type estimate as in (5.18) one obtains

$$\|\Phi_1^\tau(U)[V]\|_{C^1\mathbf{H}^s} \le \|V\|_{C^1\mathbf{H}^s}(1 + C\|U\|_{C^1\mathbf{H}^{s_0}}),$$

which implies (5.12) with $K = 1$. Iterating K times the reasoning above one gets the bound (5.12). By using Corollary 3.1 one gets that

$$\Phi_1^\tau(U)[\cdot] = \exp\{\tau \mathrm{Op}^{\mathcal{B}W}(Z(U; x))\}[\cdot] = \mathrm{Op}^{\mathcal{B}W}(\exp\{\tau Z(U; x)\})[\cdot] + Q_1^\tau(U)[\cdot], \tag{5.21}$$

with Q_1^τ belonging to $\mathcal{R}^{-\rho}_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$ for any $\rho>0$ and any $\tau\in[0,1]$. We now set $\Phi_1(U)[\cdot]:=\Phi_1^\tau(U)[\cdot]_{|\tau=1}$. In particular we have

$$\begin{aligned}&\Phi_1(U)[\cdot]=\mathrm{Op}^{\mathcal{B}W}(C(U;x))[\cdot]+Q_1^1(U)[\cdot]\\ &C(U;x):=\exp\{Z(U;x)\}:=\begin{pmatrix}c_1(U;x) & c_2(U;x)\\ \overline{c_2(U;x)} & c_1(U;x)\end{pmatrix},\quad C(U;x)-\mathbb{1}\in\mathcal{F}_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C}),\end{aligned}\tag{5.22}$$

where

$$c_1(U;x):=\cosh(|z(U;x)|),\qquad c_2(U;x):=\frac{z(U;x)}{|z(U;x)|}\sinh(|z(U;x)|).\tag{5.23}$$

Note that the function $c_2(U;x)$ above is not singular indeed

$$\begin{aligned}c_2(U;x)&=\frac{z(U;x)}{|z(U;x)|}\sinh(|z(U;x)|)=\frac{z(U;x)}{|z(U;x)|}\sum_{k=0}^{\infty}\frac{(|z|(U;x))^{2k+1}}{(2k+1)!}\\ &=z(U;x)\sum_{k=0}^{\infty}\frac{\big(z(U;x)\overline{z(U;x)}\big)^k}{(2k+1)!}.\end{aligned}$$

We note moreover that for any $x\in\mathbb{T}$ one has $\det(C(U;x))=1$, hence its inverse $C^{-1}(U;x)$ is well defined. In particular, by Propositions 3.3 and 3.4, we note that

$$\mathrm{Op}^{\mathcal{B}W}(C^{-1}(U;x))\circ\Phi_1=\mathbb{1}+\tilde{Q}(U),\qquad \tilde{Q}\in\mathcal{R}^{-\rho}_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C}),\tag{5.24}$$

for any $\rho>0$, since the expansion of $(C^{-1}(U;x)\sharp C(U;x))_\rho$ (see formula (3.30)) is equal to $C^{-1}(U;x)C(U;x)$ for any ρ. This implies that

$$(\Phi_1(U))^{-1}[\cdot]=\mathrm{Op}^{\mathcal{B}W}(C^{-1}(U;x))[\cdot]+Q_2(U)[\cdot],\tag{5.25}$$

for some $Q_2(U)$ in the class $\mathcal{R}^{-\rho}_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$ for any $\rho>0$. By setting $V_1:=\Phi_1(U)[V]$ the system (5.1) in the new coordinates reads

$$\begin{aligned}(V_1)_t&=\Phi_1(U)\Big(\mathrm{i}E(\Lambda+\mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi)))\Phi_1^{-1}(U)\Big)V_1+(\partial_t\Phi_1(U))\Phi_1^{-1}(U)V_1+\\ &\quad+\Phi_1(U)(\mathrm{i}E)R_1^{(0)}(U)\Phi_1^{-1}(U)[V_1]+\Phi_1(U)(\mathrm{i}E)R_2^{(0)}(U)[U]\\ &=\mathrm{i}\Phi_1(U)\Big[E\mathfrak{P}[\Phi_1^{-1}(U)[V_1]]\Big]+\mathrm{i}\Phi_1(U)E\mathrm{Op}^{\mathcal{B}W}\big((\mathbb{1}+A_2(U;x))(\mathrm{i}\xi)^2\big)\Phi_1^{-1}(U)[V_1]+\\ &\quad+\mathrm{i}\Phi_1(U)E\mathrm{Op}^{\mathcal{B}W}\big(A_1(U;x)(\mathrm{i}\xi)\big)\Phi_1^{-1}(U)[V_1]+(\partial_t\Phi_1)\Phi_1^{-1}(U)V_1+\\ &\quad+\Phi_1(U)(\mathrm{i}E)R_1^{(0)}(U)\Phi_1^{-1}(U)[V_1]+\Phi_1(U)(\mathrm{i}E)R_2^{(0)}(U)[U],\end{aligned}\tag{5.26}$$

where $\mathfrak{P}$ is defined in (1.19). We have that

$$\Phi_1(U)\circ E=E\circ\mathrm{Op}^{\mathcal{B}W}\begin{pmatrix}c_1(U;x) & -c_2(U;x)\\ -\overline{c_2(U;x)} & c_1(U;x)\end{pmatrix},$$

up to remainders in $\mathcal{R}^{-\rho}_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$, where $c_i(U;x)$, $i=1,2$, are defined in (5.23). Since the matrix $C(U;x)-\mathbb{1}\in\mathcal{F}_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$ (see (5.22)) then by Lemma 5.1 one has that

$$\Phi_1(U)\circ E\mathfrak{P}\circ\Phi_1^{-1}(U)[V_1]]=E\mathfrak{P}[V_1]+Q_3(U)[V_1],$$

where $Q_3(U)$ belongs to $\mathcal{R}^0_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$. The term $(\partial_t\Phi_1)$ is $\mathrm{Op}^{\mathcal{B}W}(\partial_t C(U;x))$ plus a remainder in the class $\mathcal{R}^0_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$. Note that, since $(C(U;x)-\mathbb{1})$ belongs to the class $\Gamma^0_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$, one has that $\partial_t C(U;x)$ is in $\Gamma^0_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$. Therefore, by the composition Propositions 3.3 and 3.4, Remark 3.5, and using the discussion above we have that, there exist operators $R_1^{(1)}$, $R_2^{(1)}$ belonging to $\mathcal{R}^0_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$ such that

$$
\begin{aligned}
(V_1)_t = {}& \mathrm{i}E\mathfrak{P}V_1 + \mathrm{i}\mathrm{Op}^{\mathcal{B}W}\big(C(U;x)E(\mathbb{1}+A_2(U;x))C^{-1}(U;x)(\mathrm{i}\xi)^2\big)\big)V_1 + \mathrm{i}E\mathrm{Op}^{\mathcal{B}W}(A_1^{(1)}(U;x)(\mathrm{i}\xi))V_1 + \\
& + \mathrm{i}E\Big(R_1^{(1)}(U)[V_1] + R_2^{(1)}(U)[U]\Big)
\end{aligned}
\tag{5.27}
$$

where

$$
\begin{aligned}
A_1^{(1)}(U;x) := {}& EC(U;x)E(\mathbb{1}+A_2(U;x))\partial_x C^{-1}(U;x) - (\partial_x(C)(U;x))E(\mathbb{1}+A_2(U;x))C^{-1}(U;x) \\
& + EC(U;x)A_1(U;x)C^{-1}(U;x),
\end{aligned}
\tag{5.28}
$$

with $A_1(U;x), A_2(U;x)$ defined in (5.11). Our aim is to find a symbol $z(U;x)$ such that the matrix of symbols $C(U;x)E(\mathbb{1}+A_2(U;x))C^{-1}(U;x)$ is diagonal. We reason as follows. One can note that the eigenvalues of $E(\mathbb{1}+A_2(U;x))$ are

$$
\lambda^{\pm} := \pm\sqrt{(1+a_2(U;x))^2 - |b_2(U;x)|^2}.
$$

We define the symbols

$$
\begin{aligned}
&\lambda_2^{(1)}(U;x) := \lambda^{+}, \\
&a_2^{(1)}(U;x) := \lambda_2^{(1)}(U;x) - 1 \in \mathcal{F}_{K,0}[r].
\end{aligned}
\tag{5.29}
$$

The symbol $\lambda_2^{(1)}(U;x)$ is well defined and satisfies (5.15) thanks to Hypothesis 5.3. The matrix of the normalized eigenvectors associated to the eigenvalues of $E(\mathbb{1}+A_2(U;x))$ is

$$
\begin{aligned}
S(U;x) &:= \begin{pmatrix} s_1(U;x) & s_2(U;x) \\ \overline{s_2(U;x)} & s_1(U;x) \end{pmatrix}, \\
s_1(U;x) &:= \frac{1+a_2(U;x)+\lambda_2^{(1)}(U;x)}{\sqrt{2\lambda_2^{(1)}(U;x)\big(1+a_2(U;x)+\lambda_2^{(1)}(U;x)\big)}}, \\
s_2(U;x) &:= \frac{-b_2(U;x)}{\sqrt{2\lambda_2^{(1)}(U;x)\big(1+a_2(U;x)+\lambda_2^{(1)}(U;x)\big)}}.
\end{aligned}
\tag{5.30}
$$

Note that $1+a_2(U;x)+\lambda_2^{(1)}(U;x) \geq \mathtt{c}_1 + \sqrt{\mathtt{c}_2} > 0$ by (5.4). Therefore one can check that $S(U;x) - \mathbb{1} \in \mathcal{F}_{K,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$. Therefore the matrix S is invertible and one has

$$
S^{-1}(U;x)\big[E(\mathbb{1}+A_2(U;x))\big]S(U;x) = E\begin{pmatrix} 1+a_2^{(1)}(U;x) & 0 \\ 0 & 1+a_2^{(1)}(U;x) \end{pmatrix}.
\tag{5.31}
$$

We choose $z(U;x)$ in such a way that $C^{-1}(U;x) := S(U;x)$. Therefore we have to solve the following equations

$$
\cosh(|z(U;x)|) = s_1(U;x), \qquad \frac{z(U;x)}{|z(U;x)|}\sinh(|z(U;x)|) = -s_2(U;x).
\tag{5.32}
$$

Concerning the first one we note that s_1 satisfies

$$
(s_1(U;x))^2 - 1 = \frac{|b_2(U;x)|^2}{2\lambda_2^{(1)}(U;x)(1+a_2(U;x)+\lambda_2^{(1)}(U;x))} \geq 0,
$$

indeed we remind that $1+a_2(U;x)+\lambda_2^{(1)}(U;x) \geq \mathtt{c}_1 + \sqrt{\mathtt{c}_2} > 0$ by (5.4), therefore

$$
|z(U;x)| := \operatorname{arccosh}(s_1(U;x)) = \ln\Big(s_1(U;x) + \sqrt{(s_1(U;x))^2 - 1}\Big),
$$

is well-defined. For the second equation one observes that the function

$$
\frac{\sinh(|z(U;x)|)}{|z(U;x)|} = 1 + \sum_{k\geq 0} \frac{(z(U;x)\bar{z}(U;x))^k}{(2k+1)!} \geq 1,
$$

hence we set

$$z(U;x) := s_2(U;x)\frac{|z(U;x)|}{\sinh(|z(U;x)|)}. \tag{5.33}$$

We set

$$\begin{aligned} A^{(1)}(U;x,\xi) &:= A_2^{(1)}(U;x)(\mathrm{i}\xi)^2 + A_1^{(1)}(U;x)(\mathrm{i}\xi),\\ A_2^{(1)}(U;x) &:= \begin{pmatrix} a_2^{(1)}(U;x) & 0\\ 0 & a_2^{(1)}(U;x)\end{pmatrix}\end{aligned} \tag{5.34}$$

where $a_2^{(1)}(U;x)$ is defined in (5.29) and $A_1^{(1)}(U;x)$ is defined in (5.28). Equation (5.31), together with (5.27) and (5.34) implies that (5.14) holds. By construction one has that the matrix $A^{(1)}(U;x,\xi)$ satisfies Constraint 5.1. It remains to show that $A^{(1)}(U;x,\xi)$ satisfies either Hypothesis 5.1 or Hypothesis 5.2.

If $A(U;x,\xi)$ satisfies Hypothesis 5.2 then we have that $a_2^{(1)}(U;x)$ in (5.29) is real. Moreover by construction $S(U;x)$ in (5.30) is even in x, therefore by Remark 3.6 we have that the map $\Phi_1(U)$ in (5.21) is parity preserving according to Definition 2.4. This implies that the matrix $A^{(1)}(U;x,\xi)$ satisfies Hypothesis 5.2. Let us consider the case when $A(U;x,\xi)$ satisfies Hypothesis 5.1. One can check, by an explicit computation, that the map $\Phi_1(U)$ in (5.21), is such that

$$\Phi_1^*(U)(-\mathrm{i}E)\Phi_1(U) = (-\mathrm{i}E) + \tilde{R}(U), \tag{5.35}$$

for some smoothing operators $\tilde{R}(U)$ belonging to $\mathcal{R}^{-\rho}_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$. In other words, up to a ρ-smoothing operator, the map $\Phi_1(U)$ satisfies conditions (2.10). By following essentially word by word the proof of Lemma 2.2 one obtains that, up to a smoothing operator in the class $\mathcal{R}^{-\rho}_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$, the operator $\mathrm{Op}^{\mathcal{BW}}(A^{(1)}(U;x,\xi))$ in (5.14) is self-adjoint. This implies that the matrix $A^{(1)}(U;x,\xi)$ satisfies Hypothesis 5.1. This concludes the proof. □

5.2. Diagonalization of the first order operator

In the previous Section we conjugated system (5.1) to (5.14), where the matrix $A^{(1)}(U;x,\xi)$ has the form

$$A^{(1)}(U;x,\xi) = A_2^{(1)}(U;x)(\mathrm{i}\xi)^2 + A_1^{(1)}(U;x)(\mathrm{i}\xi), \tag{5.36}$$

with $A_i^{(1)}(U;x)$ belonging to $\mathcal{F}_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$ and where $A_2^{(1)}(U;x)$ is diagonal. In this Section we show that, since the matrices $A_i^{(1)}(U;x)$ satisfy Hypothesis 5.1 (respectively Hypothesis 5.2), it is possible to diagonalize also the term $A_1^{(1)}(U;x)$ through a change of coordinates which is the identity plus a smoothing term. This is the result of the following lemma.

Lemma 5.3. *If the matrix $A^{(1)}(U;x,\xi)$ in (5.14) satisfies Hypothesis 5.1 (resp. together with P satisfy Hypothesis 5.2) then there exists $s_0>0$ (possibly larger than the one in Lemma 5.2) such that for any $s\geq s_0$ there exists an invertible map (resp. an invertible and parity preserving map)*

$$\Phi_2 = \Phi_2(U) : C^{K-1}_{*\mathbb{R}}(I,\mathbf{H}^s) \to C^{K-1}_{*\mathbb{R}}(I,\mathbf{H}^s),$$

with

$$\|(\Phi_2(U))^{\pm1}V\|_{K-1,s} \leq \|V\|_{K-1,s}(1 + C\|U\|_{K,s_0}) \tag{5.37}$$

where $C>0$ depends only on s and $\|U\|_{K,s_0}$ such that the following holds. There exists a matrix $A^{(2)}(U;x,\xi)$ satisfying Constraint 5.1 and Hypothesis 5.1 (resp. Hypothesis 5.2) of the form

$$\begin{aligned} A^{(2)}(U;x,\xi) &:= A_2^{(2)}(U;x)(\mathrm{i}\xi)^2 + A_1^{(2)}(U;x)(\mathrm{i}\xi),\\ A_2^{(2)}(U;x) &:= A_2^{(1)}(U;x);\\ A_1^{(2)}(U;x) &:= \begin{pmatrix} a_1^{(2)}(U;x) & 0\\ 0 & \overline{a_1^{(2)}(U;x)}\end{pmatrix} \in \mathcal{F}_{K,2}[r]\otimes\mathcal{M}_2(\mathbb{C}),\end{aligned} \tag{5.38}$$

and operators $R_1^{(2)}(U)$, $R_2^{(2)}(U)$ *in* $\mathcal{R}^0_{K,2}[r]\otimes\mathcal{M}_2(\mathbb{C})$*, such that by setting* $V_2=\Phi_2(U)V_1$ *the system* (5.14) *reads*

$$\partial_t V_2 = \mathrm{i}E\Big[\Lambda V_2 + \mathrm{Op}^{\mathcal{B}W}(A^{(2)}(U;x,\xi))[V_2] + R_1^{(2)}(U)[V_2] + R_2^{(2)}(U)[U]\Big]. \tag{5.39}$$

Proof. We recall that by Lemma 5.2 we have that

$$A^{(1)}(U;x,\xi) := \begin{pmatrix} a^{(1)}(U;x,\xi) & b^{(1)}(U;x,\xi) \\ \overline{b^{(1)}(U;x,-\xi)} & \overline{a^{(1)}(U;x,-\xi)} \end{pmatrix}.$$

Moreover by (5.13) we can write

$$\begin{aligned} a^{(1)}(U;x,\xi) &= a_2^{(1)}(U;x)(\mathrm{i}\xi)^2 + a_1^{(1)}(U;x)(\mathrm{i}\xi), \\ b^{(1)}(U;x,\xi) &= b_1^{(1)}(U;x)(\mathrm{i}\xi), \end{aligned}$$

with $a_2^{(1)}(U;x), a_1^{(1)}(U;x), b_1^{(1)}(U;x)\in\mathcal{F}_{K,1}[r]$. In the case that $A^{(1)}(U;x,\xi)$ satisfies Hypothesis 5.1, we can note that $b_1(U;x)\equiv 0$. Hence it is enough to choose $\Phi_2(U)\equiv\mathbb{1}$ to obtain the thesis. On the other hand, assume that $A^{(1)}(U;x,\xi)$ satisfies Hypothesis 5.2 we reason as follows.

Let us consider a symbol $d(U;x,\xi)$ in the class $\Gamma^{-1}_{K,1}[r]$ and define

$$D(U;x,\xi) := \begin{pmatrix} 0 & d(U;x,\xi) \\ \overline{d(U;x,-\xi)} & 0 \end{pmatrix} \in \Gamma^{-1}_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C}). \tag{5.40}$$

Let $\Phi_2^\tau(U)[\cdot]$ be the flow of the system

$$\begin{cases} \partial_\tau\Phi_2^\tau(U) = \mathrm{Op}^{\mathcal{B}W}(D(U;x,\xi))\Phi_2^\tau(U) \\ \Phi_2^0(U) = \mathbb{1}. \end{cases} \tag{5.41}$$

Reasoning as done for the system (5.17) one has that there exists a unique family of invertible bounded operators on $\mathbf{H}^s$ satisfying with

$$\|(\Phi_2^\tau(U))^{\pm1}V\|_{K-1,s} \le \|V\|_{K-1,s}(1+C\|U\|_{K,s_0}) \tag{5.42}$$

for $C>0$ depending on s and $\|U\|_{K,s_0}$ for $\tau\in[0,1]$.

The operator $W^\tau(U)[\cdot] := \Phi_2^\tau(U)[\cdot] - (\mathbb{1}+\tau\mathrm{Op}^{\mathcal{B}W}(D(U;x,\xi)))$ solves the following system:

$$\begin{cases} \partial_\tau W^\tau(U) = \mathrm{Op}^{\mathcal{B}W}(D(U;x,\xi))W^\tau(U) + \tau\mathrm{Op}^{\mathcal{B}W}(D(U;x,\xi))\circ\mathrm{Op}^{\mathcal{B}W}(D(U;x,\xi)) \\ W^0(U) = 0. \end{cases} \tag{5.43}$$

Therefore, by Duhamel formula, one can check that $W^\tau(U)$ is a smoothing operator in the class $\mathcal{R}^{-2}_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$ for any $\tau\in[0,1]$. We set $\Phi_2(U)[\cdot] := \Phi_2^\tau(U)[\cdot]_{|\tau=1}$, by the discussion above we have that there exists $Q(U)$ in $\mathcal{R}^{-2}_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$ such that

$$\Phi_2(U)[\cdot] = \mathbb{1} + \mathrm{Op}^{\mathcal{B}W}(D(U;x,\xi)) + Q(U).$$

Since $\Phi_2^{-1}(U)$ exists, by symbolic calculus, it is easy to check that there exists $\tilde{Q}(U)$ in $\mathcal{R}^{-2}_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$ such that

$$\Phi_2^{-1}(U)[\cdot] = \mathbb{1} - \mathrm{Op}^{\mathcal{B}W}(D(U;x,\xi)) + \tilde{Q}(U).$$

We set $V_2 := \Phi_2(U)[V_1]$, therefore the system (5.14) in the new coordinates reads

$$\begin{aligned} (V_2)_t &= \Phi_2(U)\mathrm{i}E\Big(\Lambda + \mathrm{Op}^{\mathcal{B}W}(A^{(1)}(U;x,\xi)) + R_1^{(1)}(U)\Big)(\Phi_2(U))^{-1}[V_2]+ \\ &\quad + \Phi_2(U)\mathrm{i}ER_2^{(1)}(U)[U] + \mathrm{Op}^{\mathcal{B}W}(\partial_t\Phi_2(U))(\Phi_2(U))^{-1}[V_2]. \end{aligned} \tag{5.44}$$

The summand $\Phi_2(U)\mathrm{i}ER_2^{(1)}(U)[\cdot]$ belongs to the class $\mathcal{R}^0_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$ by composition Propositions. Since $\partial_t D(U;x,\xi)$ belongs to $\Gamma^{-1}_{K,2}[r]\otimes\mathcal{M}_2(\mathbb{C})$ and $\partial_t Q$ is in $\mathcal{R}^{-2}_{K,2}[r]\otimes\mathcal{M}_2(\mathbb{C})$ then the last summand in (5.44) belongs to $\mathcal{R}^0_{K,2}[r]\otimes\mathcal{M}_2(\mathbb{C})$. We now study the first summand. First we note that $\Phi_2(U)\mathrm{i}ER_1^{(1)}(U)\Phi_2^{-1}(U)$ is a bounded remainder in $\mathcal{R}^0_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$. It remains to study the term

$$\mathrm{i}\Phi_2(U)\Big[E\mathfrak{P}\big(\Phi_2^{-1}(U)[V_2]\big)\Big]+\mathrm{i}\Phi_2(U)\Big[\mathrm{Op}^{\mathcal{B}W}\big(E(\mathbb{1}+A_2^{(1)}(U;x))(\mathrm{i}\xi)^2+EA_1^{(1)}(U;x)(\mathrm{i}\xi)\big)\Big]\Phi_2^{-1}(U)[V_2],$$

where $\mathfrak{P}$ is defined in (1.19). The first term is equal to $\mathrm{i}E(\mathfrak{P}V_2)$ up to a bounded term in $\mathcal{R}^0_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$ by Lemma 5.1. The second is equal to

$$\begin{aligned}&\mathrm{i}\mathrm{Op}^{\mathcal{B}W}\big(E(\mathbb{1}+A_2^{(1)}(U;x))(\mathrm{i}\xi)^2+EA_1^{(1)}(U;x)(\mathrm{i}\xi)\big)+\\&+\Big[\mathrm{Op}^{\mathcal{B}W}(D(U;x,\xi)),\mathrm{i}E\mathrm{Op}^{\mathcal{B}W}\big((\mathbb{1}+A_2^{(1)}(U;x))(\mathrm{i}\xi)^2\big)\Big]\end{aligned}\tag{5.45}$$

modulo bounded terms in $\mathcal{R}^0_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$. By using formula (3.27) one get that the commutator above is equal to $\mathrm{Op}^{\mathcal{B}W}(M(U;x,\xi))$ with

$$\begin{aligned}&M(U;x,\xi):=\begin{pmatrix}0 & m(U;x,\xi)\\ \overline{m(U;x,-\xi)} & 0\end{pmatrix},\\&m(U;x,\xi):=-2d(U;x,\xi)(1+a_2^{(1)}(U;x))(\mathrm{i}\xi)^2,\end{aligned}\tag{5.46}$$

up to terms in $\mathcal{R}^0_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$. Therefore the system obtained after the change of coordinates reads

$$(V_2)_t=\mathrm{i}E\Big[\Lambda V_2+\mathrm{Op}^{\mathcal{B}W}(A^{(2)}(U;x,\xi))[V_2]+Q_1(U)[V_2]+Q_2(U)[U]\Big],\tag{5.47}$$

where $Q_1(U)$ and $Q_2(U)$ are bounded terms in $\mathcal{R}^0_{K,2}[r]\otimes\mathcal{M}_2(\mathbb{C})$ and the new matrix $A^{(2)}(U;x,\xi))$ is

$$\begin{pmatrix}a_2^{(1)}(U;x) & 0\\ 0 & \overline{a_2^{(1)}(U;x)}\end{pmatrix}(\mathrm{i}\xi)^2+\begin{pmatrix}a_1^{(1)}(U;x) & b_1^{(1)}(U;x)\\ \overline{b_1^{(1)}(U;x)} & \overline{a_1^{(1)}(U;x)}\end{pmatrix}(\mathrm{i}\xi)+M(U;x,\xi).\tag{5.48}$$

Hence the elements on the diagonal are not affected by the change of coordinates, now our aim is to choose $d(U;x,\xi)$ in such a way that the symbol

$$b_1(U;x)(\mathrm{i}\xi)+m(U;x,\xi)=b_1(U;x)(\mathrm{i}\xi)-2d(U;x,\xi)(1+a_2^{(1)}(U;x))(\mathrm{i}\xi)^2,\tag{5.49}$$

belongs to $\Gamma^0_{K,2}[r]$. We split the symbol in (5.49) in low-high frequencies: let $\varphi(\xi)$ a function in $C_0^\infty(\mathbb{R};\mathbb{R})$ such that $\mathrm{supp}(\varphi)\subset[-1,1]$ and $\varphi\equiv1$ on $[-1/2,1/2]$. Trivially one has that $\varphi(\xi)(b_1(U;x)(\mathrm{i}\xi)+m(U;x,\xi))$ is a symbol in $\Gamma^0_{K,1}[r]$, so it is enough to solve the equation

$$\big(1-\varphi(\xi)\big)\Big[b_1(U;x)(\mathrm{i}\xi)-2d(U;x,\xi)\Big(1+a_2^{(1)}(U;x))(\mathrm{i}\xi)^2\Big)\Big]=0.\tag{5.50}$$

So we should choose the symbol d as

$$\begin{aligned}&d(U;x,\xi)=\left(\frac{b_1^{(1)}(U;x)}{2(1+a_2^{(1)}(U;x))}\right)\cdot\gamma(\xi)\\&\gamma(\xi)=\begin{cases}\dfrac{1}{\mathrm{i}\xi} & \text{if } |\xi|\geq\dfrac12\\ \text{odd continuation of class } C^\infty & \text{if } |\xi|\in[0,\dfrac12).\end{cases}\end{aligned}\tag{5.51}$$

Clearly the symbol $d(U;x,\xi)$ in (5.51) belongs to $\Gamma^{-1}_{K,1}[r]$, hence the map $\Phi_2(U)$ in (5.40) is well defined and estimate (5.37) holds. It is evident that, after the choice of the symbol in (5.51), the matrix $A^{(2)}(U;x,\xi)$ is

$$\begin{pmatrix}a_2^{(1)}(U;x) & 0\\ 0 & \overline{a_2^{(1)}(U;x)}\end{pmatrix}(\mathrm{i}\xi)^2+\begin{pmatrix}a_1^{(1)}(U;x) & 0\\ 0 & \overline{a_1^{(1)}(U;x)}.\end{pmatrix}(\mathrm{i}\xi)\tag{5.52}$$

The symbol $d(U;x,\xi)$ is equal to $d(U;-x,-\xi)$ because $b_1^{(1)}(U;x)$ is odd in x and $a_2^{(1)}(U;x)$ is even in x, therefore, by Remark 3.6 the map $\Phi_2(U)$ is *parity preserving*. □

5.3. Reduction to constant coefficients 1: paracomposition

Consider the diagonal matrix of functions $A_2^{(2)}(U;x) \in \mathcal{F}_{K,2}[r] \otimes \mathcal{M}_2(\mathbb{C})$ defined in (5.38). In this section we shall reduce the operator $\mathrm{Op}^{\mathcal{B}W}(A_2^{(2)}(U;x)(\mathrm{i}\xi)^2)$ to a constant coefficient one, up to bounded terms (see (5.65)). For these purposes we shall use a paracomposition operator (in the sense of Alinhac [3]) associated to the diffeomorphism $x \mapsto x + \beta(x)$ of $\mathbb{T}$. We follow Section 2.5 of [10] and in particular we shall use their alternative definition of paracomposition operator.

Consider a real symbol $\beta(U;x)$ in the class $\mathcal{F}_{K,K'}[r]$ and the map

$$\Phi_U : x \mapsto x + \beta(U;x). \tag{5.53}$$

We state the following.

Lemma 5.4. *Let $0 \le K' \le K$ be in $\mathbb{N}$, $r > 0$ and $\beta(U;x) \in \mathcal{F}_{K,K'}[r]$ for U in the space $C^K_{*\mathbb{R}}(I, \mathbf{H}^{s_0})$. If s_0 is sufficiently large and β is 2π-periodic in x and satisfies*

$$1 + \beta_x(U;x) \ge \Theta > 0, \quad x \in \mathbb{R}, \tag{5.54}$$

for some constant Θ depending on $\sup_{t\in I} \|U(t)\|_{\mathbf{H}^{s_0}}$, then the map Φ_U in (5.53) is a diffeomorphism of $\mathbb{T}$ to itself, and its inverse may be written as

$$(\Phi_U)^{-1} : y \mapsto y + \gamma(U;y) \tag{5.55}$$

for γ in $\mathcal{F}_{K,K'}[r]$.

Proof. Under condition (5.54) there exists $\gamma(U;y)$ such that

$$x + \beta(U;x) + \gamma(U; x + \beta(U;x)) = x, \qquad x \in \mathbb{R}. \tag{5.56}$$

One can prove the bound (3.3) on the function $\gamma(U;y)$ by differentiating in x equation (5.56) and using that $\beta(U;x)$ is a symbol in $\mathcal{F}_{K,K'}[r]$. □

Remark 5.2. The Lemma above is very similar to Lemma 2.5.2 of [10]. The authors use a smallness assumption on r to prove the result. Here this assumption is replaced by (5.54) in order to treat big sized initial conditions.

Remark 5.3. By Lemma 5.4 one has that $x \mapsto x + \tau\beta(U;x)$ is a diffeomorphism of $\mathbb{T}$ for any $\tau \in [0,1]$. Indeed

$$1 + \tau\beta_x(U;x) = 1 - \tau + \tau(1 + \beta_x(U;x)) \ge (1-\tau) + \tau\Theta \ge \min\{1,\Theta\} > 0,$$

for any $\tau \in [0,1]$. Hence the (5.54) holds true with $\mathtt{c} = \min\{1,\Theta\}$ and Lemma 5.4 applies.

With the aim of simplifying the notation we set $\beta(x) := \beta(U;x)$, $\gamma(y) := \gamma(U;x)$ and we define the following quantities

$$\begin{aligned} B(\tau;x,\xi) = B(\tau,U;x,\xi) &:= -\mathrm{i}b(\tau;x)(\mathrm{i}\xi), \\ b(\tau;x) &:= \frac{\beta(x)}{(1+\tau\beta_x(x))}. \end{aligned} \tag{5.57}$$

Then one defines the paracomposition operator associated to the diffeomorphism (5.53) as $\Omega_{B(U)}(1)$, where $\Omega_{B(U)}(\tau)$ is the flow of the linear paradifferential equation

$$\begin{cases} \dfrac{d}{d\tau}\Omega_{B(U)}(\tau) = \mathrm{i}\mathrm{Op}^{\mathcal{B}W}(B(\tau;U,\xi))\Omega_{B(U)}(\tau) \\ \Omega_{B(U)}(0) = \mathrm{id}. \end{cases} \tag{5.58}$$

We state here a Lemma which asserts that the problem (5.58) is well posed and whose solution is a one parameter family of bounded operators on H^s, which is one of the main properties of a paracomposition operator. For the proof of the result we refer to Lemma 2.5.3 in [10].

Lemma 5.5. *Let $0 \leq K' \leq K$ be in $\mathbb{N}$, $r > 0$ and $\beta(U; x) \in \mathcal{F}_{K,K'}[r]$ for U in the space $C^K_{*\mathbb{R}}(I, \mathbf{H}^s)$. The system (5.58) has a unique solution defined for $\tau \in [-1, 1]$. Moreover for any s in $\mathbb{R}$ there exists a constant $C_s > 0$ such that for any U in $B^K_{s_0}(I, r)$ and any W in H^s*

$$C_s^{-1}\|W\|_{H^s} \leq \|\Omega_{B(U)}(\tau)W\|_{H^s} \leq C_s\|W\|_{H^s}, \quad \forall\, \tau \in [-1, 1], \quad W \in H^s, \tag{5.59}$$

and

$$\|\Omega_{B(U)}(\tau)W\|_{K-K',s} \leq (1 + C\|U\|_{K,s_0})\|W\|_{K-K',s}, \tag{5.60}$$

where $C > 0$ is a constant depending only on s and $\|U\|_{K,s_0}$.

Remark 5.4. As pointed out in Remark 3.2, our classes of symbols are slightly different from the ones in [10]. For this reason the authors in [10] are more precise about the constant C in (5.60). However the proof can be adapted straightforward.

Remark 5.5. In the following we shall study how symbols $a(U; x, \xi)$ changes under conjugation through the flow $\Omega_{B(U)}(\tau)$ introduced in Lemma 5.5. In order to do this we shall apply Theorem 2.5.8 in [10]. Such result requires that $x \mapsto x + \tau\beta(U; x)$ is a path of diffeomorphism for $\tau \in [0, 1]$. In [10] this fact is achieved by using the smallness of r, here it is implied by Remark 5.3.

We now study how the convolution operator $P*$ changes under the flow $\Omega_{B(U)}(\tau)$ introduced in Lemma 5.5.

Lemma 5.6. *Let $P : \mathbb{T} \to \mathbb{R}$ be a C^1 function, let us define $P_*[h] = P * h$ for $h \in H^s$, where $*$ denote the convolution between functions, and set $\Phi(U)[\cdot] := \Omega_{B(U)}(\tau)_{|\tau=1}$. There exists R belonging to $\mathcal{R}^0_{K,K'}[r]$ such that*

$$\Phi(U) \circ P_* \circ \Phi^{-1}(U)[\cdot] = P_*[\cdot] + R(U)[\cdot]. \tag{5.61}$$

Moreover if $P(x)$ is even in x and $\Phi(U)$ is parity preserving according to Definition 2.4 then the remainder $R(U)$ in (5.61) is parity preserving.

Proof. Using equation (5.58) and estimate (3.18) one has that, for $0 \leq k \leq K - K'$, the following holds true

$$\|\partial_t^k\big(\Phi^{\pm 1}(U) - \mathrm{Id}\big)h\|_{H^{s-1-2k}} \leq \sum_{k_1+k_2=k} C\|U\|_{K'+k_1,s_0}\|h\|_{k_2,s} \tag{5.62}$$

where $C > 0$ depends only on $\|U\|_{K,s_0}$ and Id is the identity map on H^s. Therefore we can write

$$\Phi(U)\Big[P * \big[\Phi^{-1}(U)h\big]\Big] = P * h + \Big((\Phi(U) - \mathrm{Id})(P * h)\Big) + \Phi\Big[P * \Big((\Phi^{-1}(U) - \mathrm{Id})h\Big)\Big]. \tag{5.63}$$

Using estimate (5.62) and the fact that the function P is of class $C^1(\mathbb{T})$ we can estimate the last two summands in the r.h.s. of (5.63) as follows

$$\Big\|\partial_t^k(\Phi(U) - \mathrm{Id})(P * h)\Big\|_{H^{s-2k}} \leq \sum_{k_1+k_2=k} C\,\|U\|_{K'+k_1,s_0}\,\|P * h\|_{k_2,s+1} \leq \sum_{k_1+k_2=k} C\,\|U\|_{K'+k_1,s_0}\,\|h\|_{k_2,s}$$

$$\Big\|\partial_t^k\Big(\Phi(U)\Big[P * \Big((\Phi^{-1}(U) - \mathrm{Id})h\Big)\Big]\Big)\Big\|_{s-2k} \leq \sum_{k_1+k_2=k} C\,\|U\|_{K'+k_1,s_0}\,\Big\|(\Phi^{-1}(U) - \mathrm{Id})h\Big\|_{k_2,s-1}$$

$$\leq \sum_{k_1+k_2=k} C\,\|U\|_{K'+k_1,s_0}\,\|h\|_{k_2,s}\,,$$

for $0 \leq k \leq K - K'$ and where C is a constant depending on $\|P\|_{C^1}$ and $\|U\|_{K,s_0}$. Hence they belong to the class $\mathcal{R}^0_{K,K'}[r]$. Finally if $P(x)$ is even in x then the operator P_* is parity preserving according to Definition 2.4, therefore if in addiction $\Phi(U)$ is parity preserving so must be $R(U)$ in (5.61). □

We are now in position to prove the following.

Lemma 5.7. *If the matrix $A^{(2)}(U;x,\xi)$ in (5.39) satisfies Hypothesis 5.1 (resp. together with P satisfy Hypothesis 5.2) then there exists $s_0>0$ (possibly larger than the one in Lemma 5.3) such that for any $s\geq s_0$ there exists an invertible map (resp. an invertible and parity preserving map)*

$$\Phi_3=\Phi_3(U):C^{K-2}_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T},\mathbb{C}^2))\to C^{K-2}_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T},\mathbb{C}^2)),$$

with

$$\|(\Phi_3(U))^{\pm1}V\|_{K-2,s}\leq\|V\|_{K-2,s}(1+C\|U\|_{K,s_0}) \tag{5.64}$$

where $C>0$ depends only on s and $\|U\|_{K,s_0}$ such that the following holds. There exists a matrix $A^{(3)}(U;x,\xi)$ satisfying Constraint 5.1 and Hypothesis 5.1 (resp. Hypothesis 5.2) of the form

$$\begin{aligned}
&A^{(3)}(U;x,\xi):=A_2^{(3)}(U)(\mathrm{i}\xi)^2+A_1^{(3)}(U;x)(\mathrm{i}\xi),\\
&A_2^{(3)}(U):=\begin{pmatrix} a_2^{(3)}(U) & 0\\ 0 & a_2^{(3)}(U)\end{pmatrix},\quad a_2^{(3)}\in\mathcal{F}_{K,3}[r],\quad \text{independent of } x\in\mathbb{T},\\
&A_1^{(3)}(U;x):=\begin{pmatrix} a_1^{(3)}(U;x) & 0\\ 0 & \overline{a_1^{(3)}(U;x)}\end{pmatrix}\in\mathcal{F}_{K,3}[r]\otimes\mathcal{M}_2(\mathbb{C}),
\end{aligned} \tag{5.65}$$

and operators $R_1^{(3)}(U)$, $R_2^{(3)}(U)$ in $\mathcal{R}^0_{K,3}[r]\otimes\mathcal{M}_2(\mathbb{C})$, such that by setting $V_3=\Phi_3(U)V_2$ the system (5.39) reads

$$\partial_tV_3=\mathrm{i}E\Big[\Lambda V_3+\mathrm{Op}^{\mathcal{B}W}(A^{(3)}(U;x,\xi))[V_3]+R_1^{(3)}(U)[V_3]+R_2^{(3)}(U)[U]\Big]. \tag{5.66}$$

Proof. Let $\beta(U;x)$ be a real symbol in $\mathcal{F}_{K,2}[r]$ to be chosen later such that condition (5.54) holds. Set moreover $\gamma(U;x)$ the symbol such that (5.56) holds. Consider accordingly to the hypotheses of Lemma 5.5 the system

$$\dot W=\mathrm{i}EMW,\quad W(0)=\mathbb{1},\quad M:=\mathrm{Op}^{\mathcal{B}W}\begin{pmatrix}B(\tau,x,\xi) & 0\\ 0 & \overline{B(\tau,x,-\xi)}\end{pmatrix}, \tag{5.67}$$

where B is defined in (5.57). Note that $\overline{B(\tau,x,-\xi)}=-B(\tau,x,\xi)$. By Lemma 5.5 the flow exists and is bounded on $\mathbf{H}^s(\mathbb{T},\mathbb{C}^2)$ and moreover (5.64) holds. We want to conjugate the system (5.39) through the map $\Phi_3(U)[\cdot]=W(1)[\cdot]$. Set $V_3=\Phi_3(U)V_2$. The system in the new coordinates reads

$$\begin{aligned}
\frac{d}{dt}V_3=\Phi_3(U)\Big[\mathrm{i}E(\mathfrak{P}[\Phi_3^{-1}(U)V_3)\Big]&+(\partial_t\Phi_3(U))\Phi_3^{-1}(U)[V_3]\\
&+\Phi_3(U)\Big[\mathrm{i}E\mathrm{Op}^{\mathcal{B}W}((\mathbb{1}+A_2^{(2)}(U;x))(\mathrm{i}\xi)^2)\Big]\Phi_3^{-1}(U)[V_3]\\
&+\Phi_3(U)\Big[\mathrm{i}E\mathrm{Op}^{\mathcal{B}W}(A_1^{(2)}(U;x)(\mathrm{i}\xi))\Big]\Phi_3^{-1}(U)[V_3]\\
&+\Phi_3(U)\Big[\mathrm{i}ER_1^{(2)}(U)\Big]\Phi_3^{-1}(U)[V_3]+\Phi_3(U)\mathrm{i}ER_2^{(2)}(U)[U],
\end{aligned} \tag{5.68}$$

where $\mathfrak{P}$ is defined in (1.19). We now discuss each term in (5.68). The first one, by Lemma 5.6, is equal to $\mathrm{i}E(\mathfrak{P}V_3)$ up to a bounded remainder in the class $\mathcal{R}^0_{K,2}[r]\otimes\mathcal{M}_2(\mathbb{C})$. The last two terms also belongs to the latter class because the map Φ_3 is a bounded operator on $\mathbf{H}^s$. For the term $(\partial_t\Phi_3(U))\Phi_3^{-1}(U)[V_3]$ we apply Proposition 2.5.9 of [10] and we obtain that

$$(\partial_t\Phi_3(U))\Phi_3^{-1}(U)[V_3]=\mathrm{Op}^{\mathcal{B}W}\begin{pmatrix}e(U;x)(\mathrm{i}\xi) & 0\\ 0 & \overline{e(U;x)(\mathrm{i}\xi)}\end{pmatrix}[V_3]+\tilde R(U)[V_3], \tag{5.69}$$

where $\tilde R$ belongs to $\mathcal{R}^{-1}_{K,3}[r]\otimes\mathcal{M}_2(\mathbb{C})$ and $e(U;x)$ is a symbol in $\mathcal{F}_{K,3}[r]\otimes\mathcal{M}_2(\mathbb{C})$ such that $\mathrm{Re}(e(U;x))=0$. It remains to study the conjugate of the paradifferential terms in (5.68). We note that

$$\begin{aligned}
&\Phi_3(U)\Big[\mathrm{i}E\mathrm{Op}^{\mathcal{B}W}((\mathbb{1}+A_2^{(2)}(U;x))(\mathrm{i}\xi)^2)\Big]\Phi_3^{-1}(U)[V_3]+\Phi_3(U)\Big[\mathrm{i}E\mathrm{Op}^{\mathcal{B}W}(A_1^{(2)}(U;x)(\mathrm{i}\xi))\Big]\Phi_3^{-1}(U)[V_3]\\
&\quad=\begin{pmatrix}T & 0\\ 0 & \overline{T}\end{pmatrix}
\end{aligned}$$

where T is the operator

$$T = \Omega_{B(U)}(1)\mathrm{Op}^{\mathcal{B}W}\Big((1+a_2^{(2)}(U;x))(\mathrm{i}\xi)^2 + a_1^{(2)}(U;x)(\mathrm{i}\xi)\Big)\Omega^{-1}_{B(U)}(1). \tag{5.70}$$

The Paracomposition Theorem 2.5.8 in [10], which can be used thanks to Remarks 5.3 and 5.5, guarantees that

$$T = \mathrm{Op}^{\mathcal{B}W}(\tilde{a}_2^{(3)}(U;x,\xi) + a_1^{(3)}(U;x)(\mathrm{i}\xi))[\cdot] \tag{5.71}$$

up to a bounded term in $\mathcal{R}^0_{K,3}[r]$ and where

$$\begin{aligned} \tilde{a}_2^{(3)}(U;x,\xi) &= \big(1+a_2^{(2)}(U;y)\big)\big(1+\gamma_y(1,y)\big)^2_{|y=x+\beta(x)}(\mathrm{i}\xi)^2, \\ a_1^{(3)}(U;x) &= a_1^{(2)}(U;y)\big(1+\gamma_y(1,y)\big)_{|y=x+\beta(x)}. \end{aligned} \tag{5.72}$$

Here $\gamma(1,x) = \gamma(\tau,x)_{|\tau=1} = \gamma(U;\tau,x)_{|\tau=1}$ with

$$y = x + \tau\beta(U;x) \ \Leftrightarrow x = y + \gamma(\tau,y), \ \ \tau \in [0,1],$$

where $x + \tau\beta(U;x)$ is the path of diffeomorphism given by Remark 5.3.

By Lemma 2.5.4 of Section 2.5 of [10] one has that the new symbols $\tilde{a}_2^{(3)}(U;x,\xi), a_1^{(3)}(U;x)$ defined in (5.72) belong to the class $\Gamma^2_{K,3}[r]$ and $\mathcal{F}_{K,3}[r]$ respectively. At this point we want to choose the symbol $\beta(x)$ in such a way that $\tilde{a}_2^{(3)}(U;x,\xi)$ does not depend on x. One can proceed as follows. Let $a_2^{(3)}(U)$ a x-independent function to be chosen later, one would like to solve the equation

$$\big(1+a_2^{(2)}(U;y)\big)\big(1+\gamma_y(1,y)\big)^2_{|y=x+\beta(x)}(\mathrm{i}\xi)^2 = (1+a_2^{(3)}(U))(\mathrm{i}\xi)^2. \tag{5.73}$$

The solution of this equation is given by

$$\gamma(U;1,y) = \partial_y^{-1}\left(\sqrt{\frac{1+a_2^{(3)}(U)}{1+a_2^{(2)}(U;y)}} - 1\right). \tag{5.74}$$

In principle this solution is just formal because the operator ∂_y^{-1} is defined only for function with zero mean, therefore we have to choose $a_2^{(3)}(U)$ in such a way that

$$\int_{\mathbb{T}}\left(\sqrt{\frac{1+a_2^{(3)}(U)}{1+a_2^{(2)}(U;y)}} - 1\right)dx = 0, \tag{5.75}$$

which means

$$1 + a_2^{(3)}(U) := \left[2\pi\left(\int_{\mathbb{T}}\frac{1}{\sqrt{1+a_2^{(2)}(U;y)}}dy\right)^{-1}\right]^2. \tag{5.76}$$

Note that everything is well defined thanks to the positivity of $1+a_2^{(2)}$. Indeed $a_2^{(2)} = a_2^{(1)}$ by (5.38), and $a_2^{(1)}$ satisfies (5.15). Indeed every denominator in (5.74), (5.75) and in (5.76) stays far away from 0. Note that $\gamma(U;y)$ belongs to $\mathcal{F}_{K,2}[r]$ and so does $\beta(U;x)$ by Lemma 5.4. By using (5.56) one can deduce that

$$1 + \beta_x(U;x) = \frac{1}{1+\gamma_y(U;1,y)} \tag{5.77}$$

where

$$1+\gamma_y(U;1,y) = 2\pi\left(\int_{\mathbb{T}}\frac{1}{\sqrt{1+a_2^{(2)}(U;y)}}dy\right)^{-1}\frac{1}{\sqrt{1+a_2^{(2)}(U;y)}}, \tag{5.78}$$

thanks to (5.74) and (5.76). Since the matrix $A_2^{(2)}$ satisfies Hypothesis 5.3 one has that there exists a universal constant $\mathtt{c} > 0$ such that $1 + a_2^{(2)}(U; y) \geq \mathtt{c}$. Therefore one has

$$1 + \beta_x(U; x) = \frac{1}{1 + \gamma_y(U; 1, y)} \geq \sqrt{\mathtt{c}}\frac{1}{2\pi}\int_{\mathbb{T}} \frac{1}{\sqrt{1 + a_2^{(2)}(U; y)}}dy$$
$$\geq \frac{1}{2\pi}\frac{\sqrt{\mathtt{c}}}{1 + C\|U\|_{0,s_0}} := \Theta > 0,$$

for some C depending only on $\|U\|_{K,s_0}$, where we used the fact that $a_2^{(2)}(U; y)$ belongs to the class $\mathcal{F}_{K,2}[r]$ (see Definition 3.2). This implies that $\beta(U; x)$ satisfies condition (5.54). We have written system (5.39) in the form (5.66) with matrices defined in (5.65).

It remains to show that the new matrix $A^{(3)}(U; x, \xi)$ satisfies either Hypothesis 5.1 or 5.2. If $A^{(2)}(U; x, \xi)$ is selfadjoint, i.e. satisfies Hypothesis 5.1, then one has that the matrix $A^{(3)}(U; x, \xi)$ is selfadjoint as well thanks to the fact that the map $W(1)$ satisfies the hypotheses (condition (2.10)) of Lemma 2.2, by using Lemma 2.1. In the case that $A^{(2)}(U; x, \xi)$ is parity preserving, i.e. satisfies Hypothesis 5.2, then $A^{(3)}(U; \xi)$ has the same properties for the following reasons. The symbols $\beta(U; x)$ and $\gamma(U; x)$ are odd in x if the function U is even in x. Hence the flow map $W(1)$ defined by equation (5.67) is parity preserving. Moreover the matrix $A^{(3)}(U; x, \xi)$ satisfies Hypothesis 5.2 by explicit computation. □

5.4. Reduction to constant coefficients 2: first order terms

Lemmata 5.2, 5.3, 5.7 guarantee that one can conjugate the system (5.1) to the system (5.66) in which the matrix $A^{(3)}(U; x, \xi)$ (see (5.65)) has the form

$$A^{(3)}(U; x, \xi) = A_2^{(3)}(U)(\mathrm{i}\xi)^2 + A_1^{(3)}(U; x)(\mathrm{i}\xi), \tag{5.79}$$

where the matrices $A_2^{(3)}(U)$, $A_1^{(3)}(U; x)$ are diagonal and belong to $\mathcal{F}_{K,3}[r] \otimes \mathcal{M}_2(\mathbb{C})$, for $i = 1, 2$. Moreover $A_2^{(3)}(U)$ does not depend on $x \in \mathbb{T}$. In this Section we show how to eliminate the x dependence of the symbol $A_1^{(3)}(U; x)$ in (5.65). We prove the following.

Lemma 5.8. *If the matrix $A^{(3)}(U; x, \xi)$ in (5.66) satisfies Hypothesis 5.1 (resp. together with P satisfy Hypothesis 5.2) then there exists $s_0 > 0$ (possibly larger than the one in Lemma 5.7) such that for any $s \geq s_0$ there exists an invertible map (resp. an invertible and parity preserving map)*

$$\Phi_4 = \Phi_4(U) : C^{K-3}_{*\mathbb{R}}(I, \mathbf{H}^s(\mathbb{T}, \mathbb{C}^2)) \to C^{K-3}_{*\mathbb{R}}(I, \mathbf{H}^s(\mathbb{T}, \mathbb{C}^2)),$$

with

$$\|(\Phi_4(U))^{\pm 1} V\|_{K-3,s} \leq \|V\|_{K-3,s}(1 + C\|U\|_{K,s_0}) \tag{5.80}$$

where $C > 0$ depends only on s and $\|U\|_{K,s_0}$ such that the following holds. Then there exists a matrix $A^{(4)}(U; \xi)$ independent of $x \in \mathbb{T}$ of the form

$$A^{(4)}(U; \xi) := \begin{pmatrix} a_2^{(3)}(U) & 0 \\ 0 & \overline{a_2^{(3)}(U)} \end{pmatrix}(\mathrm{i}\xi)^2 + \begin{pmatrix} a_1^{(4)}(U) & 0 \\ 0 & \overline{a_1^{(4)}(U)} \end{pmatrix}(\mathrm{i}\xi), \tag{5.81}$$

where $a_2^{(3)}(U)$ is defined in (5.65) and $a_1^{(4)}(U)$ is a symbol in $\mathcal{F}_{K,4}[r]$, independent of x, which is purely imaginary in the case of Hypothesis 5.1 (resp. is equal to 0). There are operators $R_1^{(4)}(U)$, $R_2^{(4)}(U)$ in $\mathcal{R}^0_{K,4}[r] \otimes \mathcal{M}_2(\mathbb{C})$, such that by setting $V_4 = \Phi_4(U)V_3$ the system (5.66) reads

$$\partial_t V_4 = \mathrm{i}E\Big[\Lambda V_4 + \mathrm{Op}^{\mathcal{B}W}(A^{(4)}(U; \xi))[V_4] + R_1^{(4)}(U)[V_4] + R_2^{(4)}(U)[U]\Big]. \tag{5.82}$$

Proof. Consider a symbol $s(U;x)$ in the class $\mathcal{F}_{K,3}[r]$ and define

$$S(U;x) := \begin{pmatrix} s(U;x) & 0 \\ 0 & \overline{s(U;x)} \end{pmatrix}.$$

Let $\Phi_4^\tau(U)[\cdot]$ be the flow of the system

$$\begin{cases} \partial_\tau \Phi_4^\tau(U)[\cdot] = \mathrm{Op}^{\mathcal{B}W}(S(U;x))\Phi_4^\tau(U)[\cdot] \\ \Phi_4^0(U)[\cdot] = \mathbb{1}. \end{cases} \tag{5.83}$$

Again one can reason as done for the system (5.17) to check that there exists a unique family of invertible bounded operators on $\mathbf{H}^s$ satisfying

$$\|(\Phi_4^\tau(U))^{\pm 1} V\|_{K-3,s} \le \|V\|_{K-3,s}(1 + C\|U\|_{K,s_0}) \tag{5.84}$$

for $C > 0$ depending on s and $\|U\|_{K,s_0}$ for $\tau \in [0,1]$. We set

$$\Phi_4(U)[\cdot] = \Phi_4^\tau(U)[\cdot]_{|\tau=1} = \exp\{\mathrm{Op}^{\mathcal{B}W}(S(U;x))\}. \tag{5.85}$$

By Corollary 3.1 we get that there exists $Q(U)$ in the class of smoothing remainder $\mathcal{R}^{-\rho}_{K,3}[r] \otimes \mathcal{M}_2(\mathbb{C})$ for any $\rho > 0$ such that

$$\Phi_4(U)[\cdot] := \mathrm{Op}^{\mathcal{B}W}(\exp\{S(U;x)\})[\cdot] + Q(U)[\cdot]. \tag{5.86}$$

Since $\Phi_4^{-1}(U)$ exists, by symbolic calculus, it is easy to check that there exists $\tilde{Q}(U)$ in $\mathcal{R}^{-\rho}_{K,3}[r] \otimes \mathcal{M}_2(\mathbb{C})$ such that

$$\Phi_4^{-1}(U)[\cdot] = \mathrm{Op}^{\mathcal{B}W}(\exp\{-S(U;x)\})[\cdot] + \tilde{Q}(U)[\cdot].$$

We set $G(U;x) = \exp\{S(U;x)\}$ and $V_4 = \Phi_4(U)V_3$. Then the system (5.66) becomes

$$\begin{aligned} (V_4)_t = {} & \Phi_4(U)\mathrm{i}E\Big(\Lambda + \mathrm{Op}^{\mathcal{B}W}(A^{(3)}(U;x,\xi)) + R_1^{(3)}(U)\Big)(\Phi_4(U))^{-1}[V_4] + \\ & + \Phi_4(U)\mathrm{i}ER_2^{(3)}(U)[U] + \mathrm{Op}^{\mathcal{B}W}(\partial_t G(U;x,\xi))(\Phi_4(U))^{-1}[V_4]. \end{aligned} \tag{5.87}$$

Recalling that $\Lambda = \mathfrak{P} + \frac{d^2}{dx^2}$ (see (1.20)) we note that by Lemma 5.1 the term $\mathrm{i}\Phi_4(U)\big[E\mathfrak{P}\big(\Phi_4^{-1}(U)[V_4]\big]$ is equal to $\mathrm{i}E\mathfrak{P}V_4$ up to a remainder in $\mathcal{R}^0_{K,4}[r] \otimes \mathcal{M}_2(\mathbb{C})$. Secondly we note that the operator

$$\hat{Q}(U)[\cdot] := \Phi_4(U)\mathrm{i}ER_1^{(3)}(U)\Phi_4^{-1}(U)[\cdot] + \Phi_4(U)\mathrm{i}ER_2^{(3)}(U)[U] + \mathrm{Op}^{\mathcal{B}W}(\partial_t G(U;x)) \circ \Phi_4^{-1}(U)[\cdot] \tag{5.88}$$

belongs to the class of operators $\mathcal{R}^0_{K,4}[r] \otimes \mathcal{M}_2(\mathbb{C})$. This follows by applying Propositions 3.3, 3.4, Remark 3.5 and the fact that $\partial_t G(U;x)$ is a matrix in $\mathcal{F}_{K,4}[r] \otimes \mathcal{M}_2(\mathbb{C})$. It remains to study the term

$$\Phi_4(U)\mathrm{i}E\Big(\mathrm{Op}^{\mathcal{B}W}\big((\mathbb{1} + A_2^{(3)}(U))(\mathrm{i}\xi)^2\big) + \mathrm{Op}^{\mathcal{B}W}(A_1^{(3)}(U;x)(\mathrm{i}\xi))\Big)(\Phi_4(U))^{-1}. \tag{5.89}$$

By using formula (3.27) and Remark 3.5 one gets that, up to remainder in $\mathcal{R}^0_{K,4}[r] \otimes \mathcal{M}_2(\mathbb{C})$, the term in (5.89) is equal to

$$\mathrm{i}E\mathrm{Op}^{\mathcal{B}W}\big((\mathbb{1} + A_2^{(3)}(U))(\mathrm{i}\xi)^2\big) + \mathrm{i}E\mathrm{Op}^{\mathcal{B}W}\begin{pmatrix} r(U;x)(\mathrm{i}\xi) & 0 \\ 0 & \overline{r(U;x)(\mathrm{i}\xi)} \end{pmatrix} \tag{5.90}$$

where

$$r(U;x) := a_1^{(3)}(U;x) + 2(1 + a_2^{(3)}(U))\partial_x s(U;x). \tag{5.91}$$

We look for a symbol $s(U;x)$ such that, the term of order one has constant coefficient in x. This requires to solve the equation

$$a_1^{(3)}(U;x) + 2(1 + a_2^{(3)}(U))\partial_x s(U;x) = a_1^{(4)}(U), \tag{5.92}$$

for some symbol $a_1^{(4)}(U)$ constant in x to be chosen. Equation (5.92) is equivalent to

$$\partial_x s(U;x) = \frac{-a_1^{(3)}(U;x) + a_1^{(4)}(U)}{2(1+a_2^{(3)}(U))}. \tag{5.93}$$

We choose the constant $a_1^{(4)}(U)$ as

$$a_1^{(4)}(U) := \frac{1}{2\pi}\int_{\mathbb{T}} a_1^{(3)}(U;x)dx, \tag{5.94}$$

so that the r.h.s. of (5.93) has zero average, hence the solution of (5.93) is given by

$$s(U;x) := \partial_x^{-1}\left(\frac{-a_1^{(3)}(U;x) + a_1^{(4)}(U)}{2(1+a_2^{(3)}(U))}\right). \tag{5.95}$$

It is easy to check that $s(U;x)$ belongs to $\mathcal{F}_{K,4}[r]$. Using equation (5.91) we get (5.82) with $A^{(4)}(U;\xi)$ as in (5.81).

It remains to prove that the constant $a_1^{(4)}(U)$ in (5.94) is purely imaginary. On one hand, if $A^{(3)}(U;x,\xi)$ satisfies Hypothesis 5.1, we note the following. The coefficient $a_1^{(3)}(U;x)$ must be purely imaginary hence the constant $a_1^{(4)}(U)$ in (5.94) is purely imaginary.

On the other hand, if $A^{(3)}(U;x,\xi)$ satisfies Hypothesis 5.2, we note that the function $a_1^{(3)}(U;x)$ is *odd* in x. This means that the constants $a_1^{(4)}(U)$ in (5.94) is zero. Moreover the symbol $s(U;x)$ in (5.95) is even in x, hence the map $\Phi_4(U)$ in (5.83) is parity preserving according to Definition 2.4 thanks to Remark 3.6. This concludes the proof. □

Proof of Theorem 5.1. It is enough to choose $\Phi(U) := \Phi_4(U)\circ\cdots\circ\Phi_1(U)$. The estimates (5.5) follow by collecting the bounds (5.12), (5.37), (5.64) and (5.80). We define the matrix of symbols $L(U;\xi)$ as

$$L(U;\xi) := \begin{pmatrix} \mathtt{m}(U,\xi) & 0 \\ 0 & \mathtt{m}(U,-\xi)\end{pmatrix}, \quad \mathtt{m}(U,\xi) := a_2^{(3)}(U)(\mathrm{i}\xi)^2 + a_1^{(4)}(U)(\mathrm{i}\xi) \tag{5.96}$$

where the coefficients $a_2^{(3)}(U), a_1^{(4)}(U)$ are x-independent (see (5.81)). One concludes the proof by setting $R_1(U) := R_1^{(4)}(U)$ and $R_2(U) := R_2^{(4)}(U)$. □

An important consequence of Theorem 5.1 is that system (5.1) admits a regular and unique solution. More precisely we have the following.

Proposition 5.1. *Let s_0 given by Theorem 5.1 with $K = 4$. For any $s \geq s_0 + 2$ let $U = U(t,x)$ be a function in $B_s^4([0,T),\theta)$ for some $T > 0$, $r > 0$, $\theta \geq r$ with $\|U(0,\cdot)\|_{\mathbf{H}^s} \leq r$ and consider the system*

$$\begin{cases} \partial_t V = \mathrm{i}E\Big[\Lambda V + \mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))[V] + R_1^{(0)}(U)[V] + R_2^{(0)}(U)[U]\Big], \\ V(0,x) = U(0,x) \in \mathbf{H}^s, \end{cases} \tag{5.97}$$

where the matrix $A(U;x,\xi)$, the operators $R_1^{(0)}(U)$ and $R_2^{(0)}(U)$ satisfy the hypotheses of Theorem 5.1. Then the following holds true.

(i) There exists a unique solution $\psi_U(t)U(0,x)$ of the system (5.97) defined for any $t\in[0,T)$ such that

$$\|\psi_U(t)U(0,x)\|_{4,s} \leq \mathtt{C}\,\|U(0,x)\|_{\mathbf{H}^s}(1+t\mathtt{C}\,\|U\|_{4,s})e^{t\mathtt{C}\|U\|_{4,s}} + t\mathtt{C}\,\|U\|_{4,s}\,e^{t\mathtt{C}\|U\|_{4,s}} + \mathtt{C}, \tag{5.98}$$

where $\mathtt{C}$ is constant depending on s, r, $\sup_{t\in[0,T]}\|U\|_{4,s-2}$ and $\|P\|_{C^1}$.

(ii) In the case that U is even in x, the matrix $A(U;x,\xi)$ and the operator Λ satisfy Hypothesis 5.2, the operator $R_1^{(0)}(U)[\cdot]$ is parity preserving according to Definition 2.4 and $R_2^{(0)}(U)[U]$ is even in x, then the solution $\Psi_U(t)U(0,x)$ is even in $x\in\mathbb{T}$.

Proof. We apply to system (5.97) Theorem 5.1 defining $W = \Phi(U)V$. The system in the new coordinates reads

$$\begin{cases} \partial_t W - \mathrm{i}E\Big[\Lambda W + \mathrm{Op}^{\mathcal{B}W}(L(U;\xi))W + R_1(U)W + R_2(U)[U]\Big] = 0 \\ W(0,x) = \Phi(U(0,x))U(0,x) := W^{(0)}(x), \end{cases} \tag{5.99}$$

where $L(U;\xi)$ is a diagonal, self-adjoint and constant coefficient in x matrix in $\Gamma^2_{4,4}[r]\otimes\mathcal{M}_2(\mathbb{C})$, $R_1(U)$, $R_2(U)$ are in $\mathcal{R}^0_{4,4}[r]\otimes\mathcal{M}_2(\mathbb{C})$. Therefore the solution of the linear problem

$$\begin{cases} \partial_t W - \mathrm{i}E\Big[\Lambda W + \mathrm{Op}^{\mathcal{B}W}(L(U;\xi))\Big]W = 0 \\ W(0,x) = W^{(0)}(x), \end{cases} \tag{5.100}$$

is well defined as long as U is well defined, moreover it is an isometry of $\mathbf{H}^s$. We denote by ψ^t_L the flow at time t of such equation. Then one can define the operator

$$T_{W^{(0)}}(W)(t,x) = \psi^t_L\big(W^{(0)}(x)\big) + \psi^t_L\int_0^t (\psi^s_L)^{-1}\mathrm{i}E\Big(R_1(U(s,x))W(s,x) + R_2(U(s,x))U(s,x)\Big)ds. \tag{5.101}$$

Thanks to (5.5) and by the hypothesis on $U(0,x)$ one has that $\|W^{(0)}\|_{\mathbf{H}^s} \le (1+cr)r$ for some constant $c>0$ depending only on r. In order to construct a fixed point for the operator $T_{W^{(0)}}(W)$ in (5.101) we consider the sequence of approximations defined as follows:

$$\begin{cases} W_0(t,x) = \psi^t_L W^{(0)}(x), \\ W_n(t,x) = T_{W^{(0)}}(W_{n-1})(t,x), \qquad n\ge 1, \end{cases}$$

for $t\in[0,T)$. For the rest of the proof we will denote by $\mathtt{C}$ any constant depending on r, s, $\sup_{t\in[0,T)}\|U(t,\cdot)\|_{4,s-2}$ and $\|P\|_{C^1}$. Using estimates (3.17) one gets for $n\ge 1$

$$\|(W_{n+1}-W_n)(t,\cdot)\|_{\mathbf{H}^s} \le \mathtt{C}\,\|U(t,\cdot)\|_{\mathbf{H}^s}\int_0^t \|(W_n - W_{n-1})(\tau,\cdot)\|_{\mathbf{H}^s}d\tau.$$

Arguing by induction over n, one deduces

$$\|(W_{n+1}-W_n)(t,\cdot)\|_{\mathbf{H}^s} \le \frac{(\mathtt{C}\,\|U(t,\cdot)\|_{\mathbf{H}^s})^n t^n}{n!}\|(W_1-W_0)(t,\cdot)\|_{\mathbf{H}^s}, \tag{5.102}$$

which implies that $W(t,x)=\sum_{n=1}^{\infty}(W_{n+1}-W_n)(t,x)+W_0(t,x)$ is a fixed point of the operator in (5.101) belonging to the space $C^0_{*\mathbb{R}}([0,T);\mathbf{H}^s(\mathbb{T};\mathbb{C}^2))$. Therefore by Duhamel principle the function W is the unique solution of the problem (5.99). Moreover, by using (3.17), we have that the following inequality holds true

$$\|W_1(t,\cdot)-W_0(t,\cdot)\|_{\mathbf{H}^s} \le t\mathtt{C}\big(\|U\|_{\mathbf{H}^s}\left\|W^{(0)}\right\|_{\mathbf{H}^s} + \|U\|_{\mathbf{H}^{s-2}}\|U\|_{\mathbf{H}^s}\big),$$

from which, together with estimates (5.102), one deduces that

$$\begin{aligned} \|W(t,\cdot)\|_{\mathbf{H}^s} &\le \sum_{n=0}^{\infty}\|(W_{n+1}-W_n)(t,\cdot)\|_{\mathbf{H}^s} + \left\|W^{(0)}\right\|_{\mathbf{H}^s} \\ &\le \left\|W^{(0)}\right\|_{\mathbf{H}^s}\Big(1 + t\mathtt{C}\,\|U\|_{\mathbf{H}^s}\sum_{n=0}^{\infty}\frac{(t\mathtt{C}\,\|U\|_{\mathbf{H}^s})^n}{n!}\Big) + t\mathtt{C}\,\|U\|_{\mathbf{H}^s}\sum_{n=0}^{\infty}\frac{(t\mathtt{C}\,\|U\|_{\mathbf{H}^s})^n}{n!} \\ &= \left\|W^{(0)}\right\|_{\mathbf{H}^s}\big(1 + t\mathtt{C}\,\|U\|_{\mathbf{H}^s}e^{t\mathtt{C}\|U\|_{\mathbf{H}^s}}\big) + t\mathtt{C}\,\|U\|_{\mathbf{H}^s}e^{t\mathtt{C}\|U\|_{\mathbf{H}^s}} \\ &\le \mathtt{C}\left\|U^{(0)}\right\|_{\mathbf{H}^s}\big(1 + t\mathtt{C}\,\|U\|_{\mathbf{H}^s}e^{t\mathtt{C}\|U\|_{\mathbf{H}^s}}\big) + t\mathtt{C}\,\|U\|_{\mathbf{H}^s}e^{t\mathtt{C}\|U\|_{\mathbf{H}^s}} \end{aligned}$$

Applying the inverse transformation $V=\Phi^{-1}(U)W$ and using (5.5) we find a solution V of the problem (5.97) such that

$$\|V\|_{\mathbf{H}^s} \le \mathtt{C}\left\|U^{(0)}\right\|_{\mathbf{H}^s}\big(1 + t\mathtt{C}\,\|U\|_{\mathbf{H}^s}e^{t\mathtt{C}\|U\|_{\mathbf{H}^s}}\big) + t\mathtt{C}\,\|U\|_{\mathbf{H}^s}e^{t\mathtt{C}.\|U\|_{\mathbf{H}^s}}$$

We now prove a similar estimate for $\partial_t V$. More precisely one has

$$\begin{aligned}\|\partial_t V\|_{\mathbf{H}^{s-2}} &\leq \|\Lambda V + {\rm Op}^{\mathcal{B}W}(A(U;x,\xi))V\|_{\mathbf{H}^{s-2}} + \|R_1^{(0)}(U)V\|_{\mathbf{H}^{s-2}} + \|R_2^{(0)}(U)U\|_{\mathbf{H}^{s-2}}\\ &\leq \mathtt{C}\,\|V\|_{\mathbf{H}^s} + \mathtt{C}\,\|V\|_{\mathbf{H}^{s-2}} + \mathtt{C}\\ &\leq \mathtt{C}\,\|U(0,x)\|_{\mathbf{H}^s}\,(1+t\mathtt{C}\,\|U\|_{4,s})e^{t\mathtt{C}\|U\|_{4,s}} + t\mathtt{C}\,\|U\|_{4,s}\,e^{t\mathtt{C}\|U\|_{4,s}} + \mathtt{C},\end{aligned} \tag{5.103}$$

where we used estimates (3.19) and (3.17). By differentiating equation (5.97) and arguing as done in (5.103) one can bound the terms $\|\partial_t^k V\|_{\mathbf{H}^{s-2k}}$, for $2\leq k\leq 4$, and hence obtain the (5.98).

In the case that U is even in x, Λ, $A(U;x,\xi)$ satisfy Hypothesis 5.2, $R_1(U)[\cdot]$ is parity preserving according to Definition 2.4 and $R_2^{(0)}(U)[U]$ is even in x we have, by Theorem 5.1, that the map $\Phi(U)$ is parity-preserving. Hence the flow of the system (5.99) preserves the subspace of even functions. This follows by Lemma 2.3. Hence the solution of (5.97) defined as $V=\Phi^{-1}(U)W$ is even in x. This concludes the proof. □

Remark 5.6. In the notation of Proposition 5.1 the following holds true.

- If $R_2^{(0)}\equiv 0$ in (5.97), then the estimate (5.98) may be improved as follows:

$$\|\psi_U(t)U(0,x)\|_{4,s} \leq \mathtt{C}\,\|U(0,x)\|_{\mathbf{H}^s}\,(1+t\mathtt{C}\,\|U\|_{4,s})e^{t\mathtt{C}\|U\|_{4,s}}. \tag{5.104}$$

  This follows straightforward from the proof of Proposition 5.1.
- If $R_2^{(0)}\equiv R_1^{(0)}\equiv 0$ then the flow $\psi_U(t)$ of (5.97) is invertible and $(\psi_U(t))^{-1}U(0,x)$ satisfies an estimate similar to (5.104). To see this one proceed as follows. Let $\Phi(U)[\cdot]$ the map given by Theorem 5.1 and set $\Gamma(t):=\Phi(U)\psi_U(t)$. Thanks to Theorem 5.1, $\Gamma(t)$ is the flow of the linear para-differential equation

$$\begin{cases}\partial_t\Gamma(t) = {\rm i}E{\rm Op}^{\mathcal{B}W}(L(U;\xi))\Gamma(t) + R(U)\Gamma(t),\\ \Gamma(0)={\rm Id},\end{cases}$$

  where $R(U)$ is a remainder in $\mathcal{R}^0_{K,4}[r]$ and ${\rm Op}^{\mathcal{B}W}(L(U;\xi))$ is diagonal, self-adjoint and constant coefficients in x. Then, if $\psi_L(t)$ is the flow generated by ${\rm iOp}^{\mathcal{B}W}(L(U;\xi))$ (which exists and is an isometry of $\mathbf{H}^s$), we have that $\Gamma(t)=\psi_L(t)\circ F(t)$, where $F(t)$ solves the Banach space ODE

$$\begin{cases}\partial_t F(t) = \big((\psi_L(t))^{-1}R(U)\psi_L(t)\big)F(t),\\ F(0)={\rm Id}.\end{cases}$$

  To see this one has to use the fact that the operators ${\rm iOp}^{\mathcal{B}W}(L(U;\xi))$ and $\psi_L(t)$ commutes. Standard theory of Banach spaces ODE implies that $F(t)$ exists and is invertible, therefore $\psi_U(t)$ is invertible as well and $(\psi_U(t))^{-1}=(F(t))^{-1}\circ(\psi_L(t))^{-1}\circ\Phi(U)$. To deduce the estimate satisfied by $(\psi_U(t))^{-1}$ one has to use (3.17) to control the contribution coming from $R(U)$, the fact that $\psi_L(t)$ is an isometry and (5.5).

6. Local existence

In this Section we prove Theorem 1.1. By previous discussions we know that (1.1) is equivalent to the system (4.15) (see Proposition 4.1). Our method relies on an iterative scheme. Namely we introduce the following sequence of linear problems. Let $U^{(0)}\in\mathbf{H}^s$ such that $\|U^{(0)}\|_{\mathbf{H}^s}\leq r$ for some $r>0$. For $n=0$ we set

$$\mathcal{A}_0 := \begin{cases}\partial_t U_0 - {\rm i}E\Lambda U_0 = 0,\\ U_0(0)=U^{(0)}.\end{cases} \tag{6.1}$$

The solution of this problem exists and is unique, defined for any $t\in\mathbb{R}$ by standard linear theory, it is a group of isometries of $\mathbf{H}^s$ (its k-th derivative is a group of isometries of $\mathbf{H}^{s-2k}$) and hence satisfies $\|U_0\|_{4,s}\leq r$ for any $t\in\mathbb{R}$.

For $n\geq 1$, assuming $U_{n-1}\in B^K_{s_0}(I,r)\cap C^K_{*\mathbb{R}}(I,H^s(\mathbb{T},\mathbb{C}^2))$ for some s_0, $K>0$ and $s\geq s_0$, we define the Cauchy problem

$$\mathcal{A}_n := \begin{cases}\partial_t U_n - {\rm i}E\Big[\Lambda U_n + {\rm Op}^{\mathcal{B}W}(A(U_{n-1};x,\xi))U_n + R(U_{n-1})[U_{n-1}]\Big] = 0,\\ U_n(0)=U^{(0)},\end{cases} \tag{6.2}$$

where the matrix of symbols $A(U; x, \xi)$ and the operator $R(U)$ are defined in Proposition 4.1 (see (4.15)).

One has to show that each problem $\mathcal{A}_n$ admits a unique solution U_n defined for $t \in I$. We use Proposition 5.1 in order to prove the following lemma.

Lemma 6.1.

Let f be a C^∞ function from $\mathbb{C}^3$ in $\mathbb{C}$ satisfying Hypothesis 1.1 (resp. Hypothesis 1.2). Let $r > 0$ and consider $U^{(0)}$ in the ball of radius r of $\mathbf{H}^s$ (resp. of $\mathbf{H}^s_e$) centered at the origin. Consider the operators Λ, $R(U)$ and the matrix of symbols $A(U; x, \xi)$ given by Proposition 4.1 with $K = 4$, $\rho = 0$. If f satisfies Hypothesis 1.3, or r is sufficiently small, then there exists $s_0 > 0$ such that for all $s \geq s_0$ the following holds. There exists a time T and a constant θ, both of them depending on r and s, such that for any $n \geq 0$ one has:

$(S1)_n$ *for $0 \leq m \leq n$ there exists a function U_m in*

$$U_m \in B^4_s([0, T), \theta), \tag{6.3}$$

which is the unique solution of the problem $\mathcal{A}_m$; in the case of parity preserving Hypothesis 1.2 the functions U_m for $0 \leq m \leq n$ are even in $x \in \mathbb{T}$;

$(S2)_n$ *for $0 \leq m \leq n$ one has*

$$\|U_m - U_{m-1}\|_{4,s'} \leq 2^{-m} r, \quad s_0 \leq s' \leq s - 2, \tag{6.4}$$

where $U_{-1} := 0$.

Proof. We argue by induction. The $(S1)_0$ and $(S2)_0$ are true thanks to the discussion following the equation (6.1). Suppose that $(S1)_{n-1}$, $(S2)_{n-1}$ hold with a constant $\theta = \theta(s, r, \|P\|_{C^1}) \gg 1$ and a time $T = T(s, r, \|P\|_{C^1}, \theta) \ll 1$. We show that $(S1)_n$, $(S2)_n$ hold with the same constant θ and T.

The Hypothesis 1.1, together with Lemma 4.2 (resp. Hypothesis 1.2 together with Lemma 4.3) implies that the matrix $A(U; x, \xi)$ satisfies Hypothesis 5.1 (resp. Hypothesis 5.2) and Constraint 5.1. The Hypothesis 1.3, together with Lemma 4.4, (or r small enough) implies that $A(U; x, \xi)$ satisfies also the Hypothesis 5.3. Therefore the hypotheses of Theorem 5.1 are fulfilled. In particular, in the case of Hypothesis 1.2, Lemma 4.3 guarantees also that the matrix of operators $R(U)[\cdot]$ is parity preserving according to Definition 2.4.

Moreover by (6.3), we have that $\|U_{n-1}\|_{4,s} \leq \theta$, hence the hypotheses of Proposition 5.1 are fulfilled by system (6.2) with $R_1^{(0)} = 0$, $R_2^{(0)} = R$, $U \rightsquigarrow U_{n-1}$ and $V \rightsquigarrow U_n$ in (5.97). We note that, by $(S2)_{n-1}$, one has that the constant $\mathtt{C}$ in (5.98) does not depend on θ, but it depend only on $r > 0$. Indeed (6.4) implies

$$\|U_{n-1}\|_{4,s-2} \leq \sum_{m=0}^{n-1} \|U_m - U_{m-1}\|_{4,s-2} \leq r \sum_{m=0}^{n-1} \frac{1}{2^m} \leq 2r, \quad \forall t \in [0, T]. \tag{6.5}$$

Proposition 5.1 provides a solution $U_n(t)$ defined for $t \in [0, T]$. By (5.98) one has that

$$\|U_n(t)\|_{4,s} \leq \mathtt{C} \left\| U^{(0)} \right\|_{\mathbf{H}^s} (1 + t\mathtt{C}\,\|U_{n-1}\|_{4,s}) e^{t\mathtt{C}\|U_{n-1}\|_{4,s}} + t\mathtt{C}\,\|U_{n-1}\|_{4,s}\, e^{t\mathtt{C}\|U_{n-1}\|_{4,s}} + \mathtt{C}, \tag{6.6}$$

where $\mathtt{C}$ is a constant depending on $\|U_{n-1}\|_{4,s-2}$, r, s and $\|P\|_{C^1}$, hence, thanks to (6.5), it depends only on r, s, $\|P\|_{C^1}$. We deduce that, if

$$T\mathtt{C}\theta \ll 1, \qquad \theta > \mathtt{C}r2e + e + \mathtt{C}, \tag{6.7}$$

then $\|U_n\|_{4,s} \leq \theta$. If $A(U_{n-1}; x, \xi)$ and Λ satisfy Hypothesis 5.2, $R(U_{n-1})$ is parity preserving then the solution U_n is even in $x \in \mathbb{T}$. Indeed by the inductive hypothesis U_{n-1} is even, hence item (ii) of Proposition 5.1 applies. This proves $(S1)_n$.

Let us check $(S2)_n$. Setting $V_n = U_n - U_{n-1}$ we have that

$$\begin{cases} \partial_t V_n - \mathrm{i}E\Big[\Lambda V_n + \mathrm{Op}^{\mathcal{B}W}(A(U_{n-1}; x, \xi))V_n + f_n\Big] = 0, \\ V_n(0) = 0, \end{cases} \tag{6.8}$$

where

$$f_n := \mathrm{Op}^{\mathcal{B}W}\Big(A(U_{n-1};x,\xi) - A(U_{n-2};x,\xi)\Big)U_{n-1} + R(U_{n-1})U_{n-1} - R(U_{n-2})U_{n-2}. \tag{6.9}$$

Note that, by (4.28), (4.29), we have

$$\begin{aligned}\|f_n\|_{4,s'} &\leq \Big\|\mathrm{Op}^{\mathcal{B}W}\Big(A(U_{n-1};x,\xi) - A(U_{n-2};x,\xi)\Big)U_{n-1}\Big\|_{4,s'} + \|R(U_{n-1})U_{n-1} - R(U_{n-2})U_{n-2}\|_{4,s'} \\ &\leq C\Big[\,\|V_{n-1}\|_{4,s_0}\,\|U_{n-1}\|_{4,s'+2} + (\|U_{n-1}\|_{4,s'} + \|U_{n-2}\|_{4,s'})\,\|V_{n-1}\|_{4,s'}\Big] \\ &\leq C\Big(\,\|U_{n-1}\|_{4,s'+2} + \|U_{n-2}\|_{4,s'+2}\Big)\,\|V_{n-1}\|_{4,s'}\,,\end{aligned} \tag{6.10}$$

where $C > 0$ depends only on s, $\|U_{n-1}\|_{4,s_0}$, $\|U_{n-2}\|_{4,s_0}$. Recalling the estimate (6.5) we can conclude that the constant C in (6.10) depends only on s, r.

The system (6.8) with $f_n = 0$ has the form (5.97) with $R_2^{(0)} = 0$ and $R_1^{(0)} = 0$. Let $\psi_{U_{n-1}}(t)$ be the flow of system (6.8) with $f_n = 0$, which is given by Proposition 5.1. The Duhamel formulation of (6.8) is

$$V_n(t) = \psi_{U_{n-1}}(t)\int_0^t (\psi_{U_{n-1}}(\tau))^{-1}\mathrm{i}Ef_n(\tau)d\tau. \tag{6.11}$$

Then using the inductive hypothesis (6.3), inequality (5.104) and the second item of Remark 5.6 we get

$$\|V_n\|_{4,s'} \leq \theta \mathtt{K}_1 T\|V_{n-1}\|_{4,s'}, \quad \forall\, t\in[0,T], \tag{6.12}$$

where $\mathtt{K}_1 > 0$ is a constant depending r, s and $\|P\|_{C^1}$. If $\mathtt{K}_1\theta T \leq 1/2$ then we have $\|V_n\|_{4,s'} \leq 2^{-n}r$ for any $t\in[0,T)$ which is the $(S2)_n$. □

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Consider the equation (1.1). By Lemma 4.1 we know that (1.1) is equivalent to the system (4.15). Since f satisfies Hypothesis 1.1 (resp. Hypothesis 1.2) and Hypothesis 1.3, then Lemmata 4.2 (resp 4.3) and 4.4 imply that the matrix $A(U;x,\xi)$ satisfies Constraint 5.1 and Hypothesis 5.1 (resp. Hypothesis 5.2 and $R(U)$ is parity preserving according to Definition 2.4). According to this setting consider the problem $\mathcal{A}_n$ in (6.2).

By Lemma 6.1 we know that the sequence U_n defined by (6.2) converges strongly to a function U in $C^0_{*\mathbb{R}}([0,T),\mathbf{H}^{s'})$ for any $s' \leq s-2$ and, up to subsequences,

$$\begin{aligned}&U_n(t) \rightharpoonup U(t), \quad \text{in } \mathbf{H}^s, \\ &\partial_t U_n(t) \rightharpoonup \partial_t U(t), \quad \text{in } \mathbf{H}^{s-2},\end{aligned} \tag{6.13}$$

for any $t\in[0,T)$, moreover the function U is in $L^\infty([0,T),\mathbf{H}^s)\cap \mathrm{Lip}([0,T),\mathbf{H}^{s-2})$. In order to prove that U solves (4.15) it is enough to show that

$$\Big\|\mathrm{Op}^{\mathcal{B}W}(A(U_{n-1};x,\xi))]U_n + R(U_{n-1})[U_{n-1}] - \mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))]U - R(U)[U]\Big\|_{\mathbf{H}^{s-2}}$$

goes to 0 as n goes to ∞. Using (4.28) and (3.19) we obtain

$$\begin{aligned}&\|\mathrm{Op}^{\mathcal{B}W}(A(U_{n-1};x,\xi)U_n - \mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi)U)\|_{\mathbf{H}^{s-2}} \leq \\ &\|\mathrm{Op}^{\mathcal{B}W}(A(U_{n-1};x,\xi) - A(U;x,\xi))U_n\|_{\mathbf{H}^{s-2}} + \|\mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))(U-U_n)\|_{\mathbf{H}^{s-2}} \leq \\ &C\Big(\,\|U-U_n\|_{\mathbf{H}^{s-2}}\,\|U\|_{\mathbf{H}^{s_0}} + \|U-U_{n-1}\|_{\mathbf{H}^{s_0}}\,\|U_n\|_{\mathbf{H}^s}\Big),\end{aligned}$$

which tends to 0 since $s-2 \geq s'$. In order to show that $R(U_{n-1})[U_{n-1}]$ tends to $R(U)[U]$ in $\mathbf{H}^{s-2}$ it is enough to use (4.29). Using the equation (4.15) and the discussion above the solution U has the following regularity:

$$\begin{aligned}&U\in B^4_{s'}([0,T);\theta)\cap L^\infty([0,T),\mathbf{H}^s)\cap \mathrm{Lip}([0,T),\mathbf{H}^{s-2}), \quad \forall\, s_0\leq s'\leq s-2, \\ &\qquad\qquad \|U\|_{L^\infty([0,T),\mathbf{H}^s)} \leq \theta,\end{aligned} \tag{6.14}$$

where θ and s_0 are given by Lemma 6.1. We show that U actually belongs to $C^0_{*\mathbb{R}}([0,T),\mathbf{H}^s)$. Let us consider the problem

$$\begin{cases} \partial_t V - \mathrm{i}E\Big[\Lambda V + \mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))V + R(U)[U]\Big] = 0, \\ V(0) = U^{(0)}, \quad U^{(0)} \in \mathbf{H}^s, \end{cases} \tag{6.15}$$

where the matrices A and R are defined in Proposition 4.1 (see (4.15)) and U is defined in (6.13) (hence satisfies (6.14)). Theorem 5.1 applies to system (6.15) and provides a map

$$\Phi(U)[\cdot] : C^0_{*\mathbb{R}}([0,T),\mathbf{H}^{s'}(\mathbb{T},\mathbb{C}^2)) \to C^0_{*\mathbb{R}}([0,T),\mathbf{H}^{s'}(\mathbb{T},\mathbb{C}^2)), \tag{6.16}$$

which satisfies (5.5) with $K = 4$ and s' as in (6.14). One has that the function $W := \Phi(U)[U]$ solves, the problem

$$\begin{cases} \partial_t W - \mathrm{i}E\Big[\Lambda + \mathrm{Op}^{\mathcal{B}W}(L(U;\xi))\Big]W + R_2(U)[U] + R_1(U)W = 0 \\ W(0) = \Phi(U^{(0)})U^{(0)} := W^{(0)}, \end{cases} \tag{6.17}$$

where $L(U)$ is a diagonal, self-adjoint and constant coefficient in x matrix of symbols in $\Gamma^2_{K,4}[\theta]$, and $R_1(U)$, $R_2(U)$ are matrices of bounded operators (see eq. (5.6)). We prove that W is weakly-continuous in time with values in $\mathbf{H}^s$. First of all note that $U \in C^0([0,T);\mathbf{H}^{s'})$ with s' given in (6.14), therefore W belongs to the same space thanks to (6.16). Moreover W is in $L^\infty([0,T),\mathbf{H}^s)$ (again by (6.14) and (6.16)). Consider a sequence τ_n converging to τ as $n\to\infty$. Let $\phi \in \mathbf{H}^{-s}$ and $\phi_\varepsilon \in C^\infty_0(\mathbb{T};\mathbb{C}^2)$ such that $\|\phi-\phi_\varepsilon\|_{\mathbf{H}^{-s}} \le \varepsilon$. Then we have

$$\begin{aligned} \left|\int_{\mathbb{T}} \big(W(\tau_n) - W(\tau)\big)\phi dx\right| &\le \left|\int_{\mathbb{T}} \big(W(\tau_n) - W(\tau)\big)\phi_\varepsilon dx\right| + \left|\int_{\mathbb{T}} (W(\tau_n) - W(\tau))(\phi-\phi_\varepsilon) dx\right| \\ &\le \|W(\tau_n) - W(\tau)\|_{\mathbf{H}^{s'}}\|\phi_\varepsilon\|_{\mathbf{H}^{-s'}} + \|W(\tau_n) - W(\tau)\|_{\mathbf{H}^s}\|\phi-\phi_\varepsilon\|_{\mathbf{H}^{-s}} \\ &\le C\varepsilon + 2\|W\|_{L^\infty\mathbf{H}^s}\varepsilon \end{aligned} \tag{6.18}$$

for n sufficiently large and where $s' \le s-2$ as above.

Therefore W is weakly continuous in time with values in $\mathbf{H}^s$. In order to prove that W is in $C^0_{*\mathbb{R}}([0,T),\mathbf{H}^s)$, we show that the map $t \mapsto \|W(t)\|_{\mathbf{H}^s}$ is continuous on $[0,T)$. We introduce, for $0<\epsilon\le 1$, the Friedrichs mollifier $J_\epsilon := (1-\epsilon\partial_{xx})^{-1}$ and the Fourier multiplier $\Lambda^s := (1-\partial_{xx})^{s/2}$. Using the equation (6.17) and estimates (3.17) one gets

$$\frac{d}{dt}\left\|\Lambda^s J_\epsilon W(t)\right\|^2_{\mathbf{H}^0} \le C\Big[\|U(t)\|^2_{\mathbf{H}^s}\|W(t)\|_{\mathbf{H}^s} + \|W(t)\|^2_{\mathbf{H}^s}\|U(t)\|_{\mathbf{H}^s}\Big], \tag{6.19}$$

where the right hand side is independent of ϵ and the constant C depends on s and $\|U\|_{\mathbf{H}^{s_0}}$. Moreover, since U, W belong to $L^\infty([0,T),\mathbf{H}^s)$, the right hand side of inequality (6.19) is bounded from above by a constant independent of t. Therefore the function $t\mapsto \|J_\epsilon W(t)\|_{\mathbf{H}^0}$ is Lipschitz continuous in t, uniformly in ϵ. As $J_\epsilon W(t)$ converges to $W(t)$ in the $\mathbf{H}^s$-norm, the function $t\mapsto\|W(t)\|_{\mathbf{H}^0}$ is Lipschitz continuous as well. Therefore W belongs to $C^0_{*\mathbb{R}}([0,T),\mathbf{H}^s)$ and so does U. To recover the regularity of $\frac{d}{dt}U$ one may use equation (4.15).

Let us show the uniqueness. Suppose that there are two solution U and V in $C^0_{*\mathbb{R}}([0,T),\mathbf{H}^s)$ of the problem (4.15). Set $H := U - V$, then H solves the problem

$$\begin{cases} \partial_t H - \mathrm{i}E\Big[\Lambda H + \mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))[H] + R(U)[H]\Big] + \mathrm{i}EF = 0 \\ H(0) = 0, \end{cases} \tag{6.20}$$

where

$$F := \mathrm{Op}^{\mathcal{B}W}\big(A(U;x,\xi) - A(V;x,\xi)\big)V + \big(R(U) - R(V)\big)[V].$$

Thanks to estimates (4.28) and (4.29) we have the bound

$$\|F\|_{\mathbf{H}^{s-2}} \le C\|H\|_{\mathbf{H}^{s-2}}\Big(\|U\|_{\mathbf{H}^s} + \|V\|_{\mathbf{H}^s}\Big). \tag{6.21}$$

By Proposition 5.1, using Duhamel principle and (6.21), it is easy to show the following:

$$\|H(t)\|_{\mathbf{H}^{s-2}} \leq C(r)\int_0^t \|H(\sigma)\|_{\mathbf{H}^{s-2}} d\sigma.$$

Thus by Gronwall Lemma the solution is equal to zero for almost everywhere time t in $[0, T)$. By continuity one gets the unicity. □

Proof of Theorem 1.2. The proof is the same of the one of Theorem 1.1, one only has to note that the matrix $A(U; x, \xi)$ satisfies Hypothesis 5.3 thanks to the smallness of the initial datum instead of Hypothesis 1.3. □

Conflict of interest statement

We wish to confirm that there are no conflicts of interest concerning associated with the publication "Local well-posedness for quasi-linear NLS with large Cauchy data on the circle" by Roberto Feola, Felice Iandoli and there has been no significant financial support for this work that could have influenced its outcome.

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