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Stable ground states for the HMF Poisson model

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Abstract

In this paper we prove the nonlinear orbital stability of a large class of steady state solutions to the Hamiltonian Mean Field (HMF) system with a Poisson interaction potential. These steady states are obtained as minimizers of an energy functional under *one, two or infinitely many constraints*. The singularity of the Poisson potential prevents from a direct run of the general strategy in [19,16] which was based on generalized rearrangement techniques, and which has been recently extended to the case of the usual (smooth) cosine potential [17]. Our strategy is rather based on variational techniques. However, due to the boundedness of the space domain, our variational problems do not enjoy the usual scaling invariances which are, in general, very important in the analysis of variational problems. To replace these scaling arguments, we introduce new transformations which, although specific to our context, remain somehow in the same spirit of rearrangements tools introduced in the references above. In particular, these transformations allow for the incorporation of an arbitrary number of constraints, and yield a stability result for a large class of steady states.
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1. Introduction and main results

1.1. The HMF Poisson model

The Hamiltonian mean-field (HMF) model [22,1] describes the evolution of particles moving on a circle under the action of a given potential. The most popular model is the HMF system with an infinite range attractive cosine potential. Although this model has no direct physical relevance, it is commonly used in the physics literature as a toy model to describe some gravitational systems. In particular, it is involved in the study of non-equilibrium phase transitions [9,26,2,25], of traveling clusters [7,29] or of relaxation processes [28,3,10]. Many results exist concerning the stability of steady state solutions to the HMF system with a cosine potential. Some are about the dynamics of perturbations of inhomogeneous steady states [4,5] and others deal with the linear stability of steady states [9,24,6].

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In [17], the nonlinear stability of inhomogeneous steady states that satisfy an explicit criterion is proved. In the case of homogeneous (i.e. with dependence in velocity only) steady states and a cosine interaction potential, a nonlinear Landau damping analysis has been investigated for the HMF model in Sobolev spaces [14].

There exist other kinds of potentials for the HMF model like the Poisson potential or the screened Poisson potential [11,23]. In this paper, we study the orbital stability of ground states of a HMF model with a Poisson potential. This model is closer to the Vlasov–Poisson system than the HMF model with a cosine potential. The Poisson interaction potential is however more singular, which induces serious technical difficulties and prevent from a complete application of the strategy introduced in [19] for the Vlasov–Poisson system or in [17] for the HMF model with a cosine potential. For this reason, our analysis is based on variational methods. A general approach is introduced allowing to prove the nonlinear stability of a large class of steady states thanks to the study of variational problems with one, two or infinitely many constraints. Notice that, in our case, since the domain of the position is bounded and since the number of constraints may be infinite, scaling arguments like in [20,18] cannot be used. New transformations will be introduced to bypass these technical difficulties.

The HMF Poisson system reads

$$\begin{cases} \partial_t f + v\partial_\theta f - \partial_\theta \phi_f \partial_v f = 0, \quad (t,\theta,v) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}, \\ f(t=0,\theta,v) = f_{init}(\theta,v) \geq 0, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is the flat torus $\mathbb{R}/2\pi\mathbb{Z}$ and $f = f(t,\theta,v)$ is the nonnegative distribution function. The self-consistent potential ϕ_f associated to a distribution function f is defined for $\theta \in \mathbb{T}$ by

$$\partial_\theta^2 \phi_f = \rho_f - \frac{\|f\|_{\mathrm{L}^1}}{2\pi}, \qquad \rho_f(\theta) = \int_{\mathbb{R}} f(\theta,v)\mathrm{d}v \tag{1.2}$$

or, equivalently,

$$\phi_f(\theta) = \int_0^{2\pi} W(\theta - \tilde\theta)\rho_f(\tilde\theta)\mathrm{d}\tilde\theta, \tag{1.3}$$

where the function W is defined on $\mathbb{R}$ by

$$W \text{ is } 2\pi\text{-periodic}, \quad \forall \theta \in [-\pi,\pi], \quad W(\theta) = -\frac{\theta^2}{4\pi} + \frac{|\theta|}{2} - \frac{\pi}{6}.$$

Note that W has a zero average, is continuous on $\mathbb{R}$ and that ϕ_f is 2π-periodic with zero average: $\int_0^{2\pi} \phi_f(\theta)\mathrm{d}\theta = 0$.

Some quantities are invariant during the evolution:

- the Casimir functions: $\iint j(f(\theta,v))\mathrm{d}\theta\mathrm{d}v$, for any function $j \in \mathcal{C}^1(\mathbb{R}_+)$ such that $j(0) = 0$;
- the nonlinear energy:

$$\mathcal{H}(f) = \iint \frac{v^2}{2} f(\theta,v)\mathrm{d}\theta\mathrm{d}v - \frac{1}{2}\int_0^{2\pi} \phi_f'(\theta)^2\mathrm{d}\theta; \tag{1.4}$$

- the total momentum: $\iint vf(\theta,v)\mathrm{d}\theta\mathrm{d}v$.

Moreover, the HMF system satisfies the Galilean invariance, that is, if $f(t,\theta,v)$ is a solution, then so is $f(t,\theta + v_0t, v + v_0)$, for all $v_0 \in \mathbb{R}$.

In Section 2, we prove the orbital stability of stationary states which are minimizers of a one-constraint variational problem. It is obtained for two kinds of steady states: the compactly supported ones and the Maxwell–Boltzmann (non-compactly supported) distributions [10]. In Section 3, we prove the orbital stability of compactly supported steady states which are minimizers of a two constraints problem. In particular, this covers the case of compactly supported steady states which are minimizers of a one constraint problem. Lastly, in Section 4, we prove the orbital stability of the set of all the minimizers of a problem with an infinite number of constraints. This set of minimizers contains the minimizers of one and two constraints problems. However, at this stage, our strategy only provides a collective stability result (stability of the set of minimizers) for the minimizers of this problem with infinite number of constraints, instead of the individual stability of each minimizer which is only obtained for the one and two constraints variational problems.

1.2. Statement of the results

1.2.1. One-constraint problem

First, in Section 2, we will show the orbital stability of stationary states which are minimizers of the following variational problem

$$\mathcal{I}(M) = \inf_{f \in E_j, \|f\|_{L^1}=M} \mathcal{H}(f) + \iint j(f(\theta, v))d\theta dv. \tag{1.5}$$

The constant $M > 0$ is given and E_j is the energy space:

$$E_j = \left\{ f \geq 0, \|(1+v^2)f\|_{L^1} < +\infty, \left| \iint j(f(\theta, v))d\theta dv \right| < +\infty \right\}, \tag{1.6}$$

where $j : \mathbb{R}_+ \to \mathbb{R}$ is either the function defined by $j(t) = t\ln(t)$ for $t > 0$ and $j(0) = 0$ or a function j satisfying the following assumptions

(H1) $j \in \mathcal{C}^2(\mathbb{R}_+^*)$; $j(0) = j'(0) = 0$ and $j''(t) > 0$ for all $t > 0$,
(H2) $\lim_{t\to+\infty} \frac{j(t)}{t} = +\infty$.

Note that $j(t) = t\ln(t)$ satisfies (H2) but not (H1) since $j'(0) \neq 0$ in this case.

Definition 1.1. We shall say that a sequence f_n converges to f in E_j and we shall write $f_n \xrightarrow{E_j} f$ if $\|(1+v^2)(f_n - f)\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0$ and $\iint j(f_n(\theta, v))d\theta\, dv \underset{n\to+\infty}{\longrightarrow} \iint j(f(\theta, v))\, d\theta dv$.

In our first result, we establish the existence of ground states for the HMF Poisson model (1.1) which are minimizers of the variational problem (1.5). This theorem will be proved in Section 2.1.2.

Theorem 1 *(Existence of ground states). Let j be the function $j(t) = t\ln(t)$ or a function satisfying (H1) and (H2). We have:*

(1) *In both cases, the infimum (1.5) exists and is achieved at a minimizer f_0 which is a steady state of (1.1).*
(2) *If j satisfies (H1) and (H2), any minimizer f_0 of (1.5) is continuous, compactly supported, piecewise $\mathcal{C}^1$ and takes the form*

$$f_0(\theta, v) = (j')^{-1}\left(\lambda_0 - \frac{v^2}{2} - \phi_{f_0}(\theta)\right)_+ \quad \textit{for some } \lambda_0 \in \mathbb{R}.$$

The function $(.)_+$ is defined by $(x)_+ = x$ if $x \geq 0$, 0 else.
(3) *If $j(t) = t\ln(t)$, any minimizer f_0 of (1.5) is a $\mathcal{C}^\infty$ function which takes the form*

$$f_0(\theta, v) = \exp\left(\lambda_0 - \frac{v^2}{2} - \phi_{f_0}(\theta)\right) \textit{ for some } \lambda_0 \in \mathbb{R}.$$

Our second result concerns the orbital stability of the above constructed ground states under the action of the HMF Poisson flow. But first and foremost, we need to prove the uniqueness of the minimizers under equimeasurability condition. To do that, first recall the definition of the equimeasurability of two functions.

Definition 1.2. Let f_1 and f_2 be two nonnegative functions in $L^1([0, 2\pi] \times \mathbb{R})$. The functions f_1 and f_2 are said to be equimeasurable, if, and only if, $\mu_{f_1} = \mu_{f_2}$ where μ_f denotes the distribution function of f, defined by

$$\mu_f(s) = |\{(\theta, v) \in [0, 2\pi] \times \mathbb{R} : f(\theta, v) > s\}|, \quad \text{for all } s \geq 0, \tag{1.7}$$

and $|A|$ stands for the Lebesgue measure of a set A.

Lemma 1.1 *(Uniqueness of the minimizer under equimeasurability condition). Let f_1 and f_2 be two equimeasurable steady states of (1.1) which minimize (1.5) with $j(t) = t \ln(t)$ or with j given by a function satisfying (H1) and (H2). Then the steady states f_1 and f_2 are equal up to a shift in θ.*

This lemma will be proved in Section 2.2.1. Now, using the compactness of all the minimizing sequences of (1.5) (which will be obtained along the proof of Theorem 2 in Section 2.2.2) and the uniqueness result given by Lemma 1.1, we can get the following stability result. It will be proved in Section 2.2.2.

Theorem 2 *(Orbital stability of ground states). Consider the variational problem (1.5) with $j(t) = t \ln(t)$ or with j given by a function satisfying (H1) and (H2). In both cases, we have the following result. For all $M > 0$, any steady state f_0 of (1.1) which minimizes (1.5) is orbitally stable under the flow (1.1). More precisely for all $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that the following holds true. Consider $f_{init} \in E_j$ satisfying $\|(1 + v^2)(f_{init} - f_0)\|_{L^1} < \eta(\varepsilon)$ and $|\iint j(f_{init}) - \iint j(f_0)| < \eta(\varepsilon)$. Let $f(t)$ be a weak global solution to (1.1) on $\mathbb{R}_+$ with initial data f_{init} such that the Casimir functions are preserved during the evolution and that $\mathcal{H}(f(t)) \leq \mathcal{H}(f_{init})$. Then there exists a translation shift $\theta(.)$ with values in $[0, 2\pi]$ such that $\forall t \in \mathbb{R}_+^*$, we have*

$$\|(1 + v^2)(f(t, \theta + \theta(t), v) - f_0(\theta, v))\|_{L^1} < \varepsilon.$$

1.2.2. Two-constraints problem

In Section 3, we will show the orbital stability of stationary states which are minimizers of the following variational problem

$$\mathcal{I}(M_1, M_j) = \inf_{\substack{f \in E_j \\ \|f\|_{L^1} = M_1, \|j(f)\|_{L^1} = M_j}} \mathcal{H}(f) \tag{1.8}$$

where E_j is the same energy space as above and the function j satisfies (H1) and (H2) together with the following additional assumption

(H3) There exist $p, q > 1$ such that $p \leq \frac{tj'(t)}{j(t)} \leq q$, for $t > 0$.

Note that j is a nonnegative function. The first result of this part is the following theorem which will be proved in Section 3.2.2.

Theorem 3 *(Existence of ground states). Let j be a function satisfying (H1), (H2) and (H3). We have:*

(1) *The infimum (1.8) exists and is achieved at a minimizer f_0 which is a steady state of (1.1);*
(2) *Any steady state f_0 obtained as a minimizer of (1.8) is continuous, compactly supported, piecewise $\mathcal{C}^1$ and takes the form*

$$f_0(\theta, v) = (j')^{-1}\left(\frac{\frac{v^2}{2} + \phi_{f_0}(\theta) - \lambda_0}{\mu_0}\right)_+ \quad \text{where } (\lambda_0, \mu_0) \in \mathbb{R} \times \mathbb{R}_-^*; \tag{1.9}$$

(3) *The associated density ρ_{f_0} is continuous and the associated potential ϕ_{f_0} is $\mathcal{C}^2$ on $\mathbb{T}$.*

Since the existence of ground states is established, the natural second result is the uniqueness of these ground states. For the two constraints cases, we are only able to obtain a local uniqueness for the ground states under equimeasurability condition. A steady state f will be said to be homogeneous if $\phi_f = 0$ and inhomogeneous is $\phi_f \neq 0$. We have the following lemma which will be proved in Section 3.3.1.

Lemma 1.2 *(Local uniqueness of the minimizer under equimeasurability condition). Let $f_0 \in E_j$ be a steady state of (1.1) and a minimizer of (1.8). It can be written in the form (1.9) with $(\lambda_0, \mu_0) \in \mathbb{R} \times \mathbb{R}_-^*$. We have the following cases:*

- *f_0 is a homogeneous steady state. Then it is the only steady state minimizer of (1.8) under equimeasurability condition.*
- *f_0 is an inhomogeneous steady state. Then, there exists $\delta_0 > 0$ such that for all $f \in E_j$ inhomogeneous steady state of (1.1) and minimizer of (1.8) equimeasurable to f_0 which can be written as (1.9) with $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}_-^*$, we have*
  - *either $\mu_0 \neq \mu$ and $||\mu_0| - |\mu|| > \delta_0$,*
  - *or $\mu_0 = \mu$ and $f_0 = f$ up to a translation shift in θ.*

Then, similarly to the one-constraint problem, we will show the following result concerning the orbital stability of the ground states under the action of the HMF Poisson flow. It will be proved in Section 3.3.2.

Theorem 4 *(Orbital stability of ground states). Let $M_1, M_j > 0$. Then any steady state f_0 of (1.1) which minimizes (1.8) is orbitally stable under the flow (1.1). It means that given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that the following holds true. Consider $f_{init} \in E_j$ with $\|(1+v^2)(f_{init} - f_0)\|_{L^1} < \eta(\varepsilon)$ and with $\left|\iint j(f_{init}) - \iint j(f_0)\right| < \eta(\varepsilon)$. Let $f(t)$ be a weak global solution to (1.1) on $\mathbb{R}_+$ with initial data f_{init} such that the Casimir functions are preserved during the evolution and that $\mathcal{H}(f(t)) \leq \mathcal{H}(f_{init})$. Then there exists a translation shift $\theta(.)$ with values in $[0, 2\pi]$ such that $\forall t \in \mathbb{R}_+^*$, we have*

$$\|(1+v^2)(f(t, \theta + \theta(t), v) - f_0(\theta, v))\|_{L^1} < \varepsilon.$$

1.2.3. Infinite number of constraints problem

Finally, in Section 4, we will show the orbital stability of stationary states which are minimizers of a problem with an infinite number of constraints. In this Section, the energy space is the following

$$\mathcal{E} = \{f \geq 0, \|(1+v^2) f\|_{L^1} < +\infty, \|f\|_{L^\infty} < +\infty\}. \tag{1.10}$$

Let $f_0 \in \mathcal{E} \cap \mathcal{C}^0([0, 2\pi] \times \mathbb{R})$. We will denote by $Eq(f_0)$ the set of equimeasurable functions to f_0. The variational problem is

$$H_0 = \inf_{f \in Eq(f_0), f \in \mathcal{E}} \mathcal{H}(f). \tag{1.11}$$

This is a variational problem with infinitely many constraints since the equimeasurability condition on f is equivalent to say that f has the same Casimirs as f_0: $\|j(f)\|_{L^1} = \|j(f_0)\|_{L^1}, \forall j$.

Definition 1.3. We shall say that a sequence f_n converges to f in $\mathcal{E}$ and we shall write $f_n \xrightarrow{\mathcal{E}} f$ if $(f_n)_n$ is uniformly bounded and satisfies $\|(1+v^2)(f_n - f)\|_{L^1} \underset{n \to +\infty}{\longrightarrow} 0$.

We start by showing in Section 4.2.2 the existence of ground states for the HMF Poisson model (1.1) which are minimizers of the variational problem (1.11).

Theorem 5 *(Existence of ground states). The infimum (1.11) is finite and is achieved at a minimizer $\bar{f} \in \mathcal{E}$ which is a steady state of (1.1).*

Our second result concerns the orbital stability of the above constructed ground states under the action of the HMF flow. As we do not have the uniqueness of the minimizers under constraint of equimeasurability, we can just get the orbital stability of the set of minimizers and not the orbital stability of each minimizer. It will be proved in Section 4.3.1.

Theorem 6 *(Orbital stability of ground states). Let $f_0 \in \mathcal{E} \cap \mathcal{C}^0([0, 2\pi] \times \mathbb{R})$. Then the set of steady states of (1.1) which minimize (1.11) is orbitally stable under the flow (1.1). More precisely given f_{i_0} minimizer of (1.11), for all $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that the following holds true. Consider $f_{init} \in \mathcal{E}$ with $\|(1+v^2)(f_{init} - f_{i_0})\|_{L^1} < \eta(\varepsilon)$. Let $f(t)$ be a weak global solution to (1.1) on $\mathbb{R}^+$ with initial data f_{init} such that the Casimir functions are preserved during the evolution and that $\mathcal{H}(f(t)) \le \mathcal{H}(f_{init})$. Then there exist f_{i_1} minimizer of (1.11) and a translation shift $\theta(.)$ with values in $[0, 2\pi]$ such that $\forall t \in \mathbb{R}_+^*$, we have*

$$\|(1+v^2)(f(t, \theta + \theta(t), v) - f_{i_1}(\theta, v))\|_{L^1} < \varepsilon.$$

2. Minimization problem with one constraint

2.1. Existence of ground states

This section is devoted to the proof of Theorem 1.

2.1.1. Properties of the infimum

For convenience, we set for $f \in E_j$, the below functional

$$J(f) = \mathcal{H}(f) + \iint j(f) = \iint \frac{v^2}{2} f(\theta, v) \mathrm{d}\theta \mathrm{d}v - \frac{1}{2} \int_0^{2\pi} \phi_f'(\theta)^2 \mathrm{d}\theta + \iint j(f(\theta, v)) \mathrm{d}\theta \mathrm{d}v. \tag{2.1}$$

Lemma 2.1. *The variational problem (1.5) satisfies the following statements.*

(1) *Let j be a function satisfying (H1) and (H2) or $j(t) = t\ln(t)$, in both cases, the infimum (1.5) exists, i.e. $\mathcal{I}(M) > -\infty$ for all $M > 0$.*
(2) *For any minimizing sequence $(f_n)_n$ of the variational problem (1.5), we have the following properties:*
   (a) *The minimizing sequence $(f_n)_n$ is weakly compact in $L^1([0, 2\pi] \times \mathbb{R})$, i.e. there exists $\bar{f} \in L^1([0, 2\pi] \times \mathbb{R})$ such that $f_n \underset{n\to+\infty}{\rightharpoonup} \bar{f}$ weakly in L^1.*
   (b) *We have $\|\phi_{f_n} - \phi_{\bar{f}}\|_{H^1} \underset{n\to+\infty}{\longrightarrow} 0$.*

Proof. Let us start with the proof of item (1). Let $f \in E_j$ such that $\|f\|_{L^1} = M$. If j satisfies (H1) and (H2), then j is nonnegative and we have

$$J(f) \ge -\frac{1}{2} \int_0^{2\pi} \phi_f'(\theta)^2 \mathrm{d}\theta \ge -\pi \|W'\|_{L^\infty}^2 M^2$$

and this term is finite for $f \in E_j$. Note that

$$\|\phi_f'\|_{L^\infty} \le \|W'\|_{L^\infty} \|f\|_{L^1}. \tag{2.2}$$

If $j(t) = t\ln(t)$, the sign of j is not constant and we have to bound from below the term $\iint j(f(\theta, v))\mathrm{d}\theta\mathrm{d}v$. With Jensen's inequality and the convexity of $t \mapsto t\ln(t)$, we get

$$\iint f \ln\left(\frac{f}{f_1}\right) \ge \left(\iint f\right)\left[\ln\left(\iint f\right) - \ln\left(\iint f_1\right)\right]. \tag{2.3}$$

Taking $f_1(\theta, v) = e^{-\frac{v^2}{2}}$ and let $C_1 = \ln\left(\iint f_1\right)$, we obtain

$$J(f) \ge -\frac{1}{2} \int_0^{2\pi} \phi_f'^2(\theta)\mathrm{d}\theta + M[\ln(M) - C_1] \ge -\pi \|W'\|_{L^\infty}^2 M^2 + M[\ln(M) - C_1]. \tag{2.4}$$

Each term is finite for $f \in E_j$. Thus $\mathcal{I}(M)$ exists for both functions j.

Then let us continue with the proof of item (2). Let $(f_n)_n$ be a minimizing sequence of (1.5). By the Dunford–Pettis theorem (see [13]), if $\|f_n\|_{L^1}$, $\|v^2 f_n\|_{L^1}$ and $\iint j(f_n(\theta,v))d\theta dv$ are bounded from above, the sequence of functions $(f_n)_n$ is weakly compact in L^1. Indeed, let us show that the set

$$\mathcal{P} = \{f \geq 0, \|f\|_{L^1} = M, \iint v^2 f \leq K_1, \left|\iint j(f)\right| \leq K_2 \text{ with } K_1, K_2 \text{ which do not depend on } f\}$$

satisfies the hypothesis of Dunford–Pettis theorem. Since $\|f\|_{L^1} = M$, it is clear that

$$\sup_{f\in\mathcal{P}} \left\{ \iint |f| d\theta dv \right\} < +\infty.$$

The boundedness of $\|v^2 f\|_{L^1}$ gives the vanishing at infinity. Indeed, we have for $\varepsilon > 0$, there exists $K = [0, 2\pi] \times [-R, R]$ with $R = \sqrt{\frac{K_1}{\varepsilon}}$ a compact subset of $[0, 2\pi] \times \mathbb{R}$ such that

$$\iint_{K^c} |f| d\theta dv \leq \frac{1}{R^2} \iint v^2 f(\theta, v) d\theta dv \leq \varepsilon.$$

Hence

$$\sup_{f\in\mathcal{P}} \left\{ \iint_{K^c} |f| d\theta dv \right\} < \varepsilon.$$

The equi-integrability is given by the boundedness of $\iint j(f)$. Let $0 < \varepsilon < 1$ and $R > 0$ be such that for all $t > R$, $\frac{j(t)}{t} > \frac{2K_2}{\varepsilon}$, there exists $\delta = \frac{\varepsilon}{2R}$ such that for $A \subset [0, 2\pi] \times \mathbb{R}$, $|A| \leq \delta$, we have:

$$\begin{aligned}
\iint_A |f| d\theta dv &= \iint_{\{A, f<R\}} f d\theta dv + \iint_{\{A, f>R\}} f d\theta dv, \\
&\leq R|A| + \iint_{\left\{A, \frac{\varepsilon j(f)}{2K_2} > f\right\}} f d\theta dv, \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2K_2} \iint j(f) d\theta dv \leq \varepsilon.
\end{aligned}$$

Hence

$$\sup_{f\in\mathcal{P}} \left\{ \iint_A |f| d\theta dv \right\} < \varepsilon.$$

Notice that the domain in θ is bounded thus contrary to the Vlasov–Poisson system, there is no loss of mass at infinity. Let us then show that $\|v^2 f_n\|_{L^1}$ is bounded. We have from equality (2.1)

$$\|v^2 f_n\|_{L^1} = 2J(f_n) + \int_0^{2\pi} \phi'_{f_n}(\theta)^2 d\theta - 2 \iint j(f_n(\theta, v) d\theta dv.$$

If j satisfies the hypotheses (H1) and (H2), this equality becomes

$$\|v^2 f_n\|_{L^1} \leq 2J(f_n) + 2\pi \|W'\|^2_{L^\infty} M^2.$$

Since $J(f_n)$ is bounded, we deduce in this case that $\|v^2 f_n\|_{L^1}$ is bounded. If $j(t) = t\ln(t)$, we have

$$\begin{aligned}
\|v^2 f_n\|_{L^1} &\leq 2J(f_n) + 2\pi \|W'\|^2_{L^\infty} M^2 - 2 \iint f_n(\theta, v) \ln(f_n(\theta, v)) d\theta dv \\
&\leq 2J(f_n) + 2\pi \|W'\|^2_{L^\infty} M^2 - 2M[\ln(M) - C_1] + \frac{1}{2} \|v^2 f_n\|_{L^1}
\end{aligned}$$

using Jensen's inequality (2.3) with $f_1(\theta, v) = e^{-\frac{v^2}{4}}$ and $C_1 = \ln(\iint f_1)$. Thus

$$\|v^2 f_n\|_{L^1} \leq 4J(f_n) + 4\pi \|W'\|^2_{L^\infty} M^2 - 4M[\ln(M) - C_1]$$

and this quantity is bounded. Let us then show that $\iint j(f_n(\theta, v))\mathrm{d}\theta\mathrm{d}v$ is bounded from above. Let j be a function satisfying (H1) and (H2) or $j(t) = t\ln(t)$, we have

$$\iint j(f_n(\theta, v))\mathrm{d}\theta\mathrm{d}v \leq J(f_n) + \pi \|W'\|^2_{L^\infty} M^2.$$

Each term of this inequality is bounded, therefore this quantity is bounded. Hence by Dunford–Pettis theorem, there exists $\bar{f} \in L^1$ such that $f_n \underset{n\to+\infty}{\rightharpoonup} \bar{f}$ in L^1_w. This concludes the proof of item (1) of Lemma 2.1. Then, let us prove the last result. Since

$$\phi_{f_n}(\theta) - \phi_{\bar{f}}(\theta) = \int_{\mathbb{R}} \int_0^{2\pi} W(\theta - \tilde{\theta})[f_n(\tilde{\theta}, v) - \bar{f}(\tilde{\theta}, v)]\mathrm{d}\tilde{\theta}\mathrm{d}v,$$

and

$$\phi'_{f_n}(\theta) - \phi'_{\bar{f}}(\theta) = \int_{\mathbb{R}} \int_0^{2\pi} W'(\theta - \tilde{\theta})[f_n(\tilde{\theta}, v) - \bar{f}(\tilde{\theta}, v)]\mathrm{d}\tilde{\theta}\mathrm{d}v,$$

we immediately deduce applying dominated convergence and from the weak convergence of f_n in $L^1([0, 2\pi] \times \mathbb{R})$ that $\|\phi_{f_n} - \phi_{\bar{f}}\|_{H^1} \underset{n\to+\infty}{\longrightarrow} 0$. □

The following lemma is the analog for $j(t) = t\ln(t)$ of a well-known result about the lower semicontinuity properties of convex nonnegative functions see [15]. The proof is not a direct consequence of the lower semicontinuity properties of convex positive functions since $j(t) = t\ln(t)$ changes sign on $\mathbb{R}_+$. It will be detailed in the appendix.

Lemma 2.2. *Let $(f_n)_n$ be a sequence of nonnegative functions converging weakly in L^1 to $\bar{f}$ such that $\|f_n\|_{L^1} = M$, $\|v^2 f_n\|_{L^1} \leq C_1$ and $|\iint f_n \ln(f_n)| \leq C_2$ where M, C_1 and C_2 do not depend on n, we have the following inequality*

$$\iint \bar{f}\ln(\bar{f})\mathrm{d}\theta\mathrm{d}v \leq \liminf_{n\to+\infty} \iint f_n \ln(f_n)\mathrm{d}\theta\mathrm{d}v.$$

2.1.2. *Proof of Theorem 1*

We are now ready to prove Theorem 1.

Step 1: Existence of a minimizer.

Let $M > 0$. From item (1) of Lemma 2.1, we know that $\mathcal{I}(M)$ is finite for functions j satisfying (H1) and (H2) or $j(t) = t\ln(t)$. Let us show that there exists a function $\bar{f} \in E_j$ which minimizes the variational problem (1.5). Let $(f_n)_n \in E_j^{\mathbb{N}}$ be a minimizing sequence of $\mathcal{I}(M)$. Thus $J(f_n) \underset{n\to+\infty}{\longrightarrow} \mathcal{I}(M)$ and $\|f_n\|_{L^1} = M$ where J is defined by (2.1). From item (2) of Lemma 2.1, we know that there exists $\bar{f} \in L^1([0, 2\pi] \times \mathbb{R})$ such that $f_n \underset{n\to+\infty}{\rightharpoonup} \bar{f}$ weakly in $L^1([0, 2\pi] \times \mathbb{R})$. The L^1-weak convergence implies $\|\bar{f}\|_{L^1} = M$ and $\bar{f} \geq 0$ a.e. In the case where j satisfies (H1) and (H2), from lower semicontinuity properties of nonnegative convex functions (see [15]) and from item (b) of Lemma 2.1, we get $\bar{f} \in E_j$. For $j(t) = t\ln(t)$, from lower semicontinuity properties of nonnegative convex functions and item (b) of Lemma 2.1, we get $\|v^2 \bar{f}\|_{L^1} < +\infty$ and from Lemma 2.2 and item (b) of Lemma 2.1, we get $\iint \bar{f}\ln(\bar{f}) < +\infty$. Using Jensen's inequality (2.3) with $f_1(\theta, v) = e^{-\frac{v^2}{2}}$, we get

$$M(\ln(M) - C_1) - \iint \frac{v^2}{2}\bar{f}\mathrm{d}\theta\mathrm{d}v \leq \iint \bar{f}\ln(\bar{f})\mathrm{d}\theta\mathrm{d}v,$$

and we conclude that $|\iint j(f(\theta,v))\mathrm{d}\theta\mathrm{d}v| < +\infty$ and that $\bar{f} \in E_j$. Therefore, in both cases, we have $\mathcal{I}(M) \le J(\bar{f})$. Moreover from item (2) of Lemma 2.1 and classical inequalities about the lower semicontinuity properties of convex nonnegative functions see [15] for j satisfying (H1) and (H2) and Lemma 2.2 for $j(t) = t\ln(t)$, we have the followings inequalities:

$$\mathcal{I}(M) = \lim_{n\to+\infty} J(f_n) \ge \iint \frac{v^2}{2}\bar{f}(\theta,v)\mathrm{d}\theta\mathrm{d}v - \frac{1}{2}\int_0^{2\pi} \phi'_{\bar{f}}(\theta)^2\mathrm{d}\theta + \iint j(\bar{f}(\theta,v))\mathrm{d}\theta\mathrm{d}v.$$

Thus $\mathcal{I}(M) \ge J(\bar{f})$. To recap, we have proved that $\mathcal{I}(M) = J(\bar{f})$ with $\bar{f} \in E_j$ and $\|\bar{f}\|_{\mathrm{L}^1} = M$ thus $\mathcal{I}(M)$ is achieved.

Step 2: Euler–Lagrange equation for the minimizers.

Let $M > 0$ and $\bar{f}$ be a minimizer of $\mathcal{I}(M)$, let us write Euler–Lagrange equations satisfied by $\bar{f}$. For this purpose, for any given potential ϕ, we introduce a new distribution function F^ϕ having mass M and displaying nice monotonicity property for the energy–Casimir functional.

Lemma 2.3. *Let j be a function verifying (H1) and (H2) or $j(t) = t\ln(t)$ and let $M > 0$. For all $\phi : [0,2\pi] \longrightarrow \mathbb{R}$ continuous function, there exists a unique $\lambda \in]\min\phi, +\infty[$ for j satisfying (H1), (H2) and $\lambda \in \mathbb{R}$ for $j(t) = t\ln(t)$ such that the function $F^\phi : [0,2\pi] \times \mathbb{R} \longrightarrow \mathbb{R}_+$ defined by*

$$\begin{cases} F^\phi(\theta,v) = (j')^{-1}\left(\lambda - \frac{v^2}{2} - \phi(\theta)\right)_+ \text{ for } j \text{ satisfying (H1), (H2)} \\ F^\phi(\theta,v) = \exp\left(\lambda - \frac{v^2}{2} - \phi(\theta)\right) \text{ for } j(t) = t\ln(t), \end{cases} \tag{2.5}$$

satisfies $\|F^\phi\|_{\mathrm{L}^1} = M$.

Proof. Letting $\lambda \in \mathbb{R}$, we define

$$\begin{cases} K(\lambda) = \int_0^{2\pi}\int_{\mathbb{R}} (j')^{-1}\left(\lambda - \frac{v^2}{2} - \phi(\theta)\right)_+ \mathrm{d}\theta\mathrm{d}v \text{ for } j \text{ satisfying (H1), (H2)} \\ K(\lambda) = \int_0^{2\pi}\int_{\mathbb{R}} \exp\left(\lambda - \frac{v^2}{2} - \phi(\theta)\right)\mathrm{d}\theta\mathrm{d}v \text{ for } j(t) = t\ln(t). \end{cases} \tag{2.6}$$

Since in both cases, j is strictly convex and $\left|\left\{\frac{v^2}{2} + \phi(\theta) < \lambda\right\}\right|$ is strictly increasing in λ, the map K is strictly increasing on $[\min\phi, +\infty[$ for j satisfying (H1), (H2) and on $\mathbb{R}$ for $j(t) = t\ln(t)$. Note that for j satisfying (H1), (H2), $K(\lambda) = 0$ for $\lambda \le \min\phi$, then we have the following limit: $\lim_{\lambda\to\min\phi} K(\lambda) = 0$ by using the monotone convergence theorem. For $j(t) = t\ln(t)$, we have $\lim_{\lambda\to-\infty} K(\lambda) = 0$. For both functions, we have $\lim_{\lambda\to+\infty} K(\lambda) = +\infty$ by using Fatou's lemma. Hence, there exists a unique λ such that $\|F^\phi\|_{\mathrm{L}^1} = M$. $\square$

We introduce a second problem of minimization, we set $M > 0$. Let $j(t) = t\ln(t)$ or j given by a function satisfying (H1) and (H2).

$$\mathcal{J}_0 = \inf_{\int_0^{2\pi}\phi=0} \mathcal{J}(\phi) \text{ where } \mathcal{J}(\phi) = \iint \left(\frac{v^2}{2} + \phi(\theta)\right) F^\phi(\theta,v)\mathrm{d}\theta\mathrm{d}v + \frac{1}{2}\int_0^{2\pi}\phi'(\theta)^2\mathrm{d}\theta + \iint j(F^\phi), \tag{2.7}$$

where F^ϕ is defined by Lemma 2.3.

Lemma 2.4. *We have the following inequalities:*

(1) *For all $\phi \in H^2([0,2\pi])$ such that $\phi(0) = \phi(2\pi)$ and $\int_0^{2\pi}\phi = 0$, we have $J(F^\phi) \le \mathcal{J}(\phi)$.*

(2) *For all* $f \in E_j$ *with* $\|f\|_{L^1} = M_1$, *we have*

$$\mathcal{I}(M) \le J(F^{\phi_f}) \le \mathcal{J}(\phi_f) \le J(f).$$

Besides $\mathcal{I}(M) = \mathcal{J}_0$.

Proof. First we will show item (1) of this lemma. Let $\phi \in H^2([0, 2\pi])$ such that $\phi(0) = \phi(2\pi)$ and $\int_0^{2\pi} \phi = 0$, we have

$$\mathcal{J}(\phi) = J(F^\phi) - \frac{1}{2}\|\phi'_{F^\phi}\|^2_{L^2} + \frac{1}{2}\|\phi'\|^2_{L^2} + \iint (\phi(\theta) - \phi_{F^\phi}(\theta))F^\phi(\theta, v)\mathrm{d}\theta\mathrm{d}v$$
$$= J(F^\phi) - \frac{1}{2}\|\phi'_{F^\phi}\|^2_{L^2} + \frac{1}{2}\|\phi'\|^2_{L^2} + \int_0^{2\pi} (\phi - \phi_{F^\phi})(\phi''_{F^\phi} + \frac{\|F^\phi\|_{L^1}}{2\pi})\mathrm{d}\theta,$$

since ϕ_{F^ϕ} satisfies the Poisson equation (1.2). Then, after integrating by parts and gathering the terms, we get

$$\mathcal{J}(\phi) = J(F^\phi) + \frac{1}{2}\|\phi'_{F^\phi} - \phi'\|^2_{L^2}.$$

Hence $\mathcal{J}(\phi) \ge J(F^\phi)$. Then, let us show the right inequality of item (2). Let $f \in E_j$ such that $\|f\|_{L^1} = M$. Using $\|F^\phi\|_{L^1} = M$, using the equality (2.5), the functional can be written as

$$J(f) = \mathcal{J}(\phi_f) + \iint \left(\frac{v^2}{2} + \phi_f(\theta)\right)(f(\theta, v) - F^{\phi_f}(\theta, v))\mathrm{d}\theta\mathrm{d}v + \iint j(f) - \iint j(F^\phi)$$
$$= \mathcal{J}(\phi_f) + \iint (\lambda - j'(F^{\phi_f}))(f(\theta, v) - F^{\phi_f}(\theta, v))\mathrm{d}\theta\mathrm{d}v + \iint j(f) - \iint j(F^\phi).$$

We get

$$J(f) = \mathcal{J}(\phi_f) + \iint (j(f) - j(F^{\phi_f}) - j'(F^{\phi_f})(f - F^{\phi_f}))\mathrm{d}\theta\mathrm{d}v. \tag{2.8}$$

The convexity of j gives us the desired inequality. The other inequalities are straightforward. □

We are now ready to get Euler–Lagrange equations. According to Lemma 2.4, if $\bar{f}$ is a minimizer of $\mathcal{I}(M)$, $\bar{\phi} := \phi_{\bar{f}}$ is a minimizer of $\mathcal{J}_0$ and $J(\bar{f}) = \mathcal{J}(\bar{\phi})$. Using (2.8), we get

$$\iint (j(\bar{f}) - j(F^{\bar{\phi}}) - j'(F^{\bar{\phi}})(\bar{f} - F^{\bar{\phi}}))\mathrm{d}\theta\mathrm{d}v = 0.$$

Then writing the Taylor's formula for the function $j(\bar{f})$ and integrating over $[0, 2\pi] \times \mathbb{R}$, we get

$$\iint (\bar{f} - F^{\bar{\phi}})^2 \int_0^1 (1-u)j''(u(\bar{f} - F^{\bar{\phi}}) + F^{\bar{\phi}})\mathrm{d}u = \iint j(\bar{f}) - \iint j(F^{\bar{\phi}}) - \iint (\bar{f} - F^{\bar{\phi}})j'(F^{\bar{\phi}}).$$

Thus $\iint (\bar{f} - F^{\bar{\phi}})^2 \int_0^1 (1-u)j''(u(\bar{f} - F^{\bar{\phi}}) + F^{\bar{\phi}})\mathrm{d}u\mathrm{d}\theta\mathrm{d}v = 0$. As $j'' > 0$, we deduce that $\bar{f} = F^{\bar{\phi}}$. Hence, in the case where j satisfies (H1) and (H2), the minimizer $\bar{f}$ has the following expression

$$\bar{f}(\theta, v) = (j')^{-1}\left(\bar{\lambda} - \frac{v^2}{2} - \phi_{\bar{f}}(\theta)\right)_+ \quad \text{where } \bar{\lambda} \in \mathbb{R}.$$

In the case where $j(t) = t\ln(t)$, we have

$$\bar{f}(\theta, v) = \exp\left(\bar{\lambda} - \frac{v^2}{2} - \phi_{\bar{f}}(\theta)\right), \qquad \text{where } \bar{\lambda} \in \mathbb{R}.$$

Notice that in the case of j satisfying (H1) and (H2), the minimizer is continuous, piecewise $\mathcal{C}^1$ and compactly supported in v. In the case of $j(t) = t\ln(t)$, $\bar{f}$ is a function of class $\mathcal{C}^\infty$. We have shown that any minimizer of (1.5) takes the above form and is at least piecewise $\mathcal{C}^1$ thus clearly any minimizer is a steady state of (1.1). The proof of Theorem 1 is complete.

2.2. Orbital stability of the ground states

To prove the orbital stability result stated in Theorem 2, we first need to prove the uniqueness of the minimizers under equimeasurability condition.

2.2.1. Uniqueness of the minimizers under equimeasurability condition

This section is devoted to the proof of Lemma 1.1. Let f_1 and f_2 be two equimeasurable minimizers of $\mathcal{I}(M)$. In the case where j satisfies (H1) and (H2), they have the following expressions

$$f_1(\theta,v)=(j')^{-1}\left(\lambda_1-\frac{v^2}{2}-\phi_{f_1}(\theta)\right)_+,\quad f_2(\theta,v)=(j')^{-1}\left(\lambda_2-\frac{v^2}{2}-\phi_{f_2}(\theta)\right)_+.$$

In the case where $j(t)=t\ln(t)$, they have the following expressions

$$f_1(\theta,v)=\exp\left(\lambda_1-\frac{v^2}{2}-\phi_{f_1}(\theta)\right),\quad f_2(\theta,v)=\exp\left(\lambda_2-\frac{v^2}{2}-\phi_{f_2}(\theta)\right).$$

They can be written in the form

$$f_1(\theta,v)=G\left(\frac{v^2}{2}+\psi_1(\theta)\right),\quad f_2(\theta,v)=G\left(\frac{v^2}{2}+\psi_2(\theta)\right); \tag{2.9}$$

where $G(t)=(j')^{-1}((-t)_+)$ or $G(t)=\exp(-t)$ with $\psi_i(\theta)=\phi_{f_i}(\theta)-\lambda_i$. In both cases, G is a continuous, strictly decreasing and piecewise $\mathcal{C}^1$ function. The functions f_1 and f_2 are equimeasurable so $\|f_1\|_{L^\infty}=\|f_2\|_{L^\infty}$. Since G is a decreasing function, this means that $G(\min\psi_1)=G(\min\psi_2)$. Besides, G being strictly decreasing and continuous on $\mathbb{R}$, it is one-to-one from $\mathbb{R}$ to $\mathbb{R}_+$ then $\min\psi_1=\min\psi_2=\alpha$. Thus, there exist θ_1 and θ_2 such that

$$\psi_1(\theta_1)=\psi_2(\theta_2)=\alpha,\quad \psi_1'(\theta_1)=\psi_2'(\theta_2)=0.$$

Therefore, ψ_i satisfies

$$\begin{cases}\Psi''(\theta)=\mathcal{G}(\Psi(\theta)),\\ \Psi'(\theta_i)=0,\\ \Psi(\theta_i)=\psi_1(\theta_1)=\psi_2(\theta_2)=\alpha,\end{cases}$$

for $i=1$ or 2 and where $\mathcal{G}(e)=\int_{\mathbb{R}}G(\frac{v^2}{2}+e)\mathrm{d}v-\frac{M}{2\pi}$. In both cases, $\mathcal{G}$ is locally Lipschitz thus according to Cauchy–Lipschitz theorem, $\psi_1=\psi_2$ up to the translation shift $\theta_2-\theta_1$. From (2.9), we get $f_1=f_2$ up to a translation shift in θ.

2.2.2. Proof of Theorem 2

We will prove the orbital stability of steady states of (1.1) which are minimizers of (1.5) in two steps. First, we will assume that all minimizing sequences of $\mathcal{I}(M)$ are compact and deduce that all minimizer is orbitally stable. Then, we will show the compactness of all minimizing sequence.

Step 1: Proof of the orbital stability.

Assume that all minimizing sequences are compact. Let us argue by contradiction. Let f_0 be a minimizer and assume that f_0 is orbitally unstable. Then there exist $\varepsilon_0>0$, a sequence $(f_{init}^n)_n\in E_j^{\mathbb{N}}$ and a sequence $(t_n)_n\in\mathbb{R}_*^+$ such that $\lim\limits_{n\to+\infty}\|(1+v^2)(f_{init}^n-f_0)\|_{L^1}=0$ and $\lim\limits_{n\to+\infty}\left|\iint j(f_{init}^n)-\iint j(f_0)\right|=0$ and for all n, for all $\theta_0\in[0,2\pi]$

$$\begin{cases}\|f^n(t_n,\theta+\theta_0,v)-f_0(\theta,v)\|_{L^1}>\varepsilon_0,\\ \text{or } \|v^2(f^n(t_n,\theta+\theta_0,v)-f_0(\theta,v))\|_{L^1}>\varepsilon_0,\end{cases} \tag{2.10}$$

where $f^n(t_n,\theta,v)$ is a solution to (1.1) with initial data f_{init}^n. Letting $g_n(\theta,v)=f^n(t_n,\theta,v)$, we have $J(g_n)-J(f_0)\le J(f_{init}^n)-J(f_0)\underset{n\to+\infty}{\longrightarrow}0$ since the system (1.1) preserves the Casimir functionals and $\mathcal{H}(f^n(t_n))\le\mathcal{H}(f_{init}^n)$.

Introduce $\tilde{g_n}(\theta, v) = g_n(\theta, \frac{v}{\lambda_n})$ with $\lambda_n = \frac{M}{\|g_n\|_{L^1}}$. This function $\tilde{g_n}$ satisfies $\|\tilde{g_n}\|_{L^1} = M$, thus $0 \leq J(\tilde{g_n}) - J(f_0)$. Notice that

$$J(f_0) \leq J(\tilde{g_n}) \leq \lambda_n[(\lambda_n^2 - 1) \iint \frac{v^2}{2} g_n(\theta, v) d\theta dv - \frac{\lambda_n - 1}{2} \int_0^{2\pi} \phi_{g_n}'^2(\theta) d\theta + J(f_{init}^n)].$$

It is clear that $\lambda_n \underset{n\to+\infty}{\longrightarrow} 1$. Moreover using inequality (2.2), we show that $\left(\int_0^{2\pi} \phi_{g_n}'^2(\theta) d\theta\right)_n$ is a bounded sequence. Then, arguing as in the proof of item (2) of Lemma 2.1, we get $\left(\|v^2 g_n\|_{L^1}\right)_n$ is bounded sequence. Thus, $J(f_0) \leq \lim_{n\to+\infty} J(\tilde{g_n}) \leq J(f_0)$. Hence $(\tilde{g_n})_n$ is a minimizing sequence of $\mathcal{I}(M)$. According to our assumption, it is a compact sequence in E_j: there exists $\tilde{g} \in E_j$ such that, up to an extraction of a subsequence, we have

$$\|g_n - \tilde{g}\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0, \quad \|v^2(g_n - \tilde{g})\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0, \quad \left|\iint j(g_n) - \iint j(\tilde{g})\right| \underset{n\to+\infty}{\longrightarrow} 0. \tag{2.11}$$

According to the conservation properties of HMF Poisson system, we have

$$|\{(\theta, v) \in [0, 2\pi] \times \mathbb{R}, g_n(\theta, v) > t\}| = |\{(\theta, v) \in [0, 2\pi] \times \mathbb{R}, f_{init}^n(\theta, v) > t\}|.$$

Letting $\varepsilon > 0$, we notice that $\forall\, 0 < t < \varepsilon$

$$\begin{cases} \{g_n > t\} \subset \{\{|g_n - \tilde{g}| < \varepsilon\} \cap \{\tilde{g} > t - \varepsilon\}\} \cup \{|g_n - \tilde{g}| \geq \varepsilon\}, \\ \{g_n > t\} \supset \{|g_n - \tilde{g}| < \varepsilon\} \cap \{\tilde{g} > t + \varepsilon\}. \end{cases}$$

Passing to the limit, we get

$$\limsup_{n\to+\infty} |\{g_n > t\}| \leq |\{\tilde{g} > t - \varepsilon\}|, \qquad \liminf_{n\to+\infty} |\{g_n > t\}| \geq |\{\tilde{g} > t + \varepsilon\}|.$$

Then we pass to the limit as $\varepsilon \to 0$ and we get up to an extraction of a subsequence;

$$\lim_{n\to+\infty} |\{g_n > t\}| = |\{\tilde{g} > t\}| \quad \text{for almost all } t > 0.$$

In the same way, we obtain up to an extraction of a subsequence

$$\lim_{n\to+\infty} |\{f_{init}^n > t\}| = |\{f_0 > t\}| \quad \text{for almost all } t > 0.$$

Noticing that the functions $t \to |\{f_0 > t\}|$ and $t \to |\{\tilde{g} > t\}|$ are right-continuous, we get

$$|\{f_0 > t\}| = |\{\tilde{g} > t\}|, \quad \forall t \geq 0.$$

Thus f_0 and g are two equimeasurable minimizers of $\mathcal{I}(M)$ but according to the previous uniqueness result stated in Lemma 1.1, $f_0 = \tilde{g}$ up to a translation shift. To conclude, (2.11) contradicts (2.10) and we have proved that f_0 is orbitally stable.

Step 2: Compactness of the minimizing sequences.

Let j satisfy (H1) and (H2) or $j(t) = t\ln(t)$. Let $(f_n)_n$ be a minimizing sequence of $\mathcal{I}(M)$. Let us show that $(f_n)_n$ is compact in E_j, i.e. that there exists $f_0 \in E_j$ such that $\lim_{n\to+\infty} \|(1 + v^2)(f_n - f_0)\|_{L^1} = 0$ and $\lim_{n\to+\infty} \left|\iint j(f_{init}^n) - \iint j(f_0)\right| = 0$ up to an extraction of a subsequence. Arguing as before in Section 2.1.2, there exists $f_0 \in E_j$ such that $\|f_0\|_{L^1} = M$, $f_n \underset{n\to+\infty}{\rightharpoonup} f_0$ in L_w^1 up to an extraction of a subsequence and $J(f_0) = \mathcal{I}(M)$. From this last equality and the strong convergence in L^2 of the potential established in item (b) of Lemma 2.1, we deduce that

$$\lim_{n\to+\infty} \left(\iint \frac{v^2}{2} f_n(\theta, v) d\theta dv + \iint j(f_n)\right) = \iint \frac{v^2}{2} f_0(\theta, v) d\theta dv + \iint j(f_0). \tag{2.12}$$

From equality (2.12), from lower semicontinuity properties of nonnegative convex functions (see [15]) and from Lemma 2.2, we get

$$\iint j(f_n) \underset{n\to+\infty}{\longrightarrow} \iint j(f_0), \quad \text{and} \quad \iint \frac{v^2}{2} f_n(\theta,v)\mathrm{d}\theta\mathrm{d}v \underset{n\to+\infty}{\longrightarrow} \iint \frac{v^2}{2} f_0(\theta,v)\mathrm{d}\theta\mathrm{d}v. \tag{2.13}$$

There remains to show that $\|v^2(f_n - f_0)\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0$ and $\|f_n - f_0\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0$.

In the case of $j(t) = t\ln(t)$, the Csiszar–Kullback's inequality, see [27], gives us the strong convergence in $L^1([0,2\pi]\times\mathbb{R})$. In our case, this Csiszar–Kullback's inequality writes

$$\|f_n - f_0\|_{L^1}^2 \le 2M \iint f_n \ln\left(\frac{f_n}{f_0}\right). \tag{2.14}$$

Hence, to prove the strong convergence in $L^1([0,2\pi]\times\mathbb{R})$, it is sufficient to prove that

$$\iint f_n \ln\left(\frac{f_n}{f_0}\right)\mathrm{d}\theta\mathrm{d}v \underset{n\to+\infty}{\longrightarrow} 0.$$

Since $f_0(\theta,v) = \exp\left(\lambda_0 - \frac{v^2}{2} - \phi_{f_0}(\theta)\right)$, we have

$$\iint f_n \ln\left(\frac{f_n}{f_0}\right)\mathrm{d}\theta\mathrm{d}v = J(f_n) - J(f_0) + \frac{1}{2}(\|\phi'_{f_n}\|_{L^2}^2 - \|\phi'_{f_0}\|_{L^2}^2) + \iint \phi_{f_0}(f_n - f_0). \tag{2.15}$$

Note that

(1) $J(f_n) - J(f_0) \underset{n\to+\infty}{\longrightarrow} 0$ since $(f_n)_n$ is a minimizing sequence of $\mathcal{I}(M)$,
(2) $\|\phi'_{f_n}\|_{L^2}^2 - \|\phi'_{f_0}\|_{L^2}^2 \underset{n\to+\infty}{\longrightarrow} 0$ since of the strong convergence in $L^2([0,2\pi]\times\mathbb{R})$ of the potential established in item (b) of Lemma 2.1,
(3) $\iint \phi_{f_0}(\theta)(f_n(\theta,v) - f_0(\theta,v))\mathrm{d}\theta\mathrm{d}v \underset{n\to+\infty}{\longrightarrow} 0$ since of the weak convergence of f_n to f_0 in $L^1([0,2\pi]\times\mathbb{R})$.

Hence with (2.14) and (2.15), we get $\|f_n - f_0\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0$. From this strong convergence in $L^1([0,2\pi]\times\mathbb{R})$, we deduce the a.e. convergence of f_n and with Brezis–Lieb's lemma, and the second limit in (2.13), we get the strong convergence of $v^2 f_n$ in $L^1([0,2\pi]\times\mathbb{R})$. Hence the sequence $(f_n)_n$ is compact in E_j.

In the case of j satisfying (H1) and (H2), we again use Brezis–Lieb's lemma, see [8], to get the strong convergence of f_n in L^1. We already have that $\|f_n\|_{L^1} \underset{n\to+\infty}{\longrightarrow} \|f_0\|_{L^1}$. Hence, with Brezis–Lieb's lemma, it is sufficient to show that $f_n \underset{n\to+\infty}{\longrightarrow} f_0$ a.e. Writing the Taylor formula for the function $j(f_n)$ and integrating over $[0,2\pi]\times\mathbb{R}$, we get

$$\iint (f_n - f_0)^2 \int_0^1 (1-u)j''(u(f_n - f_0) + f_0)\mathrm{d}u = \iint j(f_n) - \iint j(f_0) - \iint (f_n - f_0)j'(f_0). \tag{2.16}$$

Note also that

(1) $\iint j(f_n) \underset{n\to+\infty}{\longrightarrow} \iint j(f_0)$,
(2) $\iint j'(f_0)(f_n - f_0) \underset{n\to+\infty}{\longrightarrow} 0$ since $f_n \underset{n\to+\infty}{\rightharpoonup} f_0$ L^1_w. Note that $j'(f_0) \in L^\infty$ since $f_0 \in L^\infty$.

Hence with Fubini–Tonelli's theorem, we get

$$\iint (f_n - f_0)^2 j''((f_n - f_0)u + f_0) \underset{n\to+\infty}{\longrightarrow} 0 \text{ for almost all } u \in [0,1].$$

Let $u_0 \in [0,1]$ such that $\iint (f_n - f_0)^2 j''((f_n - f_0)u_0 + f_0) \underset{n\to+\infty}{\longrightarrow} 0$. Up to an extraction of a subsequence, we have

$$(f_n - f_0)^2 j''((f_n - f_0)u_0 + f_0) \underset{n\to+\infty}{\longrightarrow} 0 \text{ for almost all } (\theta, v) \in [0, 2\pi] \times \mathbb{R}.$$

This means there exists Ω_{u_0} such that $|\Omega_{u_0}| = 0$ and $\forall (\theta, v) \in [0, 2\pi] \times \mathbb{R} \setminus \Omega_{u_0}$,

$$(f_n(\theta, v) - f_0(\theta, v))^2 j''(u_0(f_n(\theta, v) - f_0(\theta, v)) + f_0(\theta, v)) \underset{n\to+\infty}{\longrightarrow} 0. \tag{2.17}$$

Let us show that, up to a subsequence, $f_n(\theta, v) \underset{n\to+\infty}{\longrightarrow} f_0(\theta, v)$ for $(\theta, v) \in [0, 2\pi] \times \mathbb{R} \setminus \Omega_{u_0}$. If $u_0 = 0$, we directly have the wanted convergence. Then let $u_0 \in]0, 1]$ and let $l(\theta, v)$ be a limit point of $(f_n(\theta, v))_n$. Assume that $l(\theta, v) \neq f_0(\theta, v)$.

- First case: $l(\theta, v) < +\infty$. As j'' is continuous and $j'' > 0$, we have

$$\begin{aligned}&(f_n(\theta, v) - f_0(\theta, v))^2 j''(u_0(f_n(\theta, v) - f_0(\theta, v)) + f_0(\theta, v))\\ &\qquad \underset{n\to+\infty}{\longrightarrow} (l(\theta, v) - f_0(\theta, v))^2 j''(u_0(l(\theta, v) - f_0(\theta, v)) + f_0(\theta, v)) > 0.\end{aligned}$$

  This contradicts (2.17).
- Second case: $l(\theta, v) = +\infty$. Thus:

$$(f_n(\theta, v) - f_0(\theta, v))^2 \underset{n\to+\infty}{\longrightarrow} +\infty \text{ and } u_0(f_n(\theta, v) - f_0(\theta, v)) + f_0(\theta, v) \underset{n\to+\infty}{\longrightarrow} +\infty. \tag{2.18}$$

  However the hypothesis (H2) implies that $t^2 j''(t)$ does not converge to 0 when t goes to infinity. Indeed, arguing by contradiction, integrating twice over $[x_0, x]$ and taking the limit for $x \to +\infty$, we get

$$\forall \varepsilon > 0,\ \exists M > 0,\ \text{such that } \forall x > M,\ 0 \le \frac{j(x)}{x} \le \frac{\varepsilon}{x_0} + j'(x_0).$$

  This inequality contradicts (H2) then $t^2 j''(t)$ does not converge to 0 when t goes to infinity and (2.18) contradicts (2.17).

Hence $f_n \underset{n\to+\infty}{\longrightarrow} f_0$ a.e. and we conclude using the Brezis–Lieb's lemma. The minimizing sequence is compact in E_j.

3. Problem with two constraints

3.1. Toolbox for the two constraints problem

In this section, we define a new function denoted by F^ϕ. Note that the function F^ϕ of (3.1) differs from the one of Section 2.1.2. However it can be seen as an equivalent of (2.5) in the sense that both functions F^ϕ satisfy the constraints of the one and two constraints problem respectively. There will be no possible confusion since the function F^ϕ of Section 2.1.2 will no longer be used. First, thank to this new function, the existence of minimizers is shown. Indeed the sequence $(F^{\phi_{f_n}})_n$ has better compactness properties than the sequence $(f_n)_n$. Then, we get the compactness of the sequence $(f_n)_n$ via the sequence $(F^{\phi_{f_n}})_n$ thanks to monotonicity properties of $\mathcal{H}$ with respect to the transformation F^ϕ. These properties will be detailed in Lemma 3.2. More precisely, we have the following lemma:

Lemma 3.1. *Let j be a function verifying (H1), (H2) and (H3) and let $M_1, M_j > 0$. For all $\phi : [0, 2\pi] \longrightarrow \mathbb{R}$ continuous function, there exists a unique pair $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}_-^*$ such that the function $F^\phi : [0, 2\pi] \times \mathbb{R} \longrightarrow \mathbb{R}_+$ defined by*

$$F^\phi(\theta, v) = (j')^{-1}\left(\frac{\frac{v^2}{2} + \phi(\theta) - \lambda}{\mu}\right)_+ \ \textit{satisfies}\ \|F^\phi\|_{L^1} = M_1,\ \|j(F^\phi)\|_{L^1} = M_j. \tag{3.1}$$

Proof. Letting $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}_-^*$, we define

$$K(\lambda, \mu) = \int_0^{2\pi} \int_{\mathbb{R}} (j')^{-1} \left(\frac{\frac{v^2}{2} + \phi(\theta) - \lambda}{\mu} \right)_+ \mathrm{d}\theta \mathrm{d}v.$$

We set $\mu \in \mathbb{R}_-^*$, since j is strict convex and $\left| \left\{ \frac{v^2}{2} + \phi(\theta) < \lambda \right\} \right|$ is strictly increasing in λ, the map $\lambda \to K(\lambda, \mu)$ is strictly increasing on $[\min \phi, +\infty[$. Note that $K(\lambda, \mu) = 0$ for $\lambda \le \min \phi$. We also have the following limits: $\lim\limits_{\lambda \to \min \phi} K(\lambda, \mu) = 0$ using the monotone convergence theorem and $\lim\limits_{\lambda \to +\infty} K(\lambda, \mu) = +\infty$ using Fatou's lemma. Therefore, there exists a unique $\lambda = \lambda(\mu) \in\,]\min \phi, +\infty[$ such that $\|F^\phi\|_{L^1} = M_1$. We now define the map:

$$G : \begin{cases} \mathbb{R}_-^* \longrightarrow \mathbb{R}_+ \\ \mu \to \int_0^{2\pi} \int_{\mathbb{R}} j \circ (j')^{-1} \left(\frac{\frac{v^2}{2} + \phi(\theta) - \lambda(\mu)}{\mu} \right)_+ \mathrm{d}\theta \mathrm{d}v. \end{cases}$$

Our purpose is to show that G is continuous, strictly increasing on $\mathbb{R}_-^*$ and that $\lim\limits_{\mu \to -\infty} G(\mu) = 0$ and $\lim\limits_{\mu \to 0} G(\mu) = +\infty$. This claim would imply that there exists a unique $\mu \in \mathbb{R}_-^*$ such that $G(\mu) = M_j$ and the proof of the lemma will be ended.

To get the monotony of G and the continuity of λ on $\mathbb{R}_-^*$, we first have to show the decrease of λ. Since $K(\lambda(\mu), \mu) = M_1$, using that both functions $\lambda \mapsto K(\lambda, \mu)$ and $\mu \mapsto K(\lambda, \mu)$ are increasing, we get that the map λ is nonincreasing on $\mathbb{R}_-^*$. According to the definition of G, it is sufficient to show that $\mu \to \lambda(\mu)$ is continuous on $\mathbb{R}_-^*$ to get the continuity of G on $\mathbb{R}_-^*$. To prove the continuity of λ, we argue by contradiction. Assume that $\mu \to \lambda(\mu)$ is discontinuous at $\mu_0 < 0$. Assume on the one hand that λ is left-discontinuous, i.e. there exist $\varepsilon_0 > 0$ and an increasing sequence $(\mu_n)_n \in (\mathbb{R}_-^*)^{\mathbb{N}}$ converging to μ_0 such that $|\lambda(\mu_n) - \lambda(\mu_0)| > \varepsilon_0$. λ being nonincreasing and j being convex, we get

$$M_1 \ge K(\lambda(\mu_0) + \varepsilon_0, \mu_n).$$

Applying Fatou's lemma, we have

$$K(\lambda(\mu_0) + \varepsilon_0, \mu_n) \ge K(\lambda(\mu_0) + \varepsilon_0, \mu_0).$$

Since $K(\lambda(\mu_0) + \varepsilon_0, \mu_0) > M_1$, we get a contradiction and λ is left-continuous. On the other hand, assume that λ is right-discontinuous at $\mu_0 < 0$, i.e. there exist $\varepsilon_0 > 0$ and a decreasing sequence $(\mu_n)_n \in (\mathbb{R}_-^*)^{\mathbb{N}}$ converging to μ_0 such that $|\lambda(\mu_n) - \lambda(\mu_0)| > \varepsilon_0$. λ being nonincreasing and j being convex, we get

$$M_1 \le K(\lambda(\mu_0) - \varepsilon_0, \mu_n).$$

Using a generalization of the Beppo Levi's theorem for the decreasing functions, we get

$$K(\lambda(\mu_0) - \varepsilon_0, \mu_n) \le K(\lambda(\mu_0) - \varepsilon_0, \mu_0).$$

Since $K(\lambda(\mu_0) - \varepsilon_0, \mu_0) < M_1$, we get a contradiction and λ is right-continuous. We conclude that the map λ is continuous on $\mathbb{R}_-^*$. Let us show the increase of G. Before that, notice that $K(\lambda, \mu)$ can be written as

$$K(\lambda, \mu) = 2\sqrt{2} \int_0^{2\pi} \int_0^{+\infty} \frac{1}{j'' \circ (j')^{-1}(t)} \sqrt{(\mu t + \lambda - \phi(\theta))_+} \mathrm{d}t \mathrm{d}\theta, \tag{3.2}$$

by performing a change of variables: $t = \frac{\frac{v^2}{2} + \phi(\theta) - \lambda}{\mu}$ and an integration by parts. By doing the exact same thing for G, we can also write

$$G(\mu) = 2\sqrt{2} \int_0^{2\pi} \int_0^{+\infty} \frac{t}{j'' \circ (j')^{-1}(t)} \sqrt{(\mu t + \lambda(\mu) - \phi(\theta))_+} \mathrm{d}t \mathrm{d}\theta. \tag{3.3}$$

Let $\mu_1, \mu_2 \in \mathbb{R}_-^*$ be such that $\mu_1 \neq \mu_2$. Thanks to the previous step, there exists, for $i = 1, 2$, $\lambda_i := \lambda(\mu_i) \in]\min\phi, +\infty[$ such that $K(\lambda_i, \mu_i) = M_1$. Hence, by using the equality (3.2) and by setting for $i = 1, 2$, $A_{\mu_i} := \mu_i t + \lambda_i - \phi(\theta)$, we get

$$K(\lambda_1, \mu_1) - K(\lambda_2, \mu_2) = 2\sqrt{2}\int_0^{2\pi}\int_0^{+\infty} \frac{1}{j'' \circ (j')^{-1}(t)}[(A_{\mu_1})_+^{\frac{1}{2}} - (A_{\mu_2})_+^{\frac{1}{2}}]\mathrm{d}t\mathrm{d}\theta = 0. \tag{3.4}$$

Then, by using (3.3) and (3.4), we have for all $C \in \mathbb{R}$

$$G(\mu_1) - G(\mu_2) = 2\sqrt{2}\int_0^{2\pi}\int_0^{+\infty} \frac{t + C}{j'' \circ (j')^{-1}(t)}[(A_{\mu_1})_+^{\frac{1}{2}} - (A_{\mu_2})_+^{\frac{1}{2}}]\mathrm{d}t\mathrm{d}\theta.$$

We set $C_0 := \frac{\lambda_1 - \lambda_2}{\mu_1 - \mu_2}$ and we get

$$(\mu_1 - \mu_2)(G(\mu_1) - G(\mu_2)) = 2\sqrt{2}\int_0^{2\pi}\int_0^{+\infty} \frac{(A_{\mu_1} - A_{\mu_2})}{j'' \circ (j')^{-1}(t)}[(A_{\mu_1})_+^{\frac{1}{2}} - (A_{\mu_2})_+^{\frac{1}{2}}]\mathrm{d}t\mathrm{d}\theta. \tag{3.5}$$

Since the function $t \mapsto (t)_+^{\frac{1}{2}}$ is nondecreasing, we have $(A_{\mu_1} - A_{\mu_2})[(A_{\mu_1})_+^{\frac{1}{2}} - (A_{\mu_2})_+^{\frac{1}{2}}] \geq 0$. Hence G is a nondecreasing function. We now notice that $(A_{\mu_1} - A_{\mu_2})[(A_{\mu_1})_+^{\frac{1}{2}} - (A_{\mu_2})_+^{\frac{1}{2}}] > 0$ for $\theta \in \{\phi < \lambda_1\}$ and $t \in]0, \frac{\phi(\theta)-\lambda_1}{\mu_1}[$. Besides the measure of the set $\{\phi < \lambda_1\}$ is strictly positive because $\lambda_1 > \min\phi$. Thus, the function G is strictly increasing on $\mathbb{R}_-^*$.

It remains to compute the limits of G. First let us prove that $\lim_{\mu\to-\infty} \lambda(\mu) = +\infty$. The function λ being nonincreasing, $\lim_{\mu\to-\infty} \lambda(\mu)$ exists and we denote it by λ_∞. Assume that $\lambda_\infty < \infty$. We have

$$M_1 = K(\lambda(\mu), \mu) \leq K(\lambda_\infty, \mu) \underset{\mu\to-\infty}{\longrightarrow} 0.$$

This is a contradiction then $\lim_{\mu\to-\infty} \lambda(\mu) = +\infty$. Then let us prove that $\lim_{\mu\to 0^-} \lambda(\mu) = \min\phi$. λ being nonincreasing, $\lim_{\mu\to 0^-} \lambda(\mu)$ exists and we denote it by λ_0. We have to deal with three cases. First, notice that (H2) and (H3) imply $\lim_{t\to+\infty} (j')^{-1}(t) = +\infty$, then we get

$$\begin{cases} \text{if } \lambda_0 > \min\phi : M_1 = K(\lambda(\mu), \mu) > K(\lambda_0, \mu) \underset{\mu\to 0^-}{\longrightarrow} +\infty, & \text{applying Fatou's lemma,} \\ \text{if } \lambda_0 < \min\phi : M_1 = K(\lambda(\mu), \mu) < K(\frac{\min\phi+\lambda_0}{2}, \mu) = 0 & \text{since } \frac{\min\phi+\lambda_0}{2} < \min\phi. \end{cases}$$

Hence only the third case can occur, i.e. $\lim_{\mu\to 0^-} \lambda(\mu) = \min\phi$.

Let us continue with the computation of $\lim_{\mu\to 0^-} G(\mu)$. Performing the change of variables: $u = \frac{v}{\sqrt{2(\lambda(\mu)-\phi(\theta))_+}}$, we get

$$G(\mu) = 2\sqrt{2}\int_0^{2\pi}\int_0^{1} \sqrt{(\lambda(\mu) - \phi(\theta))_+}\, j \circ (j')^{-1}\left(\frac{(\lambda(\mu) - \phi(\theta))_+}{|\mu|}(1 - u^2)\right)\mathrm{d}\theta\mathrm{d}u$$

and

$$M_1 = 2\sqrt{2}\int_0^{2\pi}\int_0^{1} \sqrt{(\lambda(\mu) - \phi(\theta))_+}\,(j')^{-1}\left(\frac{(\lambda(\mu) - \phi(\theta))_+}{|\mu|}(1 - u^2)\right)\mathrm{d}\theta\mathrm{d}u. \tag{3.6}$$

Then applying Jensen's inequality to the convex function j, we obtain

$$j\left(\frac{M_1}{\int_0^{2\pi} 2\sqrt{2}\sqrt{(\lambda(\mu)-\phi(\theta))_+}\mathrm{d}\theta}\right)_+ \le \frac{G(\mu)}{\int_0^{2\pi} 2\sqrt{2}\sqrt{(\lambda(\mu)-\phi(\theta))_+}\mathrm{d}\theta}.$$

Hence

$$G(\mu) \ge \frac{j\left(\frac{M_1}{\alpha(\mu)}\right)}{\frac{M_1}{\alpha(\mu)}} M_1 \text{ with } \alpha(\mu) = 2\sqrt{2}\int_0^{2\pi} \sqrt{(\lambda(\mu)-\phi(\theta))_+}\mathrm{d}\theta. \tag{3.7}$$

Using the dominated convergence theorem, we show that $\alpha(\mu) \underset{\mu\to 0^-}{\longrightarrow} 0$. But j satisfies (H2) therefore

$$\frac{j\left(\frac{M_1}{\alpha(\mu)}\right)}{\frac{M_1}{\alpha(\mu)}} \underset{\mu\to 0^-}{\longrightarrow} +\infty \text{ and } \lim_{\mu\to 0^-} G(\mu) = +\infty.$$

Let us continue with the computation of $\lim_{\mu\to-\infty} G(\mu)$. The hypothesis (H3) implies the following inequality:

$$\frac{t(j')^{-1}(t)}{q} \le j\circ(j')^{-1}(t) \le \frac{t(j')^{-1}(t)}{p}. \tag{3.8}$$

Thanks to (3.8), we can estimate

$$0 \le G(\mu) \le \frac{M_1}{p}\frac{(\lambda(\mu)-\min\phi)_+}{|\mu|} \tag{3.9}$$

Let us show that $\frac{M_1}{p}\frac{(\lambda(\mu)-\min\phi)_+}{|\mu|} \underset{\mu\to-\infty}{\longrightarrow} 0$. Using the expression of M_1 given by (3.6), we get

$$M_1 \ge \sqrt{(\lambda(\mu)-\max\phi)_+}4\pi\sqrt{2}\int_0^1 (j')^{-1}\left(\frac{(\lambda(\mu)-\max\phi)_+}{|\mu|}(1-u^2)\right)\mathrm{d}u \ge 0.$$

For $|\mu|$ sufficiently large, we have $(\lambda(\mu)-\max\phi)_+ > 0$. Therefore, we have

$$\frac{M_1}{\sqrt{(\lambda(\mu)-\max\phi)_+}}\frac{1}{4\pi\sqrt{2}} \ge \int_0^1 (j')^{-1}\left(\frac{(\lambda(\mu)-\max\phi)_+}{|\mu|}(1-u^2)\right)\mathrm{d}u \ge 0,$$

the term on the left side converges to 0. Hence using Fatou's lemma, we get

$$\int_0^1 \liminf_{\mu\to-\infty}(j')^{-1}\left(\frac{(\lambda(\mu)-\max\phi)_+}{|\mu|}(1-u^2)\right)\mathrm{d}u = 0.$$

We deduce that $\frac{(\lambda(\mu)-\max\phi)_+}{|\mu|} \underset{\mu\to-\infty}{\longrightarrow} 0$ and we conclude with (3.9) that $\lim_{\mu\to-\infty} G(\mu) = 0$. The proof is complete. □

As mentioned before the sequence $(F^{\phi_{f_n}})_n$ will be used to show the existence of minimizers of (1.8) and the compactness of minimizing sequences. To do that, we need to link $\mathcal{H}(f_n)$ and $\mathcal{H}(F^{\phi_{f_n}})$. For this purpose, we introduce a second problem of minimization and we set $M_1, M_j > 0$.

$$\mathcal{J}_0 = \inf_{\int_0^{2\pi}\phi=0} \mathcal{J}(\phi) \text{ where } \mathcal{J}(\phi) = \iint \left(\frac{v^2}{2}+\phi(\theta)\right)F^{\phi}(\theta,v)\mathrm{d}\theta\mathrm{d}v + \frac{1}{2}\int_0^{2\pi}\phi'(\theta)^2\mathrm{d}\theta, \tag{3.10}$$

where F^{ϕ} is defined by Lemma 3.1.

Lemma 3.2. *We have the following inequalities:*

(1) *For all $\phi \in H^2([0,2\pi])$ such that $\phi(0)=\phi(2\pi)$ and $\int_0^{2\pi}\phi = 0$, we have $\mathcal{H}(F^{\phi}) \le \mathcal{J}(\phi)$.*
(2) *For all $f \in E_j$ with $\|f\|_{L^1} = M_1$ and $\|j(f)\|_{L^1} = M_j$, we have*

$$\mathcal{I}(M_1, M_j) \le \mathcal{H}(F^{\phi_f}) \le \mathcal{J}(\phi_f) \le \mathcal{H}(f).$$

Besides $\mathcal{I}(M_1, M_j) = \mathcal{J}_0$.

Proof. First, let us show item (1) of this lemma. Let $\phi \in H^2([0,2\pi])$ such that $\phi(0)=\phi(2\pi)$ and $\int_0^{2\pi}\phi = 0$, we have

$$\mathcal{J}(\phi) = \mathcal{H}(F^{\phi}) - \frac{1}{2}\|\phi'_{F^\phi}\|_{L^2}^2 + \frac{1}{2}\|\phi'\|_{L^2}^2 + \iint (\phi(\theta) - \phi_{F^\phi}(\theta)) F^{\phi}(\theta, v)\mathrm{d}\theta\mathrm{d}v$$

$$= \mathcal{H}(F^{\phi}) - \frac{1}{2}\|\phi'_{F^\phi}\|_{L^2}^2 + \frac{1}{2}\|\phi'\|_{L^2}^2 + \int_0^{2\pi} (\phi - \phi_{F^\phi})(\phi''_{F^\phi} + \frac{\|F^{\phi}\|_{L^1}}{2\pi})\mathrm{d}\theta,$$

since ϕ_{F^ϕ} satisfies the Poisson equation (1.2). Then, after integrating by parts and gathering the terms, we get

$$\mathcal{J}(\phi) = \mathcal{H}(F^{\phi}) + \frac{1}{2}\|\phi'_{F^\phi} - \phi'\|_{L^2}^2. \tag{3.11}$$

Hence $\mathcal{J}(\phi) \ge \mathcal{H}(F^{\phi})$. Then, let us show the right inequality of item (2). Let $f \in E_j$ such that $\|f\|_{L^1} = M_1$ and $\|j(f)\|_{L^1} = M_j$. Using $\|F^{\phi}\|_{L^1} = M_1$ and $\|j(F^{\phi})\|_{L^1} = M_j$, using equality (1.4), the Hamiltonian can be written in the form

$$\mathcal{H}(f) = \mathcal{J}(\phi_f) + \iint \left(\frac{v^2}{2} + \phi_f(\theta)\right)(f(\theta,v) - F^{\phi_f}(\theta,v))\mathrm{d}\theta\mathrm{d}v$$

$$= \mathcal{J}(\phi_f) + \iint (\mu j'(F^{\phi_f}) + \lambda)(f(\theta,v) - F^{\phi_f}(\theta,v))\mathrm{d}\theta\mathrm{d}v.$$

We get

$$\mathcal{H}(f) = \mathcal{J}(\phi_f) - \mu \iint (j(f) - j(F^{\phi_f}) - j'(F^{\phi_f})(f - F^{\phi_f}))\mathrm{d}\theta\mathrm{d}v. \tag{3.12}$$

The convexity of j gives us the desired inequality. The other inequalities are straightforward. □

3.2. Existence of ground states

This section is devoted to the proof of Theorem 3.

3.2.1. Properties of the infimum

Lemma 3.3. *The variational problem (1.8) satisfies the following statements.*

(1) *The infimum (1.8) exists, i.e. $\mathcal{I}(M_1, M_j) > -\infty$ for $M_1, M_j > 0$.*
(2) *For any minimizing sequence $(f_n)_n$ of the variational problem (1.8), we have the following properties:*
   (a) *The minimizing sequence $(f_n)_n$ is weakly compact in $L^1([0,2\pi]\times\mathbb{R})$, i.e. there exists $\bar{f} \in L^1([0,2\pi]\times\mathbb{R})$ such that $f_n \underset{n\to+\infty}{\rightharpoonup} \bar{f}$ weakly in L^1.*
   (b) *We have $\|\phi_{f_n} - \phi_{\bar{f}}\|_{H^1} \underset{n\to+\infty}{\longrightarrow} 0$.*

The proof of Lemma 3.3 is similar to the one of Lemma 2.1.

Lemma 3.4. *Let $(f_n)_n$ be a minimizing sequence of the variational problem (1.8) and let $\phi_n := \phi_{f_n}$ be the associated potential. Using Lemma 3.1, there exists a unique pair $(\lambda_n, \mu_n) \in \mathbb{R} \times \mathbb{R}_-^*$ such that $F^{\phi_n}(\theta, v) = (j')^{-1}\left(\frac{\frac{v^2}{2}+\phi_n(\theta)-\lambda_n}{\mu_n}\right)_+$ verifies $\|F^{\phi_n}\|_{L^1} = M_1$ and $\|j(F^{\phi_n})\|_{L^1} = M_j$. The sequences $(\lambda_n)_n$ and $(\mu_n)_n$ are bounded.*

Proof. Let us first prove that the sequence $(\lambda_n)_n$ is bounded. We argue by contradiction. Hence up to an extraction of a subsequence, $\lambda_n \underset{n\to+\infty}{\longrightarrow} +\infty$. According to the expression (1.3) of the potential ϕ_n, we have $\|\phi_n\|_{L^\infty} \le 2\pi\|W\|_{L^\infty} M_1 := C$. Using the expression of M_1 given by (3.6), we get

$$M_1 \ge \sqrt{(\lambda_n - C)_+}4\pi\sqrt{2}\int_0^1 (j')^{-1}\left(\frac{(\lambda_n - C)_+}{|\mu_n|}(1-u^2)\right) \mathrm{d}u \ge 0.$$

Then, we argue as at the end of the proof of Lemma 3.1 and we deduce that $\frac{(\lambda_n - C)_+}{|\mu_n|} \underset{n\to+\infty}{\longrightarrow} 0$. With the hypothesis (H3) and $\|\phi_n\|_{L^\infty} \le C$, we can estimate M_j as follows:

$$0 \le M_j \le \frac{M_1}{p}\frac{(\lambda_n + C)_+}{|\mu_n|}.$$

The term of the right side converges to 0 then we get a contradiction. The sequence $(\lambda_n)_n$ is hence bounded. Now, we shall prove that the sequence $(\mu_n)_n$ is bounded. Using the expression (3.6) of M_1 and the fact that λ_n is bounded, we have

$$\frac{M_1}{4\pi\sqrt{2}\tilde{C}} \le (j')^{-1}\left(\frac{\tilde{C}}{|\mu_n|}\right) \quad \text{where } \tilde{C} \text{ is a constant.}$$

Therefore we obtain

$$0 \le |\mu_n| \le \frac{\tilde{C}}{j'\left(\frac{M_1}{4\pi\sqrt{2}\tilde{C}}\right)}$$

and we deduce that the sequence $(\mu_n)_n$ is bounded. This achieves the proof of this lemma. □

3.2.2. Proof of Theorem 3

We are now ready to prove Theorem 3.

Step 1: Existence of a minimizer.

Let $M_1, M_j > 0$. From Lemma 3.3, we know that $\mathcal{I}(M_1, M_j)$ is finite. Let us show that there exists a function of E_j which minimizes the variational problem (1.8). Let $(f_n)_n \in E_j^{\mathbb{N}}$ be a minimizing sequence of $\mathcal{I}(M_1, M_j)$. Thus $\mathcal{H}(f_n) \underset{n\to+\infty}{\longrightarrow} \mathcal{I}(M_1, M_j)$, $\|f_n\|_{L^1} = M_1$ and $\|j(f_n)\|_{L^1} = M_j$. From item (2) of Lemma 3.3, there exists $\bar{f} \in L^1([0, 2\pi] \times \mathbb{R})$ such that $f_n \underset{n\to+\infty}{\rightharpoonup} \bar{f}$ weakly in L^1. In what follows, we will denote by ϕ_n the potential ϕ_{f_n} defined by (1.3). Thanks to the weak convergence in L^1, we only get that $\|\bar{f}\|_{L^1} = M_1$ and $\|j(\bar{f})\|_{L^1} \le M_j$. The idea is to introduce a new sequence which is a minimizing sequence of (1.8) and which has better compactness properties. For this purpose, we define

$$F^{\phi_n}(\theta, v) = (j')^{-1}\left(\frac{\frac{v^2}{2} + \phi_n(\theta) - \lambda_n}{\mu_n}\right)_+ \tag{3.13}$$

where (λ_n, μ_n) is the unique pair of $\mathbb{R} \times \mathbb{R}_-^*$ such that $\|F^{\phi_n}\|_{L^1} = M_1$ and $\|j(F^{\phi_n})\|_{L^1} = M_j$. According to Lemma 3.1, F^{ϕ_n} is well-defined and notice that the pair (λ_n, μ_n) depends on ϕ_n this is why we will denote by

$\lambda_n = \lambda(\phi_n)$ and $\mu_n = \mu(\phi_n)$. Besides, using Lemma 3.2, we see that $(F^{\phi_n})_n$ is a minimizing sequence of (1.8). According to item (b) of Lemma 3.3, ϕ_n converges to $\bar\phi := \phi_{\bar f}$ strongly in $L^2([0,2\pi]\times\mathbb{R})$. Thus, up to an extraction of a subsequence, ϕ_n converges to $\bar\phi$ a.e. Let us prove that the sequences $(\lambda_n)_n$ and $(\mu_n)_n$ converge. Using Lemma 3.4, we get that the sequences $(\lambda_n)_n$ and $(\mu_n)_n$ are bounded. Therefore, there exist λ_0 and μ_0 such that, up to an extraction of a subsequence, $\lambda_n \underset{n\to+\infty}{\longrightarrow} \lambda_0$ and $\mu_n \underset{n\to+\infty}{\longrightarrow} \mu_0$. Let us prove that $\mu_0 < 0$. Assume that $\mu_n \underset{n\to+\infty}{\longrightarrow} 0$. First assume that $\lambda_n \underset{n\to+\infty}{\longrightarrow} \lambda_0 \neq \min\bar\phi$. From assumptions on j, this implies

$$(j')^{-1}\left(\frac{\lambda_n - \frac{v^2}{2} - \phi_n(\theta)}{|\mu_n|}\right)_+ \underset{n\to+\infty}{\longrightarrow} +\infty \text{ for almost all } (\theta,v)\in[0,2\pi]\times\mathbb{R}.$$

And using Fatou's lemma, we get a contradiction. Then assume that $\lambda_n \underset{n\to+\infty}{\longrightarrow} \min\bar\phi$, using inequality (3.7), we get

$$M_j \ge \frac{j\left(\frac{M_1}{\alpha_n}\right)}{\frac{M_1}{\alpha_n}} \quad \text{with} \quad \alpha_n = 2\sqrt{2}\int_0^{2\pi}\sqrt{(\lambda_n - \phi_n(\theta))_+}\,\mathrm{d}\theta. \tag{3.14}$$

Using the dominated convergence theorem, we show that $\alpha_n \underset{n\to+\infty}{\longrightarrow} 0$. But j satisfies (H2) thus $\frac{j\left(\frac{M_1}{\alpha_n}\right)}{\frac{M_1}{\alpha_n}} \underset{n\to+\infty}{\longrightarrow} +\infty$ and we get a contradiction with (3.14). Besides $\lambda_0 \neq \min\bar\phi$ since otherwise F^{ϕ_n} converges to 0 and we get a contradiction with $\|F^{\phi_n}\|_{L^1} = M_1$. Hence we have proved that F^{ϕ_n} converges to $(j')^{-1}\left(\frac{\frac{v^2}{2}+\bar\phi(\theta)-\lambda_0}{\mu_0}\right)_+$ a.e. Now let us show that $\lambda_0 = \lambda(\bar\phi)$ and $\mu_0 = \mu(\bar\phi)$ to get that $(j')^{-1}\left(\frac{\frac{v^2}{2}+\bar\phi(\theta)-\lambda_0}{\mu_0}\right)_+$ satisfies the two constraints. For this purpose, we first prove by the dominated convergence theorem, $\|\phi_n\|_{L^\infty}$ being bounded, that

$$\begin{cases} \|F^{\phi_n}\|_{L^1} \underset{n\to+\infty}{\longrightarrow} \int_0^{2\pi}\int_{\mathbb{R}}(j')^{-1}\left(\frac{\frac{v^2}{2}+\bar\phi(\theta)-\lambda_0}{\mu_0}\right)_+ \mathrm{d}\theta\mathrm{d}v, \\ \|j(F^{\phi_n})\|_{L^1} \underset{n\to+\infty}{\longrightarrow} \int_0^{2\pi}\int_{\mathbb{R}} j\circ(j')^{-1}\left(\frac{\frac{v^2}{2}+\bar\phi(\theta)-\lambda_0}{\mu_0}\right)_+ \mathrm{d}\theta\mathrm{d}v. \end{cases} \tag{3.15}$$

But $(\|F^{\phi_n}\|_{L^1}, \|j(F^{\phi_n})\|) = (M_1, M_j)$ then

$$M_1 = \int_0^{2\pi}\int_{\mathbb{R}}(j')^{-1}\left(\frac{\frac{v^2}{2}+\bar\phi(\theta)-\lambda_0}{\mu_0}\right)_+ \mathrm{d}\theta\mathrm{d}v,\quad M_j = \int_0^{2\pi}\int_{\mathbb{R}} j\circ(j')^{-1}\left(\frac{\frac{v^2}{2}+\bar\phi(\theta)-\lambda_0}{\mu_0}\right)_+ \mathrm{d}\theta\mathrm{d}v.$$

According to Lemma 3.1, the couple $(\lambda(\bar\phi),\mu(\bar\phi))$ is unique, so $\lambda_0 = \lambda(\bar\phi)$ and $\mu_0 = \mu(\bar\phi)$. Hence F^{ϕ_n} converges to $F^{\bar\phi}$ a.e. But $\|F^{\bar\phi}\|_{L^1} = \|F^{\phi_n}\|_{L^1} = M_1$ then according to Brezis–Lieb's lemma, $F^{\phi_n} \underset{n\to+\infty}{\longrightarrow} F^{\bar\phi}$ strongly in $L^1([0,2\pi]\times\mathbb{R})$. We already know that $F^{\bar\phi}$ satisfies the two constraints, there remains to show that $\mathcal{H}(F^{\bar\phi}) = \mathcal{I}(M_1,M_j)$. The strong convergence in $L^1([0,2\pi]\times\mathbb{R})$ of F^{ϕ_n} to $F^{\bar\phi}$ implies that $\phi'_{F^{\phi_n}} \underset{n\to+\infty}{\longrightarrow} \phi'_{F^{\bar\phi}}$ strongly in L^2. Therefore using classical inequalities about the lower semicontinuity properties of convex nonnegative functions see [15] and the convergence in $L^2([0,2\pi])$ of $\phi'_{F^{\phi_n}}$, we get

$$\mathcal{I}(M_1,M_j) \ge \iint \frac{v^2}{2}F^{\bar\phi}(\theta,v)\mathrm{d}\theta\mathrm{d}v - \frac{1}{2}\int_0^{2\pi}\phi'_{F^{\bar\phi}}(\theta)^2\mathrm{d}\theta.$$

Thus $\mathcal{I}(M_1,M_j) \ge \mathcal{H}(F^{\bar\phi})$. As $F^{\bar\phi}$ satisfies the two constraints and belongs to E_j, we have $\mathcal{I}(M_1,M_j) \le \mathcal{H}(F^{\bar\phi})$. Therefore we get the equality and we have shown the existence of a minimizer.

Step 2: The minimizer is a steady state of (1.1).

To prove that the minimizer $F^{\bar{\phi}}$ is a stationary state of the system (1.1), it is sufficient to show that $\bar{\phi} = \phi_{F^{\bar{\phi}}}$. First, $(F^{\phi_n})_n$ being a minimizing sequence of (1.8), we have $\mathcal{H}(F^{\phi_n}) \underset{n\to+\infty}{\longrightarrow} \mathcal{I}(M_1, M_j)$. Then, using Lemma 3.2, we know that $\mathcal{J}_0 = \mathcal{I}(M_1, M_j)$ and that $\mathcal{I}(M_1, M_j) \le \mathcal{J}(\phi_n) \le \mathcal{H}(f_n)$. Hence $(\phi_n)_n$ is a minimizing sequence of $\mathcal{J}_0$: we have $\mathcal{J}(\phi_n) \underset{n\to+\infty}{\longrightarrow} \mathcal{I}(M_1, M_j) = \mathcal{J}_0$. Hence using the equality (3.11), we get

$$\|\phi'_{F^{\phi_n}} - \phi'_n\|^2_{L^2} \underset{n\to+\infty}{\longrightarrow} 0.$$

Passing to the limit $n \to +\infty$ and knowing that $\bar{\phi}$ has a zero average, we deduce that $\bar{\phi} = \phi_{F^{\bar{\phi}}}$ a.e.

Step 3: Euler–Lagrange equation for minimizers.

There remains to prove part (2) of Theorem 3. We obtain Euler–Lagrange equation for the minimizer in the same way as in the proof of Theorem 1 in Section 2.1.2. Indeed, according to Lemma 3.2, if $\bar{f}$ is a minimizer of $\mathcal{I}(M_1, M_j)$, $\bar{\phi} := \phi_{\bar{f}}$ is a minimizer of $\mathcal{J}_0$ and $\mathcal{H}(\bar{f}) = \mathcal{J}(\bar{\phi})$. Using (3.12), we get

$$\iint (j(\bar{f}) - j(F^{\bar{\phi}}) - j'(F^{\bar{\phi}})(\bar{f} - F^{\bar{\phi}}))\mathrm{d}\theta\mathrm{d}v = 0.$$

Then writing the Taylor's formula for j and using $j'' > 0$, we can deduce as in Section 2.1.2 that $\bar{f} = F^{\bar{\phi}}$.

Step 4: Regularity of the potential ϕ_f.

First, we will show that $\phi_f \in \mathcal{C}^1([0, 2\pi])$. Thanks to the Sobolev embedding

$$W^{2,3}([0, 2\pi]) \hookrightarrow \mathcal{C}^{1,\frac{2}{3}}([0, 2\pi]),$$

it is sufficient to show that $\phi_f \in W^{2,3}([0, 2\pi])$. We know that $f \in L^1([0, 2\pi] \times \mathbb{R})$, then with expression (1.3), we get $\phi_f \in L^\infty([0, 2\pi]) \subset L^3([0, 2\pi])$. In the same way, $\phi'_f \in L^3([0, 2\pi])$. Besides ϕ_f satisfies (1.2), then let us show that $\rho_f \in L^3([0, 2\pi])$. According to the previous step, f is compactly supported and since $\phi_f \in L^\infty$, we get $f \in L^\infty([0, 2\pi] \times \mathbb{R})$. We also have $v^2 f \in L^1([0, 2\pi] \times \mathbb{R})$. Therefore with a classical argument, we show $\rho_f \in L^3([0, 2\pi])$ and we get $\phi_f \in \mathcal{C}^1([0, 2\pi])$. Then, according to its expression (1.3), ρ_f is continuous. Hence $\phi''_f \in \mathcal{C}^0([0, 2\pi])$ and $\phi'_f \in W^{1,3}([0, 2\pi]) \cap \mathcal{C}^0([0, 2\pi])$, then we can write for $x, y \in [0, 2\pi]$

$$\phi'_f(y) - \phi'_f(x) = \int_x^y \phi''_f(t)\mathrm{d}t. \tag{3.16}$$

We deduce from (3.16) that $\phi'_f \in \mathcal{C}^1([0, 2\pi])$ then $\phi_f \in \mathcal{C}^2([0, 2\pi])$.

3.3. Orbital stability of the ground states

To prove the orbital stability result stated in Theorem 4, we first need to prove the local uniqueness of the minimizers under equimeasurability condition.

3.3.1. Local uniqueness of the minimizers under equimeasurability condition

In this section, we prove Lemma 1.2. To this purpose, we first need to prove some preliminary lemmas.

Lemma 3.5. *Let f_1, f_2 be two equimeasurable steady states of (1.1) which minimizes (1.8), they can be written in the form (1.9) with $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathbb{R} \times \mathbb{R}^*_-$, we have for all $e \ge 0$*

$$|\mu_1|^{\frac{1}{2}}\int_0^{2\pi}(\psi_1(\theta)-e)_+^{\frac{1}{2}}\mathrm{d}\theta = |\mu_2|^{\frac{1}{2}}\int_0^{2\pi}(\psi_2(\theta)-e)_+^{\frac{1}{2}}\mathrm{d}\theta \quad \textit{where} \quad \psi_i = \frac{\phi_{f_i}-\lambda_i}{\mu_i},\ i=1,2. \tag{3.17}$$

Besides, if f_1 and f_2 are inhomogeneous then there exist $p_1 = p_1(\phi_{f_1}) \in \mathbb{N}^*$ and $p_2 = p_2(\phi_{f_2}) \in \mathbb{N}^*$ such that

$$\frac{p_1|\mu_1|^{\frac{3}{4}}}{\sqrt{\left|a(e_0)-\frac{1}{|\mu_1|^{\frac{1}{2}}}c_0\right|}} = \frac{p_2|\mu_2|^{\frac{3}{4}}}{\sqrt{\left|a(e_0)-\frac{1}{|\mu_2|^{\frac{1}{2}}}c_0\right|}}, \tag{3.18}$$

where

$$\begin{cases} a(e_0) = \int_{\mathbb{R}}(j')^{-1}\left(e_0-\frac{v^2}{2}\right)_+ & \textit{with } e_0 = \max\left(\frac{\phi_{f_i}-\lambda_i}{\mu_i}\right), \quad i=1,2; \\ c_0 = \frac{M_1}{2\pi}. \end{cases}$$

Lemma 3.6. *Let $\psi \in \mathcal{C}^2([0,2\pi])$ such that there exists a finite number p of values $\xi \in [0,2\pi]$ satisfying $\psi(\xi) = \max(\psi) := e_0$. We will denote them by ξ_i for $i \in \{1,..,p\}$. Besides we assume that for all $i \in \{1,..,p\}$, we have $\psi''(\xi_i) \neq 0$ thus we have*

$$\int_0^{2\pi}(\psi(\theta)-e)_+^{\frac{1}{2}}\mathrm{d}\theta = \varepsilon\sum_{i=1}^{p}\frac{\sqrt{2}}{\sqrt{|\psi''(\xi_i)|}}\int_0^1 s^{-\frac{1}{2}}(1-s)^{\frac{1}{2}}\mathrm{d}s + o(\varepsilon) \quad \textit{with } \varepsilon = e_0 - e.$$

We first show Lemma 1.2 using Lemmas 3.5 and 3.6 then Lemmas 3.5 and 3.6 will be proved.

Proof of Lemma 1.2. Let f_0 be a homogeneous steady state of (1.1) and a minimizer of (1.8). It can be written in the form (1.9) with $(\lambda_0,\mu_0) \in \mathbb{R}\times\mathbb{R}_-^*$. First, let f be a homogeneous steady state of (1.1) and a minimizer of (1.8) equimeasurable to f_0. It can be written in the form (1.9) with $(\lambda,\mu) \in \mathbb{R}\times\mathbb{R}_-^*$. We can also write

$$\begin{cases} f_0(\theta,v) = (j')^{-1}\left(\frac{-v^2}{2|\mu_0|}+\psi_0(\theta)\right)_+ & \text{with} \quad \psi_0(\theta) = \frac{\phi_{f_0}(\theta)-\lambda_0}{|\mu_0|}, \\ f(\theta,v) = (j')^{-1}\left(\frac{-v^2}{2|\mu|}+\psi(\theta)\right)_+ & \text{with} \quad \psi(\theta) = \frac{\phi_f(\theta)-\lambda}{|\mu|}. \end{cases} \tag{3.19}$$

The homogeneity and equimeasurability of f_0 and f implies $\frac{\lambda_0}{|\mu_0|} = \frac{\lambda}{|\mu|}$. Besides replacing in equality (3.17) of Lemma 3.5, we get $\mu_0 = \mu$ and then $\lambda_0 = \lambda$. Thus $f_0 = f$. Then let f be an inhomogeneous steady state (1.1) and a minimizer of (1.8) equimeasurable to f_0. The minimizer f can be written in the form (3.19). The equimeasurability of f_0 and f implies $\max(\psi_0) = \max(\psi)$. We note this value e_0 and we notice that $\psi_0(\theta) = e_0$ for all $\theta \in [0,2\pi]$. Replacing in equality (3.17) of Lemma 3.5, we get

$$2\pi|\mu_1|^{\frac{1}{2}}(e_0-e)_+^{\frac{1}{2}} = |\mu_2|^{\frac{1}{2}}\int_0^{2\pi}(\psi_2(\theta)-e)_+^{\frac{1}{2}}\mathrm{d}\theta.$$

To estimate the right term of this equality, we will apply Lemma 3.6 and we get

$$2\pi|\mu_1|^{\frac{1}{2}}\sqrt{\varepsilon} = \left(|\mu_2|^{\frac{1}{2}}\sum_{j=1}^{p_2}\frac{\sqrt{2}}{\sqrt{|\psi_2''(\xi_j)|}}\int_0^1 s^{-\frac{1}{2}}(1-s)^{\frac{1}{2}}\mathrm{d}s\right)\varepsilon + o(\varepsilon).$$

This last equality shows us that this case cannot occur. Thus f_0 is the only homogeneous steady state of (1.1) and minimizer of (1.8) under equimeasurability condition.

Let f_0 be an inhomogeneous steady state of (1.1) and a minimizer of (1.8), it can be written in the form (1.9) with $(\lambda_0,\mu_0) \in \mathbb{R}\times\mathbb{R}_-^*$. Let f be an inhomogeneous steady state of (1.1) and a minimizer of (1.8) equimeasurable to

f_0. It can be written in the form (1.9) with $(\lambda,\mu)\in\mathbb{R}\times\mathbb{R}_-^*$. Let us assume that $\mu_0=\mu$ then we can write our two minimizers like that

$$f_0(\theta,v)=G\left(\frac{v^2}{2}+\psi_0(\theta)\right),\quad f(\theta,v)=G\left(\frac{v^2}{2}+\psi(\theta)\right),$$

with $G(t)=(j')^{-1}\left(\left(\frac{t}{\mu}\right)_+\right)$ and $\psi_i(\theta)-\lambda_i$. Arguing as the one constraint case, we get $f_0=f$ up to a translation shift in θ. Let us assume that $\mu_0\neq\mu$ and let us show that μ_0 is isolated. Since f_0 and f are inhomogeneous, they verify (3.18) according to Lemma 3.5. Define for $x>0$, $F(x)=\frac{x^{\frac{3}{4}}}{\sqrt{|a(e_0)-x^{-\frac{1}{2}}c_0|}}$ and introduce the set

$$E=\underset{p\in\mathbb{N}}{\cup}\{\mu\text{ s.t. } pF(|\mu|)=A_0\}.$$

If E is finite, the result is trivial. Otherwise E is countable, it can be written in the form $E=(\mu_n)_n$ with μ_n injective and satisfying for all $n\in\mathbb{N}$, there exists p_n such that $p_nF(|\mu_n|)=A_0$. Let μ_1 be a limit point of the sequence $(\mu_n)_n$, it verifies $F(|\mu_1|)=0$. Indeed, the sequence $(p_n)_n$ cannot take an infinity of times the same value since in equality (3.18), for p fixed, there are at the most 4 μ. Therefore $p_n\underset{n\to+\infty}{\longrightarrow}+\infty$. Thus $\mu_1=0$. As $\mu_0<0$, it is isolated. Thus there exists $\delta_0>0$ such that for all $f\neq f_0$ inhomogeneous steady state of (1.1) and minimizer of (1.8), we have $||\mu|-|\mu_0||>\delta_0$. □

Now, let us prove Lemma 3.6.

Proof of Lemma 3.6. Let $\psi\in\mathcal{C}^2([0,2\pi])$ satisfying the assumptions noted above, we have

$$\int_0^{2\pi}(\psi(\theta)-e)_+^{\frac{1}{2}}\mathrm{d}\theta=\frac{1}{2}\int_0^{2\pi}\int_0^{(\psi(\theta)-e)_+}s^{-\frac{1}{2}}\mathrm{d}s\mathrm{d}\theta=\frac{1}{2}\int_0^{e_0-e}s^{-\frac{1}{2}}\,|\{e+s\le\psi\le e_0\}|\,\mathrm{d}s$$
$$=\frac{\varepsilon^{\frac{1}{2}}}{2}\int_0^1 s^{-\frac{1}{2}}\,|\{e_0-\varepsilon(1-s)\le\psi\le e_0\}|\,\mathrm{d}s,$$

using Fubini's theorem, putting $\varepsilon=e_0-e$ and performing a change of variables $\tilde{s}=\frac{s}{\varepsilon}$.

We define $E_\varepsilon=\{\theta\in[0,2\pi],e_0-\varepsilon(1-s)\le\psi\le e_0\}$. We can write $[0,2\pi]=\cup_{i=1}^pE_i$ with

$$\begin{cases}E_1=[0,\frac{\xi_1+\xi_2}{2}]\\ E_i=[\frac{\xi_{i-1}+\xi_i}{2},\frac{\xi_i+\xi_{i+1}}{2}]\text{ for }i\in\{2,..,p-1\}\\ E_p=[\frac{\xi_{p-1}+\xi_p}{2},2\pi].\end{cases}$$

Thus $E_\varepsilon=\cup_{i=1}^pE_\varepsilon^i$ with $E_\varepsilon^i=\{\theta\in E_i,-\varepsilon(1-s)\le\psi(\theta)-e_0\le 0\}$ and we get

$$\int_0^{2\pi}(\psi(\theta)-e)_+^{\frac{1}{2}}\mathrm{d}\theta=\sum_{i=1}^p\frac{\varepsilon^{\frac{1}{2}}}{2}\int_0^1 s^{-\frac{1}{2}}|E_\varepsilon^i|\mathrm{d}s.$$

The next step is to compute for $i\in\{1...p\}$ the limit of $|E_\varepsilon^i|$ when ε goes to 0. Notice that there is a unique ξ_i in each interval E_i for $i\in\{1...p\}$, and use the Taylor formula for ψ, to get

$$E_\varepsilon^i=\left\{\theta\in E_i,-\varepsilon(1-s)\le(\theta-\xi_i)^2\int_0^1(1-u)\psi''(u(\theta-\xi_i)+\xi_i)\mathrm{d}u\le 0\right\}.$$

Letting $A(\theta,\xi)=\int_0^1(1-u)\psi''(u(\theta-\xi)+\xi)\mathrm{d}u$, we can write

$$E_\varepsilon^i=\left\{\theta\in E_i,\frac{|\theta-\xi_i|}{\sqrt{\varepsilon}}\sqrt{|A(\theta,\xi_i)|}\le\sqrt{1-s}\right\}.$$

Then we have

$$|E^i_\varepsilon| = 2\sqrt{\varepsilon}\left|\left\{\theta \in E_i, \theta\sqrt{|B(\theta,\xi_i)|} \le \sqrt{1-s}\right\}\right| \text{ where } B(\theta,\xi_i) = \int_0^1 (1-u)\psi''(u\sqrt{\varepsilon}\theta + \xi_i)\mathrm{d}u.$$

Recall that $\psi''(\xi_i) \neq 0$ hence by continuity of ψ'', we have $\psi'' \neq 0$ on a neighborhood of ξ_i. Thus for e close to e_0, i.e. for ε sufficiently small, we have $B(\theta,\xi_i) \neq 0$. Thus we can write

$$\frac{|E^i_\varepsilon|}{\sqrt{\varepsilon}} = 2\int_0^{2\pi} \mathbb{1}_{\left\{0 \le \theta \le \frac{\sqrt{1-s}}{\sqrt{|B(\theta,\xi_i)|}}\right\}}\mathrm{d}\theta.$$

Applying the dominated convergence theorem, we get for $i \in \{1 \ldots p\}$

$$\lim_{\varepsilon\to 0}\frac{|E^i_\varepsilon|}{\sqrt{\varepsilon}} = 2\frac{\sqrt{2(1-s)}}{\sqrt{|\psi''(\xi_i)|}}.$$

This ends the proof of Lemma 3.6. □

To prove Lemma 3.5, we need a last technical lemma.

Lemma 3.7. *Let f be an inhomogeneous minimizer of the variational problem (1.8) given by (1.9) with $(\lambda,\mu) \in \mathbb{R} \times \mathbb{R}^*_-$. We denote by $e_0 := \max\psi$ where $\psi(\theta) = \frac{\phi_f(\theta)-\lambda}{\mu}$. Then there is only a finite number of values ξ satisfying $\psi(\xi) = e_0$.*

Proof. Let us argue by contradiction. Assume there is an infinite number of different values ξ satisfying $\psi(\xi) = e_0$. We define a strictly increasing sequence $(\xi_n)_n$ such that for all n, $\psi(\xi_n) = e_0$. In particular we have $\psi'(\xi_n) = 0$. Then we apply Rolle's theorem on each interval $[\xi_n, \xi_{n+1}]$ and we build a new sequence $(\tilde{\xi}_n)_n$ such that $\psi''(\tilde{\xi}_n) = 0$. We have $(\tilde{\xi}_n)_n \in [0,2\pi]^{\mathbb{N}}$ thus there exists $\tilde{\xi}$ such that $\tilde{\xi}_n \underset{n\to+\infty}{\longrightarrow} \tilde{\xi}$ up to an extraction of a subsequence. With the continuity of ψ'' and Theorem 3, we get $\psi''(\tilde{\xi}) = 0$. By construction, we have for all n, $\tilde{\xi}_{n-1} < \xi_n < \tilde{\xi}_n$. Thus up to an extraction of a subsequence $\xi_n \underset{n\to+\infty}{\longrightarrow} \tilde{\xi}$ and the limit satisfies $\psi'(\tilde{\xi}) = 0$ and $\psi(\tilde{\xi}) = e_0$. Besides we know that

$$\psi'' = \frac{\phi''_f}{\mu} = \frac{\rho_f - \frac{M_1}{2\pi}}{\mu},$$

then $\rho_f(\tilde{\xi}) = \frac{M_1}{2\pi}$. Using the expression of ρ_f, we get for all $\theta \in [0,2\pi]$, $\rho_f(\theta) \le \rho_f(\tilde{\xi})$ and $\max(\rho_f) = \rho_f(\tilde{\xi}) = \frac{M_1}{2\pi}$. Since $\int_{\mathbb{R}} \rho_f = M_1$, we deduce that for all $\theta \in [0,2\pi]$, $\rho_f(\theta) = \frac{M_1}{2\pi}$. Thus for all θ, $\phi''_f(\theta) = 0$. Since ϕ_f has a zero average and $\phi_f(0) = \phi_f(2\pi)$, we get $\phi_f = 0$. Contradiction. □

We are now ready to prove Lemma 3.5.

Proof. Let f_1 and f_2 be two steady states of (1.1) and two minimizers of (1.8) equimeasurable. They can be written in the form (1.9) and we can write

$$f_1(\theta,v) = (j')^{-1}\left(\frac{v^2}{2\mu_1} + \psi_1(\theta)\right)_+, \qquad f_2(\theta,v) = (j')^{-1}\left(\frac{v^2}{2\mu_2} + \psi_2(\theta)\right)_+$$

where $\psi_i(\theta) = \frac{\phi_{f_i}(\theta)-\lambda_i}{\mu_i}$ for $i = 1$ or 2. Since f_1 and f_2 are equimeasurable, we know that for all $t \ge 0$

$$\left|\left\{(j')^{-1}\left(\frac{-v^2}{2|\mu_1|} + \psi_1(\theta)\right)_+ > t\right\}\right| = \left|\left\{(j')^{-1}\left(\frac{-v^2}{2|\mu_2|} + \psi_2(\theta)\right)_+ > t\right\}\right|.$$

We have for $i = 1$ or 2,

$$\left|\left\{(j')^{-1}\left(\frac{-v^2}{2|\mu_i|} + \psi_i(\theta)\right)_+ > t\right\}\right| 1 = \left|\left\{\frac{v^2}{2} - |\mu_i|\psi_i(\theta) < -|\mu_i| j'(t)\right\}\right|$$

$$= 2\sqrt{2}|\mu_i|^{\frac{1}{2}} \int_0^{2\pi} (\psi_i(\theta) - j'(t))_+^{\frac{1}{2}} \mathrm{d}\theta.$$

Thus for all $e \geq 0$, we have equality (3.17). Then let us assume that $\phi_{f_1} \neq 0$ and $\phi_{f_2} \neq 0$. According to the third point of Theorem 3, $\psi_1, \psi_2 \in \mathcal{C}^2([0, 2\pi])$. Besides, according to Lemma 3.7, there exists, for $i = 1$ or 2, $p_i = p_i(\phi_{f_i})$ such that ψ_i has p_i values ξ satisfying $\psi_i(\xi) = e_0$. We denote them $\{\xi_{i,1}, .., \xi_{i,p_i}\}$. In order to apply Lemma 3.6, let us show that $\psi_i''(\xi_{i,j}) \neq 0$ for $j \in \{1, .., p_i\}$ and $i = 1$ or 2. If $\psi_i''(\xi_{i,j}) = 0$, since $\xi_{i,j}$ is a maximum of ψ too, we are in the same case as the end of the proof of Lemma 3.7 and we get a contradiction. Hence we are allowed to use Lemma 3.6 and get

$$|\mu_1|^{\frac{1}{2}} \sum_{j=1}^{p_1} \frac{1}{\sqrt{|\psi_1''(\xi_{1,j})|}} = |\mu_2|^{\frac{1}{2}} \sum_{j=1}^{p_1} \frac{1}{\sqrt{|\psi_2''(\xi_{1,j})|}}.$$

Notice that we have for $i = 1$ or 2

$$\psi_i''(\theta) = \phi_{f_i}''(\theta) = \frac{\rho_{f_i}(\theta) - \frac{M_1}{2\pi}}{\mu_i} = \frac{1}{\mu_i}\left(\int_{\mathbb{R}} (j')^{-1}\left(\frac{-v^2}{2|\mu_i|} + \psi_i(\theta)\right)_+ \mathrm{d}v - \frac{M_1}{2\pi}\right)$$

$$= -|\mu_i|^{-\frac{1}{2}}\left(\int_{\mathbb{R}} (j')^{-1}\left(\frac{-v^2}{2} + \psi_i(\theta)\right)_+ \mathrm{d}v - \frac{1}{|\mu_i|^{\frac{1}{2}}}\frac{M_1}{2\pi}\right).$$

Thus we have

$$\psi_i''(\xi_{i,j}) = -|\mu_i|^{-\frac{1}{2}}\left(a(e_0) - \frac{1}{|\mu_i|}\frac{M_1}{2\pi}\right)$$

with $a(e_0) = \int_{\mathbb{R}} (j')^{-1}\left(e_0 - \frac{v^2}{2}\right)_+ \mathrm{d}v$, and therefore equality (3.18) is proved. □

3.3.2. Proof of Theorem 4

We will prove the orbital stability of steady states of (1.1) which are minimizers of (1.8) in two steps. First we will show that any minimizing sequence is compact.

Step 1: Compactness of the minimizing sequences.

Let $(f_n)_n$ be a minimizing sequence of $\mathcal{I}(M_1, M_j)$. Let us show that $(f_n)_n$ is compact in E_j, i.e. there exists $f_0 \in E_j$ such that $f_n \xrightarrow{E_j} f_0$ up to an extraction of a subsequence. Using item (2) of Lemma 3.3, there exists $f_0 \in \mathrm{L}^1([0, 2\pi] \times \mathbb{R})$ such that $f_n \underset{n\to+\infty}{\rightharpoonup} f_0$ weakly in $\mathrm{L}^1([0, 2\pi] \times \mathbb{R})$ and we denote by $\phi_0 := \phi_{f_0}$. In the same way as the proof of Theorem 3 in Section 3.2.2, we introduce the function F^{ϕ_n} defined by (3.13). According to Step 1 of the proof of Theorem 3 in Section 3.2.2, it is a minimizing sequence of (1.8), F^{ϕ_n} converges to F^{ϕ_0} strongly in $\mathrm{L}^1([0, 2\pi] \times \mathbb{R})$ and F^{ϕ_0} is a minimizer of $\mathcal{I}(M_1, M_j)$. Our goal is to prove that $f_0 = F^{\phi_0}$ and $f_n \xrightarrow{E_j} f_0$.

In order to do that, let us start with the proof of the strong convergence in $\mathrm{L}^1([0, 2\pi] \times \mathbb{R})$ of f_n to F^{ϕ_0}. First, we notice that $\|f_n\|_{\mathrm{L}^1} = \|F^{\phi_0}\|_{\mathrm{L}^1} = M_1$, then thanks to Brezis–Lieb's lemma, it is sufficient to show that f_n converges to F^{ϕ_0} a.e. in order to get the strong convergence in $\mathrm{L}^1([0, 2\pi] \times \mathbb{R})$. To this purpose, let us write

$$f_n - F^{\phi_0} = f_n - F^{\phi_n} + F^{\phi_n} - F^{\phi_0}.$$

As the a.e. convergence of F^{ϕ_n} to F^{ϕ_0} is already known, the next step is to show that $f_n - F^{\phi_n}$ converges to 0 a.e. For this purpose, we will argue as in the proof of Theorem 2 in Section 2.2.2. We notice that we have

$$\iint (j(f_n) - j(F^{\phi_n}) - j'(F^{\phi_n})(f_n - F^{\phi_n}))\mathrm{d}\theta \mathrm{d}v \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.20}$$

Indeed, using equality (3.12), we get

$$\iint (j(f_n) - j(F^{\phi_n}) - j'(F^{\phi_n})(f_n - F^{\phi_n}))\mathrm{d}\theta \mathrm{d}v = \frac{\mathcal{J}(\phi_n) - \mathcal{H}(f_n)}{\mu}.$$

There remains to argue as in Step 2 of the proof of Theorem 3 in Section 3.2.2 to get the desired limit. Then writing the Taylor's formula for the function $j(f_n)$ and integrating over $[0, 2\pi] \times \mathbb{R}$, we get

$$\iint (f_n - F^{\phi_n})^2 \int_0^1 (1-u) j''(u(f_n - F^{\phi_n}) + F^{\phi_n})\mathrm{d}u = \iint j(f_n) - \iint j(F^{\phi_n}) - \iint (f_n - F^{\phi_n}) j'(F^{\phi_n}).$$

Thus $\iint (f_n - F^{\phi_n})^2 \int_0^1 (1-u) j''(u(f_n - F^{\phi_n}) + F^{\phi_n})\mathrm{d}u \underset{n\to+\infty}{\longrightarrow} 0$. Arguing in the same way as the proof of Theorem 2 in Section 2.2.2, we get $f_n - F^{\phi_n} \underset{n\to+\infty}{\longrightarrow} 0$ a.e. To recap, we have obtained that $\|f_n - F^{\phi_0}\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0$. But $f_n \underset{n\to+\infty}{\longrightarrow} f_0$ weakly in $\mathrm{L}^1([0, 2\pi] \times \mathbb{R})$ then by uniqueness of the limit, we have $F^{\phi_0} = f_0$. Therefore $\|f_n - f_0\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0$. To show the convergence in E_j, there remains to show that

$$\|v^2(f_n - f_0)\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0, \quad \text{and} \quad \|j(f_n)\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} \|j(f_0)\|_{\mathrm{L}^1}.$$

The second limit clearly comes from the fact that $f_0 = F^{\phi_0}$ satisfies the constraints. For the first limit, we write

$$\iint v^2(f_n(\theta, v) - f_0(\theta, v))\mathrm{d}\theta \mathrm{d}v = 2(\mathcal{H}(f_n) - \mathcal{H}(f_0)) + \|\phi_n'\|_{\mathrm{L}^2}^2 - \|\phi_0'\|_{\mathrm{L}^2}^2.$$

Then $\|v^2 f_n\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} \|v^2 f_0\|_{\mathrm{L}^1}$. Besides the strong convergence in $\mathrm{L}^1([0, 2\pi] \times \mathbb{R})$ of f_n to f_0 implies that $v^2 f_n \underset{n\to+\infty}{\longrightarrow} v^2 f_0$ a.e. up to an extraction of a subsequence. We conclude with Brezis–Lieb's lemma. Hence the minimizing sequence is compact in E_j.

Step 2: Proof of the orbital stability.

Before starting the proof of Theorem 4, notice the following fact. As mentioned in Section 3.2.2, it is possible to obtain Euler–Lagrange equations for the minimizers in the same way as in the proof of Theorem 1. This method provides the expressions of λ and μ. In particular, we have

$$\mu = -\frac{\|v^2 f\|_{\mathrm{L}^1}}{C_f} \text{ with } C_f = \iint f j'(f)\mathrm{d}\theta \mathrm{d}v - M_j. \tag{3.21}$$

If f_1 and f_2 are equimeasurable, then $C_{f_1} = C_{f_2}$. Hence, we can rewrite the first point of Lemma 1.2 as follows.

Lemma 3.8. *Let f_0 be an inhomogeneous steady state of (1.1) which is a minimizer of (1.8). Let $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}_-^*$ be the Lagrange multipliers associated with f_0 according to (1.9). There exists $\delta_0 > 0$ such that for all $f \in E_j$ inhomogeneous steady state of (1.1) which is minimizer of (1.8) and which is equimeasurable to f_0 with $\mu_0 \neq \mu$, where μ is the Lagrange constant associated with f in the expression (1.9), we have*

$$\left| \|v^2 f_0\|_{\mathrm{L}^1} - \|v^2 f\|_{\mathrm{L}^1} \right| > \delta_0. \tag{3.22}$$

This characterization will be used in the proof of the orbital stability of steady states.

Before proving the orbital stability of minimizers, we need to prove a preliminary lemma.

Lemma 3.9. *Let f_0 be an inhomogeneous steady state of (1.1) which minimizes (1.8). We denote by δ_0 the constant associated with f_0 as defined in Lemma 1.2. We have: $\forall \varepsilon > 0$, $\exists \eta > 0$ such that $\forall f_{init} \in E_j$*

$$\|(1+v^2)(f_{init} - f_0)\|_{L^1} \le \eta \text{ and } \left| \iint j(f_{init}) - \iint j(f_0) \right| \le \eta$$
$$\Rightarrow \left[\forall t > 0, \left[\left| \|v^2 f(t)\|_{L^1} - \|v^2 f_0\|_{L^1} \right| \le \frac{\delta_0}{2} \Rightarrow \left| \|v^2 f(t)\|_{L^1} - \|v^2 f_0\|_{L^1} \right| \le \varepsilon \right]\right],$$

where $f(t)$ is a solution to (1.1) with initial data f_{init}.

With this lemma, we are able to prove Theorem 4. We will prove Lemma 3.9 after the proof of Theorem 4.

Proof of Theorem 4. Let us argue by contradiction, let f_0 be an inhomogeneous minimizer of (1.8). Assume that f_0 is orbitally unstable. Then there exist $\varepsilon_0 > 0$, a sequence $(f^n_{init})_n \in E_j^{\mathbb{N}}$ and a sequence $(t_n)_n \in (\mathbb{R}^+_*)^{\mathbb{N}}$ such that $f^n_{init} \xrightarrow{E_j} f_0$ and for all n, for all $\theta_0 \in [0, 2\pi]$

$$\begin{cases} \|f^n(t_n, \theta+\theta_0, v) - f_0(\theta, v)\|_{L^1} > \varepsilon_0, \\ \text{or } \|v^2(f^n(t_n, \theta+\theta_0, v) - f_0(\theta, v))\|_{L^1} > \varepsilon_0, \end{cases} \tag{3.23}$$

where $f^n(t_n, \theta, v)$ is a solution to (1.1) with initial data f^n_{init}. Letting $g_n(\theta, v) = f^n(t_n, \theta, v)$, we have $\mathcal{H}(g_n) \le \mathcal{H}(f^n_{init})$ from the conservation property of the flow (1.1). Introduce $\bar{g_n}(\theta, v) = \gamma_n g_n\left(\theta, \frac{\gamma_n}{\lambda_n} v\right)$ where (γ_n, λ_n) is the unique pair such that $\|\bar{g_n}\|_{L^1} = M_1$ and $\|j(\bar{g_n})\|_{L^1} = M_j$. Besides γ_n and λ_n satisfy

$$\lambda_n = \frac{M_1}{\|g_n\|_{L^1}} \text{ and } \gamma_n \text{ is such that } \frac{\|j(\gamma_n g_n)\|_{L^1}}{\gamma_n} = \frac{M_j \|g_n\|_{L^1}}{M_1}. \tag{3.24}$$

The existence and uniqueness of such (γ_n, λ_n) can be proved exactly the same way as Lemma A.1 in [20]. As $\bar{g_n}$ satisfies the two constraints of the minimization problem (1.8), we have $\mathcal{H}(f_0) \le \mathcal{H}(\bar{g_n})$. Besides we have

$$\mathcal{H}(f_0) \le \mathcal{H}(\bar{g_n}) \le \lambda_n^2 \left(\left(\frac{\lambda_n}{\gamma_n^2} - 1 \right) \|\frac{v^2}{2} g_n\|_{L^1} + \mathcal{H}(f^n_{init}) \right). \tag{3.25}$$

Notice that

$$\|g_n\|_{L^1} = \|f^n_{init}\|_{L^1} \underset{n\to+\infty}{\longrightarrow} M_1 \text{ since } \|f^n_{init} - f_0\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0 \text{ and } \|f_0\|_{L^1} = M_1.$$

Hence the sequence $(g_n)_n$ is bounded in L^1. We also have

$$\left\| \frac{v^2}{2} g_n \right\|_{L^1} = \mathcal{H}(g_n) + \frac{1}{2} \int_0^{2\pi} \phi'^2_{g_n}(\theta) d\theta \le C + \pi \|W'\|^2_{L^\infty} \|g_n\|^2_{L^1} \text{ where } C \text{ is a constant,}$$

and therefore the sequence $(\|\frac{v^2}{2} g_n\|_{L^1})_n$ is bounded too. Let us then show that λ_n and γ_n converge to 1. With (3.24), we get $\lambda_n \underset{n\to+\infty}{\longrightarrow} 1$. To deal with the case of γ_n, we will use the fact that the hypothesis (H3) is equivalent to the hypothesis (H3bis)

$$\text{(H3bis)} : b^p j(t) \le j(bt) \le b^q j(t), \forall b \ge 1, t \ge 0 \text{ and } b^q j(t) \le j(bt) \le b^p j(t), \forall b \le 1, t \ge 0.$$

Therefore using (H3bis), we get

$$\min(C_n^{\frac{1}{p-1}}, C_n^{\frac{1}{q-1}}) \le \gamma_n \le \max(C_n^{\frac{1}{p-1}}, C_n^{\frac{1}{q-1}}), \text{ where } C_n = \left(\frac{M_j}{M_1} \frac{\|g_n\|_{L^1}}{\|j(g_n)\|_{L^1}} \right)^{\frac{1}{p-1}}.$$

But $\|j(g_n)\|_{L^1} = \|j(f^n_{init})\|_{L^1} \underset{n\to+\infty}{\longrightarrow} \|j(f_0)\|_{L^1}$ and therefore $C_n \underset{n\to+\infty}{\longrightarrow} 1$. Thus $\gamma_n \underset{n\to+\infty}{\longrightarrow} 1$. We deduce with (3.25) that $\lim_{n\to+\infty} \mathcal{H}(\bar{g_n}) = \mathcal{H}(f_0)$ and thus $(\bar{g_n})_n$ is a minimizing sequence of (1.8). According to the previous step, this sequence is compact, hence, up to an extraction of a subsequence, there exists $\bar{g} \in E_j$ such that $\bar{g_n} \underset{n\to+\infty}{\longrightarrow} \bar{g}$ in E_j. It is easy to show with Brezis–Lieb's lemma that $g_n \underset{n\to+\infty}{\longrightarrow} \bar{g}$ in E_j up to an extraction of a subsequence. This implies that

$$\|g_n - \bar{g}\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0,\ \|v^2(g_n - \bar{g})\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0 \text{ and } |\iint j(g_n) - \iint j(\bar{g})| \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.26}$$

Then we deduce of this convergence that $\mathcal{H}(g_n) \underset{n\to+\infty}{\longrightarrow} \mathcal{H}(\bar{g})$, but $\mathcal{H}(g_n) \underset{n\to+\infty}{\longrightarrow} \mathcal{I}(M_1, M_j)$ and $\mathcal{I}(M_1, M_j) = \mathcal{H}(\bar{g})$. Besides $\bar{g}$ satisfies the two constraints therefore $\bar{g}$ is a minimizer of (1.8). Furthermore in the same way as the proof of Theorem 2 in Section 2.2.2, we prove that $\bar{g}$ and f_0 are equimeasurable. In summary, f_0 and $\bar{g}$ are equimeasurable minimizers of $\mathcal{I}(M_1, M_j)$. According to Lemma 1.2, g cannot be a homogeneous steady state. Thus g is an inhomogeneous minimizer and has the form (1.9) with $(\lambda_{\bar{g}}, \mu_{\bar{g}}) \in \mathbb{R} \times \mathbb{R}^*_-$. The inhomogeneous minimizer f_0 also has the form (1.9) with $(\lambda_0, \mu_0) \in \mathbb{R} \times \mathbb{R}^*_-$. If $\mu_{\bar{g}} = \mu_0$, according to Lemma 1.2, $f_0 = \bar{g}$ up to a translation in θ. Then (3.26) contradicts (3.23) and we have proved that f_0 is an orbitally stable steady state. Otherwise, $\mu_{\bar{g}} \neq \mu_0$ and according to Lemma 3.8, there exists δ_0 such that (3.22) holds. Now, let us show that $|\|v^2\bar{g}\|_{L^1} - \|v^2 f_0\|_{L^1}| \leq \delta_0$. In order to do that, let us prove that for all n,

$$|\|v^2 g_n\|_{L^1} - \|v^2 f_0\|_{L^1}| \leq \frac{\delta_0}{2}. \tag{3.27}$$

We will show that $\forall t \geq 0$, $|\|v^2 f^n(t)\|_{L^1} - \|v^2 f_0\|_{L^1}| \leq \frac{\delta_0}{2}$. Let us argue by contradiction and assume there exists $t \geq 0$ such that $|\|v^2 f^n(t)\|_{L^1} - \|v^2 f_0\|_{L^1}| > \frac{\delta_0}{2}$. As $\|(1+v^2)(f^n_{init} - f_0)\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0$, we can assume $\forall n$, $\|(1+v^2)(f^n_{init} - f_0)\|_{L^1} \leq \frac{\delta_0}{4}$. This implies $\forall n$, $|\|v^2 f^n_{init}\|_{L^1} - \|v^2 f_0\|_{L^1}| \leq \frac{\delta_0}{4}$. Thus we have

$$|\|v^2 f^n(0)\|_{L^1} - \|v^2 f_0\|_{L^1}| \leq \frac{\delta_0}{4} \text{ and } \exists t > 0 \text{ s.t. } |\|v^2 f^n(t)\|_{L^1} - \|v^2 f_0\|_{L^1}| > \frac{\delta_0}{2}.$$

By continuity of the map $t \mapsto \|v^2 f^n(t)\|_{L^1}$, there exists $t_0 >$ such that

$$|\|v^2 f^n(t_0)\|_{L^1} - \|v^2 f_0\|_{L^1}| = \frac{\delta_0}{3} < \frac{\delta_0}{2},$$

therefore according to Lemma 3.9, for all $\varepsilon > 0$, we have $|\|v^2 f^n(t_0)\|_{L^1} - \|v^2 f_0\|_{L^1}| \leq \varepsilon$. For instance with $\varepsilon = \frac{\delta_0}{5}$, we get a contradiction. Hence: $\forall t \geq 0$, $|\|v^2 f^n(t)\|_{L^1} - \|v^2 f_0\|_{L^1}| \leq \frac{\delta_0}{2}$ and we deduce (3.27). Recall that we have $\|v^2(g_n - \bar{g})\| \underset{n\to+\infty}{\longrightarrow} 0$, hence with (3.27), we deduce that $|\|v^2 f_0\|_{L^1} - \|v^2\bar{g}\|_{L^1}| \leq \delta_0$. We get a contradiction with (3.22) and $\mu_0 = \mu_{\bar{g}}$ then $f_0 = \bar{g}$ up to a translation shift in θ. Then (3.26) contradicts (3.23) and we have proved that f_0 is an orbitally stable steady state.

If f_0 is a homogeneous minimizer of (1.8). We follow the same reasoning by contradiction and we build an other equimeasurable minimizer $\bar{g}$. Two cases arise: firstly, $\bar{g}$ is inhomogeneous and in fact, this case cannot occur according to the third point of Lemma 3.5. Hence we get a contradiction. Secondly, $\bar{g}$ is homogeneous and we have $f_0 = \bar{g}$ according to the first point of Lemma 1.2. We get the same kind of contradiction as in the case of f_0 inhomogeneous. Hence, we have proved that f_0 is an orbitally stable steady state. □

To end this section, let us prove the preliminary Lemma 3.9.

Proof of Lemma 3.9. Let us argue contradiction. Then there exist $\varepsilon_0 > 0$, a sequence $(f^n_{init})_n \in E_j^{\mathbb{N}}$ and a sequence $(t_n)_n \in \mathbb{R}^+_*$ such that $f^n_{init} \overset{E_j}{\longrightarrow} f_0$ and for all n,

$$|\|v^2 f^n(t_n)\|_{L^1} - \|v^2 f_0\|_{L^1}| \leq \frac{\delta_0}{2} \quad \text{and} \quad |\|v^2 f^n(t_n)\|_{L^1} - \|v^2 f_0\|_{L^1}| > \varepsilon_0, \tag{3.28}$$

where $f^n(t_n)$ is a solution to (1.1) with initial data f^n_{init}. Let $g_n(\theta, v) = f^n(t_n, \theta, v)$, exactly like in the proof of Theorem 4, we introduce $\bar{g_n}(\theta, v) = \gamma_n g_n\left(\theta, \frac{\gamma_n}{\lambda_n} v\right)$ where (γ_n, λ_n) is the unique pair such that $\|\bar{g_n}\|_{L^1} = M_1$ and

$\|j(\bar{g_n})\|_{L^1} = M_j$. In the same way as the proof of Theorem 4 in Section 3.3.2, we prove that $\bar{g}$ is a minimizer of (1.8) and as in the proof of Theorem 2 in Section 2.2.2, we show that $\bar{g}$ and f_0 are equimeasurable. Using the first inequality of (3.28) and the convergence of $\|v^2 g_n\|_{L^1}$ to $\|v^2\bar{g}\|_{L^1}$, we get

$$||\|v^2 f_0\|_{L^1} - \|v^2\bar{g}\|_{L^1}| \le \delta_0 \tag{3.29}$$

Therefore according to Lemma 1.2, we deduce that $f_0 = \bar{g}$ up to a translation in θ and we get a contradiction with the second inequality of (3.28) and the convergence in E_j of g_n to $\bar{g}$. □

4. Problem with an infinite number of constraints

4.1. Generalized rearrangement with respect to the microscopic energy

In the same way as in the two-constraints problem, we introduce a new function denoted by $f^{*\phi}$. The sequence $(f^{*\phi_n})_n$ has better compactness properties than the sequence $(f_n)_n$. We get the compactness of $(f_n)_n$ via the compactness of $(f^{*\phi_n})_n$ thanks to monotonicity properties of $\mathcal{H}$ with respect to the transformation $f^{*\phi}$ which will be detailed in Lemma 4.3. To define this new function, we use the generalization of symmetric rearrangement with respect to the microscopic energy $e = \frac{v^2}{2} + \phi(\theta)$ introduced in [17]. For more generalized results, see also [16]. We first recall the usual notion of rearrangement which is adapted here to functions defined on the domain $\mathbb{T}\times\mathbb{R}$. For more details on this subject see [15] and [21]. For any nonnegative function $f \in L^1(\mathbb{T}\times\mathbb{R})$, we define its distribution function with (1.7). Let $f^\#$ be the pseudo-inverse of the function μ_f defined by (1.7):

$$f^\#(s) = \inf\{t \ge 0, \mu_f(t) \le s\} = \sup\{t \ge 0, \mu_f(t) > s\}, \quad \text{for all } s \ge 0. \tag{4.1}$$

We notice that $f^\#(0) = \|f\|_{L^\infty} \in \mathbb{R}\cup\{+\infty\}$ and $f^\#(+\infty) = 0$. It is well known that μ_f is right-continuous and that for all $s \ge 0$, $t \ge 0$,

$$f^\#(s) > t \quad \Longleftrightarrow \quad \mu_f(t) > s. \tag{4.2}$$

Next, we define the rearrangement f^* of f by

$$f^*(\theta, v) = f^\#\left(\left|B(0, \sqrt{\theta^2 + v^2}) \cap \mathbb{T}\times\mathbb{R}\right|\right), \tag{4.3}$$

where $B(0, R)$ denotes the open ball in $\mathbb{R}^2$ centered at 0 with radius R. Then in order to generalize the rearrangements, we introduce for $\phi \in \mathcal{C}^2(\mathbb{T})$ the quantity

$$a_\phi(e) = \left|\left\{(\theta, v) \in [0, 2\pi]\times\mathbb{R} : \frac{v^2}{2} + \phi(\theta) < e\right\}\right|. \tag{4.4}$$

From this quantity, we can adapt the proofs in Section 2.1 of [17] to the case of $\phi \in \mathcal{C}^2$ and we are able to define the generalized rearrangement with respect to the microscopic energy. We get the following properties gathered in Lemma 4.1. The last item of this lemma is proved in the Step 2 of the proof of Proposition 2.3 in [17].

Lemma 4.1 *(Properties of a_ϕ). We have the following statements.*

(1) *The function a_ϕ is continuous on $\mathbb{R}$, vanishes on $]-\infty, \min\phi]$ and is strictly increasing from $[\min\phi, +\infty[$ to $[0, +\infty[$.*
(2) *The function a_ϕ is invertible from $[\min\phi, +\infty[$ to $[0, +\infty[$, we denote its inverse by a_ϕ^{-1}. This inverse satisfies*

$$\frac{s^2}{32\pi^2} + \min\phi \le a_\phi^{-1}(s) \le \frac{s^2}{32\pi^2} + \max\phi, \quad \forall s \in \mathbb{R}_+. \tag{4.5}$$

(3) *Let $\phi \in \mathcal{C}^2([0, 2\pi])$ and let a_ϕ be the function defined by (4.4). Let f be a nonnegative function in $L^1([0, 2\pi]\times\mathbb{R})$. Then the function*

$$f^{*\phi}(\theta, v) = f^\#\left(a_\phi\left(\frac{v^2}{2} + \phi(\theta)\right)\right), \quad (\theta, v) \in [0, 2\pi]\times\mathbb{R}$$

is equimeasurable to f, that is $\mu_{f^{*\phi}} = \mu_f$ where μ_f is defined by (1.7). The function $f^{*\phi}$ is called the decreasing rearrangement with respect to the microscopic energy $\frac{v^2}{2} + \phi(\theta)$.

(4) *Let $f \in \mathrm{L}^1([0, 2\pi] \times \mathbb{R})$ and ϕ_f is the potential associated to f defined by (1.3), we have*

$$\iint \left(\frac{v^2}{2} + \phi_f(\theta)\right)(f(\theta, v) - f^{*\phi_f}(\theta, v))\mathrm{d}\theta\mathrm{d}v \geq 0. \tag{4.6}$$

The next lemma, proved in Section 3.1 of [16], is a technical lemma about rearrangements which will be used in Lemma 4.5.

Lemma 4.2. *Let $\phi \in \mathcal{C}^2([0, 2\pi])$ and $f \in \mathrm{L}^1([0, 2\pi] \times \mathbb{R})$, we have the following identity*

$$\int_0^{2\pi}\int_{\mathbb{R}} \left(\frac{v^2}{2} + \phi(\theta)\right) f^{*\phi}(\theta, v)\mathrm{d}\theta\mathrm{d}v = \int_0^{+\infty} a_\phi^{-1}(s) f^{\#}(s)\mathrm{d}s.$$

In the rest of this Section, we adopt the following definition of minimizing sequences.

Definition 4.1 *(Minimizing sequence).* We shall say that $(f_n)_n$ is a minimizing sequence of (1.11) if $(f_n)_n$ is uniformly bounded and

$$\mathcal{H}(f_n) \underset{n\to+\infty}{\longrightarrow} H_0 \quad \text{and} \quad \|f_n^* - f_0^*\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0.$$

As mentioned at the beginning of this section, we need to link $\mathcal{H}(f_n)$ and $\mathcal{H}(f^{*\phi_n})$ to get compactness for f_n. Hence, we introduce a second problem of minimization

$$\mathcal{J}_{f^*}^0 = \inf_{\int_0^{2\pi}\phi=0} \mathcal{J}_{f^*}(\phi) \text{ where } \mathcal{J}_{f^*}(\phi) = \iint \left(\frac{v^2}{2} + \phi(\theta)\right) f^{*\phi}(\theta, v)\mathrm{d}\theta\mathrm{d}v + \frac{1}{2}\int_0^{2\pi} \phi'(\theta)^2\mathrm{d}\theta. \tag{4.7}$$

Lemma 4.3 *(Monotonicity properties of $\mathcal{H}$ with respect to the transformation $f^{*\phi}$). We have the following inequalities:*

(1) *Let $f \in \mathcal{E}$, for all $\phi \in H^2([0, 2\pi])$ such that $\phi(0) = \phi(2\pi)$ and $\int_0^{2\pi}\phi = 0$, we have $\mathcal{H}(f^{*\phi}) \leq \mathcal{J}_{f^*}(\phi)$.*
(2) *For all $f \in \mathcal{E}$, $H_0 \leq \mathcal{H}(f^{*\phi_f}) \leq \mathcal{J}_{f^*}(\phi_f) \leq \mathcal{H}(f)$ where H_0 is defined by (1.11). Besides $H_0 = \mathcal{J}_{f^*}^0$.*

Proof. The first item of this lemma is proved exactly like item (2) of Lemma 3.2. Hence we have

$$\mathcal{J}_{f^*}(\phi) = \mathcal{H}(f^{*\phi}) + \frac{1}{2}\|\phi'_{f^{*\phi}} - \phi'\|_{\mathrm{L}^2}^2 \tag{4.8}$$

Then, let us prove the right inequality of item (2). Let $f \in \mathcal{E}$, the Hamiltonian can be written as

$$\begin{aligned}\mathcal{H}(f) &= \int_0^{2\pi}\int_{\mathbb{R}} \left(\frac{v^2}{2} + \phi_f(\theta)\right) f^{*\phi_f}(\theta, v)\mathrm{d}\theta\mathrm{d}v + \frac{1}{2}\int_0^{2\pi} \phi_f'(\theta)^2\mathrm{d}\theta \\ &+ \int_0^{2\pi}\int_{\mathbb{R}} \left(\frac{v^2}{2} + \phi_f(\theta)\right)(f(\theta, v) - f^{*\phi_f}(\theta, v))\mathrm{d}\theta\mathrm{d}v \\ &= \mathcal{J}_{f^*}(\phi_f) + \int_0^{2\pi}\int_{\mathbb{R}} \left(\frac{v^2}{2} + \phi_f(\theta)\right)(f(\theta, v) - f^{*\phi_f}(\theta, v))\mathrm{d}\theta\mathrm{d}v.\end{aligned}$$

Using (4.6), we get that $\mathcal{H}(f^{*\phi}) \leq \mathcal{J}_{f^*}(\phi)$. Thanks to the two above inequalities, we easily deduce $H_0 = \mathcal{J}_{f^*}^0$. □

4.2. Existence of ground states

This section is devoted to the proof of Theorem 5.

4.2.1. Properties of the infimum

Lemma 4.4. *The variational problem (1.11) satisfies the following statements.*

(1) *The infimum (1.11) exists, i.e.* $H_0 > -\infty$.
(2) *For any minimizing sequence* $(f_n)_n$ *of the variational problem (1.11), we have the following properties:*
  (a) *There exists* $\bar{f} \in \mathrm{L}^1([0,2\pi]\times\mathbb{R})$ *such that* $f_n \underset{n\to+\infty}{\longrightarrow} \bar{f}$ *weakly in* L^1.
  (b) *We have* $\|\phi_{f_n} - \phi_{\bar{f}}\|_{H^1} \underset{n\to+\infty}{\longrightarrow} 0$.

The proof of item (1) from Lemma 4.4 is similar to the one of Lemma 2.1. In the spirit of Lemma 2.1, noticing that $\|f_n\|_{\mathrm{L}^1} = \|f_n^*\|_{\mathrm{L}^1}$ is bounded and using Dunford–Pettis's theorem, we get the weak convergence of $(f_n)_n$ in $\mathrm{L}^1([0,2\pi]\times\mathbb{R})$. The proof of item (b) is similar to the one of item (2) in Lemma 2.1.

4.2.2. Proof of Theorem 5

We are now ready to prove Theorem 5.

Step 1: Existence of a minimizer.

From item (1) of Lemma 4.4, we know that H_0 is finite. Let us show that there exists a function which minimizes the variational problem (1.11). Let $(f_n)_n \in \mathcal{E}^{\mathbb{N}}$ be a minimizing sequence of (1.11). From item (a) of Lemma 4.4, there exists $\bar{f} \in \mathrm{L}^1([0,2\pi]\times\mathbb{R})$ such that $f_n \underset{n\to+\infty}{\rightharpoonup} \bar{f}$ in L^1_w. From item (b) of Lemma 4.4, ϕ_{f_n} strongly converges to $\phi_{\bar{f}}$ in $\mathrm{L}^2([0,2\pi]\times\mathbb{R})$ and ϕ'_{f_n} strongly converges to $\phi'_{\bar{f}}$ in $\mathrm{L}^2([0,2\pi]\times\mathbb{R})$.

In the following paragraphs, we will note $\phi_n := \phi_{f_n}$ and $\bar{\phi} := \phi_{\bar{f}}$. Notice using item (2) of Lemma 4.3 that $(\phi_n)_n$ is a minimizing sequence of (4.7). As in the proof of Theorem 3, we introduce a new minimizing sequence which has better compactness properties than $(f_n)_n$. The sequence $(f_0^{*\phi_n})_n$ is well-defined according to Lemma 4.1. Since $(\phi_n)_n$ is a minimizing sequence of (4.7) and using the second item of Lemma 4.3, we directly get $\mathcal{H}(f_0^{*\phi_n}) \underset{n\to+\infty}{\longrightarrow} H_0$. The next step is to prove that $\mathcal{H}(f_0^{*\phi_n}) \underset{n\to+\infty}{\longrightarrow} \mathcal{H}(f_0^{*\bar{\phi}})$. In order to do that, let us show that $f_0^{*\phi_n} \underset{n\to+\infty}{\longrightarrow} f_0^{*\bar{\phi}}$ strongly in $\mathrm{L}^1([0,2\pi]\times\mathbb{R})$. From general properties of rearrangements, see [15] and [21], we have $\|f_0^{*\phi_n}\|_{\mathrm{L}^1} = \|f_0\|_{\mathrm{L}^1}$ and $\|f_0^{*\bar{\phi}}\|_{\mathrm{L}^1} = \|f_0\|_{\mathrm{L}^1}$ and therefore using Brezis–Lieb, see [8], it is sufficient to show that $f_0^{*\phi_n} \underset{n\to+\infty}{\longrightarrow} f_0^{*\bar{\phi}}$ a.e. to get the strong convergence in $\mathrm{L}^1([0,2\pi]\times\mathbb{R})$. Using the dominated convergence theorem, we easily get that

$$a_{\phi_n}\left(\frac{v^2}{2} + \phi_n(\theta)\right) \underset{n\to+\infty}{\longrightarrow} a_{\bar{\phi}}\left(\frac{v^2}{2} + \bar{\phi}(\theta)\right) \text{ a.e. up to a subsequence.}$$

As by hypothesis, $f_0 \in \mathcal{E} \cap \mathcal{C}^0([0,2\pi]\times\mathbb{R})$, $f_0^{\#}$ is continuous then $f_0^{*\phi_n} \underset{n\to+\infty}{\longrightarrow} f_0^{*\bar{\phi}}$ a.e. up to an extraction of a subsequence. Thus, we get $\|f_0^{*\phi_n} - f_0^{*\bar{\phi}}\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0$. Then, from classical inequality about lower semicontinuous functions (see [15]) and the convergence in $\mathrm{L}^2([0,2\pi]\times\mathbb{R})$ of ϕ_n, we deduce that

$$H_0 \ge \iint \frac{v^2}{2} f_0^{*\bar{\phi}}(\theta,v)\mathrm{d}\theta\mathrm{d}v - \frac{1}{2}\int_0^{2\pi} \phi'_{f_0^{*\bar{\phi}}}(\theta)^2\mathrm{d}\theta = \mathcal{H}(f_0^{*\bar{\phi}}) \tag{4.9}$$

Since $f_0^{*\bar{\phi}} \in \mathcal{E}$ and is equimeasurable to f_0, we get $H_0 \le \mathcal{H}(f_0^{*\bar{\phi}})$. Hence with the inequality (4.9), we deduce $H_0 = \mathcal{H}(f_0^{*\bar{\phi}})$ and $f_0^{*\bar{\phi}}$ is a minimizer of (1.11).

Step 2: The minimizer is a steady state of (1.1).

The minimizer $f_0^{*\bar\phi}$ is a stationary state of the system (1.1) and to prove that it is sufficient to show that $\bar\phi = \phi_{f_0^{*\bar\phi}}$. The proof is similar to the one of two-constraints case in Section 3.2.2, we use Lemma 4.3 and equality (4.8) to get the result.

4.3. Orbital stability of the ground states

4.3.1. Proof of Theorem 6

This section is devoted to the proof of Theorem 6. As we do not have the uniqueness of the minimizers under constraint of equimeasurability, we can only get the orbital stability of the set of minimizers and not the orbital stability of each minimizer.

First, we need to the following lemma which is at the heart of the proof of the compactness of minimizing sequences. This lemma will be proved at the end of the proof of Theorem 6.

Lemma 4.5. *Let $f_0 \in \mathcal{E} \cap \mathcal{C}^0([0,2\pi]\times\mathbb{R})$ and let $(f_n)_n$ be a minimizing sequence of (1.11). Then $(f_n)_n$ has a weak limit $\bar f$ in $\mathrm{L}^1([0,2\pi]\times\mathbb{R})$. Denoting $\bar\phi := \phi_{\bar f}$, we have*

$$\int_0^{\|f_0\|_{\mathrm{L}^\infty}} B_{\bar\phi}(\mu_{f_0}(t) + \beta_{f_n, f_0^{*\bar\phi}}(t)) + B_{\bar\phi}(\mu_{f_0}(t) - \beta_{f_n, f_0^{*\bar\phi}}(t)) - 2B_{\bar\phi}(\mu_{f_0}(t))\mathrm{d}t \underset{n\to+\infty}{\longrightarrow} 0$$

where

$$\begin{cases} \beta_{f,g}(t) = |\{(\theta, v) \in [0,2\pi]\times\mathbb{R} : f(\theta,v) \le t < g(\theta,v)\}|, \\ B_{\bar\phi}(\mu) = \iint_{\{a_{\bar\phi}(\frac{v^2}{2}+\bar\phi(\theta))<\mu\}} \frac{v^2}{2} + \bar\phi(\theta)\mathrm{d}\theta\mathrm{d}v. \end{cases} \tag{4.10}$$

Step 1: Compactness of the minimizing sequences.

Let $(f_n)_n$ be a minimizing sequence of (1.11), let us show that $(f_n)_n$ is compact in $\mathcal{E}$. Using Lemma 4.4, there exists $\bar f \in \mathrm{L}^1$ such that $f_n \underset{n\to+\infty}{\rightharpoonup} \bar f$ weakly in $\mathrm{L}^1([0,2\pi]\times\mathbb{R})$ and $\phi_n \underset{n\to+\infty}{\longrightarrow} \bar\phi$ strongly in $\mathrm{L}^2([0,2\pi]\times\mathbb{R})$ where $\bar\phi := \phi_{\bar f}$. Arguing as in the proof of Theorem 5 in Section 4.2.2, we also get $f_0^{*\phi_n} \underset{n\to+\infty}{\longrightarrow} f_0^{*\bar\phi}$ strongly in $\mathrm{L}^1([0,2\pi]\times\mathbb{R})$. Our aim is now to show that $\|f_n - f_0^{*\bar\phi}\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0$. In order to do that, we will use some techniques about rearrangements introduced in [16]. In particular, we will use the following equality established in the proof of Theorem 1 in Section 2.3 in [16]

$$\|f_n - f_0^{*\bar\phi}\|_{\mathrm{L}^1} = 2\int_0^{+\infty} \beta_{f_n, f_0^{*\bar\phi}}(t)\mathrm{d}t + \|f_n\|_{\mathrm{L}^1} - \|f_0\|_{\mathrm{L}^1} \tag{4.11}$$

where $\beta_{f,g}$ is defined in (4.10). The second term of (4.11): $\|f_n\|_{\mathrm{L}^1} - \|f_0\|_{\mathrm{L}^1}$ goes to 0 when n goes to infinity. Indeed, according to Definition 4.1 of a minimizing sequence, we have: $\|f_n^* - f_0^*\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0$ then $\|f_n^*\|_{\mathrm{L}^1} = \|f_n\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} \|f_0^*\|_{\mathrm{L}^1} = \|f_0\|_{\mathrm{L}^1}$ using rearrangements properties, see [15]. Hence to prove that: $\|f_n - f_0^{*\bar\phi}\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0$, we need to prove that $\int_0^{+\infty} \beta_{f_n, f_0^{*\bar\phi}}(t)\mathrm{d}t \underset{n\to+\infty}{\longrightarrow} 0$. For this purpose, it is sufficient to show that $\beta_{f_n, f_0^{*\bar\phi}}(t) \underset{n\to+\infty}{\longrightarrow} 0$. Indeed, this a direct application of the dominated convergence theorem

- $\beta_{f_n, f_0^{*\bar{\phi}}}(t) \underset{n\to+\infty}{\longrightarrow} 0$,
- $0 \le \beta_{f_n, f_0^{*\bar{\phi}}}(t) \le \mu_{f_0}(t)$ and $\int_0^{+\infty} \mu_{f_0}(t)\mathrm{d}t = \|f_0\|_{\mathrm{L}^1}$ using Fubini's theorem.

To get the a.e. convergence to 0 of $\beta_{f_n, f_0^{*\bar{\phi}}}(t)$, we will use Lemma 4.5. By convexity of $B_{\bar{\phi}}$ given by Theorem 1 in [16],

$$B_{\bar{\phi}}(\mu_{f_0}(t) + \beta_{f_n, f_0^{*\bar{\phi}}}(t)) + B_{\bar{\phi}}(\mu_{f_0}(t) - \beta_{f_n, f_0^{*\bar{\phi}}}(t)) - 2B_{\bar{\phi}}(\mu_{f_0}(t)) \ge 0$$

therefore Lemma 4.5 implies that

$$B_{\bar{\phi}}(\mu_{f_0}(t) + \beta_{f_n, f_0^{*\bar{\phi}}}(t)) + B_{\bar{\phi}}(\mu_{f_0}(t) - \beta_{f_n, f_0^{*\bar{\phi}}}(t)) - 2B_{\bar{\phi}}(\mu_{f_0}(t)) \underset{n\to+\infty}{\longrightarrow} 0 \text{ for almost } t \ge 0.$$

Notice that $\beta_{f_n, f_0^{*\bar{\phi}}}(0) = 0$ and for all $t > 0$,

$$0 < \beta_{f_n, f_0^{*\bar{\phi}}}(t) \le \frac{1}{t}\|f\|_{\mathrm{L}^1}.$$

Thus the sequence $(\beta_{f_n, f_0^{*\bar{\phi}}}(t))_n$ is bounded and has a convergent subsequence. Let us suppose that $\beta_{f_n, f_0^{*\bar{\phi}}}(t) \underset{n\to+\infty}{\longrightarrow} l \ne 0$, then by strict convexity of $B_{\bar{\phi}}$,

$$\begin{aligned} &B_{\bar{\phi}}(\mu_{f_0}(t) + \beta_{f_n, f_0^{*\bar{\phi}}}(t)) + B_{\bar{\phi}}(\mu_{f_0}(t) - \beta_{f_n, f_0^{*\bar{\phi}}}(t)) - 2B_{\bar{\phi}}(\mu_{f_0}(t)) \\ &\qquad \underset{n\to+\infty}{\longrightarrow} B_{\bar{\phi}}(\mu_{f_0}(t) + l) + B_{\bar{\phi}}(\mu_{f_0}(t) - l) - 2B_{\bar{\phi}}(\mu_{f_0}(t)) > 0. \end{aligned}$$

This yields a contradiction, and therefore $\beta_{f_n, f_0^{*\bar{\phi}}}(t) \underset{n\to+\infty}{\longrightarrow} 0$ for almost $t \ge 0$. Hence $\|f_n - f_0^{*\bar{\phi}}\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0$. Besides we have proved that $f_n \underset{n\to+\infty}{\rightharpoonup} \bar{f}$ weakly in $\mathrm{L}^1([0, 2\pi] \times \mathbb{R})$, hence by uniqueness of the limit, we get $f_0^{*\bar{\phi}} = \bar{f}$. Since by definition, a minimizing sequence is uniformly bounded, to prove the compactness of the sequence $(f_n)_n$ in the energy space $\mathcal{E}$, there remains show that

$$\|v^2(f_n - \bar{f})\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0.$$

Notice that

$$\iint v^2(f_n(\theta, v) - \bar{f}(\theta, v))\mathrm{d}\theta\mathrm{d}v = 2(\mathcal{H}(f_n) - \mathcal{H}(\bar{f})) + \|\phi_n'\|_{\mathrm{L}^2}^2 - \|\bar{\phi}'\|_{\mathrm{L}^2}^2,$$

thus $\|v^2 f_n\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} \|v^2 \bar{f}\|_{\mathrm{L}^1}$ since $(f_n)_n$ is a minimizing sequence and $\bar{f}$ is a minimizer. Moreover $v^2 f_n \underset{n\to+\infty}{\longrightarrow} v^2 \bar{f}$ up to an extraction of a subsequence since $f_n \underset{n\to+\infty}{\longrightarrow} \bar{f}$ strongly in L^1. Thanks to Brezis–Lieb's lemma (see [8]), we deduce that $\|v^2(f_n - \bar{f})\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0$. To conclude, we have proved that the sequence $(f_n)_n$ is compact in $\mathcal{E}$.

Step 2: Proof of the orbital stability.

Let us argue by contradiction, let f_{i_0} be a steady state of (1.1) which minimizes (1.11). Assume that f_{i_0} is orbitally unstable. Then there exist $\varepsilon_0 > 0$, a sequence $(f_{init}^n)_n \in \mathcal{E}^{\mathbb{N}}$ and a sequence $(t_n)_n \in (\mathbb{R}_*^+)^{\mathbb{N}}$ such that $f_{init}^n \overset{\mathcal{E}}{\longrightarrow} f_{i_0}$ and for all n, for all $\theta_0 \in [0, 2\pi]$, for all f_i minimizer of (1.11),

$$\begin{cases} \|f^n(t_n, \theta + \theta_0, v) - f_i(\theta, v)\|_{\mathrm{L}^1} > \varepsilon_0, \\ \text{or } \|v^2(f^n(t_n, \theta + \theta_0, v) - f_i(\theta, v))\|_{\mathrm{L}^1} > \varepsilon_0, \end{cases} \tag{4.12}$$

where $f^n(t_n, \theta, v)$ is a solution to (1.1) with initial data f_{init}^n. Let $g_n(\theta, v) = f^n(t_n, \theta, v)$. Notice that

$$\begin{aligned} \|(f_{init}^n)^* - f_0^*\|_{\mathrm{L}^1} &= \|(f_{init}^n)^* - f_{i_0}^*\|_{\mathrm{L}^1} \quad \text{since } f_{i_0} \in Eq(f_0), \\ &\le \|f_{init}^n - f_{i_0}\|_{\mathrm{L}^1} \quad \text{by contractivity of rearrangement (see [15])}, \end{aligned}$$

but from conservation properties of the flow (1.1), we have $g_n^* = (f_{init}^n)^*$ together with $\|g_n\|_{L^\infty} = \|f_{init}^n\|_{L^\infty}$. Therefore $g_n^* \underset{n\to+\infty}{\longrightarrow} f_0^*$ strongly in L^1 and $(g_n)_n$ is uniformly bounded. Finally, from item (2) of Lemma 4.3 and from the conservation property of the flow (1.1), we have

$$H_0 \le \mathcal{H}(f_0^{*\phi_{g_n}}) \le \mathcal{H}(g_n) \le \mathcal{H}(f_{init}^n) \underset{n\to+\infty}{\longrightarrow} H_0.$$

Thus $\mathcal{H}(g_n) \underset{n\to+\infty}{\longrightarrow} H_0$ and the sequence $(g_n)_n$ is a minimizing sequence of (1.11). According to the previous step, this sequence is compact, hence, up to an extraction of a subsequence, there exists $f_I \in \mathcal{E}$ such that $g_n \overset{\mathcal{E}}{\longrightarrow} f_I$. This implies that

$$\|g_n - f_I\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0 \quad \text{and} \quad \|v^2(g_n - f_I)\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.13}$$

Arguing as in the proof of Theorem 2 in Section 2.2.2, we prove that $\mathcal{H}(f_I) = H_0$ and that f_I is equimeasurable to f_{i_0}. We deduce that f_I is equimeasurable to f_0 and hence this is a minimizer of (1.11). We get a contradiction with (4.13) and (4.12). There remains to show Lemma 4.5.

Proof of Lemma 4.5. The existence of the weak limit $\bar{f}$ is given by item (3) of Lemma 4.4. Many techniques in this proof have been introduced in [16]. By convexity of $B_{\bar{\phi}}$, see Theorem 1 in [16], we have

$$\int_0^{\|f_0\|_{L^\infty}} B_{\bar{\phi}}(\mu_{f_0}(t) + \beta_{f_n, f_0^{*\bar{\phi}}}(t)) + B_{\bar{\phi}}(\mu_{f_0}(t) - \beta_{f_n, f_0^{*\bar{\phi}}}(t)) - 2B_{\bar{\phi}}(\mu_{f_0}(t))\mathrm{d}t \ge 0.$$

Using the remark following Theorem 1 in [16], we have

$$\int_0^{\|f_0\|_{L^\infty}} B_{\bar{\phi}}(\mu_{f_0}(t) + \beta_{f_n, f_0^{*\bar{\phi}}}(t)) + B_{\bar{\phi}}(\mu_{f_0}(t) - \beta_{f_n, f_0^{*\bar{\phi}}}(t)) - 2B_{\bar{\phi}}(\mu_{f_0}(t))\mathrm{d}t \le A_n + B_n$$

where

$$\begin{cases} A_n = \int_0^{2\pi} \int_{\mathbb{R}} \left(\frac{v^2}{2} + \bar{\phi}(\theta)\right)(f_n(\theta, v) - f_0^{*\bar{\phi}}(\theta, v))\mathrm{d}\theta \mathrm{d}v, \\ B_n = \int_0^{+\infty} [a_{\bar{\phi}}^{-1}(2\mu_{f_0}(s))\beta_{f_n^*, f_0^*}(s) - a_{\bar{\phi}}^{-1}(\mu_{f_0}(s))\beta_{f_0^*, f_n^*}(s)]\mathrm{d}s. \end{cases}$$

Then let us show that $A_n \underset{n\to+\infty}{\longrightarrow} 0$. After integrating by parts, we get

$$A_n = \int_0^{2\pi}\int_{\mathbb{R}} \left(\frac{v^2}{2} + \bar{\phi}(\theta)\right)(f_n(\theta, v) - f_0^{*\bar{\phi}}(\theta, v))\mathrm{d}\theta \mathrm{d}v = \mathcal{H}(f_n) - \mathcal{H}(f_0^{*\bar{\phi}}) + \frac{1}{2}\|\phi_n' - \bar{\phi}'\|_{L^2}^2.$$

We have seen in Step 1 of the proof of Theorem 5 in Section 4.2.2 that $\mathcal{H}(f_n) - \mathcal{H}(f_0^{*\bar{\phi}})$ converges to 0 and $\|\phi_n' - \bar{\phi}'\|_{L^2}^2 \underset{n\to+\infty}{\longrightarrow} 0$; therefore $A_n \underset{n\to+\infty}{\longrightarrow} 0$. Finally let us show that $B_n \underset{n\to+\infty}{\longrightarrow} 0$. We have the following inequality using inequality (4.5)

$$\begin{aligned} B_n &= \int_0^{+\infty} [a_{\bar{\phi}}^{-1}(2\mu_{f_0}(s))\beta_{f_n^*, f_0^*}(s) - a_{\bar{\phi}}^{-1}(\mu_{f_0}(s))\beta_{f_0^*, f_n^*}(s)]\mathrm{d}s \\ &\le \int_0^{+\infty} \left(\frac{4\mu_{f_0}(s)^2}{32\pi^2} + \max\bar{\phi}\right)\beta_{f_n^*, f_0^*}(s) - \left(\frac{\mu_{f_0}(s)^2}{32\pi^2} + \min\bar{\phi}\right)\beta_{f_0^*, f_n^*}(s)\mathrm{d}s. \end{aligned}$$

Using the following identity, see the proof of Proposition 4.1 in [17],

$$\int_0^{+\infty} \beta_{f_0^*, f_n^*}(s)\mathrm{d}s + \int_0^{+\infty} \beta_{f_n^*, f_0^*}(s)\mathrm{d}s = \|f_n^* - f_0^*\|_{\mathrm{L}^1},$$

we get

$$B_n \le \frac{1}{8\pi^2} \int_0^{+\infty} \mu_{f_0}(s)^2 \beta_{f_n^*, f_0^*}(s)\mathrm{d}s + (\max\bar{\phi} + \min\bar{\phi}) \int_0^{+\infty} \beta_{f_n^*, f_0^*}(s)\mathrm{d}s - \min\bar{\phi}\|f_n^* - f_0^*\|_{\mathrm{L}^1}.$$

Notice that $\min\bar{\phi}\|f_n^* - f_0^*\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0$ since $(f_n)_n$ is a minimizing sequence of (1.11). Besides

$$\begin{aligned}(\max\bar{\phi} + \min\bar{\phi}) \int_0^{+\infty} \beta_{f_n^*, f_0^*}(s)\mathrm{d}s &\le \max\bar{\phi} \int_0^{+\infty} \beta_{f_n^*, f_0^*}(s)\mathrm{d}s \\ &\le \max\bar{\phi} \int_0^{+\infty} (f_n^* - f_0^*)_+ \mathrm{d}s \\ &\le \max\bar{\phi}\|f_n^* - f_0^*\|_{\mathrm{L}^1} \underset{n\to+\infty}{\longrightarrow} 0.\end{aligned}$$

Finally, let us prove that

$$\frac{1}{8\pi^2} \int_0^{+\infty} \mu_{f_0}(s)^2 \beta_{f_n^*, f_0^*}(s)\mathrm{d}s \underset{n\to+\infty}{\longrightarrow} 0.$$

First notice that $\beta_{f_n^*, f_0^*}(s) \underset{n\to+\infty}{\longrightarrow} 0$. Indeed we shall apply the dominated convergence theorem to $\beta_{f_n^*, f_0^*}(s) = \iint \mathbb{1}_{\{f_n^*(\theta,v) \le s < f_0^*(\theta,v)\}}\mathrm{d}\theta\mathrm{d}v$ for $s > 0$. We first have

- $\mathbb{1}_{\{f_n^*(\theta,v) \le s < f_0^*(\theta,v)\}} \underset{n\to+\infty}{\longrightarrow} \mathbb{1}_{\{f_0^*(\theta,v) \le s < f_0^*(\theta,v)\}}$ a.e. since $f_n^* \underset{n\to+\infty}{\longrightarrow} f_0^*$ strongly in $\mathrm{L}^1([0, 2\pi] \times \mathbb{R})$,
- $\mathbb{1}_{\{f_n^*(\theta,v) \le s < f_0^*(\theta,v)\}} \le \mathbb{1}_{\{s < f_0^*(\theta,v)\}}$. But $\iint \mathbb{1}_{\{s < f_0^*(\theta,v)\}}\mathrm{d}\theta\mathrm{d}v = \mu^*_{f_0}(s) = \mu_{f_0}(s) < \infty$ since $f_0 \in \mathrm{L}^1([0, 2\pi] \times \mathbb{R})$.

Hence by the dominated convergence theorem, we get for all $s > 0$, $\beta_{f_n^*, f_0^*}(s) \underset{n\to+\infty}{\longrightarrow} 0$. For $s = 0$, $\beta_{f_n^*, f_0^*}(0) = |\varnothing| = 0$, thus for all $s \ge 0$, $\beta_{f_n^*, f_0^*}(s) \underset{n\to+\infty}{\longrightarrow} 0$. There remains to dominate the term $\mu_{f_0}(s)^2\beta_{f_n^*, f_0^*}(s)$. Notice that $\mu_{f_0}(s)^2\beta_{f_n^*, f_0^*}(s) \le \mu_{f_0}(s)^3$. However we have

$$\int_0^{+\infty} s^2 f_0^\#(s)\mathrm{d}s = \int_0^{+\infty} \left(\int_{0 \le s < \mu_{f_0}(t)} s^2 \mathrm{d}s \right) \mathrm{d}t = \frac{1}{3} \int_0^{+\infty} \mu_{f_0}(t)^3 \mathrm{d}t.$$

So to prove the integrability of $s \to \mu_{f_0}(s)^3$, it is sufficient to show that $\int_0^{+\infty} s^2 f_0^\#(s)\mathrm{d}s < \infty$. Using equality (4.5), identity $\int f_0^\#(s)\mathrm{d}s = \|f_0\|_{\mathrm{L}^1}$ and Lemma 4.2, we get

$$\begin{aligned}\int_0^{+\infty} s^2 f_0^\#(s)\mathrm{d}s &\lesssim \int_0^{+\infty} (a_{\bar{\phi}}^{-1}(s) + 1) f_0^\#(s)\mathrm{d}s \\ &= \int_0^{+\infty} a_{\bar{\phi}}^{-1}(s) f_0^\#(s)\mathrm{d}s + \|f_0\|_{\mathrm{L}^1}\end{aligned}$$

$$= \iint \left(\frac{v^2}{2} + \bar{\phi}(\theta)\right) f_0^{*\bar{\phi}}(\theta, v)\mathrm{d}\theta\mathrm{d}v + \|f_0\|_{\mathrm{L}^1} < +\infty$$

since $f_0^{*\bar{\phi}}$ satisfies $H_0 = \mathcal{H}(f_0^{*\bar{\phi}})$ and $f_0 \in \mathrm{L}^1([0, 2\pi] \times \mathbb{R})$. Hence $\int_0^{+\infty} \mu_{f_0}(t)^3 \mathrm{d}t < +\infty$. We conclude by dominated convergence that

$$\int_0^{+\infty} \mu_{f_0}(s)^2 \beta_{f_n^*, f_0^*}(s)\mathrm{d}s \underset{n\to+\infty}{\longrightarrow} 0.$$

Therefore $B_n \underset{n\to+\infty}{\longrightarrow} 0$ and the lemma is proved. □

4.3.2. Expression of the minimizers

From the proof of compactness of minimizing sequences in Section 4.3.1, we can deduce the expression of the steady states of (1.1) which minimizes (1.11). Indeed, we have proved that any minimizing sequences $(f_n)_n$ converge to a minimizer $\bar{f}$ in $\mathcal{E}$ which satisfies $\bar{f} = f_0^{*\bar{\phi}}$. Hence any minimizer $\bar{f}$ of (1.11) has the following expression:

$$\bar{f} = f_0^{\#}\left(a_{\bar{\phi}}\left(\frac{v^2}{2} + \bar{\phi}(\theta)\right)\right).$$

Conflict of interest statement

There is no conflict of interest.

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Appendix A

Proof of Lemma 2.2. Let $(f_n)_n$ be a sequence of nonnegative functions converging weakly in $\mathrm{L}^1([0, 2\pi] \times \mathbb{R})$ to $\bar{f}$ such that $\|f_n\|_{\mathrm{L}^1} = M$, $\|v^2 f_n\|_{\mathrm{L}^1} \leq C_1$ and $\iint f_n \ln(f_n) \leq C_2$ where M, C_1 and C_2 do not depend on n. Let $\lambda \in \mathbb{R}_+$ and $f_1(\theta, v) = e^{-|v|}$, we have

$$\begin{aligned}\iint f_n \ln(f_n) &= \iint f_n \ln\left(\frac{f_n}{\lambda f_1}\right) + \ln(\lambda)\iint f_n + \iint f_n \ln(f_1)\\ &= \iint_{\{0\leq f_n \leq \lambda f_1\}} f_n \ln\left(\frac{f_n}{\lambda f_1}\right) + \iint \left(f_n \ln\left(\frac{f_n}{\lambda f_1}\right)\right)_+ + \ln(\lambda)M + \iint f_n \ln(f_1).\end{aligned}$$

First by using the lower semicontinuity properties of convex positive functions, we get

$$\liminf_{n\to+\infty} \iint \left(f_n \ln\left(\frac{f_n}{\lambda f_1}\right)\right)_+ \geq \iint \left(\bar{f} \ln\left(\frac{\bar{f}}{\lambda f_1}\right)\right)_+ .$$

At this stage, we have the following identity

$$\begin{aligned}\liminf_{n\to+\infty} \iint f_n \ln(f_n) \geq &\left[\iint \left(\bar{f} \ln\left(\frac{\bar{f}}{\lambda f_1}\right)\right)_+ + \ln(\lambda)M\right] + \liminf_{n\to+\infty} \iint f_n \ln(f_1)\\ &+ \liminf_{n\to+\infty} \iint_{\{0\leq f_n\leq\lambda f_1\}} f_n \ln\left(\frac{f_n}{\lambda f_1}\right).\end{aligned} \tag{A.1}$$

Let us then show that

$$\lim_{\lambda\to 0}\sup_n \left| \iint_{\{0\le f_n\le \lambda f_1\}} f_n \ln\left(\frac{f_n}{\lambda f_1}\right)\right| = 0. \tag{A.2}$$

This term can be written as

$$\iint_{\{0\le f_n\le \lambda f_1\}} f_n \ln\left(\frac{f_n}{\lambda f_1}\right) = \iint_{\{0\le f_n\le \lambda f_1\}} f_n \ln\left(\frac{f_n}{f_1}\right) - \ln(\lambda) \iint_{\{0\le f_n\le \lambda f_1\}} f_n = T_1 + T_2.$$

We have $|T_2| \le \lambda|\ln(\lambda)|M_1 \underset{\lambda\to 0}{\longrightarrow} 0$ uniformly in n where $M_1 = \|f_1\|_{L^1}$. Since for λ sufficiently small, the function $x \to x|\ln(x)|$ is increasing on $[0, \lambda f_1]$, we have for T_1

$$\begin{aligned}|T_1| &\le \iint_{\{0\le f_n\le \lambda f_1\}} f_n|\ln(f_1)| + \iint_{\{0\le f_n\le \lambda f_1\}} f_n|\ln(f_n)|\\ &\le \lambda \iint f_1|\ln(f_1)| + \lambda \iint f_1|\ln(\lambda f_1)| \le 2\lambda \iint f_1|\ln(f_1)| + \lambda|\ln(\lambda)|M_1.\end{aligned}$$

Clearly, we have $\iint f_1|\ln(f_1)| < +\infty$ so $|T_1| \underset{\lambda\to 0}{\longrightarrow} 0$ uniformly in n. So far, we have

$$\liminf_{n\to+\infty} \iint f_n \ln(f_n) \ge \lim_{\lambda\to 0}\left[\iint \left(\bar f \ln\left(\frac{\bar f}{\lambda f_1}\right)\right)_+ + \ln(\lambda)M\right] + \liminf_{n\to+\infty} \iint f_n \ln(f_1). \tag{A.3}$$

The next step is to show that $\lim_{\lambda\to 0}\left[\iint \left(\bar f \ln\left(\frac{\bar f}{\lambda f_1}\right)\right)_+ + \ln(\lambda)M\right] = \iint \bar f \ln\left(\frac{\bar f}{f_1}\right)$. We have

$$\left|\iint \left(\bar f \ln\left(\frac{\bar f}{\lambda f_1}\right)\right)_+ + \ln(\lambda)M - \iint \bar f \ln\left(\frac{\bar f}{f_1}\right)\right| \le \left| \iint_{\{\bar f\ge \lambda f_1\}} \bar f \ln\left(\frac{\bar f}{f_1}\right) - \iint \bar f \ln\left(\frac{\bar f}{f_1}\right)\right| + \lambda|\ln(\lambda)|M_1. \tag{A.4}$$

Let us show, using the dominated convergence theorem, that the first term of (A.4) converges to 0 when λ goes to 0. The term $\bar f \ln(\frac{\bar f}{f_1})\mathbb{1}_{\{\bar f\ge\lambda f_1\}}$ clearly converges to $\bar f \ln(\frac{\bar f}{f_1})$. So it remains to show that $\iint |\bar f \ln(\frac{\bar f}{f_1})| \mathrm{d}\theta \mathrm{d}v < +\infty$. We have

$$\begin{aligned}\iint \left|\bar f \ln\left(\frac{\bar f}{f_1}\right)\right| \mathrm{d}\theta\mathrm{d}v &\le \iint |\bar f \ln(\bar f)|\mathrm{d}\theta\mathrm{d}v + \iint |\bar f \ln(f_1)|\mathrm{d}\theta\mathrm{d}v\\ &\le \iint |\bar f \ln(\bar f)|\mathrm{d}\theta\mathrm{d}v + M + \|v^2 \bar f\|_{L^1}.\end{aligned}$$

It is well known, see [12], that for $\bar f \ge 0$, if $\|\bar f\|_{L^1} < +\infty$, $\|v^2\bar f\|_{L^1} < +\infty$, $|\iint \bar f \ln(\bar f)\mathrm{d}\theta\mathrm{d}v| < +\infty$, we have $\iint |\bar f \ln(\bar f)|\mathrm{d}\theta\mathrm{d}v < +\infty$. We already have that $\|\bar f\|_{L^1} < +\infty$, $\|v^2\bar f\|_{L^1} < +\infty$, so let us show that $|\iint \bar f \ln(\bar f)\mathrm{d}\theta\mathrm{d}v| < +\infty$. Thanks to Jensen's inequality (2.3), we have

$$\iint \bar f \ln(\bar f)\mathrm{d}\theta\mathrm{d}v \ge M(\ln(M) - \ln(M_1)) - \iint |v|\bar f > -\infty.$$

By hypothesis, we know that $\liminf_{n\to+\infty} \iint f_n \ln(f_n)\mathrm{d}\theta\mathrm{d}v \le C_2$ and with inequality (A.1) and limit (A.2), we get for all $\lambda \in \mathbb{R}_+$

$$C_2 \ge \iint_{\{\bar f\ge\lambda f_1\}} \bar f \ln(\bar f)\mathrm{d}\theta\mathrm{d}v + \ln(\lambda) \iint_{\{\bar f\le\lambda f_1\}} \bar f\mathrm{d}\theta\mathrm{d}v - \iint |v|\bar f.$$

The two last terms are bounded so $\iint_{\{\bar f \geq \lambda f_1\}} \bar f \ln(\bar f) \mathrm{d}\theta \mathrm{d}v$ is bounded from above and we deduce that $\iint \bar f \ln(\bar f) \mathrm{d}\theta \mathrm{d}v$ is bounded from above. So the dominated convergence theorem gives the limit. Then the second term of (A.4) clearly converges to 0. So

$$\liminf_{n\to+\infty} \iint f_n \ln(f_n) \geq \iint \bar f \ln(\bar f) + \liminf_{n\to+\infty} \iint (f_n - \bar f)\ln(f_1).$$

To conclude, it is sufficient to show that $\iint (f_n - \bar f)\ln(f_1) \underset{n\to+\infty}{\longrightarrow} 0$. Let $\varepsilon > 0$ and $R > 0$ such that $\frac{2C_1}{R} \leq \varepsilon$, we have

$$\begin{aligned}
\left|\iint (f_n - \bar f)\ln(f_1)\right| &\leq \left|\iint_{\{|v|\leq R\}} (f_n - \bar f)|v| \mathrm{d}\theta \mathrm{d}v\right| + \iint_{\{|v|>R\}} (f_n + \bar f)|v| \mathrm{d}\theta \mathrm{d}v \\
&\leq \left|\iint_{\{|v|\leq R\}} (f_n - \bar f)|v| \mathrm{d}\theta \mathrm{d}v\right| + \frac{1}{R}\iint v^2 (f_n + \bar f) \mathrm{d}\theta \mathrm{d}v \\
&\leq \left|\iint_{\{|v|\leq R\}} (f_n - \bar f)|v| \mathrm{d}\theta \mathrm{d}v\right| + \frac{2C_1}{R}.
\end{aligned}$$

The first term converges to 0 when n goes to infinity thanks to the weak convergence in $\mathrm{L}^1([0, 2\pi] \times \mathbb{R})$ of f_n to $\bar f$ and R is chosen such that the second term is smaller than ε. □

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