Ann. Scient. Éc. Norm. Sup.,
4e série, t. 38, 2005, p. 93 à 115.

COUNTING RATIONAL POINTS ON HYPERSURFACES OF LOW DIMENSION

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ABSTRACT. – Let $N(X,B)$ be the number of rational points of height at most B on a variety $X\subset \mathbf{P}^n$ defined over $\mathbf{Q}$. We establish new upper bounds for $N(X,B)$ for hypersurfaces of dimension at most four. We also study $N(X',B)$ for the open complements X' of all lines or all planes on such hypersurfaces. One of the goals is to show that $N(X,B)=O_d(B^{7/2-\delta}),\ \delta>0$ for smooth hypersurfaces in $\mathbf{P}^5$ defined by a form $F(x_0,\dots,x_5)$ of degree $d\geqslant 9$. This improves upon a classical upper estimate of Hua for the form $x_0^d+x_1^d+x_2^d-x_3^d-x_4^d-x_5^d$.

RÉSUMÉ. – Soit $N(X,B)$ le nombre de points rationnels de hauteur inférieure ou égale à B sur une variété $X\subset \mathbf{P}^n$ définie sur $\mathbf{Q}$. Nous établissons des nouvelles majorations pour $N(X,B)$ pour les hypersurfaces de dimension $\leqslant 4$. Nous étudions aussi $N(X',B)$ pour les compléments X' de toutes les droites ou tous les plans sur ces hypersurfaces. Un des buts est de démontrer que $N(X,B)=O_d(B^{7/2-\delta}),\ \delta>0$ pour les hypersurfaces lisses dans $\mathbf{P}^5$ definies par une forme $F(x_0,\dots,x_5)$ de degré $d\geqslant 9$. C´est une amélioration d´une majoration classique de Hua pour la forme $x_0^d+x_1^d+x_2^d-x_3^d-x_4^d-x_5^d$.

0. Introduction

Let Z be a locally closed subset of $\mathbf{P}^n$ defined over $\mathbf{Q}$. To count rational points on Z, we shall use the height $H(x)=\max(|x_0|,\dots,|x_n|)$ for a rational point x on $\mathbf{P}^n$ represented by a primitive integral $(n+1)$-tuple $(x_0,\dots,x_n)$. We denote by $N(Z,B)$ the number of rational points of height at most B on Z. This is clearly an abuse of notation since $N(Z,B)$ depends both on the choice of the embedding $Z\subset \mathbf{P}^n$ and the choice of coordinates of $\mathbf{P}^n$.

We want to investigate the asymptotic behaviour of $N(Z,B)$ when $B\to\infty$. The following result is due to Pila (see [23] and [24]).

0.1. THEOREM. – *Let $X\subset \mathbf{P}^n$ be a geometrically irreducible projective variety of degree d defined over $\mathbf{Q}$. Then,*

$$N(X,B)=O_{d,n,\varepsilon}(B^{\dim X+1/d+\varepsilon}).$$

It is remarkable that the implied constant does not depend on X apart from its dependence on the degree and the dimension of X. The starting point and the most subtle ingredient of Pila's proof is a uniform bound for curves due to Bombieri and Pila [3] (cf. also [24]). It is also possible to reduce to the case of hypersurfaces by means of finite birational projections.

We shall in this paper give estimates for hypersurfaces of dimension at most four which are stronger than Theorem 0.1 and other known results. To obtain these, we apply the methods of

Heath-Brown's fundamental paper on rational points on curves and surfaces [19]. The most important new tool in [19] is Theorem 14, where determinants are used to construct auxiliary hypersurfaces containing the rational points of bounded height. We shall use the technique with auxiliary hypersurfaces when we count points on curves and surfaces. In higher dimensions we use a number of new arguments from algebraic geometry. We thus make essential use of the Kodaira dimension in our estimates for smooth hypersurfaces. We obtain in particular improvements of a well-known bound of Hua from 1938 concerning the number of non-trivial positive solutions to the diophantine equation $x_0^d + x_1^d + x_2^d - x_3^d - x_4^d - x_5^d = 0$. This was also the aim of the paper of Browning and Heath-Brown [8], but our estimates are sharper and valid for general non-singular forms.

This paper is organised as follows. In Section i, $1 \leqslant i \leqslant 4$, we give uniform estimates for hyper-surfaces of dimension i. The approach is inductive and the proofs based on the estimates in lower dimensions. We shall apart from the determinant method also apply Siegel's lemma in the inductive process. We use thereby an idea from [19, p. 581] to get non-trivial savings when summing over the "non-degenerate" hyperplane sections. To avoid unnecessary repetition, we give in Section 5 a more abstract and general formulation of this argument. In Section 6, we collect some lemmas from algebraic geometry which we use to prove the estimates for threefolds and fourfolds.

1. Curves

The following result is due to Heath-Brown [19] in the case where C is geometrically irreducible and $n = 2$ or 3. The general case is due to Broberg [4], who also proves the same uniform estimate for curves over arbitrary number fields.

1.1. THEOREM. – *Let $C \subset \mathbf{P}^n$ be an irreducible projective curve over $\mathbf{Q}$ of degree d. Then,*

$$N(C,B) = O_{d,n,\varepsilon}(B^{2/d+\varepsilon}).$$

Note that all rational points on C are singular if C is irreducible but not geometrically irreducible. We have thus at most $O_d(1)$ rational points on such curves.

There are also results on the number of rational points in boxes with sides of different lengths in [19] and [4]. We now deduce some consequences of Theorem 3 in [19] for plane curves.

1.2. *Notation.* – Let $C \subset \mathbf{P}^2$ be a plane curve over $\mathbf{Q}$ and B_0, B_1, B_2 three positive integers.

(a) $N(C, B_0, B_1, B_2)$ denotes the number of $\mathbf{Q}$-points on C with a primitive integral representative (x_0, x_1, x_2) satisfying $|x_i| \leqslant B_i$ for $i = 0, 1, 2$.

(b) $V = B_0 B_1 B_2$.

1.3. THEOREM. – *Let C be an irreducible plane curve over $\mathbf{Q}$ of degree d. Then,*

$$N(C, B_0, B_1, B_2) = O_{d,\varepsilon}(V^{1/(2d-m)+\varepsilon})$$

for the maximum multiplicity $m \leqslant \max(d-1, 1)$ of rational points on C. (If $C(\mathbf{Q})$ is empty, then m is defined to be zero.)

If there are no rational points on C of multiplicity $> d/2$, then

$$N(C, B_0, B_1, B_2) = O_{d,\varepsilon}(V^{2/3d+\varepsilon}).$$

Proof. – Let $F(x_0,x_1,x_2)=\sum a_{e,f,g}x_0^e x_1^f x_2^g$ be a form defining $C\subset\mathbf{P}^2$ and let T be the maximum of all $B_0^e B_1^f B_2^g$ with $a_{e,f,g}\neq 0$. Then Heath-Brown [19, Th. 3] proved that (cf. also [4, Cor. 2]):

$$N(C,B_0,B_1,B_2)=O_{d,\varepsilon}(V^{1/d+\varepsilon}/T^{1/d^2}). \tag{1.4}$$

It thus suffices to show the following inequalities:

$$\begin{aligned} &\text{(a)}\quad T\geqslant V^{d/3}\quad \text{if } m\leqslant d/2,\\ &\text{(b)}\quad T\geqslant V^{(d-m)d/(2d-m)}\quad \text{if } m>d/2. \end{aligned} \tag{1.5}$$

We may also assume that $B_0\geqslant B_1\geqslant B_2$. Suppose first that $P_0=(1,0,0)\notin C$. Then $T=B_0^d\geqslant V^{d/3}$. Suppose next that $P_0\in C$ and let m_0 be the multiplicity of C at P_0. Then $a_{e,f,g}\neq 0$ for some triple (e,f,g) with $e\geqslant d-m_0$. Hence $T\geqslant B_0^e B_1^f B_2^g\geqslant B_0^{d-m}B_2^m$. Also, since F is irreducible, we get that $a_{e,f,g}\neq 0$ for some triple (e,f,g) with $g=0$. Therefore, $T\geqslant B_0^e B_1^f\geqslant B_1^d$. We have thus:

$$\begin{aligned} T\geqslant \max\{B_0^{d-m}B_2^m,B_1^d\} &= B_2^d\max\bigl\{(B_0/B_2)^{d-m},(B_1/B_2)^d\bigr\}\\ &\geqslant B_2^d\bigl[(B_0/B_2)(B_1/B_2)\bigr]^{d(d-m)/(2d-m)}\\ &=(B_0^{d-m}B_1^{d-m}B_2^m)^{d/(2d-m)}. \end{aligned}$$

Also, since $B_0\geqslant B_1\geqslant B_2\geqslant 1$ we conclude that:

$$\begin{aligned} &B_0^{d-m}B_1^{d-m}B_2^m\geqslant V^{(2d-m)/3}\quad \text{if } m\leqslant d/2,\\ &B_0^{d-m}B_1^{d-m}B_2^m\geqslant V^{d-m}\quad \text{if } m>d/2. \end{aligned}$$

This completes the proof of (1.5) and Theorem 1.3. □

One can deduce a more precise result from (1.4). If P is a rational point on C, let m_P be its multiplicity and n_P be the maximum multiplicity of the rational lines in the tangent cone of C at P.

1.6. THEOREM. – *Let C be a geometrically irreducible plane curve over $\mathbf{Q}$ of degree $d>1$. Then,*

$$N(C,B_0,B_1,B_2)=O_{d,\varepsilon}(V^{2/3d+\varepsilon}),$$

if $m_P+n_P\leqslant d$ for all rational points on C. Otherwise, there exists a rational point P on C with $m_P+n_P>d$ such that

$$N(C,B_0,B_1,B_2)=O_{d,\varepsilon}(V^{(d+n-m)/d(2d+n-2m)+\varepsilon})$$

for $m=m_P$, $n=n_P$.

The proof is similar to the proof of Theorem 1.3. We leave the details to the reader, as we shall not use this result in the sequel.

1.7. DEFINITION. – Let $\Pi\subset\mathbf{P}^n$ be a rational hyperplane given by the equation

$$y_0x_0+\cdots+y_nx_n=0$$

for a rational point $\mathbf{y}=(y_0,\dots,y_n)$ in the dual projective space $\mathbf{P}^{n\vee}$. Then the height $H(\Pi)$ of Π is defined as the height of the rational point $\mathbf{y}$ in $\mathbf{P}^{n\vee}$.

The following corollary of Theorem 1.3 will be important.

1.8. COROLLARY. – *Let $C \subset \mathbf{P}^3$ be an irreducible projective curve over $\mathbf{Q}$ of degree $d > 1$ contained in a plane $\Pi \subset \mathbf{P}^3$. Then,*

$$N(C,B) = O_{d,\varepsilon}\big((B^3/H(\Pi))^{1/(d+1)+\varepsilon} + 1\big).$$

Moreover, if there are no rational points of multiplicity $> d/2$ on C, then

$$N(C,B) = O_{d,\varepsilon}\big(\big(B^3/H(\Pi)\big)^{2/3d+\varepsilon} + 1\big).$$

Proof. – A proof of the case $d = 2$ is implicit in the proof of Lemma 10 in [19, p. 582] (cf. also [4, Lemma 8]) and we shall use a similar approach.

Let (y_0, y_1, y_2, y_3) be a primitive integral quadruple such that Π is given by the equation $y_0x_0 + y_1x_1 + y_2x_2 + y_3x_3 = 0$. This equation defines a three-dimensional sublattice $L \subset \mathbf{Z}^4$ which has a $\mathbf{Z}$-basis $(\mathbf{b}^0, \mathbf{b}^1, \mathbf{b}^2)$ with the following properties (cf. [19, Lemma 1]).

(1.9) $$|\mathbf{b}^0||\mathbf{b}^1||\mathbf{b}^2| \leqslant 4H(\Pi) \ll |\mathbf{b}^0||\mathbf{b}^1||\mathbf{b}^2|.$$

(1.10) Let $\lambda_0\mathbf{b}^0 + \lambda_1\mathbf{b}^1 + \lambda_2\mathbf{b}^2$ be an element of L which represents a rational point on $\mathbf{P}^3$ of height $\leqslant B$.Then, $|\lambda_i| \leqslant cB/|\mathbf{b}^i|$ for $i = 0, 1, 2$ and for some constant c not depending on Π.

By $|\mathbf{b}|$ we mean $\max(|b_0|, |b_1|, |b_2|, |b_3|)$ for $\mathbf{b} = (b_0, b_1, b_2, b_3) \in \mathbf{Z}^4$.

From (1.10) we get:

(1.11) $$N(C,B) = N\big(C, cB/|\mathbf{b}^0|, cB/|\mathbf{b}^1|, cB/|\mathbf{b}^2|\big),$$

where on the right hand side we regard C as a curve of degree d in $\Pi = \mathbf{P}^3$ with $(\lambda_0, \lambda_1, \lambda_2)$ as homogeneous coordinates. If $cB/|\mathbf{b}^i| < 1$ for some $i \in \{0, 1, 2\}$, then $\lambda_i = 0$ for any rational point on C of height $\leqslant B$ by (1.10). Hence $N(C,B) \leqslant d$ in that case.

If $cB/|\mathbf{b}^i| \geqslant 1$ for all $i \in \{0, 1, 2\}$, then by (1.11), Theorem 1.3 and (1.9) we get that

$$N(C,B) \ll_{d,\varepsilon} \big(B^3/|\mathbf{b}^0||\mathbf{b}^1||\mathbf{b}^2|\big)^{1/(d+1)+\varepsilon} \ll \big(B^3/H(\Pi)\big)^{1/(d+1)+\varepsilon}.$$

Moreover, if there are no rational points of multiplicity $> d/2$ on C, then

$$N(C,B) \ll_{d,\varepsilon} \big(B^3/|\mathbf{b}^0||\mathbf{b}^1||\mathbf{b}^2|\big)^{2/3d+\varepsilon} \ll \big(B^3/H(\Pi)\big)^{2/3d+\varepsilon}. \qquad \square$$

2. Surfaces

The following result is a consequence of Theorems 5 and 9 in [19].

2.1. THEOREM. – *Let $X \subset \mathbf{P}^n$ be a closed subscheme defined over $\mathbf{Q}$ where all irreducible components are of dimension at most two. Suppose that X does not contain any plane. Then,*

$$N(X,B) = O_{n,D,\varepsilon}(B^{2+\varepsilon}),$$

where D is the sum of the degrees of all irreducible components of X.

Proof. – One reduces immediately to the case where X is integral. Then the base extension $\overline{X}$ over $\overline{\mathbf{Q}}$ is equidimensional and reduced. Let $\overline{Y}$ be the intersection of all the irreducible components of $\overline{X}$. Then any rational point on X gives rise to a $\overline{\mathbf{Q}}$-point which lies on $\overline{Y}$ as the Galois group acts transitively on the irreducible components of $\overline{X}$. There is also by the theory of Galois descent a closed subscheme Y of X with base extension $\overline{Y}$ over $\overline{\mathbf{Q}}$ which must then contain all rational points on X. Now apply the Bezout theorem in [14, 8.4.6]. Then the sum of the degrees of the irreducible components of $\overline{Y}$ (endowed with their reduced scheme structures) can be bounded in terms of the degree of $\overline{X}$. This is therefore also true for the sum of the degrees of the irreducible components of Y. But $\dim Y < \dim X$ if $\overline{X}$ is not integral. It is thus by induction enough to treat the case where $\overline{X}$ is integral. If X is a curve, use Theorem 1.1. If X is a surface, combine the estimate in [19, Th. 5] with the projection argument in [7, Lemma 1]. □

2.2. LEMMA. – *Let $X \subset \mathbf{P}^3$ be a surface defined by a form F of degree d with coefficients in a field k of characteristic 0. Let $\mathbf{P}^{3\vee}$ be the dual projective space parametrising planes $\Pi \subset \mathbf{P}^3$. Then the following holds.*

(a) *There is a closed subscheme $W_e \subset \mathbf{P}^{3\vee}$ defined over k such that the K-points of W_e, $K \supset k$ corresponds to K-planes Π where $X \cap \Pi$ contains a curve of degree e. The sum of the degrees of all the irreducible components of $W_e \subset \mathbf{P}^{3\vee}$ can be bounded in terms of d and e.*

(b) *Suppose that X is geometrically irreducible and that X is not a cone. Then $W_e \subset \mathbf{P}^{3\vee}$ does not contain any plane when $e < d$.*

Proof. – (a) Let H_1 (resp. H_2) be the Hilbert scheme of all closed subschemes of $\mathbf{P}^3$ with Hilbert polynomials:

$$P_1(x) = (x+2)(x+1)/2 - (x+2-e)(x+1-e)/2 \quad \text{resp.}$$
$$P_2(x) = (x+3)(x+2)(x+1)/6 - (x+3-d)(x+2-d)(x+1-d)/6.$$

Then H_1 and H_2 are projective and the projection map π: $H_1 \times H_2 \times \mathbf{P}^{3\vee} \to H_2 \times \mathbf{P}^{3\vee}$ a proper morphism. Let $I \subset H_1 \times H_2 \times \mathbf{P}^{3\vee}$ be the closed subscheme representing triples (C, X, Π) where the curve C of degree e is contained in the surface X of degree d and the plane Π. Then the scheme-theoretic image of I under π is a closed subscheme J of $H_2 \times \mathbf{P}^{3\vee}$. Also, if $X \subset \mathbf{P}^3$ corresponds to a k-point x on H_2, then the fibre W_e of $J \to H_2$ at x, satisfies the first statement.

To bound the degree of W_e in terms of d and e, choose a projective embedding $H_2 \subset \mathbf{P}^m$ and a finite set $\{Q_1, \ldots, Q_r\}$ of bihomogeneous polynomials defining J as a closed subscheme of $\mathbf{P}^m \times \mathbf{P}^{3\vee}$. Suppose that $Q_1, \ldots, Q_r$ are of degrees $d_1, \ldots, d_r$ with respect to the coordinates for the dual projective space $\mathbf{P}^{3\vee}$. Then the sum of the degrees of all the irreducible components of $W_e \subset \mathbf{P}^{3\vee}$ is at most $d_1 d_2 \ldots d_r$ by [14, 8.4,6].

(b) Let P_0 be an arbitrary rational point on $\mathbf{P}^3$ and $\varphi : \mathbf{P}^3 \backslash P_0 \to \mathbf{P}^2$ be the morphism sending a point P on X to the line between P and P_0. Then $\varphi(X) = \mathbf{P}^2$ since X is not a cone. Therefore, $X \cap \varphi^{-1}(L)$ is geometrically irreducible for a generic line L on $\mathbf{P}^2$ by Bertini's theorem (cf. [15, Th. 2.1] and [21, Cor. 6.11.3]). This means that $X \cap \Pi$ is geometrically irreducible for the plane $\Pi = \varphi^{-1}(L) \cup P_0$. Therefore, the hyperplane in $\mathbf{P}^{3\vee}$ dual to P_0 cannot be contained in W_e, as was to be proved. □

Remark. – One can classify all surfaces $X \subset \mathbf{P}^3$ with a two-dimensional family of plane reducible sections (i.e. with $\dim W_e \geqslant 2$ for some $e < d$). Kronecker stated and Castelnuovo "proved" (see [9]) that such surfaces are either ruled by lines or isomorphic to the Steiner Roman surface.

2.3. *Notation.* – Let $X \subset \mathbf{P}^n$ be a hypersurface defined by a form over a field k. Then X' (resp. X'') is the complement of the union of all irreducible curves on X over k which split into a union of lines (resp. lines and conics) over an algebraic closure of k.

It is easy to show that X' and X'' are open subsets of X (cf. the proof of Lemma 6.1) defined over k.

The following result is inspired by [19, Th. 10].

2.4. THEOREM. – *Let $X \subset \mathbf{P}^3$ be an irreducible projective surface of degree d defined over $\mathbf{Q}$. Then,*
(a) $N(X'', B) = O_{d,\varepsilon}(B^{4/3+16/9d+\varepsilon})$.
(b) $N(X', B) = O_{d,\varepsilon}(B^{4/3+16/9d+\varepsilon})$ *if* $d \leqslant 8$,
$N(X', B) = O_{d,\varepsilon}(B^{14/9+\varepsilon})$ *if* $d > 8$.

Proof. – If X is not geometrically irreducible, then (cf. the proof of Theorem 2.1) all rational points on X lie on a subscheme Y of lower dimension where the sum of the degrees of the irreducible components of Y is bounded in terms of d. Hence $N(X', B) = O_{d,\varepsilon}(B^{1+\varepsilon})$ by Theorem 1.1 in that case. Also, if X is cone, then X' is empty.We may thus assume that X is geometrically irreducible and not a cone.

Let P be a rational point of height $\leqslant B$ on $\mathbf{P}^3$. Then, by Siegel's lemma there is a rational plane Π of height $\leqslant (4B)^{1/3}$ containing P. Let us first consider the planes for which $X \cap \Pi$ is irreducible. Then, $N(X \cap \Pi, B) = O_{d,\varepsilon}(B^{2/d+\varepsilon})$ by Theorem 1.1. There are $O(B^{4/3})$ rational planes Π of height $\leqslant (4B)^{1/3}$. Hence we get a total contribution of $O_{d,\varepsilon}(B^{4/3+2/d+\varepsilon})$ to $N(X, B)$ from the irreducible plane sections. But it is possible to save $1/9$ of the exponent $2/d$ by means of an argument in [19, p. 581]. This follows from Lemma 5.1 below in which we will take the condition (%) to be that $X \cap \Pi$ is irreducible. We thus get the bound $O_{d,\varepsilon}(B^{4/3+16/9d+\varepsilon})$ for the number of rational points of height $\leqslant B$ on X lying on the union of all irreducible plane sections $X \cap \Pi$ with Π of height $\leqslant (4B)^{1/3}$.

We now consider the contribution from the reducible plane sections. Let $W_e \subset \mathbf{P}^{3\vee}$ be the closed $\mathbf{Q}$-subscheme in Lemma 2.2(a). Then $N(W_e, (4B)^{1/3}) = O_{d,\varepsilon}(B^{2/3+\varepsilon})$ if $e < d$ by Theorem 2.1 and Lemma 2.2. This means that there are at most $O_{d,\varepsilon}(B^{2/3+\varepsilon})$ reducible plane sections $X \cap \Pi$ with Π of height $\leqslant (4B)^{1/3}$. We also have for each Π that $N(X'' \cap \Pi, B) = O_{d,\varepsilon}(B^{2/3+\varepsilon})$ by [19, Th. 5]. There are therefore $O_{d,\varepsilon}(B^{4/3+\varepsilon})$ rational points of height $\leqslant B$ on X'' lying on the union of all reducible plane sections $X \cap \Pi$ with Π of height $\leqslant (4B)^{1/3}$. This proves (a).

To prove (b), let $1 \leqslant R \leqslant (4B)^{1/3}$. Then $N(W_2, 2R) = O_d(R^{2+\varepsilon})$ by Theorem 2.1 and Lemma 2.2. We have thus $O_{d,\varepsilon}(R^{2+\varepsilon})$ rational planes Π of height $\leqslant 2R$ for which $X \cap \Pi$ contains a conic. Also, $N(C, B) = O_{d,\varepsilon}(B^{1+\varepsilon}/R^{1/3})$ for $H(\Pi) \in [R, 2R]$ by Corollary 1.8.

There are thus $O_{d,\varepsilon}(B^{1+\varepsilon}R^{5/3})$ rational points of height $\leqslant B$ on the union of all conics in plane sections $X \cap \Pi$ with $H(\Pi) \in [R, 2R]$. Now sum over all two-powers R with $1 \leqslant R \leqslant (4B)^{1/3}$. Then we get at most $O_{d,\varepsilon}(B^{14/9+\varepsilon})$ rational points on the union of all conics contained in plane sections $X \cap \Pi$ with rational planes Π of height $\leqslant (4B)^{1/3}$. This combined with the previous estimates on rational points on plane curves on X of degree $e \in \{3, \ldots, d-1\}$ implies in its turn that there are at most $O_{d,\varepsilon}(B^{14/9+\varepsilon})$ rational points of height $\leqslant B$ on X' on the union of all reducible plane sections $X \cap \Pi$ with Π of height $\leqslant (4B)^{1/3}$. We have thus proved that $N(X', B) = O_{d,\varepsilon}(B^{4/3+16/9d+\varepsilon} + B^{14/9+\varepsilon})$, as was to be shown. □

2.5. *Remarks.* – (a) From Theorem 2.4 we obtain,

$$N(X', B) = O_{d,\varepsilon}(B^{16/9+\varepsilon}) \quad \text{when } d = 4,$$
$$N(X', B) = O_{d,\varepsilon}(B^{76/45+\varepsilon}) \quad \text{when } d \geqslant 5.$$

This should be compared with the estimates on p. 558 in [19]

$$N(X',B)=O_{d,\varepsilon}(B^{52/27+\varepsilon}) \quad \text{when } d=3,$$
$$N(X',B)=O_{d,\varepsilon}(B^{17/9+\varepsilon}) \quad \text{when } d\geqslant 4.$$

(b) It is possible to extend the estimates in Theorem 2.4 to irreducible surfaces in $\mathbf{P}^n$ by means of a birational projection argument which will appear in a forthcoming paper by Browning, Heath-Brown and the author. One can also use an approach without projections as in [4, Th. 3].

2.6. THEOREM. – *Let $X\subset\mathbf{P}^3$ be an irreducible projective surface of degree d defined over $\mathbf{Q}$. Then,*

(a) $N(X'',B)=O_{d,\varepsilon}(B^{3/\sqrt{d}+2/3+\varepsilon})$.
(b) $N(X',B)=O_{d,\varepsilon}(B^{5/2\sqrt{d}+1+\varepsilon})$.

To prove Theorem 2.6, we shall use the following fundamental result of Heath-Brown [19, Th. 14] (cf. also [4, Cor. 2]).

2.7. THEOREM. – *Let $X\subset\mathbf{P}^{r+1}$ be an irreducible hypersurface of degree d over $\mathbf{Q}$ and B be a positive number. Then there exists another hypersurface Y defined over $\mathbf{Q}$ with the following properties.*

(a) *All rational points on X of height at most B lie on Y.*
(b) *X is not contained in Y.*
(c) $\deg(Y)=O_{d,\varepsilon}(B^{\rho+\varepsilon})$, $\rho=(r+1)/d^{1/r}$.
(d) *The degrees of the irreducible components of Y are bounded uniformly in terms of d,r and ε.*

Proof of Theorem 2.6. – Let $Y\subset\mathbf{P}^3$ be a surface of degree $O_{d,\varepsilon}(B^{3/\sqrt{d}+\varepsilon})$ as in Theorem 2.7. The rational points on X'' lie on irreducible components of degree $\geqslant 3$ on $X\cap Y$. By Theorem 1.1 one gets therefore the uniform bound $O_{d,\varepsilon}(B^{2/3+\varepsilon})$ for each of these components. This finishes the proof of (a) as there are at most $O_{d,\varepsilon}(B^{3/\sqrt{d}+\varepsilon})$ irreducible components of $X\cap Y$.

To prove (b), it suffices to treat the case where $d\geqslant 3$ and X is not a cone since X' is empty otherwise. Then $3/\sqrt{d}+2/3<5/2\sqrt{d}+1$. There are thus $\ll_{d,\varepsilon} B^{5/2\sqrt{d}+1+\varepsilon}$ rational points on X of height $\leqslant B$ lying on the irreducible components of degree $\geqslant 3$ of $X\cap Y$. It thus only remains to consider the contribution from the geometrically irreducible conics on $X\cap Y$.

Let $T=B^{3/2\sqrt{d}}$. By Corollary 1.8 there are $\ll_{d,\varepsilon} (B^3/T)^{1/3+\varepsilon}+1=B^{1-1/2\sqrt{d}+\varepsilon}+1$ rational points of height at most B on each conic contained in a plane Π of height $>T$. There are thus $\ll_{d,\varepsilon} B^{5/2\sqrt{d}+1+\varepsilon}$ rational points of height $\leqslant B$ on a set of $O_{d,\varepsilon}(B^{3/\sqrt{d}+\varepsilon})$ such conics. To count rational points of height $\leqslant B$ on conics contained in planes Π of height $\leqslant T$, we cover the interval $[1,T]$ by $O(\log T)$ dyadic intervals $[R,2R]$. But we have already seen in the proof of Theorem 2.4(b) that there are $O_{d,\varepsilon}(R^{2+\varepsilon})$ rational planes Π of height $\leqslant 2R$ for which $X\cap\Pi$ contains a conic C and that $N(C,B)=O_{d,\varepsilon}(B^{1+\varepsilon}/R^{1/3})$ for $H(\Pi)\in[R,2R]$. There are thus $O_{d,\varepsilon}(B^{1+\varepsilon}R^{5/3})$ rational points of height $\leqslant B$ on the union of all irreducible components of degree 2 of $X\cap Y$ spanning a plane Π of height $H(\Pi)\in[R,2R]$. Now sum over all two-powers R with $1\leqslant R\leqslant T$. Then we obtain $\ll_{d,\varepsilon} B^{1+\varepsilon}T^{5/3}=B^{5/2\sqrt{d}+1+\varepsilon}$ rational points of height $\leqslant B$ on the union of all conics spanning a plane of height $\leqslant T$. This completes the proof. □

We now restrict to smooth surfaces $X\subset\mathbf{P}^3$. Recall that a curve in $\mathbf{P}^n$ is said to be *degenerate* if it is contained in a hyperplane and *non-degenerate* otherwise. We shall need the following result of Colliot-Thélène [10] already used in [19] in the case of curves of degree $\leqslant d-2$.

2.8. THEOREM. – *Let $X \subset \mathbf{P}^3$ be a smooth projective surface of degree d over an algebraically closed field. Then there are at most finitely many irreducible curves of degree $\leqslant d-2$ on X and at most finitely many non-degenerate curves of degree $< 2d-4$ on X. Also, there exists in both cases a uniform upper bound for the number of such curves which only depends on d.*

Proof. – By Proposition 2 in [10], one has $(C.C) < 0$ for an integral curve C of degree $\leqslant d-2$ on X. But an examination of the proof reveals that $(C.C) < 0$ also for a non-degenerate integral curve of degree less than $2d-4$ on X. There are thus in both cases only finitely many such curves by Proposition 1 in (op. cit.). To get a uniform bound of the number of such curves, apply Proposition 3 in (op. cit.). □

2.9. THEOREM. – *Let $X \subset \mathbf{P}^3$ be a smooth projective surface over $\mathbf{Q}$ of degree d. Let U be the open complement of all curves on X of degree at most $d-2$. Then,*

$$N(U,B) = O_{d,\varepsilon}(B^{3/\sqrt{d}+2/d+\varepsilon}).$$

Proof. – Let $Y \subset \mathbf{P}^3$ be a surface of degree $O_{d,\varepsilon}(B^{3/\sqrt{d}+\varepsilon})$ as in Theorem 2.7. There are then $O_{d,\varepsilon}(B^{3/\sqrt{d}+\varepsilon})$ irreducible components of $X \cap Y$ and each such component is of degree $O_{d,\varepsilon}(1)$. Hence, by Theorem 1.1, there are $O_{d,\varepsilon}(B^{2/d+\varepsilon})$ rational points of height $\leqslant B$ on each irreducible component of degree $\geqslant d$ on $X \cap Y$. There are therefore $O_{d,\varepsilon}(B^{3/\sqrt{d}+2/d+\varepsilon})$ rational points of height $\leqslant B$ on X lying on the union of the irreducible components of $X \cap Y$ of degree $\geqslant d$.

To prove Theorem 2.9, it remains to count the points on the irreducible components of degree $d-1$ on $X \cap Y$. We may and shall assume that $d \geqslant 4$ (cf. e.g. Theorem 2.1). Then, by Theorem 2.8, there are at most $O_d(1)$ non-degenerate irreducible components of degree $d-1$ on $X \cap Y$. Their total contribution to $N(U,B)$ is thus $O_{d,\varepsilon}(B^{2/(d-1)+\varepsilon})$, which is satisfactory since $2/(d-1) < 3/\sqrt{d} + 2/d$.

If an irreducible curve C over $\mathbf{Q}$ of degree $d-1$ lies on $X \cap \Pi$ for a rational plane $\Pi \subset \mathbf{P}^3$, then there is a complementary line Λ over $\mathbf{Q}$ on $X \cap \Pi$. Now since there are only finitely many lines on X (see Theorem 2.8), it suffices to count rational points on degenerate components of degree $d-1$ on $X \cap Y$ with a *fixed* complementary line Λ over $\mathbf{Q}$.

The linear system $|(\Pi \cap X) - \Lambda|$ defines a morphism $f: X \to \mathbf{P}^1$ which sends a point P outside Λ to the plane spanned by P and Λ and a point P on Λ to the tangent plane at P. By a theorem of Bertini, all but finitely many fibres of f are smooth. One can even give an upper bound for the number of singular fibres in terms of d by means of the well known formula [1, III 11.4] for the Euler numbers of such fibrations. It thus suffices to show that the contribution to $N(U,B)$ from any set of $O_{d,\varepsilon}(B^{3/\sqrt{d}+\varepsilon})$ *smooth* fibres of $f: X \to \mathbf{P}^1$ is bounded by $O_{d,\varepsilon}(B^{3/\sqrt{d}+2/d+\varepsilon})$. We shall establish the sharper bound $O_{d,\varepsilon}(B^{3/\sqrt{d}+2/(d-1)-1/(d-1)\sqrt{d}+\varepsilon})$.

Let $T = B^{3/2\sqrt{d}}$. Then there are $O_{d,\varepsilon}((B^3/T)^{2/3(d-1)+\varepsilon}) = O_{d,\varepsilon}(B^{2/(d-1)-1/(d-1)\sqrt{d}+\varepsilon})$ rational points on each smooth fibre of f contained in a plane Π of height $> T$. A set of $O_{d,\varepsilon}(B^{3/\sqrt{d}+\varepsilon})$ such smooth fibres contains thus $O_{d,\varepsilon}(B^{3/\sqrt{d}+2/(d-1)-1/(d-1)\sqrt{d}+2\varepsilon})$ rational points of height $\leqslant B$. For the smooth fibres spanning planes of height $\leqslant T$, we cover the interval $[1,T]$ by $O(\log T)$ dyadic intervals $[R,2R]$. There are $O(R^2)$ fibres in planes of height $\leqslant 2R$ and $O_{d,\varepsilon}((B^3/R)^{2/3(d-1)+\varepsilon}))$ points in each fibre lying in a plane of height $\geqslant R$. There are thus $O_{d,\varepsilon}(B^{2/(d-1)+\varepsilon}R^{2-2/3(d-1)})$ rational points lying on the union of fibres contained in planes Π of height $H(\Pi) \in [R,2R]$. Hence since $T^{2-2/3(d-1)} = B^{3/\sqrt{d}-1/(d-1)\sqrt{d}}$ we get the desired bound by summing over dyadic intervals. □

This theorem is an improvement of Heath-Brown's bound [19, Th. 11]

$$N(U,B)=O_{d,\varepsilon}(B^{3/\sqrt{d}+2/(d-1)+\varepsilon}).$$

We now prove a result which is slightly stronger than Theorem 2.9.

2.10. THEOREM. – *Let $X\subset \mathbf{P}^3$ be a smooth projective surface of degree d over $\mathbf{Q}$. Let U be the open complement of the union of all curves on X of degree at most $d-2$. Then,*

(a) $N(U,B)=O_{d,\varepsilon}(B^{f(d)+\varepsilon})$*, where*

$$f(d)=3/\sqrt{d}+2/(d-1)-1/(d-1)\sqrt{d} \quad \text{if } d\leqslant 13,$$
$$f(d)=3/\sqrt{d}+2/d-1/2d\sqrt{d} \quad \text{if } d\geqslant 14.$$

(b)

$$N(X',B)=O_{d,\varepsilon}(B^{f(d)+\varepsilon}) \quad \text{if } d\leqslant 12,$$
$$N(X',B)=O_{d,\varepsilon}(B^{1+\varepsilon}) \quad \text{if } d\geqslant 13.$$

Proof. – (a) The result is already known for $d\leqslant 5$ by Theorem 2.4. Moreover,

$$2/(d-1)-1/(d-1)\sqrt{d}>2/d-1/2d\sqrt{d}>2/(2d-4) \quad \text{if } 6\leqslant d\leqslant 13,$$
$$2/d-1/2d\sqrt{d}>2/(d-1)-1/(d-1)\sqrt{d}>2/(2d-4) \quad \text{if } d\geqslant 14.$$

It thus suffices to show the following estimate for $d\geqslant 6$.

$$N(U,B)=O_{d,\varepsilon}(B^{3/\sqrt{d}+2/(d-1)-1/(d-1)\sqrt{d}+\varepsilon}+B^{3/\sqrt{d}+2/d-1/2d\sqrt{d}+\varepsilon}+B^{3/\sqrt{d}+2/(2d-4)+\varepsilon}).$$

We choose again an auxiliary surface $Y\subset\mathbf{P}^3$ of degree $\ll_{d,\varepsilon} B^{3/\sqrt{d}+\varepsilon}$ as in the proof of Theorem 2.9 and consider the irreducible components of $X\cap Y$. The total contribution to $N(U,B)$ from the non-degenerate components of degree $\delta\in(d-2,2d-4)$ is $O_{d,\varepsilon}(B^{2/(d-1)+\varepsilon})$ by Theorems 2.8 and 1.1 while the contribution from the union of all components of $X\cap Y$ of degree $\delta\geqslant 2d-4$ is $O_{d,\varepsilon}(B^{3/\sqrt{d}+2/(2d-4)+\varepsilon})$. It thus only remains to consider the degenerate irreducible components of degree $>d-2$. But we have shown in the proof of Theorem 2.9 that the union of the components of degree $d-1$ contribute with $O_{d,\varepsilon}(B^{3/\sqrt{d}+2/(d-1)-1/(d-1)\sqrt{d}+\varepsilon})$. It is therefore enough to count the rational points on the components of $X\cap Y$ which are irreducible plane sections $X\cap\Pi$.

We now count the rational points on the degenerate components of degree d. Put $T=B^{3/4\sqrt{d}}$ and cover $[1,T]$ by $O(\log T)$ dyadic intervals $[R,2R]$. As long as we consider plane sections $X\cap\Pi$ where all rational points have multiplicity $\leqslant d/2$, we proceed exactly as in the proof of Theorem 2.9. There are $O(R^4)$ planes of height $\leqslant 2R$ and $O_{d,\varepsilon}((B^3/R)^{2/3d+\varepsilon}+1)$ points in each such section with a plane of height $\geqslant R$ (see Corollary 1.8). The total number of rational points on such planes Π of height $H(\Pi)\in[R,2R]$ is thus $O_{d,\varepsilon}(B^{2/d+\varepsilon}R^{4-2/3d})$. Hence as $T^{4-2/3d}=B^{3/\sqrt{d}-1/2d\sqrt{d}}$ we get the bound $O_{d,\varepsilon}(B^{3/\sqrt{d}+2/d-1/2d\sqrt{d}+\varepsilon})$ when summing over dyadic intervals. The contribution from a section with a plane of height $>T$ is $O_{d,\varepsilon}((B^3/T)^{2/3d+\varepsilon}+1)$. We thus get the same total bound $O_{d,\varepsilon}(B^{3/\sqrt{d}+2/d-1/2d\sqrt{d}+\varepsilon})$ for the union of such plane sections with planes Π on Y. (We use here that $\deg(Y)\ll_{d,\varepsilon}B^{3/\sqrt{d}+\varepsilon}$.)

It remains to count the rational points of height $\leqslant B$ on U which lie on irreducible plane sections $X\cap\Pi$ with a rational point P of multiplicity $>d/2\geqslant 2$. Let us first consider those planes Π for which we in addition have three non-collinear rational points on $X\cap\Pi$ of height

$\leqslant B$. Then $H(\Pi) \leqslant 6B^3$ so that we may cover $[1, 6B^3]$ by $O(\log B)$ dyadic intervals $[R, 2R]$ and consider plane sections as above with $H(\Pi) \in [R, 2R]$. There are then by Lemma 2.11 below and Theorem 1.1 $O_{d,\varepsilon}(R^{2/2(d-1)+\varepsilon})$ such planes Π and $O_{d,\varepsilon}((B^3/R)^{1/(d+1)+\varepsilon})$ points (cf. Corollary 1.8) in each section $U \cap \Pi$. The contribution to $N(U, B)$ from the union of these plane sections $X \cap \Pi$ is thus $O_{d,\varepsilon}(B^{3/(d+1)+\varepsilon} R^{2/(d-1)(d+1)})$. If we sum over all the dyadic intervals, then we get a total contribution of $O_{d,\varepsilon}(B^{3/(d+1)+6/(d-1)(d+1)+\varepsilon})$ rational points, which is satisfactory.

Let us finally consider the planes Π for which all rational points on $X \cap \Pi$ of height $\leqslant B$ lie on a line. There are at most d rational points on $U \cap \Pi$ for such planes since any line with $d+1$ rational points lies on X. As we only consider irreducible components Π of Y, we therefore get at most $O_{d,\varepsilon}(B^{3/\sqrt{d}+\varepsilon})$ rational points of height $\leqslant B$ on U on the union of such planes. This completes the proof of (a). To prove (b), use Theorems 2.10(a), 1.1 and 2.8. $\square$

2.11. LEMMA. – *Let $X \subset \mathbf{P}^3$ be a smooth projective surface of degree d over a field k of characteristic 0. Then there is a closed k-subscheme $W \subset \mathbf{P}^{3\vee}$ with the following properties.*

(a) *The K-points of W, $K \supset k$ correspond to K-planes Π where $X \cap \Pi$ contains a K-point of multiplicity $> d/2$. The sum of the degrees of the irreducible components of W is bounded in terms of d.*

(b) *Suppose that $d > 3$. Then there are no irreducible components of W of dimension > 1.*

(c) *Let Π be a plane over k such that $X \cap \Pi$ contains no line. Then Π cannot belong to a one-dimensional irreducible component W_0 of W of degree less than $2(d-1)$.*

Proof. – (a) Let H be the Hilbert scheme over k of all closed subschemes of $\mathbf{P}^3$ defined by a form of degree d and $\Omega \subset H$ be the open subscheme of all smooth surfaces of degree d. Let $I \subset \mathbf{P}^3 \times \Omega \times \mathbf{P}^{3\vee}$ be the closed k-subscheme representing all triples (P, X, Π) such that P is a point of multiplicity $> d/2$ on $X \cap \Pi$. More precisely, it represents the contravariant functor $F : (\text{schemes}_{/k}) \to (\text{sets})$ that to any k-scheme B associates the set of k-morphisms from B to $\mathbf{P}^3 \times \Omega \times \mathbf{P}^{3\vee}$ for which the composition with any point $\operatorname{Spec} K \to B$ gives rise to a triple (P, X, Π) where P is a K-point of multiplicity $> d/2$ on $X \cap \Pi$ for smooth K-surfaces X, Π of degree d resp. 1 in $\mathbf{P}^3$.

The scheme-theoretic image J of I under the projection morphism from $\mathbf{P}^3 \times \Omega \times \mathbf{P}^{3\vee}$ to $\Omega \times \mathbf{P}^{3\vee}$ is a closed subscheme of $\Omega \times \mathbf{P}^{3\vee}$. It is clear from the definition of J that the fibre W of the projection $J \to \Omega$ at the k-point on Ω corresponding to $X \subset \mathbf{P}^3$ satisfies the first statement of (a). The second assertion is proved just as in Lemma 2.2(a).

(b) Let $W \subset \mathbf{P}^{3\vee}$ be the closed k-subscheme in the proof of (a). To prove that $\dim W \leqslant 1$, we may assume that k is algebraically closed. Let $F(x_0, \ldots, x_3)$ be a form over k defining $X \subset \mathbf{P}^3$ and $g : X \to \mathbf{P}^{3\vee}$ be the Gauss morphism which sends $(x_0, \ldots, x_3)$ to $(\delta F/\delta x_0, \ldots, \delta F/\delta x_3)$. It sends $P \in X(k)$ to the tangent plane $T_P \in \mathbf{P}^{3\vee}(k)$ of X at P. It is well known [26] that g is a finite birational morphism when $d > 1$. Let $X^\vee = g(X) \subset \mathbf{P}^{3\vee}$ be the dual surface and let $V \subset X^\vee$ be the largest open subset such that g induces an isomorphism between $g^{-1}(V)$ and V. Then V consists of the planes $\Pi \subset \mathbf{P}^3$ for which $X \cap \Pi$ has exactly one singular point P and where P is a quadratic non-degenerate singularity on $X \cap \Pi$ (see [22, 3.5]). Therefore, $W \subset X^\vee \backslash V$ since each point Π on W must be the tangent plane T_P of X at a point P on $X \backslash g^{-1}(V)$. In particular, $\dim W \leqslant 1$.

(c) We may again assume that k is algebraically closed. Let $Z \subset \mathbf{P}^3$ be the closed subscheme of all points such that P is a point of multiplicity $> d/2$ on $X \cap T_P$. Then the restriction of the Gauss map $g : X \to \mathbf{P}^{3\vee}$ to $Z \cap X'$ is injective as any two points P, Q of multiplicity $> d/2$ on a plane section of X must lie on a line on X and hence not belong to X'.

Now let W_0 be a one-dimensional irreducible component of W. Then W_0 is equal to $g(Z_0)$ for some one-dimensional irreducible component Z_0 of Z. If $Z_0 \subset \mathbf{P}^3$ is a line, then $W_0 \subset \mathbf{P}^{3\vee}$

must be the dual line since T_P contains any line on X through P. In particular, if Π is represented by a point on W_0, then Z_0 cannot be a line since otherwise it would lie on $X \cap \Pi$. Hence $Z_0 \cap X'$ is non-empty so that Z_0 is mapped birationally onto W_0 under g. As g is given by forms of degree $d-1$, we must therefore have $\deg(W_0) = (d-1)\deg(Z_0) \geqslant 2(d-1)$. $\square$

3. Threefolds

The following result is due to Broberg and the author [5].

3.1. THEOREM. – *Let $X \subset \mathbf{P}^4$ be a geometrically irreducible projective threefold of degree d over $\mathbf{Q}$. Then,*

(a) $N(X,B) = O_\varepsilon(B^{55/18+\varepsilon})$ *for* $d = 3$,

(b) $N(X,B) = O_{d,\varepsilon}(B^{3+\varepsilon})$ *for* $d \geqslant 4$.

3.2. THEOREM. – *Let $X \subset \mathbf{P}^n$ be a closed subscheme of dimension at most three defined over $\mathbf{Q}$. Suppose that X does not contain any irreducible three-dimensional component of degree one. Then,*

$$N(X,B) = O_{n,D,\varepsilon}(B^{55/18+\varepsilon}),$$

where D is the sum of the degrees of all irreducible components of X.

Proof. – One reduces to the case $n = 3$ by the projection argument in the proof of Lemma 1 in [7] and then to the case where X is geometrically integral by the same argument as in Theorem 2.1. If $\dim X = 1$ or 2, then the result follows from Pila's Theorem 0.1 or by an elementary projection argument onto $\mathbf{P}^1$ or $\mathbf{P}^2$. If $\dim X = 3$, then we apply Theorem 3.1. $\square$

The proof of the following lemma is almost identical to the proof of Lemma 2.2.

3.3. LEMMA. – *Let $X \subset \mathbf{P}^4$ be a threefold defined by a form F of degree $d > 2$ over a field k of characteristic 0. Let $\mathbf{P}^{4\vee}$ be the dual projective space parametrising hyperplanes $\Pi \subset \mathbf{P}^4$. Then the following holds.*

(a) *There is a closed subscheme $W_e \subset \mathbf{P}^{4\vee}$ such that the K-points of W_e, $K \supset k$ corresponds to K-planes Π where $X \cap \Pi$ contains a surface of degree e. The sum of the degrees of all the irreducible components of $W_e \subset \mathbf{P}^{4\vee}$ can be bounded in terms of d and e.*

(b) *Suppose that X is geometrically integral and not a cone. Then $W_e \subset \mathbf{P}^{3\vee}$ does not contain any hyperplane when $e < d$.*

3.4. THEOREM. – *Let $X \subset \mathbf{P}^4$ be an irreducible projective threefold over $\mathbf{Q}$ of degree d. Let $X' \subset X$ resp. $\Xi \subset X$ be the complement of the union of all lines resp. planes on X. Then,*

$$N(X',B) = O_{d,\varepsilon}(B^{5/2+5/3d+\varepsilon} + B^{65/24+\varepsilon}),$$
$$N(\Xi,B) = O_{d,\varepsilon}(B^{5/2+5/3d+\varepsilon} + B^{199/72+\varepsilon}).$$

Proof. – Suppose first that X is not geometrically irreducible. Then all rational points on X are singular. We also know from the proof of Theorem 2.1 that the sum of the degrees of the irreducible components is bounded solely in terms of d. It is therefore sufficient to count rational points of height $\leqslant B$ on $O_d(1)$ irreducible curves (cf. Theorem 2.1) and surfaces. From Theorems 1.1 and 2.1 we get that $N(\Xi,B) = O_{d,\varepsilon}(B^{2+\varepsilon})$. We may and shall thus in the rest of this proof assume that X is geometrically integral.

Let P be a rational point of height $\leqslant B$ on $\mathbf{P}^4$. Then, by Siegel's lemma there is a rational hyperplane Γ of height $\leqslant (5B)^{1/4}$ passing through P. Let us first consider the contribution from the geometrically reducible hyperplane sections $X \cap \Gamma$. If we combine Theorem 3.2

and Lemma 3.3, then we conclude that there are $O_{d,\varepsilon}(B^{55/72+\varepsilon})$ such hyperplanes of height $\leqslant (5B)^{1/4}$. By results of Heath-Brown (see Theorem 2.1 resp. Theorem 2.4), we have that $N(X' \cap \Gamma, B) = O_{d,\varepsilon}(B^{52/27+\varepsilon})$ resp. $N(\Xi \cap \Gamma, B) = O_{d,\varepsilon}(B^{2+\varepsilon})$. The contribution to $N(X', B)$ resp. $N(\Xi, B)$ from the union of the geometrically reducible hyperplane sections is thus $O_{d,\varepsilon}(B^{52/27+55/72+\varepsilon})$ resp. $O_{d,\varepsilon}(B^{2+55/72+\varepsilon})$, which are satisfactory since for the first bound we have $52/27 + 55/72 < 65/24$.

It remains to consider the hyperplanes for which $X \cap \Gamma$ is geometrically irreducible. By Theorem 2.4 we have that

$$
\begin{aligned}
N(X' \cap \Gamma, B) &= O_{d,\varepsilon}(B^{4/3+16/9d+\varepsilon}) \quad \text{when } d \leqslant 9, \\
N(X' \cap \Gamma, B) &= O_{d,\varepsilon}(B^{14/9+\varepsilon}) \quad \text{when } d > 9.
\end{aligned} \tag{3.5}
$$

There are $O(B^{5/4})$ rational hyperplanes Γ of height $\leqslant (5B)^{1/4}$. It is possible to save $1/16$ of both exponents on the right hand side of (3.5) by adapting the argument in [19, p. 581] to threefolds. This follows from Lemma 5.1 below in which we take the condition (%) to mean that $X \cap \Gamma$ is geometrically irreducible. The number of rational points of height $\leqslant B$ lying on the union of the geometrically irreducible hyperplane sections $X \cap \Gamma, H(\Gamma) \leqslant (5B)^{1/4}$ but not on a line on $X \cap \Gamma$ is thus

$$
\begin{aligned}
&O_{d,\varepsilon}(B^{15(4/3+16/9d)/16+5/4+\varepsilon}) \quad \text{when } d \leqslant 9, \\
&O_{d,\varepsilon}(B^{15(14/9)/16+5/4+\varepsilon}) \quad \text{when } d > 9.
\end{aligned}
$$

This bound can be expressed as $O_{d,\varepsilon}(B^{5/2+5/3d+\varepsilon} + B^{65/24+\varepsilon})$ for all d, thereby completing the proof of the bound for $N(X', B)$.

To estimate $N(\Xi, B)$ we must also consider the contribution from the union of the lines on all the geometrically integral intersections $X \cap \Gamma$ where $H(\Gamma) \leqslant (5B)^{1/4}$. It is proved in Section 3.2 of [5] that the contribution to $N(X, B)$ from the lines on the hyperplane sections is $\ll_{d,\varepsilon} B^{5/2+3/2d+\varepsilon} + B^{3+\varepsilon}$. But an inspection of the proof in (op. cit.) reveals that we may omit the last term $B^{3+\varepsilon}$ if we restrict to rational points on Ξ which do not belong to any of the geometrically reducible hyperplane sections $X \cap \Gamma$ with Γ of height $\leqslant (5B)^{1/4}$. This completes the proof of the bound for $N(\Xi, B)$. $\square$

3.6. THEOREM. – *Let $X \subset \mathbf{P}^4$ be a smooth projective threefold over $\mathbf{Q}$ of degree d. Let U be the complement in X of the union of all curves on X with irreducible components of degree at most three over an extension of $\mathbf{Q}$. Then,*

$$N(U, B) = O_{d,\varepsilon}(B^{15f(d)/16+5/4+\varepsilon} + B^{15/32+5/4+\varepsilon}),$$

where

$$
\begin{aligned}
f(d) &= 3/\sqrt{d} + 2/(d-1) - 1/(d-1)\sqrt{d} \quad \textit{if } d \leqslant 13, \\
f(d) &= 3/\sqrt{d} + 2/d - 1/2d\sqrt{d} \quad \textit{if } d \geqslant 14.
\end{aligned}
$$

Proof. – Let P be a rational point of height $\leqslant B$ on $\mathbf{P}^4$. By Siegel's lemma there is a rational hyperplane Γ of height $\leqslant (5B)^{1/4}$ passing through P. Also, all hyperplane sections $X \cap \Gamma$ are geometrically irreducible by Lemma 6.2. Let us first consider the hyperplanes such that $X \cap \Gamma$ is smooth. Then, we may apply Theorem 2.10 to $X \cap \Gamma$ after introducing a basis for Γ of the same kind as in the proof of Corollary 1.8. Hence by Theorems 2.10 and 1.1 we obtain that

$$N(U \cap \Gamma, B) = O_{d,\varepsilon}(B^{f(d)+\varepsilon} + B^{2/4+\varepsilon}), \tag{3.7}$$

where the last term $B^{2/4+\varepsilon}$ corresponds to rational points on the set of $O_d(1)$ curves (cf. Theorem 2.8) of degree $e \in [4, d-2]$. There are $O(B^{5/4})$ rational hyperplanes Γ of height $\leqslant (5B)^{1/4}$. This gives a total contribution of $O_{d,\varepsilon}(B^{f(d)+5/4+\varepsilon} + B^{2/4+5/4+\varepsilon})$ to $N(U,B)$ from the smooth hyperplane sections. But it is possible to save $1/16$ of both exponents on the right hand side of (3.7) by means of the same technique as in the previous proof. This follows from Lemma 5.1 and Remark 5.2(b) below in which we take the condition (%) to mean that $X \cap \Gamma$ is smooth. The outcome is the bound $O_{d,\varepsilon}(B^{15f(d)/16+5/4+\varepsilon} + B^{15/32+5/4+\varepsilon})$ for the number of a rational points of height $\leqslant B$ on U lying on the union of all smooth hyperplane sections $X \cap \Gamma$ with Γ of height $\leqslant (5B)^{1/4}$.

It remains to treat the singular (and geometrically irreducible) hyperplane sections $X \cap \Gamma$. By Theorem 2.7 there exists another surface $Y \subset \Gamma$ of degree $O_{d,\varepsilon}(B^{3/\sqrt{d}+\varepsilon})$ containing all rational points of height $\leqslant B$ on $X \cap \Gamma$ but not $X \cap \Gamma$ itself. This surface Y can be chosen such that the degrees of all irreducible components of Y are uniformly bounded in terms of d and ε. We have thus a common uniform bound $O_{d,\varepsilon}(B^{2/4+\varepsilon})$ for the number of rational points of height $\leqslant B$ of the irreducible components of $(U \cap \Gamma) \cap Y$.

This means that

$$N(U \cap \Gamma, B) = O_{d,\varepsilon}(B^{3/\sqrt{d}+2/4+\varepsilon}) \tag{3.8}$$

for all Γ.

Next, we prove that there are $O_{d,\varepsilon}(B^{(g(d)+2)/4+\varepsilon}), g(d) = 4/(d-1)d^{1/3}$ singular hyperplanes Γ of height $\leqslant (5B)^{1/4}$. To see this, we first note that the Gauss map $g : X \to X^\vee$ (cf. the proof of Lemmas 2.11 or 6.3) is a finite morphism of degree $(d-1)^3$ since $\dim X = 3$ and g is given by forms of degree $(d-1)$. Hence the dual hypersurface $X^\vee \subset \mathbf{P}^{4\vee}$ is of degree $(d-1)^3 d$. We may therefore by Theorem 2.7 find an auxiliary hypersurface $W \subset \mathbf{P}^{4\vee}$ of degree $O_{d,\varepsilon}(B^{g(d)/4+\varepsilon})$ containing all rational points of height $\leqslant (5B)^{1/4}$ on $X^\vee$ but not containing $X^\vee$ itself. This threefold W may be chosen such that the sum of the degrees of all irreducible components of $X^\vee \cap W$ is bounded in terms of d and ε. Also, there are no planes among these irreducible components by Lemma 6.3. There are therefore (cf. Theorem 2.1) $O_{d,\varepsilon}(B^{2/4+\varepsilon})$ rational points of height $\leqslant (5B)^{1/4}$ on $X^\vee \cap W$. This together with (3.8) implies that we have $O_{d,\varepsilon}(B^{3/\sqrt{d}+2/4+(g(d)+2)/4+\varepsilon})$ rational points of height $\leqslant B$ on U lying on the union of all singular hyperplane sections $X \cap \Gamma$ with Γ of height $\leqslant (5B)^{1/4}$.

One can show that:

$$3/\sqrt{d} + 2/4 + \big(g(d)+2\big)/4 < 15f(d)/16 + 5/4,$$

so that this contribution to $N(U,B)$ is smaller than the contribution from the smooth hyperplane sections. This completes the proof of Theorem 3.6. □

3.9. COROLLARY. – *Let $X \subset \mathbf{P}^4$ be a smooth projective threefold over $\mathbf{Q}$ of degree $d > 5$. Then,*

$$N(X,B) = O_{d,\varepsilon}(B^{15f(d)/16+5/4+\varepsilon} + B^{2+\varepsilon}),$$

with $f(d)$ as in Theorem 3.6.

Proof. – Let Z be the complement of the open subset $U \subset X$ in Theorem 3.6. Then Z is a *proper* subset of X of degree bounded in terms of d by Lemma 6.1. Also, since X is smooth and $d > 1$ there are no planes on X. Therefore, $N(Z,B) = O_{d,\varepsilon}(B^{2+\varepsilon})$ by Theorem 2.1. We therefore get the desired bound for $N(X,B)$ from Theorem 3.6. □

3.10. *Remark.* – Note that $f(d) < 3/\sqrt{d} + 2/d$. We have thus for $d > 5$ the bound

$$N(X,B) = O_{d,\varepsilon}(B^{45/16\sqrt{d}+15/8d+5/4+\varepsilon} + B^{2+\varepsilon}).$$

This can also be proved more directly by using Theorem 2.9 instead of Theorem 2.10.

4. Fourfolds

We shall in this section prove the following theorems.

4.1. THEOREM. – *Let $X \subset \mathbf{P}^5$ be a smooth projective fourfold over $\mathbf{Q}$ of degree d. Then the following uniform estimates hold.*

$$N(X,B) = O_{d,\varepsilon}(B^{27/10\sqrt{d}+9/5(d-1)-9/10(d-1)\sqrt{d}+12/5+\varepsilon}) \quad \textit{if } 6 < d \leqslant 13,$$
$$N(X,B) = O_{d,\varepsilon}(B^{27/10\sqrt{d}+9/5d-9/20d\sqrt{d}+12/5+\varepsilon}) \quad \textit{if } 13 < d \leqslant 25,$$
$$N(X,B) = O_{d,\varepsilon}(B^{3+\varepsilon}) \quad \textit{if } d > 25.$$

4.2. THEOREM. – *Let $X \subset \mathbf{P}^5$ be a smooth projective fourfold over $\mathbf{Q}$ of degree d. Let Ξ be the complement of the union of planes on X. Then,*

$$N(\Xi,B) = O_{d,\varepsilon}(B^{27/10\sqrt{d}+9/5(d-1)-9/10(d-1)\sqrt{d}+12/5+\varepsilon}) \quad \textit{if } 6 < d \leqslant 13,$$
$$N(\Xi,B) = O_{d,\varepsilon}(B^{27/10\sqrt{d}+9/5d-9/20d\sqrt{d}+12/5+\varepsilon}) \quad \textit{if } 13 < d \leqslant 34,$$
$$N(\Xi,B) = O_{d,\varepsilon}(B^{131/45+\varepsilon}) \quad \textit{if } d > 34.$$

4.3. *Remark.* – The bounds can also be expressed in the following way.

$$N(\Xi,B) = O_{d,\varepsilon}(B^{9f(d)/10+12/5+\varepsilon} + B^{131/45+\varepsilon}), \quad \text{where}$$

$$f(d) = 3/\sqrt{d} + 2/(d-1) - 1/(d-1)\sqrt{d} \quad \text{if } 6 < d \leqslant 13,$$
$$f(d) = 3/\sqrt{d} + 2/d - 1/2d\sqrt{d} \quad \text{if } d > 13.$$

Proof of Theorem 4.2. – Let K be an algebraic closure of $\mathbf{Q}$ and $Z \subset X$ be the reduced subscheme such that the underlying closed set is the union of all closed curves C on X where all irreducible components of $C \times K$ are of degree at most three. Then $\dim Z \leqslant 3$ and the sum of the degrees of the irreducible components of Z is bounded solely in terms of d (see Lemma 6.1). Also, by Lemma 6.2 each three-dimensional irreducible component of $Z \times K$ is of degree divisible by d. We have therefore by Theorem 3.4 that

$$N(Z \cap \Xi, B) = O_{d,\varepsilon}(B^{5/2+5/3d+\varepsilon} + B^{199/72+\varepsilon})$$

which is smaller than the desired upper bound for $N(\Xi,B)$. It is thus sufficient to count rational points on the complement $U = X \backslash Z$.

Let P be a rational point of height $\leqslant B$ on $\mathbf{P}^5$. Then, by Siegel's lemma there exist four rational hyperplanes $\Gamma, \Gamma', \Gamma'', \Gamma'''$ such that $\Gamma \cap \Gamma' \cap \Gamma'' \cap \Gamma''' = P$ and

$$H(\Gamma)H(\Gamma')H(\Gamma'')H(\Gamma''') \leqslant (6B)^{4/5}.$$

We may assume that $H(\Gamma) \leqslant H(\Gamma') \leqslant H(\Gamma'') \leqslant H(\Gamma''')$. It is then sufficient to count rational points on sections $U \cap \Gamma \cap \Gamma'$ with pairs of rational hyperplanes $\Gamma \neq \Gamma'$ where $H(\Gamma) \leqslant (6B)^{1/5}$

and $H(\Gamma)H(\Gamma')^3 \leqslant (6B)^{4/5}$. Note also that all hyperplane sections $X \cap \Gamma$ are geometrically irreducible by Lemma 6.2.

Let us first consider the hyperplanes Γ for which $X \cap \Gamma$ is smooth. Then, by Theorem 3.6 we have that

$$N(U \cap \Gamma, B) = O_{d,\varepsilon}(B^{15f(d)/16+5/4+\varepsilon} + B^{15/32+5/4+\varepsilon}). \tag{4.4}$$

There are $O(B^{6/5})$ rational hyperplanes Γ of height $\leqslant (6B)^{1/5}$. Hence by Lemma 5.1, we get a total contribution of $O_{d,\varepsilon}(B^{24(15f(d)/16+5/4)/25+6/5+\varepsilon} + B^{24(15/32+5/4)/25+6/5+\varepsilon})$ rational points of height $\leqslant B$ on U lying on the union of all smooth hyperplane sections $X \cap \Gamma$ with Γ of height $\leqslant (6B)^{1/5}$. After multiplying out, we get the bound $O_{d,\varepsilon}(B^{9f(d)/10+12/5+\varepsilon} + B^{57/20+\varepsilon})$ for the number of these rational points, which is satisfactory.

We now turn to the hyperplanes Γ for which $X \cap \Gamma$ is singular. For rational points P on such $X \cap \Gamma$ we shall make use of the second hyperplane Γ' containing P mentioned in the beginning of the proof. Let $T \leqslant T'$ and $TT'^3 \ll B^{4/5}$. We begin by estimating the number of pairs (Γ, Γ') of rational hyperplanes of heights $H(\Gamma) \in [T, 2T]$ resp. $H(\Gamma') \in [T', 2T']$ where $X \cap \Gamma \cap \Gamma'$ is a geometrically irreducible surface. Let $h(d) = 1/(d-1)d^{1/4} + 11/18 + 6/5$. We claim that there are $O_{d,\varepsilon}(B^{h(d)+\varepsilon})$ such pairs (Γ, Γ').

If $X \cap \Gamma$ is singular, then the rational hyperplane Γ may be regarded as a rational point of height $\leqslant T$ on the dual variety $X^\vee \subset \mathbf{P}^{5\vee}$. The dual hypersurface $X^\vee \subset \mathbf{P}^5$ is of degree $(d-1)^4 d$. We may therefore by Theorem 2.7 find an auxiliary hypersurface $W \subset \mathbf{P}^{5\vee}$ of degree $O_{d,\varepsilon}(T^{5/(d-1)d^{1/4}+\varepsilon})$ containing all rational points of height $\leqslant T$ on $X^\vee$ but not $X^\vee$ itself. This fourfold W may be chosen such that the degrees of all irreducible components of $X^\vee \cap W$ are uniformly bounded in terms of d and ε. Also, by Lemma 6.3, there are no three-dimensional irreducible components of degree one on $X^\vee \cap W$. There are therefore by Theorem 3.2 $O_{d,\varepsilon}(T^{55/18+\varepsilon})$ rational points of height $\leqslant T$ on each irreducible component of $X^\vee \cap W$ and hence $O_{d,\varepsilon}(T^{55/18+5/(d-1)d^{1/4}+\varepsilon})$ rational points of height $\leqslant T$ on $X^\vee$. In particular, we find that there are $O_{d,\varepsilon}(T^{55/18+5/(d-1)d^{1/4}+\varepsilon})$ possibilities for Γ. Also, there are $O_{d,\varepsilon}(T'^6)$ possibilities for Γ'. It therefore follows from the assumptions $T \leqslant T'$ and $TT'^3 \ll B^{4/5}$ that there are $O_{d,\varepsilon}(B^{h(d)+\varepsilon})$ such pairs (Γ, Γ').

To estimate $N(U \cap \Gamma \cap \Gamma', B)$, we introduce new coordinates for $\Gamma \cap \Gamma' = \mathbf{P}^3$ (cf. part (iii) of [19, Lemma 1]) such that the new height does not exceed the old height given by the embedding $\Gamma \cap \Gamma' \subset \mathbf{P}^5$. We may then by Theorem 2.7 find an auxiliary hypersurface Y of degree $O_{d,\varepsilon}(B^{3/\sqrt{d}+\varepsilon})$ on $\Gamma \cap \Gamma' = \mathbf{P}^3$ containing all the rational points on $X \cap \Gamma \cap \Gamma' \subset \mathbf{P}^3$. Also, each irreducible component of the intersection of $X \cap Y \cap \Gamma \cap \Gamma' \subset \mathbf{P}^3$ not lying in $Z = X \backslash U$ is of degree $\geqslant 4$. We thus conclude that $N(U \cap \Gamma \cap \Gamma', B) \leqslant B^{3/\sqrt{d}+2/4+\varepsilon}$ by Theorem 1.1 and the arguments in the proof of Theorem 2.6(a).

After summing over dyadic intervals $[T, 2T]$ resp. $[T', 2T']$ where $T \leqslant T'$ and $TT'^3 \ll B^{4/5}$, we obtain $O_{d,\varepsilon}(B^{h(d)+3/\sqrt{d}+2/4+\varepsilon})$ rational points on U of height $\leqslant B$ on the union of all geometrically irreducible intersections $X \cap \Gamma \cap \Gamma'$ with hyperplanes where $X \cap \Gamma$ is singular, $H(\Gamma) \leqslant H(\Gamma')$ and $H(\Gamma)H(\Gamma')^3 \leqslant (6B)^{4/5}$. This is enough, as

$$h(d) + 3/\sqrt{d} + 2/4 < 9f(d)/10 + 12/5.$$

We now consider the contribution from rational points on the sections $X \cap \Gamma \cap \Gamma'$ which are geometrically reducible. Then, by Lemma 6.4 Γ and Γ' belong to the singular locus of $X^\vee \subset \mathbf{P}^{5\vee}$. By Theorem 3.2 and Lemma 6.3 there are thus $O_{d,\varepsilon}(T^{55/18+\varepsilon})$ resp. $O_{d,\varepsilon}(T'^{55/18+\varepsilon})$ possibilities for such pairs of rational hyperplanes Γ (resp. Γ') of heights $H(\Gamma) \in [T, 2T]$ resp. $H(\Gamma') \in [T', 2T']$. After summing over dyadic intervals $[T, 2T]$ resp. $[T', 2T']$ we obtain

$O_{d,\varepsilon}(B^{55/45+\varepsilon})$ pairs (Γ,Γ') where $X\cap\Gamma\cap\Gamma'$ is geometrically reducible and where $H(\Gamma)\leqslant H(\Gamma')$ and $H(\Gamma)H(\Gamma')\leqslant(6B)^{2/5}$. Also, if Y is an irreducible component of degree >4 on $X\cap\Gamma\cap\Gamma'$, then we have by Remark 2.5 $O_{d,\varepsilon}(B^{76/45+\varepsilon})$ rational points of height $\leqslant B$ on Y outside the lines. The total contribution to $N(U,B)$ from all such subvarieties Y is thus $O_{d,\varepsilon}(B^{76/45+55/45+\varepsilon})$, which is satisfactory.

We still have to consider the contribution from the irreducible components Y of degree $\leqslant 4$ on intersections $X\cap\Gamma\cap\Gamma'$ as above. But it is shown in Lemma 6.1 that all these surfaces lie on a proper closed subset Z_2 of X of uniformly bounded degree. We therefore get the same bound $N(Z_2\cap\Xi,B)=O_{d,\varepsilon}(B^{5/2+5/3d+\varepsilon}+B^{199/72+\varepsilon})$ as we obtained for $N(Z\cap\Xi,B)$. This completes the proof of Theorem 4.2. □

Proof of Theorem 4.1. – It follows from the proof of Lemma 6.1 that all lines on X lie on a proper closed subset W of X degree bounded solely in terms of d. Also, by Lemma 6.2 we have that the degree of each three-dimensional irreducible component of W is divisible by d. Therefore, $N(W,B)=O_{d,\varepsilon}(B^{3+\varepsilon})$ by Theorem 3.1. To complete the proof, use the bound for $N(X\backslash W,B)$ given by Theorem 4.2. □

For hypersurfaces defined by diagonal forms one can give a more explicit version of Theorem 4.2 by means of the following geometric lemma.

4.5. PROPOSITION. – *Let $X\subset\mathbf{P}^5$ be a smooth fourfold over a field k of characteristic 0 given by an equation*

$$a_0x_0^d+\cdots+a_5x_5^d=0,$$

where $a_0,\ldots,a_5\in k$ and $d\geqslant 5$. Then the union Z of all planes on X is equal to the union of the fifteen closed subsets given by the equations,

$$a_0x_0^d+a_ix_i^d=a_jx_j^d+a_kx_k^d=a_lx_l^d+a_mx_m^d=0.$$

Proof. – Choose dth roots $a_i^{1/d}$ in an algebraic closure K of k. We may then write the equation as

$$(a_0^{1/d}x_0)^d+\cdots+(a_5^{1/d}x_5)^d=0$$

and reduce to the Fermat hypersurface $x_0^d+\cdots+x_5^d=0$ over K treated in [12, p. 54]. □

4.6. COROLLARY. – *Let $n_d(B)$ be the number of solutions in positive integers $x_i,y_i\leqslant B$, $0\leqslant i\leqslant 2$, to the equation*

$$x_0^d+x_1^d+x_2^d=y_0^d+y_1^d+y_2^d,$$

where $(x_0,x_1,x_2)\neq(y_i,y_j,y_k)$ for all six permutations (i,j,k) of $(0,1,2)$.

Then,

$$n_d(B)=O_{d,\varepsilon}(B^{27/10\sqrt{d}+9/5(d-1)-9/10(d-1)\sqrt{d}+12/5+\varepsilon})\quad\text{if } 6<d\leqslant 13,$$
$$n_d(B)=O_{d,\varepsilon}(B^{27/10\sqrt{d}+9/5d-9/20d\sqrt{d}+12/5+\varepsilon})\quad\text{if } 13<d\leqslant 34,$$
$$n_d(B)=O_{d,\varepsilon}(B^{131/45+\varepsilon})\quad\text{if } d>34.$$

Proof. – It is clearly sufficient to establish these estimates for the primitive solutions. Also, by Proposition 4.5 we find that the rational points represented by positive sixtuples as above all lie on Ξ. The result is thus a special case of Theorem 4.2. □

4.7. *Remarks.* – Note that $27/10\sqrt{d}+9/5d-9/20d\sqrt{d}+12/5<3$ when $d>25$. We then get an asymptotic formula

$$N_d(B)=6B^3+O_{d,\varepsilon}(B^{3-\delta})$$

for the number $N_d(B)$ of all solutions in positive integers $x_i, y_i \leqslant B$, $0\leqslant i\leqslant 2$, to the equation

$$x_0^d+x_1^d+x_2^d=y_0^d+y_1^d+y_2^d.$$

This improves upon [8] where the authors get such a result for $d>32$.

For $d\geqslant 9$ we get:

$$N_d(B)=O_{d,\varepsilon}(B^{7/2-1/80+\varepsilon}).$$

This should be compared with Hua's estimate (cf. [20,11])

$$N_d(B)=O_{d,\varepsilon}(B^{7/2+\varepsilon})$$

from 1938. This was still the best known result until the paper of Heath-Brown [19] appeared. He gives an improvement on Hua's estimate for $d\geqslant 24$. There are no improvements of Hua's bound for lower d in [8]. But after sending my paper to Browning and Heath-Brown, I immediately received a second manuscript on equal sums in which they prove that

$$n_d(B)=O_d(B^{7/2-\delta}),\quad \delta>0,$$

for $d\geqslant 11$. The main new ingredient is Lemma 6.1(a) of this paper, which I communicated to them earlier. Their method uses special properties of the form $x_0^d+x_1^d+x_2^d-x_3^d-x_4^d-x_5^d$.

5. Hyperplane sections

We shall in this section formulate a lemma about hyperplane sections that we use at several places. It is an extension of a technique used by Heath-Brown [19, p. 581] for surfaces in $\mathbf{P}^3$.

5.1. LEMMA. – *Let $n,d>1$ be two integers and suppose that we have a uniform bound*:

$$N(Y,B)=O_{n,d,\varepsilon}(B^{g(d)+\varepsilon}),$$

for all hypersurfaces $Y\subset\mathbf{P}^{n-1}$ of degree d defined over $\mathbf{Q}$ satisfying a certain property (%) *independent of the choice of coordinates.*

Let $X\subset\mathbf{P}^n$ be an irreducible closed hypersurface of degree d defined over $\mathbf{Q}$ and $n(X,B)$ be the number of rational points of height $\leqslant B$ on X which lie on the union of all intersections $X\cap\Gamma$ with rational hyperplanes $\Gamma\in\mathbf{P}^{n\vee}(\mathbf{Q})$ satisfying (%) *and $H(\Gamma)\leqslant((n+1)B)^{1/n}$. Then,*

$$n(X,B)=O_{n,d,\varepsilon}(B^{g(d)-g(d)/n^2+(n+1)/n+\varepsilon}).$$

Proof. – Let $\Gamma\subset\mathbf{P}^n$ be a rational hyperplane. Then, by [19, Lemma 1] there exists homogeneous coordinates $(y_1,\dots,y_n)$ for Γ such that the following holds.

(i) Each y_i is a linear combination $y_i=m_{i0}x_0+\cdots+m_{in}x_n$ for a primitive $(n+1)$-tuple $(m_{i0},\dots,m_{in})$ of integers.

(ii) Let $P_i, 1\leqslant i\leqslant n$, be the rational point on Γ defined by

$$y_1=\cdots=y_{i-1}=y_{i+1}=\cdots=y_n=0.$$

Then,

$$H(\Gamma) \ll H(P_1)\dots H(P_n) \ll H(\Gamma),$$

where the implied constants only depend on n. (Note that $H(P_i) = \max|m_{ij}|$, $0 \leqslant j \leqslant n$.)

(iii) Let $(y_1,\dots,y_n)$ be a primitive integral n-tuple representing a rational point P on Γ. Then $|y_i| \ll H(P)/H(P_i)$ for $1 \leqslant i \leqslant n$ with the implied constant depending only on n.

Suppose now that the rational points are ordered such that $H(P_1) \leqslant \cdots \leqslant H(P_n)$. By (iii) and the assumption we get that

$$N(X\cap\Gamma,B) \ll_{n,d,\varepsilon} \bigl(B/H(P_1)\bigr)^{g(d)+\varepsilon}$$

whenever $X\cap\Gamma$ satisfies (%).

We now consider the hyperplanes Γ spanned by n rational points $P_1,\dots,P_n$ as above and where $C_i \leqslant H(P_i) \leqslant 2C_i$ for some constants $1 \leqslant C_1 \leqslant \cdots \leqslant C_n$ with $C_1\dots C_n \ll_n B^{1/n}$. We may assume that the coordinates of P_1 are ordered such that $|m_{10}| \geqslant \cdots \geqslant |m_{1n}|$. There are then unique integers $q_i, r_i, i \in \{2,\dots,n\}$, satisfying:

$$m_{i0} = q_i m_{10} + r_i, \quad 0 \leqslant r_i < |m_{10}|.$$

Let $Q_i \neq P_1$, $i \in \{2,\dots,n\}$, be the rational point on the line between P_1 and P_i represented by the $(n+1)$-tuple where $x_j(Q_i) = m_{ij} - q_i m_{1j}$ for $j \in \{0,1,2,\dots,n\}$. Then,

$$\bigl|x_0(Q_i)\bigr| = |r_i| < |m_{10}| \leqslant 2C_1 \quad \text{and}$$

$$\bigl|x_j(Q_i)\bigr| \leqslant |m_{ij}| + |q_i||m_{1j}| \leqslant |m_{ij}| + |q_i||m_{10}| = |m_{ij}| + |m_{i0} - r_i| \leqslant 4C_i + 2C_1 \leqslant 6C_i$$

for $j \in \{1,2,\dots,n\}$.

The bounds for the x-coordinates give $O(C_1^{n+1})$ possibilities for P_1, and $O(C_1C_i^n)$ possibilities for Q_i, $i \in \{2,\dots,n\}$. There are, therefore, $O(C_1^{2n}(C_2\dots C_n)^n)$ hyperplanes Γ spanned by points $P_1,\dots,P_n$ as above. For each such Γ, we have

$$N(X\cap\Gamma,B) \ll_{n,d,\varepsilon} (B/C_1)^{g(d)+\varepsilon}$$

whenever $X\cap\Gamma$ satisfies (%).

This gives $O_{n,d,\varepsilon}(B^{g(d)+\varepsilon}C_1^{2n-g(d)}(C_2\dots C_n)^n)$ rational points of height $\leqslant B$ on the union of these hyperplane sections. But $N(Y,B) = O(B^n)$ for hypersurfaces $Y \subset \mathbf{P}^{n-1}$ over $\mathbf{Q}$. We may thus assume that $g(d) \leqslant n$ so that

$$C_1^{2n-g(d)}(C_2\dots C_n)^n \leqslant (C_1C_2\dots C_n)^{n+1-g(d)/n} \ll_n B^{(n+1)/n-g(d)/n^2}.$$

There are thus

$$O_{n,d,\varepsilon}(B^{g(d)-g(d)/n^2+(n+1)/n+\varepsilon}),$$

rational points of height $\leqslant B$ on the unions of these hyperplane sections. The same estimate remains valid after summing over all dyadic intervals $[C_i, 2C_i]$ for 2-powers C_i with $C_1\dots C_n \ll_n B^{1/n}$. This completes the proof of Lemma 5.1. $\square$

5.2. *Remarks.* – (a) We made a very specific hypothesis $H(\Gamma) \leqslant ((n+1)B)^{1/n}$ about the heights of the hyperplanes. This hypothesis is natural for the applications since any rational point on $\mathbf{P}^n$ lies on such a hyperplane Γ by Siegel's lemma.

(b) There are many variants of Lemma 5.1. It is often useful to consider hyperplane sections when counting rational points on open subsets of projective varieties like the complement X' of all lines on $X \subset \mathbf{P}^n$. It is clear from the proof above that if we start with a hypothesis for $N(X' \cap \Gamma, B)$ instead of $N(X \cap \Gamma, B)$, then we get the same bound for $N(X', B)$. We shall in this paper count points on other open subsets like the complement of all curves of degree at most three or the complement of all surfaces of degree at most four contained in a projective linear three-dimensional subspace (cf. Lemma 6.1). It is obvious that the proof of Lemma 5.1 gives the same kind of implications as above.

(c) One can formulate versions of Lemma 5.1 for closed subsets $X \subset \mathbf{P}^n$ of higher codimension. See [4, Lemma 7] for such a result for surfaces.

6. Lemmas from algebraic geometry

6.1. LEMMA. – *Let $X \subset \mathbf{P}^n$ be a smooth hypersurface of degree $d > n+1$ over a field k. Let K be an algebraic closure of k. Then there are proper closed subsets Z_1, resp. Z_2, of X with the following properties.*

(i) *Let Z_1 be the union of all closed curves C on X such that all irreducible components of $X \times K$ are of degree at most three. Then Z_1 is a proper closed subset of X.*

(ii) *Let Z_2 be the union of all closed surfaces Y on X such that all irreducible components of $X \times K$ are of degree at most four and contained in a projective linear 3-subspace of $\mathbf{P}^n$. Then Z_2 is a proper closed subset of X.*

Moreover, the sum of the degrees of the irreducible components of Z_1, resp. Z_2, can be bounded in terms of d if we endow Z_1, resp. Z_2, with their reduced scheme structures.

Proof. – (i) There are only finitely many Hilbert polynomials $P(x) = \delta x + e$ occurring among *reduced* closed curves of degree δ in $\mathbf{P}^n$. We have in fact $e \ll \delta^2$ by the inequality of Castelnuovo [2]. Let us fix one such polynomial $P(x) = \delta x + e$. Let

$$h_{P,X} : (\text{schemes}_{/K}) \to (\text{sets})$$

be the contravariant functor that associates to any K-scheme B the set of subschemes $S \subset X \times_k B$ flat over B where fibers over points of B have Hilbert polynomial $P(x)$. Then $h_{P,X}$ is representable by a projective scheme $H_{P,X}$ over K (see [13, pp. 295–296]). Let $S_{P,X} \subset X \times_k H_{P,X}$ be the corresponding universal family of curves on X with Hilbert polynomial $P(x)$ and let $Z_{P,X}$ be the scheme-theoretic image of $S_{P,X}$ under the projection $X \times H_{P,X} \to X$. Then $Z_{P,X}$ is a closed subscheme of X by the main theorem of elimination theory.

We now prove that $Z_{P,X} \neq X$ if $P(x) = \delta x + e$ and $\delta = 1, 2, 3$. It suffices to do this in the case where K is algebraically closed and there is an integral curve C on $X \subset \mathbf{P}^n$ with $P(x)$ as Hilbert polynomial. If $Z_{P,X} = X$, then there exists a family $S \subset X \times_k B$ as above where B is irreducible, $\dim S = \dim X$, and where S projects onto X. Then, by [25, pp. 550–551], we have the following relations between the Kodaira-dimensions κ of S, X, B and V.

$$\kappa(C) + \dim B \geqslant \kappa(S) \geqslant \kappa(X) = \dim X.$$

It is well known that the resolution of an integral curve of degree at most three is of genus at most one. Hence $\kappa(C) \leqslant 0$ and $\dim X - 1 = \dim B \geqslant \dim X$. We have thus shown that the assumption $Z_P = X$ leads to a contradiction.

To get (i), let Z_1 be the finite union of the $Z_{P,X}$ above with its reduced scheme structure. To prove the last assertion, it suffices to give a uniform bound for the degree of each $Z_{P,X}$. Let H_d

be the Hilbert scheme of all hypersurfaces $X \subset \mathbf{P}^n$ of degree d and Ω_d be the open subscheme of all smooth hypersurfaces of degree d. Further, let H_P be the Hilbert scheme of all closed subschemes $C \subset \mathbf{P}^n$ with Hilbert polynomial $P(x)$. Then, there is a closed subscheme $I_{d,P}$ of $H_d \times H_P$ representing pairs (C,X) where C is a closed subscheme of X. Let $S_P \subset \mathbf{P}^n \times I_{d,P}$ be the corresponding universal family of curves on $\mathbf{P}^n$ and $Z_P \subset \mathbf{P}^n \times H_d$ be the scheme-theoretic image of S_P under the projection from $\mathbf{P}^n \times H_d \times H_P$ to $\mathbf{P}^n \times H_d$. Then Z_P is a closed subscheme of $\mathbf{P}^n \times H_d$ since H_P is projective. Now choose a projective embedding $H_d \subset \mathbf{P}^m$. Then $Z_P \subset \mathbf{P}^n \times \mathbf{P}^m$ is defined by finitely many bihomogeneous polynomials. Therefore, we get a uniform bound for the degree of $Z_{P,X}$ for all $X \in \Omega_d(K)$ since $Z_{P,X}$ is the fibre at $X \in \Omega_d(K)$ of the projection from $Z_P \subset \mathbf{P}^n \times \mathbf{P}^m$ to $\mathbf{P}^m$. This completes the proof of the uniform bound for the sum of the degrees of the irreducible components of $\deg(Z_1)$ in (i).

The proof of (ii) is almost identical to the proof of (i). All surfaces of degree δ in $\mathbf{P}^n$ contained in a projective linear 3-subspace of $\mathbf{P}^n$ have the same Hilbert polynomial $P(x)$ with $\delta x^2/2$ as leading term. This time we shall not consider the same contravariant functor $h_{P,X}$ as in (i). Instead we consider the closed (contravariant) subfunctor

$$g_{P,X} : (\text{schemes}_{/K}) \to (\text{sets})$$

of $h_{P,X}$ defined as follows. To any K-scheme B, let $g_{P,X}(B)$ be the set of subschemes $S \subset X \times_k B$ flat over B where fibers over points of B have Hilbert polynomial $P(x)$ and are contained in a three-dimensional projective subspace. Then $g_{P,X}$ is representable by a closed subscheme $G_{P,X}$ of the projective scheme representing $h_{P,X}$ (cf. [13]). Let $S_{P,X} \subset X \times_k G_{P,X}$ be the corresponding universal family of surfaces on X with Hilbert polynomial $P(x)$ and $Z_{P,X}$ be the scheme-theoretic image of $S_{P,X}$ under the projection $X \times G_{P,X} \to X$. Then $Z_{P,X}$ is a closed subscheme of X. We can now proceed exactly as before and prove that $Z_{P,X} \neq X$ provided that we know that $\kappa(Y) \leqslant 1$ for any integral surface $Y \subset \mathbf{P}^3$ of degree at most four over an algebraically closed field K.

If the singular locus of Y is of dimension one, choose a generic pencil of plane sections. The generic member is then an integral singular curve C of degree $\delta \leqslant 4$. The resolution of C is therefore of genus $\leqslant 1$ and $\kappa(Y) \leqslant 1$. Suppose next that the singular locus of Y is zero-dimensional or empty. Then Y is normal and Gorenstein. The cases $\delta = 1, 2$ are trivial and the case $\delta = 3$ is a consequence of the classification of cubic surfaces with isolated singularities [6]. So it suffices to treat quartic surfaces in $\mathbf{P}^3$. Then the trivial sheaf is a dualizing sheaf. Also, since Y is normal, it follows that there are no regular functions on the smooth locus U of Y apart from the constant ones. Therefore, any plurigenus of a smooth compactification of U is at most one. Hence $\kappa(Y) \leqslant 0$ in this case and $\kappa(Y) \leqslant 1$ for any integral surface $Y \subset \mathbf{P}^3$ of degree at most four. This implies in its turn by arguments almost identical to those in (i) that $Z_2 = Z_{P,X}$ is a proper closed subscheme of X which satisfies (ii). The proof of the uniform bound for $\deg(Z_2)$ is the same as the proof for the uniform bound for $\deg(Z_1)$. $\square$

6.2. LEMMA. – *Let $X \subset \mathbf{P}^n$ be a hypersurface of dimension $r = n - 1 \geqslant 3$ over an algebraically closed field of characteristic 0. Suppose that X is smooth or of degree at least 3 with an ordinary double point as its only singularity. Then the class group $\operatorname{Cl} X = \mathbf{Z}$ and is generated by the class of a hyperplane section.*

Proof. – This is a well known theorem (Noether–Lefschetz) if X is smooth (see [18, p. 180] for an algebraic proof). It therefore suffices to consider the case where there is an ordinary double point x on X. Let $\Xi \to X$, resp. $\Pi \to P$, be the blow-ups of X, resp. $P = \mathbf{P}^n$, at x and let E, resp. π, be the exceptional loci. Then Ξ is a smooth closed subvariety of Π and E a non-singular quadric in $\pi = \mathbf{P}^{n-1}$. We may clearly assume that $x = (1, 0, \ldots, 0)$ after a coordinate change. Then Π is the closed subvariety of $\mathbf{P}^n \times \mathbf{P}^{n-1}$ with biprojective coordinates $(x_0, \ldots, x_n; y_1, \ldots, y_n)$ given by the equations $x_j y_k = x_k y_j$ for $1 \leqslant j \leqslant k \leqslant n$.

Suppose $X \subset \mathbf{P}^n$ is defined by the form $F(x_0, \ldots, x_n)$. Choose a representation of F as a finite sum $\Sigma_\lambda G_\lambda Q_\lambda$ where G_λ is a monomial of degree $d-2$ in $(x_0, \ldots, x_n)$ and Q_λ a quadratic form in $(x_1, \ldots, x_n)$ for each λ. Then $\Xi \subset \Pi$ is the closed subvariety defined by the bihomogeneous polynomial $\Sigma_\lambda G_\lambda(x_0, \ldots, x_n) Q_\lambda(y_1, \ldots, y_n)$ of bidegree $(d-2, 2)$. Hence Ξ is the hyperplane section of Π in $\mathbf{P}^m$ under the embedding $\mathbf{P}^n \times \mathbf{P}^{n-1} \subset \mathbf{P}^m$ given by all bihomogeneous monomials of bidegree $(d-2, 2)$. Therefore, the restriction map from $\operatorname{Pic} \Pi$ to $\operatorname{Pic} \Xi$ is an isomorphism by [18, p. 180]. But it is well known that $\operatorname{Pic} \Pi = \mathbf{Z} \oplus \mathbf{Z}$ and is generated by the classes of π and the image in $\operatorname{Pic} \Pi$ of the hyperplane class in $\operatorname{Pic} P$. Hence, the contravariant functorial map from $\operatorname{Pic} P = \mathbf{Z}$ to $\operatorname{Pic}(\Xi \backslash E)$ is an isomorphism. To conclude, use the canonical isomorphisms $\operatorname{Cl}(X) = \operatorname{Cl}(X \backslash x) = \operatorname{Pic}(X \backslash x) = \operatorname{Pic}(\Xi \backslash E)$. $\square$

We now study the dual variety $X^\vee \subset \mathbf{P}^{n\vee}$ of hyperplanes $\Gamma \subset \mathbf{P}^n$ for which $X \cap \Gamma$ is singular.

6.3. LEMMA. – *Let $X \subset \mathbf{P}^n$ be a smooth hypersurface of dimension $n-1 \geqslant 3$ over a field K of characteristic 0. Then $X^\vee \subset \mathbf{P}^{n\vee}$ does not contain any $(n-2)$-dimensional irreducible components of degree 1.*

Proof. – Let $F(x_0, \ldots, x_n)$ be a form of degree d over K defining $X \subset \mathbf{P}^n$. Let $\varphi : X \to X^\vee$ be the Gauss map which sends $(x_0, \ldots, x_n)$ to $(\delta F/\delta x_0, \ldots, \delta F/\delta x_n)$. Any $(n-2)$-dimensional component of $X^\vee$ of degree 1 is then the image of a closed subset W of X defined by two forms G and H of degree $d-1$ which are linear combinations of $\delta F/\delta x_0, \ldots, \delta F/\delta x_n$. We may after a coordinate change assume that the coefficients of $\delta F/\delta x_0$ (resp. $\delta F/\delta x_1$) in G (resp. H) are different from zero.

The homogeneous coordinate ring $R = K[x_0, \ldots, x_n]/(F)$ is a unique factorization domain (UFD) since $\operatorname{Cl}(X) = \mathbf{Z}$ and X is projectively normal [17, p. 147]. In particular, any homogeneous prime ideal I of height 1 of R is principal and generated by the image in R of some form $F^* \in K[x_0, \ldots, x_n]$.

We now show that W is of codimension at least two in X. Suppose not. Then there is an irreducible component Z of W of codimension one on X. Let $I \subset R$ be the homogeneous prime ideal of height 1 consisting of all elements which vanish on Z. Then, as R is an *UFD* we may find a form $F^* \in K[x_0, \ldots, x_n]$ such that the image of F^* in R generates $I \subset R$. But G and H vanish on $Z \subset \mathbf{P}^5$. By the homogeneous Nullstellensatz G and H must therefore belong to the prime ideal of $K[x_0, \ldots, x_n]$ generated by F and F^*. This implies in its turn that F^* is a common factor of G and H, since G and H are of lower degree than F. But then any common zero of the forms F^*, $\delta F/\delta x_2, \ldots, \delta F/\delta x_n$ would be a singular point on X, which is impossible by hypothesis. Hence W must be of codimension at least two in X. The image $\varphi(W)$ is therefore of dimension less than $n-2$. This completes the proof. $\square$

6.4. LEMMA. – *Let $X \subset \mathbf{P}^n$ be a smooth hypersurface of dimension $n-1 \geqslant 3$ over an algebraically closed field of characteristic 0. Let $L \subset \mathbf{P}^{n\vee}$ be a line corresponding to a projective linear subspace $V \subset \mathbf{P}^n$ of codimension two such that $X \cap V$ is reducible. Then L belongs to the singular locus of $X^\vee \subset \mathbf{P}^{n\vee}$.*

Proof. – Let Q be a point on $\mathbf{P}^{n\vee}$ and $\Gamma \subset \mathbf{P}^n$ be the hyperplane corresponding to Q. It is well known [22] that Q lies outside the singular locus of $X^\vee$ if and only if the intersection $X \cap \Gamma$ is smooth or has an ordinary double point as its only singularity. Also, by Lemma 6.2 $X \cap V$ is irreducible for any projective linear subspace $V \subset \mathbf{P}^n$ of codimension two contained in a hyperplane Γ where $X \cap \Gamma$ is smooth or has an ordinary double point as its only singularity. Hence any point Q on a line $L \subset \mathbf{P}^{n\vee}$ as in Lemma 6.4 must belong to the singular locus of $X^\vee$. $\square$

Acknowledgements

The work which led to this paper began in 2002 during my stay at the Max-Planck institute. I would thus like to thank the institute for the excellent working conditions and professor Heath-Brown for inviting me to take part in the activities in analytic number theory there. I would also like to thank the referee and T. Browning for their comments on this paper.

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(Manuscrit reçu le 1 mars 2004;
accepté, après révision, le 15 octobre 2004.)

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