

MULTILINEAR EIGENFUNCTION ESTIMATES AND GLOBAL EXISTENCE FOR THE THREE DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. – We study nonlinear Schrödinger equations, posed on a three dimensional Riemannian manifold M . We prove global existence of strong H^1 solutions on $M = S^3$ and $M = S^2 \times S^1$ as far as the nonlinearity is defocusing and sub-quintic and thus we extend results of Ginibre, Velo and Bourgain who treated the cases of the Euclidean space \mathbb{R}^3 and the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ respectively. The main ingredient in our argument is a new set of multilinear estimates for spherical harmonics.

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RÉSUMÉ. – On étudie l'équation de Schrödinger sur une variété de dimension trois M . On démontre l'existence globale en temps de solutions fortes H^1 si $M = S^3$ ou $S^2 \times S^1$, pour les non linéarités sous quintiques et défocalisantes. On étend ainsi les résultats de Ginibre et Velo et Bourgain qui ont traité les cas de l'espace euclidien \mathbb{R}^3 et du tore $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ respectivement. L'ingrédient essentiel de notre démonstration est l'obtention de nouvelles estimées multilinéaires pour les harmoniques sphériques.

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1. Introduction

Let (M, g) be a compact smooth boundary-less Riemannian manifold of dimension $d \geq 2$. Denote by Δ the Laplace operator associated to the metric g . In the case $d = 2$, we discovered in [8] a bilinear generalization of the well-known Sogge estimates [22–24] for L^p ($p \geq 2$) norms of L^2 normalized eigenfunctions of Δ . These bilinear estimates play a central role in the analysis of [8] concerning the nonlinear Schrödinger equation (NLS) posed on M . The goal of this paper is to generalize our bilinear estimate of [8] to all higher dimensions and to deduce new results regarding the global existence of solutions for NLS when $d = 3$.

We consider thus the Cauchy problem for NLS

$$(1.1) \quad iu_t + \Delta u = F(u), \quad u|_{t=0} = u_0.$$

In (1.1), u is a complex valued function on M . The nonlinear interaction F satisfies $F(0) = 0$ and is supposed of the form $F = \frac{\partial V}{\partial \bar{z}}$ with $V \in C^\infty(\mathbb{C}; \mathbb{R})$ satisfying

$$(1.2) \quad V(e^{i\theta} z) = V(z), \quad \theta \in \mathbb{R}, z \in \mathbb{C},$$

and, for some $\alpha > 1$,

$$|\partial_z^{k_1} \partial_{\bar{z}}^{k_2} V(z)| \leq C_{k_1, k_2} (1 + |z|)^{1 + \alpha - k_1 - k_2}.$$

The number α involved in the second condition on V corresponds to the “degree” of the nonlinearity $F(u)$ in (1.1). Under these assumptions on F , NLS can be seen as a Hamiltonian equation in an infinite dimensional phase space. It follows from that Hamiltonian structure that smooth solutions of (1.1) enjoy the conservation laws

$$(1.3) \quad \|u(t, \cdot)\|_{L^2} = \|u_0\|_{L^2}, \quad E(u(t)) = E(u_0),$$

where the energy functional E reads as follows,

$$(1.4) \quad E(u) = \int_M |\nabla_g u|^2 dx + \int_M V(u) dx.$$

In view of (1.3) and (1.4), the local well-posedness of (1.1) in $H^1(M)$ (with time existence depending upon the H^1 norm) is of particular importance. If for example $V \geq 0$ and $(d-2)\alpha \leq d+2$, (1.3) provides an H^1 a priori bound and thus the local well-posedness of (1.1) in H^1 implies the global well-posedness in H^1 . Let us notice, on the other hand, that the local well-posedness of (1.1) in H^s , $s > d/2$ can be obtained by the classical energy method (see [21]). If M is two dimensional, the well-posedness of (1.1) in $H^1(M)$ is established in [5]. In this case, the issue is to get an improvement of ε derivatives with respect to the energy method. In [5], this ε gain is achieved by a Strichartz type inequality (with derivative loss). Therefore, for $d=2$, the H^1 well-posedness theory for (1.1) is complete. Moreover, in the recent paper [8], we establish a sharp H^s theory in the case $M = S^2$, as far as cubic nonlinearities are concerned.

In three dimensions, the H^1 theory for (1.1) becomes much harder. In the case $d=3$, the Strichartz type inequalities established in [5] yield the local well-posedness of (1.1) in H^s , $s > 1$, as far as $\alpha \leq 3$. Notice that this is already a significant improvement with respect to the energy approach. Unfortunately, it barely misses the crucial H^1 regularity. However, in [5], we succeeded in using the conservation laws (1.3) in order to get *global* H^s , $s > 1$ strong solutions. By “strong H^s solutions”, we mean the existence, the uniqueness, the propagation of regularity and the uniform continuous dependence in bounded subsets of initial data in H^s . Moreover, the methods of [5] also yield uniqueness of H^1 weak solutions.

On the other hand, if M is the torus \mathbb{T}^3 and $\alpha < 5$, the global existence of H^1 strong solutions of (1.1) was established by Bourgain [1]. The approach in [1] is based on an ingenious use of multiple Fourier series and thus relies deeply on the particular structure of the torus. In this paper, we will prove the counterpart of this result of Bourgain to the cases of the sphere S^3 and the product manifold $S_\rho^2 \times S^1$, where S_ρ^d , $d \geq 1$ is the embedded sphere of radius ρ in \mathbb{R}^{d+1} .

THEOREM 1. – *Let $M = S^3$ or $M = S_\rho^2 \times S^1$ endowed with the standard metrics. Suppose that $\alpha < 5$ and*

$$V(z) \geq -C(1 + |z|)^\beta, \quad \beta < 10/3.$$

Then there exists a space X continuously embedded in $C(\mathbb{R}; H^1(M))$ such that for every $u_0 \in H^1(M)$ there exists a unique global solution $u \in X$ of the Cauchy problem (1.1). Moreover

1. *For every $T > 0$, the map $u_0 \mapsto u \in C([-T, T]; H^1(M))$ is Lipschitz continuous on bounded sets of $H^1(M)$.*
2. *If $u_0 \in H^\sigma(M)$, $\sigma \geq 1$, then for every $t \in \mathbb{R}$, $u(t) \in H^\sigma(M)$.*

Let us make some comments about this result. The condition

$$V(z) \geq -C(1 + |z|)^\beta, \quad \beta < \frac{10}{3},$$

is classically (see e.g. Cazenave [10]) imposed to ensure that the energy controls the $H^1(M)$ norm (defocusing case).

The space X will be defined in Section 3 as a local version of Bourgain space $X^{1,b}$. It is used to ensure the uniqueness of solutions. However, observe that if $\sigma > 3/2$, then the uniqueness holds in the class $C(\mathbb{R}; H^\sigma(M))$. In particular, our theorem implies that for any smooth data u_0 , there exists a unique global smooth solution.

In the appendix of this paper, we show that Theorem 1 cannot hold for $\alpha > 5$. The proof is based on an adaptation of an argument of a recent paper of Christ, Colliander and Tao [11] to the setting of compact Riemannian manifolds. The critical case $\alpha = 5$ is still open.

Let us recall that the result of Theorem 1 was known if we replace M with the Euclidean space \mathbb{R}^3 (see Ginibre and Velo [13] and Kato [17]). To get the $H^1(\mathbb{R}^3)$ well-posedness of (1.1), for $\alpha < 5$, it is sufficient to apply the Picard iteration scheme to the Duhamel formulation of (1.1) in the space $L_T^2 W^{1,6}(\mathbb{R}^3) \cap L_T^\infty H^1(\mathbb{R}^3)$, where T depends only on $\|u_0\|_{H^1}$. The approach on \mathbb{R}^3 breaks down in the case of a compact manifold since the corresponding Strichartz type estimates have to encounter some unavoidable derivative losses (see [1,5,6]). In order to deal with such losses, bilinear improvements of the Strichartz inequalities are very useful (see e.g. [1,18,19,28,8]). This is the approach that we will adopt in the proof of Theorem 1 when $M = S^3$. The proof in the case $M = S_\rho^2 \times S^1$ is more intricate. The bilinear Strichartz estimates that we are able to prove in the case $M = S_\rho^2 \times S^1$ are considerably weaker compared to the corresponding estimates for $M = S^3$. However, they are sufficient to treat the case $\alpha \leq 4$. The crucial new point involved in the analysis on $S_\rho^2 \times S^1$ is that we can prove a trilinear improvement of the Strichartz estimate which enables one to treat the case $\alpha = 5$ for data in $H^s(S_\rho^2 \times S^1)$, $s > 1$. A suitable interpolation (in the framework of a Littlewood–Paley analysis) between the bilinear and the trilinear approach finally completes the argument in the case $M = S_\rho^2 \times S^1$.

The results of Ginibre and Velo [13] on \mathbb{R}^3 , of Bourgain [1] on \mathbb{T}^3 (and more recently on the irrational three dimensional torus [4]), and Theorem 1 were obtained for seemingly different reasons in each case. For the torus the eigenfunctions enjoy very good algebraic properties and L^p bounds whereas the spectrum is “badly” localized. On the other hand for the sphere S^3 , the eigenfunctions present “bad” concentration properties but the spectrum is very well localized, and the manifold $S^2 \times S^1$ has an intermediate behavior. The balance between these properties (concentration of eigenfunctions and repartition of the spectrum) leads to the suggestion that a similar result might hold for *any* three dimensional manifold. The proof of this conjecture would necessitate a general analysis of the Schrödinger group, unifying these different approaches, which seems to be out of reach at the present moment.

The H^1 theory for (1.1) in dimensions $d \geq 4$ remains an open problem. The only known result in this direction is that of Bourgain [2] who gets *global* $H^s(\mathbb{T}^4)$ solutions, if $\alpha \leq 2$, $s > 1$.

It seems that the obstructions to extending our approach to high dimensions are not only of technical nature since in [6] we have shown that for no $\alpha \in]1, 2]$ (even very close to 1), the Cauchy problem (1.1), posed on S^6 can have strong H^1 solutions in the sense explained above. Interestingly, the result of [6] is in strong contrast with the situation on \mathbb{R}^6 (see [7]).

We now turn to the crucial step in the proof of Theorem 1. To that purpose, we introduce the following notation: given $\nu \geq 1$, we set

$$\Lambda(d, \nu) := \begin{cases} \nu^{1/4} & \text{if } d = 2, \\ \nu^{1/2} \log^{1/2}(\nu) & \text{if } d = 3, \\ \nu^{(d-2)/2} & \text{if } d \geq 4. \end{cases}$$

With this notation, we have the following multilinear eigenfunction estimates.

THEOREM 2. – *There exists $C > 0$ such that, if H_p and H_q are two spherical harmonics of respective degrees p and q ,*

$$(1.5) \quad \|H_p H_q\|_{L^2(S^d)} \leq C \Lambda(d, \min(p, q) + 1) \|H_p\|_{L^2(S^d)} \|H_q\|_{L^2(S^d)}.$$

Moreover for any $p \geq q \geq r \geq 0$, the following trilinear estimates hold

$$(1.6) \quad \|H_p H_q H_r\|_{L^2(S^2)} \leq C [(1+q)(1+r)]^{\frac{1}{4}} \|H_p\|_{L^2(S^2)} \|H_q\|_{L^2(S^2)} \|H_r\|_{L^2(S^2)}.$$

Estimates (1.5) and (1.6) are sharp, apart from the logarithmic loss in (1.5) for $d = 3$.

Remark 1.1. – As an easy consequence of (1.5), one can prove the corresponding estimate to (1.6), for $d \geq 3$,

$$(1.7) \quad \|H_p H_q H_r\|_{L^2(S^d)} \leq C \Lambda(d, q+1) (1+r)^{\frac{d-1}{2}} \|H_p\|_{L^2(S^d)} \|H_q\|_{L^2(S^d)} \|H_r\|_{L^2(S^d)}.$$

Indeed it suffices to use that the $L^\infty(S^d)$ norm of H_r is bounded by $(1+r)^{(d-1)/2}$ (Weyl bound) and (1.5) for the product $H_p H_q$.

In view of further possible developments, we will also prove in Section 2 that for every $\eta \in]0, 1]$ there exists C_η such that

$$(1.8) \quad \|H_p H_q H_r\|_{L^2(S^3)} \leq C_\eta (1+q)^{\frac{1}{2}+\eta} (1+r)^{1-\eta} \|H_p\|_{L^2(S^3)} \|H_q\|_{L^2(S^3)} \|H_r\|_{L^2(S^3)}.$$

In fact, we deduce Theorem 2 as a consequence of a more general statement concerning the approximated spectral projectors $\chi(\sqrt{-\Delta} - \lambda)$, $\lambda \gg 1$, $\chi \in \mathcal{S}(\mathbb{R})$, where Δ is the Laplace operator on an arbitrary compact Riemannian manifold (M, g) (see Theorem 3 below).

Notice that when $p = q = r$, apart from the log loss in $3d$, we recover some particular case of the $L^p - L^2$ linear estimates of Sogge [22–24]. In the proof of Theorem 1, we typically apply Theorem 2 for $p \gg q$ and thus estimates (1.5), (1.6) are used in their full strength.

In the case $d = 2$, estimate (1.5) has already appeared in our previous paper [8]. In [8], the proof is inspired by Hörmander’s work [16] on Carleson–Sjölin type operators. The proof we present here is different even for $d = 2$ and relies on a “bilinearization” of the arguments in [22–24]. After several preliminaries, we reduce the matters to two micro-local linear estimates of quite a different nature. The first one is applied to the higher frequency eigenfunction and is in the spirit of the L^2 boundedness of spectral projectors. The second one is applied to the smaller frequency eigenfunctions and relies on a dispersive (curvature) effect. As far as the optimality of (1.5), (1.6) is concerned, we notice that it is achieved either by testing the estimates against eigenfunctions concentrating on an equator or by testing against zonal eigenfunctions concentrating on a point.

Let us mention that estimates (1.5), (1.6) and a sketch of the proof of (1.5) appeared in [9].

The rest of this paper is organized as follows. In Section 2 we prove Theorem 2. In Section 3 we set up the framework of Bourgain’s spaces and reduce the proof of Theorem 1 to obtaining nonlinear estimates in this framework. Section 4 consists in two parts. First we prove bilinear Strichartz estimates for the linear Schrödinger group on S^3 . Then we show that Theorem 1 holds for any three dimensional manifold on which these estimates are true. Section 5 also consists in two parts. First we prove trilinear Strichartz estimates for the linear Schrödinger group on the product manifold $S^2_\rho \times S^1$ and then we show that Theorem 1 holds for any three dimensional manifold on which these estimates are true. An appendix is devoted to the proof of the optimality of the quintic threshold.

2. Multilinear eigenfunction estimates

In this section we prove Theorem 2, and more generally the corresponding result for spectral projectors on arbitrary compact manifolds.

2.1. On the optimality of the estimates

We first consider the optimality of (1.5) in the case $d = 2, 3$. Let us see S^d as a hyper-surface in \mathbb{R}^{d+1} , i.e.

$$S^d = \{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} : x_1^2 + \dots + x_{d+1}^2 = 1\}.$$

Let us define the highest weight spherical harmonics $R_p = (x_1 + ix_2)^p$ which concentrate, for $p \gg 1$, on the closed geodesic (a big circle) $x_1^2 + x_2^2 = 1$. An easy computation shows that

$$\|R_p\|_{L^2(S^d)} \approx p^{-\frac{d-1}{4}}, \quad p \gg 1.$$

Clearly $R_p R_q = R_{p+q}$ and therefore there exist constants C, \tilde{C} such that for every (p, q) ,

$$\|R_p R_q\|_{L^2(S^d)} \geq C(p+q)^{-\frac{d-1}{4}} \geq \tilde{C}(\min(p, q))^{\frac{d-1}{4}} \|R_p\|_{L^2(S^d)} \|R_q\|_{L^2(S^d)}.$$

Therefore, for $d = 2, 3$, estimate (1.5) turns out to be optimal, modulo the logarithmic loss in $3d$. In the same way, since $R_p R_q R_r = R_{p+q+r}$, estimate (1.6) is optimal by testing it on R_p, R_q and R_r .

Let us now consider the case $d \geq 4$. In this case the optimality of (1.5) is given by the zonal spherical harmonics. Let us fix a pole on S^d . If we consider functions on S^d depending only on the geodesic distance to the fixed pole, we obtain the zonal functions on S^d . The zonal functions can be expressed in terms of zonal spherical harmonics which in their turn can be expressed in terms of the classical Jacobi polynomials (see e.g. [22]). Using asymptotics for the Jacobi polynomials (see [26], [22, Lemma 2.1]) we can obtain the following representation for the zonal spherical harmonics Z_p of degree p , in the coordinate θ ,

$$(2.1) \quad Z_p(\theta) = C(\sin \theta)^{-\frac{d-1}{2}} \left\{ \cos[(p + \alpha)\theta + \beta] + \frac{\mathcal{O}(1)}{p \sin \theta} \right\}, \quad \frac{c}{p} \leq \theta \leq \pi - \frac{c}{p},$$

where α and β are some fixed constants depending only on d . Moreover, we have a point-wise concentration

$$(2.2) \quad |Z_p(\theta)| \approx p^{\frac{d-1}{2}}, \quad \theta \notin [c/p, \pi - c/p],$$

and $\|Z_p\|_{L^2(S^d)} \approx 1$. Let $q \gg p$. Then

$$\|Z_p Z_q\|_{L^2(S^d)}^2 = \int_0^\pi Z_p^2(\theta) Z_q^2(\theta) (\sin \theta)^{d-1} d\theta \geq \int_{c/q}^{c/p} Z_p^2(\theta) Z_q^2(\theta) (\sin \theta)^{d-1} d\theta.$$

Using (2.2), we get

$$\|Z_p Z_q\|_{L^2(S^d)}^2 \geq C p^{d-1} \int_{c/q}^{c/p} Z_q^2(\theta) (\sin \theta)^{d-1} d\theta.$$

In view of (2.1),

$$\|Z_p Z_q\|_{L^2(S^d)}^2 \geq C p^{d-1} [I_1 - I_2],$$

where

$$I_1 = \int_{c/q}^{c/p} \cos^2[(q + \alpha)\theta + \beta] d\theta \geq \frac{C}{p} \quad \text{and} \quad I_2 = \frac{1}{q^2} \int_{c/q}^{c/p} \frac{1}{(\sin \theta)^2} d\theta \leq \frac{C}{q} \ll \frac{C}{p}.$$

Therefore

$$(2.3) \quad \|Z_p Z_q\|_{L^2(S^d)}^2 \geq C p^{d-2} \|Z_p\|_{L^2(S^d)}^2 \|Z_q\|_{L^2(S^d)}^2,$$

if $p \ll q$. Let finally $p \approx q$. Using (2.2), we get

$$(2.4) \quad \|Z_p Z_q\|_{L^2}^2 \geq C p^{2(d-1)} \int_0^{c/p} (\sin \theta)^{d-1} d\theta \geq \tilde{C} p^{2(d-1)} p^{-d} = \tilde{C} p^{d-2}.$$

Therefore, collecting (2.3) and (2.4), we obtain

$$\|Z_p Z_q\|_{L^2(S^d)} \geq C (\min(p, q))^{\frac{d-2}{2}} \|Z_p\|_{L^2(S^d)} \|Z_q\|_{L^2(S^d)}$$

which proves the optimality of (1.5), for $d \geq 3$, modulo the logarithmic loss in $3d$. Let us finally notice that similarly we can prove that for $p \geq q \geq r$

$$\|Z_p Z_q Z_r\|_{L^2(S^d)} \geq C q^{\frac{d-2}{2}} r^{\frac{d-1}{2}} \|Z_p\|_{L^2(S^d)} \|Z_q\|_{L^2(S^d)} \|Z_r\|_{L^2(S^d)}$$

which proves the optimality of (1.7), for $d \geq 3$, apart from the logarithmic loss in $3d$, and the optimality of (1.8) apart from the η shift.

2.2. A first reduction

Let (M, g) be a compact smooth Riemannian manifold without boundary of dimension d and Δ be the Laplace operator on functions on M . It turns out that estimates (1.5), (1.6) and (1.8) can be deduced from the following more general result.

THEOREM 3. – Let $\chi \in \mathcal{S}(\mathbb{R})$. For $\lambda \in \mathbb{R}$, denote by $\chi_\lambda = \chi(\sqrt{-\Delta} - \lambda)$ the approximated spectral projector around λ . There exists C such that for any $\lambda, \mu \geq 1$, $f, g \in L^2(M)$,

$$(2.5) \quad \|\chi_\lambda f \chi_\mu g\|_{L^2(M)} \leq C \Lambda(d, \min(\lambda, \mu)) \|f\|_{L^2(M)} \|g\|_{L^2(M)}.$$

Moreover, in the case $d = 2$, for any $1 \leq \lambda \leq \mu \leq \nu$, $f, g, h \in L^2(M)$, the following trilinear estimate holds

$$(2.6) \quad \|\chi_\lambda f \chi_\mu g \chi_\nu h\|_{L^2(M)} \leq C(\lambda\mu)^{\frac{1}{4}} \|f\|_{L^2(M)} \|g\|_{L^2(M)} \|h\|_{L^2(M)}.$$

Finally, in the case $d = 3$, for any $1 \leq \lambda \leq \mu \leq \nu$, $f, g, h \in L^2(M)$, $\eta \in]0, 1]$, the following trilinear estimate holds

$$(2.7) \quad \|\chi_\lambda f \chi_\mu g \chi_\nu h\|_{L^2(M)} \leq C_\eta \lambda^{1-\eta} \mu^{\frac{1}{2}+\eta} \|f\|_{L^2(M)} \|g\|_{L^2(M)} \|h\|_{L^2(M)}.$$

Remark 2.1. – If one is only interested in estimates for single eigenfunctions, the bounds provided by Theorem 3 seem to be relevant for “sphere like manifolds” but they are far from the optimal ones in the case of the torus. For example, the classical result of Zygmund [27] says that there exists a constant C such that for every couple (f, g) of eigenfunctions of the Laplace operator on the torus \mathbb{T}^2 , one has

$$\|fg\|_{L^2(\mathbb{T}^2)} \leq C \|f\|_{L^2(\mathbb{T}^2)} \|g\|_{L^2(\mathbb{T}^2)}.$$

We refer to Bourgain [3] for further extensions of Zygmund’s result.

A first reduction in the proof of Theorem 3 is that it suffices to prove it for one fixed nontrivial function χ .

LEMMA 2.2. – Suppose that the assertion of Theorem 3 holds for a bump function $\chi \in \mathcal{S}(\mathbb{R})$ which is not identically zero. Then it holds for any other choice of the bump function.

Proof. – Suppose that (2.5) holds for a nontrivial $\chi \in \mathcal{S}(\mathbb{R})$. Then, there exists $x_0 \in \mathbb{R}$ such that $\chi(x_0) \neq 0$ and moreover there exists $\delta > 0$ such that $\chi(x) \neq 0$ for $x \in \mathbb{R}$ satisfying $|x - x_0| < 2\delta$. Using a partition of unity argument, we can find $\psi \in C_0^\infty(\mathbb{R})$ supported in $\{x \in \mathbb{R} : |x| < 3\delta/4\}$ such that

$$(2.8) \quad \sum_{n \in \mathbb{Z}} \psi(x - n\delta) = 1.$$

Thanks to the support properties of ψ and χ , we can write

$$(2.9) \quad \psi(x - n\delta - \lambda) = \chi(x + x_0 - n\delta - \lambda) \frac{\psi(x - n\delta - \lambda)}{\chi(x + x_0 - n\delta - \lambda)}.$$

Notice that the second factor in the right-hand side of (2.9) is uniformly bounded. Therefore, using that (2.5) holds for χ , we obtain the estimate

$$(2.10) \quad \begin{aligned} & \|\psi(\sqrt{-\Delta} - n\delta - \lambda)(f) \psi(\sqrt{-\Delta} - m\delta - \mu)(g)\|_{L^2} \\ & \leq C \Lambda(d, \min(|n| + \lambda, |m| + \mu)) \|f\|_{L^2} \|g\|_{L^2}. \end{aligned}$$

Let us now take an arbitrary function $\chi_1 \in \mathcal{S}(\mathbb{R})$. Using (2.8), we can write

$$(2.11) \quad \chi_1(\sqrt{-\Delta} - \lambda)f = \sum_{n \in \mathbb{Z}} \psi(\sqrt{-\Delta} - n\delta - \lambda) \chi_1(\sqrt{-\Delta} - \lambda)f.$$

Let $\tilde{\psi} \in C_0^\infty(\mathbb{R})$ be equal to one on the support of ψ . Then clearly

$$(2.12) \quad |\chi_1(x - \lambda)\tilde{\psi}(x - \lambda - n\delta)| \leq \frac{C_N}{(1 + |x - \lambda|)^N(1 + |x - \lambda - n\delta|)^N} \leq \frac{\tilde{C}_N}{(1 + |n|)^N}.$$

Using the expansion (2.11) together with (2.10) and (2.12) yields

$$\begin{aligned} & \|\chi_1(\sqrt{-\Delta} - \lambda)(f)\chi_1(\sqrt{-\Delta} - \mu)(g)\|_{L^2} \\ & \leq \sum_{(n,m) \in \mathbb{Z}^2} \frac{C_N \Lambda(d, \min(|n| + \lambda, |m| + \mu))}{(1 + |n|)^N(1 + |m|)^N} \|f\|_{L^2} \|g\|_{L^2} \\ & \leq C \Lambda(d, \min(\lambda, \mu)) \|f\|_{L^2} \|g\|_{L^2}. \end{aligned}$$

Hence (2.5) holds for χ_1 . The proof of the independence of (2.6) and (2.7) with respect to the bump function χ is very similar and thus we will omit it. \square

2.3. Reduction to oscillatory integral estimates and main properties of the phase function

Following [24, Chapter 4], thanks to Lemma 2.2, it is sufficient to prove Theorem 3 with χ such that $\hat{\chi}(\tau)$ is supported in the set

$$\{\tau \in \mathbb{R}: \varepsilon \leq \tau \leq 2\varepsilon\},$$

where $\varepsilon > 0$ is a *small* number to be determined later. We can write

$$\chi_\lambda f = \frac{1}{2\pi} \int_{\varepsilon}^{2\varepsilon} e^{-i\lambda\tau} \hat{\chi}(\tau) (e^{i\tau\sqrt{-\Delta}} f) d\tau.$$

For $\varepsilon \ll 1$ and $|\tau| \leq 2\varepsilon$, using a partition of the unity on M , we can represent $e^{i\tau\sqrt{-\Delta}}$ as a Fourier integral operator (see e.g. [15]). Therefore χ_λ can also be represented as such. After a stationary phase argument (see [24, Chapter 5]) we can represent $\chi_\lambda f$ as follows.

LEMMA 2.3. – *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in]0, \varepsilon_0[$, every $N \geq 1$, we have the splitting*

$$(2.13) \quad \chi_\lambda f = \lambda^{\frac{d-1}{2}} T_\lambda f + R_\lambda f,$$

with

$$\|R_\lambda f\|_{H^k(M)} \leq C_{N,k} \lambda^{k-N} \|f\|_{L^2(M)}, \quad k = 0, \dots, N.$$

Moreover there exist $\delta > 0$ and, for every $x_0 \in M$, a system of coordinates $V \subset \mathbb{R}^d$, containing $0 \in \mathbb{R}^d$ such for $x \in V$, $|x| \leq \delta$,

$$T_\lambda f(x) = \int_{\mathbb{R}^d} e^{i\lambda\varphi(x,y)} a(x,y,\lambda) f(y) dy,$$

where $a(x,y,\lambda)$ is a polynomial in λ^{-1} with smooth coefficients supported in the set

$$\{(x,y) \in V \times V: |x| \leq \delta \ll \varepsilon/C \leq |y| \leq C\varepsilon\}$$

and $-\varphi(x,y) = d_g(x,y)$ is the geodesic distance between x and y .

Remark 2.4. – Let us notice that one can use $\chi(-\lambda^{-1}\Delta - \lambda)$ as approximated spectral projector instead of $\chi(\sqrt{-\Delta} - \lambda)$. In that case one should use semi-classical calculus for the approximation of $\exp(it\lambda^{-1}\Delta)$, $\lambda \gg 1$, as we did in [5].

In view of Lemma 2.3, to prove (2.5), it is enough to show

$$(2.14) \quad \|T_\lambda f T_\mu g\|_{L^2} \leq C\Lambda(d, \lambda)(\lambda\mu)^{-\frac{d-1}{2}} \|f\|_{L^2} \|g\|_{L^2},$$

uniformly for $1 \leq \lambda \leq \mu$. Indeed, using (2.13), one has to evaluate in L^2 the products

$$T_\lambda f R_\mu g, \quad R_\lambda f T_\mu g, \quad R_\lambda f R_\mu g.$$

The products involving R_μ are straightforward to estimate while for $R_\lambda f T_\mu g$, using the L^2 boundedness of χ_μ , we write

$$\|R_\lambda f T_\mu g\|_{L^2} \leq C \|R_\lambda f\|_{L^\infty} \|T_\mu g\|_{L^2} \leq C_N \lambda^{-N} \mu^{-\frac{d-1}{2}} \|f\|_{L^2} \|g\|_{L^2}.$$

Furthermore, we notice that once (2.14) is proved (at least for $d = 2$), to prove (2.6) it is enough to show that for $d = 2$,

$$\|T_\lambda f T_\mu g T_\nu h\|_{L^2} \leq C(\lambda\mu)^{-\frac{1}{4}} \nu^{-\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2},$$

uniformly for $1 \leq \lambda \leq \mu \leq \nu$. In this case there are more remainder terms to estimate. The most difficult one is $R_\lambda f T_\mu g T_\nu h$. This term can be evaluated, by using (2.14) for $d = 2$, as follows

$$\|R_\lambda f T_\mu g T_\nu h\|_{L^2} \leq \|R_\lambda f\|_{L^\infty} \|T_\mu g T_\nu h\|_{L^2} \leq C_N \lambda^{-N} \mu^{-\frac{1}{4}} \nu^{-\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}.$$

Similarly, to prove (2.7), it is enough to show that for $d = 3$,

$$\|T_\lambda f T_\mu g T_\nu h\|_{L^2} \leq C \lambda^{-\eta} \mu^{-\frac{1}{2} + \eta} \nu^{-1} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2},$$

uniformly for $1 \leq \lambda \leq \mu \leq \nu$. In this case, we estimate $R_\lambda f T_\mu g T_\nu h$, by another use of (2.14), as follows

$$\begin{aligned} \|R_\lambda f T_\mu g T_\nu h\|_{L^2} &\leq \|R_\lambda f\|_{L^\infty} \|T_\mu g T_\nu h\|_{L^2} \\ &\leq C_N \lambda^{-N} \log^{1/2}(\mu) \mu^{-1/2} \nu^{-1} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2} \\ &\leq C_{N,\eta} \lambda^{-N} \mu^{-\frac{1}{2} + \eta} \nu^{-1} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}, \end{aligned}$$

where $\eta > 0$.

Next, we represent y in geodesic (polar) coordinates as $y = \exp_0(r\omega)$, $r > 0$, $\omega \in S^{d-1}$. For $|x| \leq \delta$ and $\omega \in S^{d-1}$, we define the frozen phase φ_r ,

$$\varphi_r(x, \omega) = \varphi(x, \exp_0(r\omega)).$$

We now state the main property of the phase φ_r .

LEMMA 2.5. – *There exists $\varepsilon > 0$ such that for every $r \in [\varepsilon/C, C\varepsilon]$, every*

$$\omega = (\omega_1, \dots, \omega_d) \in S^{d-1} \subset \mathbb{R}^d,$$

we have the identity,

$$\nabla_x \varphi_r(0, \omega) = \omega.$$

Proof. – The proof for $d = 2$ is given in [8]. The extension to an arbitrary d is straightforward as we explain below. For $\varepsilon \ll 1$, let $y = \exp_0(r\omega)$, $r = -\varphi(0, y)$ and $u = u(x, y) \in T_y M$ be the unique unit vector in the tangent space to M at y such that

$$\exp_y(-\varphi(x, y)u(x, y)) = x.$$

Differentiating with respect to x this identity, we get for $x = 0$, and any $h \in T_0 M$,

$$(2.15) \quad h = -g_0(\nabla_x \varphi(0, y), h) T_{ru(0, y)}(\exp_y) \cdot u(0, y) \\ + T_{ru(0, y)}(\exp_y)(r T_x u(0, y) \cdot h),$$

where T denotes the tangent map.

On the other hand, we have

$$(2.16) \quad T_{ru(0, y)}(\exp_y) \cdot u(0, y) = -\omega \quad \text{or} \quad u(0, y) = -T_{r\omega}(\exp_0)(\omega).$$

Consequently, using Gauss' Lemma (see [12, 3.70]), we get

$$(2.17) \quad g_0(T_{ru(0, y)}(\exp_y)(r T_x u(0, y) \cdot h), \omega) = 0.$$

Let us now take the scalar product of (2.15) with ω . Collecting (2.15), (2.17) and (2.16) yields

$$g_0(\omega, h) = g_0(\nabla_x \varphi(0, y), h), \quad \forall h \in T_0 M$$

which completes the proof of Lemma 2.5. \square

Let us notice that there exists a smooth positive function $\kappa(r, \omega)$ such that $dy = \kappa(r, \omega) dr d\omega$. For $r \in [\varepsilon/C, C\varepsilon]$ and $\lambda \geq 1$, we define the operator T_λ^r , acting on functions on S^{d-1} via the identity

$$(T_\lambda^r f)(x) = \int_{S^{d-1}} e^{i\lambda\varphi_r(x, \omega)} a_r(x, \omega, \lambda) f(\omega) d\omega,$$

where $a_r(x, \omega, \lambda) = \kappa(r, \omega) a(x, \exp_0(r\omega), \lambda)$. Then clearly

$$(T_\lambda f)(x) = \int_0^\infty (T_\lambda^r f_r)(x) dr,$$

where $f_r(\omega) = f(r, \omega)$. Similarly, with $g_q(\omega) = g(q, \omega)$,

$$(T_\lambda f T_\mu g)(x) = \int_{\varepsilon/C}^{C\varepsilon} \int_{\varepsilon/C}^{C\varepsilon} (T_\lambda^r f_r)(x) (T_\mu^q g_q)(x) dr dq,$$

and the Minkowski inequality shows that (2.5) will be a consequence of

$$(2.18) \quad \|T_\lambda^r f T_\mu^q g\|_{L^2} \leq C \Lambda(d, \lambda) (\lambda \mu)^{-\frac{d-1}{2}} \|f\|_{L^2(S^{d-1})} \|g\|_{L^2(S^{d-1})},$$

uniformly for $1 \leq \lambda \leq \mu$ and $r, q \in [\varepsilon/C, C\varepsilon]$.

Similarly, to prove (2.6), it is enough to show

$$(2.19) \quad \|T_\lambda^r f T_\mu^q g T_\nu^s h\|_{L^2} \leq C(\lambda\mu)^{-\frac{1}{4}} \nu^{-\frac{1}{2}} \|f\|_{L^2(S^1)} \|g\|_{L^2(S^1)} \|h\|_{L^2(S^1)},$$

uniformly for $1 \leq \lambda \leq \mu \leq \nu$ and $r, q, s \in [\varepsilon/C, C\varepsilon]$.

Finally, to prove (2.7), it is enough to show

$$(2.20) \quad \|T_\lambda^r f T_\mu^q g T_\nu^s h\|_{L^2} \leq C\lambda^{-\eta} \mu^{-\frac{1}{2}+\eta} \nu^{-1} \|f\|_{L^2(S^2)} \|g\|_{L^2(S^2)} \|h\|_{L^2(S^2)},$$

uniformly for $1 \leq \lambda \leq \mu \leq \nu$ and $r, q, s \in [\varepsilon/C, C\varepsilon]$.

Fix a point $\underline{\omega} \in S^{d-1}$. The set

$$S_x = \{ \nabla_x \varphi_r(x, \omega), \omega \in S^{d-1}, \omega \sim \underline{\omega} \}$$

is a smooth hyper-surface in \mathbb{R}^d . Indeed assuming for instance $\underline{\omega} = (1, 0, \dots, 0)$, then $(w_1 = \omega_2, \dots, w_{d-1} = \omega_d)$ is a system of coordinates on S^{d-1} and according to Lemma 2.5, $\nabla_w \nabla_x \varphi_r$ has rank $d - 1$.

Following Stein [25] and Sogge [24], we now state the crucial curvature property.

LEMMA 2.6. – *The hyper-surface S_x has nonvanishing principal curvatures: for $w \in \mathbb{R}^{d-1}$ a local coordinate system near $\underline{\omega} \in S^{d-1}$, if we denote by $\pm n(x, w)$ the normal unit vectors to the surface S_x at the point $\nabla_x \varphi_r(x, w)$, then for x close to 0,*

$$(2.21) \quad \left| \det_{i,j} \left\langle \frac{\partial^2}{\partial w_j \partial w_i} \nabla_x \varphi_r(x, w), n(x, w) \right\rangle \right| \geq c > 0.$$

Proof. – The relation (2.21) is equivalent to the fact that

$$w \mapsto n(x, w) \in S^{d-1}$$

is a local diffeomorphism. Indeed, dropping the x variable for conciseness and denoting by

$$M(w) = \nabla_x \varphi_r(x, w), \quad n(w) = n(x, w),$$

we have

$$\left\langle \frac{\partial M}{\partial w_i}, n(w) \right\rangle = 0 \quad \Rightarrow \quad \left\langle \frac{\partial^2 M}{\partial w_i \partial w_j}, n \right\rangle = - \left\langle \frac{\partial M}{\partial w_i}, \frac{\partial n}{\partial w_j} \right\rangle.$$

As a consequence, the determinant in (2.21) is nonvanishing if and only if the system of vectors $\frac{\partial n}{\partial w_j}$ is of maximal rank in $T_w S_x$. We deduce that (2.21) is independent of the choice of coordinates w and it suffices to prove it for a particular choice of a coordinate system near $\underline{\omega}$. We can suppose that $\underline{\omega} = (1, 0, \dots, 0)$ and we choose as coordinates

$$w = (w_1, \dots, w_{d-1}) := (\omega_2, \dots, \omega_d).$$

We can also assume that at the point $(x = 0)$, the metric is diagonal, $g_{i,j} = \delta_{i,j}$. Using Lemma 2.5, we get

$$(2.22) \quad \left\langle \frac{\partial^2}{\partial w_j \partial w_i} \nabla_x \varphi_r(0, w), n(0, 0) \right\rangle \Big|_{w=0} = \text{Id}$$

and consequently (2.21) follows by continuity. \square

Denote by $(T_r^\nu)^*$ the formal adjoint of T_r^ν . The kernel of the operator $T_r^\nu(T_r^\nu)^*$, $K(x, x')$, is given by the relation

$$K(x, x') = \int e^{i\nu(\varphi_r(x,w) - \varphi_r(x',w))} a_r(x, w, \nu) \bar{a}_r(x', w, \nu) dw.$$

The curvature property of the phase φ_r in Lemma 2.6 implies a dispersion inequality for the kernel K .

LEMMA 2.7. – *There exists $C > 0$ such that for any $\nu \geq 1$,*

$$(2.23) \quad |K(x, x')| \leq \frac{C}{(1 + \nu|x - x'|)^{(d-1)/2}}.$$

Proof. – Let us write a Taylor expansion

$$\varphi_r(x, w) - \varphi_r(x', w) = \langle x - x', \psi(x, x', w) \rangle,$$

where

$$\psi(x, x', w) = \int_0^1 \nabla_x \varphi_r(x' + \theta(x - x'), w) d\theta.$$

With $\sigma = \frac{x-x'}{|x-x'|}$, we can write

$$\varphi_r(x, w) - \varphi_r(x', w) = |x - x'| \Phi(x, x', \sigma, w),$$

where

$$\Phi(x, x', \sigma, w) = \langle \sigma, \psi(x, x', w) \rangle.$$

Now we want to prove, with $\lambda = \nu|x - x'|$,

$$|\tilde{K}(x, x', \sigma)| \leq \frac{C}{(1 + \lambda)^{(d-1)/2}}$$

where

$$(2.24) \quad \tilde{K}(x, x', \sigma) = \int e^{i\lambda\Phi(x, x', \sigma, w)} a_r(x, w, \nu) \bar{a}_r(x', w, \nu) dw.$$

From the definition of the normal $n(x, w)$, we have $\nabla_w \Phi = 0$ for $x = x' = 0, w = 0, \sigma = \pm n(0, 0)$. According to the curvature property (2.21), we have $\det(\nabla_w^2 \Phi) \neq 0$ for $x = x' = 0, w = 0, \sigma = \pm n(0, 0)$. From the implicit function theorem, there exists $\kappa > 0$, such that if

$$(2.25) \quad |\sigma - n(0, 0)| \leq \kappa \quad \text{or} \quad |\sigma + n(0, 0)| \leq \kappa$$

then the phase $\Phi(x, x', \sigma, w)$ has a unique nondegenerate critical point $w(x, x', \sigma)$ and, by stationary phase, under the assumption (2.25), the kernel (2.24) is bounded by $C(1 + \lambda)^{-(d-1)/2}$. Let us next assume that

$$(2.26) \quad |\sigma - n(0, 0)| > \kappa \quad \text{and} \quad |\sigma + n(0, 0)| > \kappa.$$

Then for w close to 0 and $|x|$ small enough, we obtain by continuity

$$(2.27) \quad |\sigma - n(x, w)| > \kappa/2 \quad \text{and} \quad |\sigma + n(x, w)| > \kappa/2.$$

The kernel of $\nabla_x \nabla_w \varphi_r(x, w)$ is one dimensional and spanned by $n(x, w)$. Coming back to the definition of Φ , we deduce that (2.27) implies (for $|x'|$ small enough)

$$|\nabla_w \Phi(x, x', \sigma, w)| \geq c > 0.$$

Consequently, integrating by parts in (2.24), we obtain that under the assumption (2.26) the kernel (2.24) is bounded by $C_N(1 + \lambda)^{-N}$ which is even better than needed. This completes the proof of Lemma 2.7. \square

The second property of the phase we need is the following:

LEMMA 2.8. – *Let $x = (t, z) \in \mathbb{R} \times \mathbb{R}^{d-1}$ where $t = x_1$ and $z = (x_2, \dots, x_d)$. Then for every*

$$\underline{\omega} = (\underline{\omega}_1, \dots, \underline{\omega}_d) \in S^{d-1}$$

with $\underline{\omega}_1 \neq 0$ there exist a neighborhood $U \subset S^{d-1}$ of $\underline{\omega}$, $\varepsilon > 0$ and $\delta > 0$ such that, for $\varepsilon/C \leq r \leq C\varepsilon$ and $|x| < \delta$, the phase $\varphi_r(t, z, w)$, where $w \in \mathbb{R}^{d-1}$ is a local coordinate in U , is uniformly nondegenerate with respect to (z, w) . More precisely

$$(2.28) \quad \left| \det_{i,j} \left(\frac{\partial^2 \varphi_r(t, z, w)}{\partial z_j \partial w_i} \right) \right| \geq c > 0.$$

Proof. – Since (2.28) is independent of the choice of coordinates w , it suffices to prove it for a particular choice of a coordinate system near $\underline{\omega}$.

For $\omega = (\omega_1, \omega_2, \dots, \omega_d) \in S^{d-1}$ in a small neighborhood of $\underline{\omega}$, we choose w as

$$w = (w_1, \dots, w_{d-1}) := (\omega_2, \dots, \omega_d)$$

which is a coordinate system thanks to the assumption $\underline{\omega}_1 \neq 0$. We can also assume that at the point $(t = 0, z = 0)$, the metric is diagonal $g_{i,j} = \delta_{i,j}$. Using Lemma 2.5, we get

$$(2.29) \quad \det_{i,j} \left(\frac{\partial^2 \varphi_r(t, z, w)}{\partial z_j \partial w_i} \right) \Big|_{(t,z,w)=(0,0,w)} = 1.$$

We now obtain (2.28) from (2.29) by continuity. \square

We next state a corollary of Lemma 2.8.

LEMMA 2.9. – *Let $\underline{\omega}^{(1)}, \dots, \underline{\omega}^{(N)}$ be N points on S^{d-1} . Then there exists a splitting of the variable $x = (t, z) \in \mathbb{R} \times \mathbb{R}^{d-1}$ and neighborhoods $U_j \subset S^{d-1}$, $j = 1, \dots, N$, of $\underline{\omega}^{(j)}$ such that $\varphi_r(t, z, w)$ satisfies (2.28), where w is a coordinate in $\bigcup_{j=1}^N U_j$.*

Proof. – Obviously, there exists a unit vector e such that

$$e \cdot \underline{\omega}^{(j)} \neq 0, \quad j = 1, \dots, N.$$

By performing a rotation, we can assume that $e = (1, 0, \dots, 0)$ and consequently it suffices to apply Lemma 2.8. \square

2.4. Linear estimates

The dispersion inequality of Lemma 2.7 leads to the following estimate.

LEMMA 2.10. – *Let $(t, z) \in \mathbb{R} \times \mathbb{R}^{d-1}$ be any local system of coordinate near $(0, 0)$. Then the operator*

$$g \in L^2_w \longmapsto (T^r_\nu)g(t, z) \in L^2(\mathbb{R}_t; L^\infty(\mathbb{R}^{d-1}_z))$$

is continuous with norm bounded by $C\Lambda(d, \nu)\nu^{-(d-1)/2}$.

Proof. – Recall that

$$(T^r_\nu f)(t, z) = \int e^{i\nu\varphi_r(t, z, w)} a_r(t, z, w, \nu) f(w) dw.$$

Let consider the formal adjoint of T^r_ν defined as

$$(T^r_\nu)^*(g)(w) := \int e^{-i\nu\varphi_r(t', z', w)} \overline{a_r(t', z', w, \nu)} g(t', z') dt' dz'.$$

According to the classical duality argument which reduces the study of T^r_ν to the study of $T^r_\nu(T^r_\nu)^*$, it is sufficient to show that the norm of the operator

$$T^r_\nu(T^r_\nu)^* : L^2_t L^1_z \longmapsto L^2_t L^\infty_z$$

is bounded by $C[\Lambda(d, \nu)\nu^{-(d-1)/2}]^2$. But according to Lemma 2.7, the kernel of this operator satisfies (2.23) and as a consequence, there exists $C > 0$ such that for every $\nu \geq 1$,

$$(2.30) \quad |K(t, z, t', z')| \leq \frac{C}{(1 + \nu|t - t'|)^{(d-1)/2}}.$$

Using (2.30) and the Young inequality, we get

$$\|T^r_\nu(T^r_\nu)^*g\|_{L^2_t L^\infty_z} \leq C \int_{|s| \leq c} \frac{ds}{(1 + \nu|s|)^{(d-1)/2}} \|g\|_{L^2_t L^1_z}$$

But clearly

$$\int_{|s| \leq c} \frac{ds}{(1 + \nu|s|)^{(d-1)/2}} \leq \begin{cases} C\nu^{-1/2} & \text{if } d = 2, \\ C\nu^{-1} \log(\nu) & \text{if } d = 3, \\ C\nu^{-1} & \text{if } d \geq 4. \end{cases}$$

It remains to observe that the right-hand side of the above inequality is equal to

$$C[\Lambda(d, \nu)\nu^{-(d-1)/2}]^2$$

which completes the proof of Lemma 2.10. \square

In two space dimensions, we shall need the following extension of Lemma 2.10.

LEMMA 2.11. – Let $d = 2$ and $(t, z) \in \mathbb{R} \times \mathbb{R}^{d-1}$ be any local system of coordinate near $(0, 0)$. The operator

$$g \in L_w^2 \longmapsto (T_\nu^r)g(t, z) \in L^4(\mathbb{R}_t; L^\infty(\mathbb{R}_z))$$

is continuous with norm bounded by $C\nu^{-1/4}$.

Proof. – Similarly to the proof of Lemma 2.10, it is sufficient to show that the norm of the operator

$$T_\nu^r (T_\nu^r)^* : L_t^{4/3} L_z^1 \longmapsto L_t^4 L_z^\infty$$

is bounded by $C\nu^{-1/2}$. The kernel $K(t, z, t', z')$ of $T_\nu^r (T_\nu^r)^*$ satisfies the bound (2.30) with $d = 2$. From the Hardy–Littlewood inequality,

$$(2.31) \quad \left\| \int_{-\infty}^{\infty} \frac{f(t') dt'}{(1 + \nu|t - t'|)^{1/2}} \right\|_{L^4(\mathbb{R}_t)} \leq C\nu^{-1/2} \|f\|_{L^{4/3}(\mathbb{R})}.$$

Therefore

$$\|T_\nu^r (T_\nu^r)^* g\|_{L_t^4 L_z^\infty} \leq C\nu^{-1/2} \|g\|_{L_t^{4/3} L_z^1}$$

which completes the proof of Lemma 2.11. \square

In the proof of (2.20), we need the following extension of Lemma 2.10 for $d \geq 3$.

LEMMA 2.12. – Let $d \geq 3$, $p > 2$ and $(t, z) \in \mathbb{R} \times \mathbb{R}^{d-1}$ be any local system of coordinate near $(0, 0)$. The operator

$$g \in L_w^2 \longmapsto (T_\nu^r)g(t, z) \in L^p(\mathbb{R}_t; L^\infty(\mathbb{R}_z^{d-1}))$$

is continuous with norm bounded by $C\nu^{-1/p}$.

Proof. – Let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. It is sufficient to show that the norm of the operator

$$T_\nu^r (T_\nu^r)^* : L_t^{p'} L_z^1 \longmapsto L_t^p L_z^\infty$$

is bounded by $C\nu^{-2/p}$. Since for $p > 2$,

$$\left\| \frac{1}{(1 + \nu|t|)^{(d-1)/2}} \right\|_{L^{\frac{p}{2}}(\mathbb{R}_t)} \leq C\nu^{-2/p},$$

using the Young inequality, we get the bound

$$\left\| \int_{-\infty}^{\infty} \frac{f(t') dt'}{(1 + \nu|t - t'|)^{(d-1)/2}} \right\|_{L^p(\mathbb{R}_t)} \leq C\nu^{-2/p} \|f\|_{L^{p'}(\mathbb{R})}$$

which completes the proof of Lemma 2.12 thanks to the bound (2.30) on the kernel of $T_\nu^r (T_\nu^r)^*$. \square

Remark 2.13. – Notice that for $p = 2$, the proof above still works in dimensions $d \geq 4$. In the case $p = 2$, $d = 3$, we have the same difficulty as in the case of the end point Strichartz estimates on \mathbb{R}^2 (see Remark 2.15 below).

A consequence of Lemma 2.8 is the following statement.

LEMMA 2.14. – *Under the assumptions of Lemma 2.8, the operator*

$$g \in L^2_w \longmapsto (T^r_\nu g)(t, z) \in L^\infty(\mathbb{R}_t; L^2(\mathbb{R}_z^{d-1}))$$

is continuous with norm bounded by $C\nu^{-(d-1)/2}$.

Proof. – In view of (2.28), the statement of Lemma 2.14, which can be understood as a refinement of the L^2 boundedness of the spectral projector, is an immediate consequence of the following generalization of Plancherel’s identity, which we borrow from [16].

LEMMA (Nondegenerate phase lemma). – *Let us consider $\varphi \in C^\infty(\mathbb{R}_z^n \times \mathbb{R}_w^n)$ and $a \in C_0^\infty(\mathbb{R}_z^n \times \mathbb{R}_w^n)$ such that*

$$(2.32) \quad (z, w) \in \text{supp}(a) \quad \Rightarrow \quad \det \left[\frac{\partial^2 \varphi}{\partial z \partial w}(z, w) \right] \neq 0.$$

There exists $C > 0$ such that for every $\lambda \geq 1$, the operator T_λ

$$(2.33) \quad T_\lambda f(z) = \int_{\mathbb{R}^n} e^{i\lambda\varphi(z,w)} a(z, w) f(w) dw$$

satisfies,

$$\|T_\lambda f\|_{L^2(\mathbb{R}_z^n)} \leq C\lambda^{-n/2} \|f\|_{L^2(\mathbb{R}_w^n)}. \quad \square$$

2.5. Multilinear estimates

Let us first prove (2.18). We can write

$$(T^r_\lambda f T^q_\mu g)(x) = \int_{S^{d-1}} \int_{S^{d-1}} e^{i\lambda\varphi_r(x,\omega) + i\mu\varphi_q(x,\omega')} a_r(x, \omega, \lambda) a_q(x, \omega', \mu) f(\omega) g(\omega') d\omega d\omega'.$$

We need to evaluate the above expression in L^2_x . After a partition of unity, we can suppose that on the support of

$$a_r(x, \omega, \lambda) a_q(x, \omega', \mu),$$

(ω, ω') is close to a fixed point $(\underline{\omega}^{(1)}, \underline{\omega}^{(2)})$. We can therefore use the splitting $x = (t, z)$ of Lemma 2.9 with $N = 2$. Using Hölder’s inequality, Lemmas 2.14 and 2.10, we infer

$$\|T^r_\lambda f T^q_\mu g\|_{L^2_t L^2_z} \leq \|T^r_\lambda f\|_{L^2_t L^\infty_z} \|T^q_\mu g\|_{L^\infty_t L^2_z} \leq C\Lambda(d, \lambda) (\lambda\mu)^{-\frac{d-1}{2}} \|f\|_{L^2_w} \|g\|_{L^2_w}.$$

This completes the proof of (2.18).

We next prove (2.19). Let us write

$$\begin{aligned} (T^r_\lambda f T^q_\mu g T^s_\nu h)(x) &= \int_{S^1} \int_{S^1} \int_{S^1} e^{i\lambda\varphi_r(x,\omega) + i\mu\varphi_q(x,\omega') + i\nu\varphi_s(x,\omega'')} \\ &\quad \times a_r(x, \omega, \lambda) a_q(x, \omega', \mu) a_s(x, \omega'', \nu) f(\omega) g(\omega') h(\omega'') d\omega d\omega' d\omega''. \end{aligned}$$

After a partition of unity we can assume that $(\omega'', \omega', \omega)$ is close to a fixed point $(\underline{\omega}^{(1)}, \underline{\omega}^{(2)}, \underline{\omega}^{(3)})$. With the splitting $x = (t, z)$ of Lemma 2.9 with $N = 3$, using Lemmas 2.14, 2.11 and the Hölder inequality, we get

$$\begin{aligned} \|T_\lambda^r f T_\mu^q g T_\nu^s h\|_{L_t^2 L_z^2} &\leq \|T_\lambda^r f\|_{L_t^4 L_z^\infty} \|T_\mu^q g\|_{L_t^4 L_z^\infty} \|T_\nu^s h\|_{L_t^\infty L_z^2} \\ &\leq C \lambda^{-\frac{1}{4}} \mu^{-\frac{1}{4}} \nu^{-\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}. \end{aligned}$$

This completes the proof of (2.19).

We finally prove (2.20). We can again use the splitting $x = (t, z)$ of Lemma 2.9 with $N = 3$. For $p \gg 2$ and $q > 2$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, a use of Lemma 2.12 gives the bound

$$\begin{aligned} \|T_\lambda^r f T_\mu^q g T_\nu^s h\|_{L_t^2 L_z^2} &\leq \|T_\lambda^r f\|_{L_t^p L_z^\infty} \|T_\mu^q g\|_{L_t^q L_z^\infty} \|T_\nu^s h\|_{L_t^\infty L_z^2} \\ &\leq C \lambda^{-\eta} \mu^{-\frac{1}{2} + \eta} \nu^{-1} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}, \end{aligned}$$

where $\eta = \frac{1}{p}$. This ends the proof (2.20) and completes the proof of Theorem 2. \square

Remark 2.15. – As pointed to us by Koch and Tataru [20], another approach to these multilinear estimates would be, after a suitable micro-localization, to particularize one variable (t in the exposition above) and see the equation satisfied by the approximated spectral projector

$$(\Delta + \lambda^2)\chi_\lambda(u) = \mathcal{O}_{L^2}(\lambda)$$

as a *semi-classical* evolution equation of the type

$$(ih\partial_t + Q(t, z, hD_z))\chi_\lambda(u) = \mathcal{O}_{L^2}(h), \quad h = \lambda^{-1}.$$

Then Lemmas 2.10, 2.11 are simply the (semi-classical) Strichartz estimates which can be proved by using the approach in [5].

3. Preliminaries to the proof of Theorem 1

In this section (M, g) is an arbitrary Riemannian manifold of dimension d . Our first purpose is to introduce the basic localization operators Δ_N and Δ_{NL} which are naturally related to the Sobolev spaces and the Bourgain spaces on M respectively. We establish some basic bounds related to Δ_N and Δ_{NL} . The main purpose of this section is to show that the well-posedness of the Cauchy problem (1.1) in the Sobolev space $H^s(M)$ is a consequence of nonlinear estimates in the Bourgain spaces associated to the Laplace operator Δ . This reduction is now classical (see e.g. [14]).

3.1. Bourgain spaces and basic localization operators

Since M is compact, Δ has a compact resolvent and thus the spectrum of Δ is discrete. Let $e_k \in L^2(M)$, $k \in \mathbb{N}$, be an orthonormal basis of eigenfunctions of $-\Delta$ associated to eigenvalues λ_k . Denote by P_k the orthogonal projector on e_k . The Sobolev space $H^s(M)$ is therefore equipped with the norm (with $\langle x \rangle = (1 + |x|^2)^{1/2}$),

$$\|u\|_{H^s(M)}^2 = \sum_k \langle \lambda_k \rangle^s \|P_k u\|_{L^2(M)}^2.$$

The Bourgain space (or conormal Sobolev space) $X^{s,b}(\mathbb{R} \times M)$ is equipped with the norm

$$\|u\|_{X^{s,b}(\mathbb{R} \times M)}^2 = \sum_k \langle \lambda_k \rangle^s \|\langle \tau + \lambda_k \rangle^b \widehat{P_k u}(\tau)\|_{L^2(\mathbb{R}_\tau \times M)}^2 = \|e^{-it\Delta} u(t, \cdot)\|_{H^b(\mathbb{R}_t; H^s(M))}^2,$$

where $\widehat{P_k u}(\tau)$ denotes the Fourier transform of $P_k u$ with respect to the time variable.

Let us first recall that for $b > 1/2$ the space $X^{s,b}(\mathbb{R} \times M)$ is continuously embedded in $C(\mathbb{R}; H^s(M))$. For $u \in C_0^\infty(\mathbb{R} \times M)$, we write

$$P_k u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\langle \tau + \lambda_k \rangle^b}{\langle \tau + \lambda_k \rangle^b} \widehat{P_k u}(\tau) e^{it\tau} d\tau.$$

For $b > 1/2$, we get by the Cauchy–Schwarz inequality, applied in τ ,

$$(3.1) \quad \langle \lambda_k \rangle^{s/2} |P_k u(t)| \leq C \left\{ \int_{-\infty}^{\infty} \langle \lambda_k \rangle^s \langle \tau + \lambda_k \rangle^{2b} |\widehat{P_k u}(\tau)|^2 d\tau \right\}^{1/2}.$$

Squaring (3.1), integrating over M and summing over k yield,

$$(3.2) \quad \|u\|_{L^\infty(\mathbb{R}; H^s(M))} \leq C \|u\|_{X^{s,b}(\mathbb{R} \times M)}, \quad b > \frac{1}{2}.$$

For $u \in C^\infty(M)$ and $N \geq 1$, we define the projector Δ_N as

$$\Delta_N(u) := \sum_{k: N \leq \langle \lambda_k \rangle^{1/2} < 2N} P_k u.$$

We now state a basic bound for Δ_N .

LEMMA 3.1. – *There exists a constant C such that for every $q \in [2, \infty]$, every $u \in L^2(M)$,*

$$\|\Delta_N(u)\|_{L^q(M)} \leq CN^{\frac{d}{2} - \frac{d}{q}} \|\Delta_N(u)\|_{L^2(M)}.$$

Proof. – The assertion clearly holds for $q = 2$. We next prove it for $q = \infty$. Let us write $\Delta_N = \sum_{j=0}^{N-1} \Delta_{N,j}$, where

$$(3.3) \quad \Delta_{N,j} := \sum_{k: N+j \leq \langle \lambda_k \rangle^{1/2} < N+j+1} P_k.$$

Due to the Weyl estimate (see Lemma 2.3)

$$\|\Delta_{N,j}(u)\|_{L^\infty(M)} \leq CN^{\frac{d-1}{2}} \|\Delta_{N,j}(u)\|_{L^2(M)}$$

and due to the triangle and Cauchy–Schwarz inequalities

$$\begin{aligned} & \|\Delta_N(u)\|_{L^\infty(M)} \\ & \leq CN^{\frac{d-1}{2}} \left(\sum_{j=0}^{N-1} \|\Delta_{N,j}(u)\|_{L^2(M)}^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{N-1} 1^2 \right)^{\frac{1}{2}} = CN^{\frac{d}{2}} \|\Delta_N(u)\|_{L^2(M)}. \end{aligned}$$

By Hölder’s inequality, this completes the proof of Lemma 3.1. \square

For $N \geq 1$ a dyadic integer, i.e. $N = 2^n$, $n \in \mathbb{N}$, we define the operator S_N as

$$S_N := \sum_{N_1 \leq N} \Delta_{N_1},$$

where the sum is taken over all dyadic integers N_1 smaller or equal to N . We also define $S_{1/2}$ by $S_{1/2}(u) := 0$.

Next, for $u \in C_0^\infty(\mathbb{R} \times M)$ and N, L positive integers, we define the localization operators Δ_{NL} as

$$\Delta_{NL}(u) := \frac{1}{2\pi} \sum_{k: N \leq \langle \lambda_k \rangle^{1/2} < 2N} \int_{L \leq \langle \tau + \lambda_k \rangle \leq 2L} \widehat{P}_k u(\tau) e^{it\tau} d\tau.$$

It is easy to check that Δ_{NL} is a projector. It follows from this definition that for every s, b there exists $C > 0$ such that

$$(3.4) \quad \frac{1}{C} \|\Delta_{NL}(u)\|_{X^{s,b}(\mathbb{R} \times M)} \leq L^b N^s \|\Delta_{NL}(u)\|_{L^2(\mathbb{R} \times M)} \leq C \|\Delta_{NL}(u)\|_{X^{s,b}(\mathbb{R} \times M)},$$

and

$$(3.5) \quad \frac{1}{C} \sum_{N,L} L^{2b} N^{2s} \|\Delta_{NL}(u)\|_{L^2}^2 \leq \|u\|_{X^{s,b}}^2 \leq C \sum_{N,L} L^{2b} N^{2s} \|\Delta_{NL}(u)\|_{L^2}^2,$$

where the sums is taken over all dyadic values of N and L , i.e. $N = 2^n$, $L = 2^l$, $n, l \in \mathbb{N}$. We now state a basic bound for Δ_{NL} .

LEMMA 3.2. – *There exists a constant C such that for every $p, q \in [2, \infty]$, every $u \in L^2(\mathbb{R} \times M)$,*

$$\|\Delta_{NL}(u)\|_{L^p(\mathbb{R}; L^q(M))} \leq CL^{\frac{1}{2} - \frac{1}{p}} N^{\frac{d}{2} - \frac{d}{q}} \|\Delta_{NL}(u)\|_{L^2(\mathbb{R} \times M)}.$$

Proof. – Since $\Delta_N \Delta_{NL} = \Delta_{NL}$, a use of Lemma 3.1 yields

$$\|\Delta_{NL}(u)\|_{L^p(\mathbb{R}; L^q(M))} \leq CN^{\frac{d}{2} - \frac{d}{q}} \|\Delta_{NL}(u)\|_{L^p(\mathbb{R}; L^2(M))}.$$

Therefore, we only need to consider the case $q = 2$. Using that $\widehat{P}_k u(\tau) = P_k \widehat{u}(\tau)$, we can write

$$\|\Delta_{NL}(u)(t)\|_{L^2(M)}^2 \leq C \sum_{k: N \leq \langle \lambda_k \rangle^{1/2} < 2N} \int_M \left| \int_{L \leq \langle \tau + \lambda_k \rangle \leq 2L} \widehat{P}_k u(\tau) e^{it\tau} d\tau \right|^2.$$

Since the integration over τ is on a region of size L , using the Cauchy–Schwarz inequality in τ and the Plancherel identity yields

$$\|\Delta_{NL}(u)(t)\|_{L^2(M)}^2 \leq CL \sum_{k: N \leq \langle \lambda_k \rangle^{1/2} < 2N} \int_M \int_{-\infty}^{\infty} |\widehat{P}_k u(\tau)|^2 d\tau \leq CL \|u\|_{L^2(\mathbb{R} \times M)}^2.$$

Applying the last inequality to $\Delta_{NL}(u)$ instead of u and using that Δ_{NL} is a projector give

$$\|\Delta_{NL}(u)\|_{L^\infty(\mathbb{R}; L^2(M))} \leq CL^{\frac{1}{2}} \|\Delta_{NL}(u)\|_{L^2(\mathbb{R} \times M)}.$$

The assertion of the lemma trivially holds for $p = q = 2$ and therefore the proof of Lemma 3.2 is completed by Hölder's inequality. \square

3.2. Reduction to nonlinear estimates in Bourgain spaces

The starting point is to consider the integral equation (Duhamel form)

$$(3.6) \quad u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau.$$

At least for classical (smooth) solutions u the integral equation (3.6) is equivalent to (1.1). For that reason we solve (3.6) by the Picard iteration in a suitable functional setting and thus we get solutions of (3.6). Notice that this is achieved classically if $s > 3/2$ by taking $X = C([-T, T]; H^s(M))$. Therefore we shall restrict the study to the case $s \in [1, 3/2]$. In this case of low regularity solutions of (3.6) the information we have for u and $F(u)$ should be strong enough to conclude that we get solutions of (1.1) too.

For $T > 0$, we define the restriction space $X_T^{s,b} := X^{s,b}([-T, T] \times M)$, equipped with the norm

$$\|u\|_{X_T^{s,b}} = \inf \{ \|w\|_{X^{s,b}}, w \in X^{s,b} \text{ with } w|_{[-T,T]} = u \}.$$

For $b > 1/2$, the space $X_T^{s,b}$ is continuously embedded in $C([-T, T]; H^s(M))$ (see (3.2)) and $X_T^{s,b}$ will be the space for the solutions of (3.6) on $[-T, T]$. The next proposition contains the basic reduction to nonlinear estimates in $X^{s,b}$.

PROPOSITION 3.3. – *Assume that there exists $(b, b') \in \mathbb{R}^2$ satisfying*

$$(3.7) \quad 0 < b' < \frac{1}{2} < b, \quad b + b' < 1$$

such that for every $s \in [1, 2)$ there exists a constant C and p such that for every $u \in X^{s,b}$,

$$(3.8) \quad \|F(u)\|_{X^{s,-b'}(\mathbb{R} \times M)} \leq C(1 + \|u\|_{X^{1,b}(\mathbb{R} \times M)}^p) \|u\|_{X^{s,b}(\mathbb{R} \times M)},$$

and for every $u, v \in X^{s,b}$,

$$(3.9) \quad \|F(u) - F(v)\|_{X^{s,-b'}(\mathbb{R} \times M)} \leq C(1 + \|u\|_{X^{s,b}(\mathbb{R} \times M)}^p + \|v\|_{X^{s,b}(\mathbb{R} \times M)}^p) \|u - v\|_{X^{s,b}(\mathbb{R} \times M)}.$$

Then

1. *For every bounded subset B of $H^1(M)$ there exists $T > 0$ such that if $u_0 \in B$ then there exists a unique solution u of (3.6) on $[-T, T]$ in the class $X_T^{1,b}$. Moreover the map $u_0 \mapsto u \in C([-T, T]; H^1(M))$ is Lipschitz continuous on B .*
2. *If in addition $u_0 \in H^s(M)$ then $u \in C([-T, T]; H^s(M))$.*
3. *The function u is a solution of (1.1) in the distributional sense.*
4. *If in addition we suppose that $V(z) \geq -C(1 + |z|)^\beta$, $\beta < 2 + 4/d$, then the results above hold for any arbitrarily large T .*
5. *For $u_0 \in H^s(M)$, $s > 3/2$, the solution is unique in $C([-T, T]; H^s(M))$.*

Proof. – Let $\psi \in C_0^\infty(\mathbb{R})$ be equal to 1 on $[-1, 1]$. The identity

$$\|\psi(t)e^{it\Delta}u_0\|_{X^{s,b}(\mathbb{R}\times M)} = \|\psi\|_{H^b(\mathbb{R})}\|u_0\|_{H^s(M)}$$

follows from the definition of $X^{s,b}(\mathbb{R}\times M)$ and therefore for $T \leq 1$

$$(3.10) \quad \|e^{it\Delta}u_0\|_{X_T^{s,b}} \leq C\|u_0\|_{H^s(M)}.$$

The bound

$$(3.11) \quad \left\| \psi(t/T) \int_0^t e^{i(t-\tau)\Delta} F(\tau) d\tau \right\|_{X^{s,b}(\mathbb{R}_t \times M)} \leq CT^{1-b-b'} \|F\|_{X^{s,-b'}(\mathbb{R}\times M)},$$

holds for $T \leq 1$ and (b, b') satisfying (3.7). Indeed (see [8, Proposition 2.11]) estimate (3.11) follows from the *one dimensional* inequality

$$(3.12) \quad \left\| \psi(t/T) \int_0^t g(\tau) d\tau \right\|_{H^b(\mathbb{R})} \leq CT^{1-b-b'} \|g\|_{H^{-b'}(\mathbb{R})}.$$

A proof of (3.12) can be found in [14].

Using (3.11) and the assumptions of the proposition we obtain the estimates

$$(3.13) \quad \left\| \int_0^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau \right\|_{X_T^{s,b}} \leq CT^{1-b-b'} (1 + \|u\|_{X_T^{1,b}}^p) \|u\|_{X_T^{s,b}}$$

and

$$(3.14) \quad \left\| \int_0^t e^{i(t-\tau)\Delta} (F(u(\tau)) - F(v(\tau))) d\tau \right\|_{X_T^{s,b}} \leq CT^{1-b-b'} (1 + \|u\|_{X_T^{s,b}}^p + \|v\|_{X_T^{s,b}}^p) \|u - v\|_{X_T^{s,b}},$$

provided $T \leq 1$ and (b, b') satisfying (3.7). Let B be a bounded subset of $H^1(M)$. It results from (3.10), (3.13) and (3.14) with $s = 1$ that there exists $T \ll 1$ such that for every $u_0 \in B$ the right-hand side of (3.6) is a contraction in a suitable ball of $X_T^{1,b}$ with a unique fixed point which is the solution of (3.6). The uniqueness in the class $X_T^{1,b}$ and the Lipschitz continuity of the flow map follow from (3.14). Suppose now that $u_0 \in H^s(M)$. Then as before it follows from (3.10), (3.13) and (3.14) that we can find $\tilde{T} \leq T$ such that we can identify $u|_{[-\tilde{T}, \tilde{T}]}$ as the unique solution of (3.6) on $[-\tilde{T}, \tilde{T}]$ in the class $X_{\tilde{T}}^{s,b} \subset X_{\tilde{T}}^{1,b}$. In particular $u(t, \cdot) \in H^s(M)$ for $t \in [-\tilde{T}, \tilde{T}]$. Then by a bootstrap and the tame estimate (3.13) we conclude that $u(t, \cdot) \in H^s(M)$ for $t \in [-T, T]$. Thanks to (3.8), we obtain that $F(u(t)) \in X_T^{1,-b'}$ and since $b' < 1/2$, we infer that

$$\partial_t \left[\int_0^t e^{-i\tau\Delta} F(u(\tau)) d\tau \right] = e^{-it\Delta} F(u(t))$$

in the distributional sense which implies that u is a solution of the original PDE (1.1) in the distributional sense. If $u_0 \in H^2(M)$ then, thanks to the propagation of the H^s regularity assertion, one can take the scalar product of (1.1) with u and u_t and it results that the conservations laws (1.3) hold. If $u_0 \in H^1(M)$, we can approximate in $H^1(M)$ the function u_0 with a sequence $(u_{0,n})$ such that $u_{0,n} \in H^2(M)$. If we denote by $u_n(t)$ the corresponding solutions of (1.1), thanks to the propagation of the regularity we obtain then that $u_n(t)$ enjoys the conservation laws (1.3) for t on the time of existence of $u(t)$. Finally we can pass to the limit $n \rightarrow \infty$ and thanks to the H^1 continuity of the conservation laws functionals, we deduce that $u(t)$ satisfies the conservation laws (1.3). If we suppose that $V(z)$ satisfies $V(z) \geq -C(1+|z|)^\beta$, $\beta < 2 + 4/d$, using the Gagliardo–Nirenberg inequalities, we obtain that there exists $\theta \in]0, 2[$ such that

$$\int_M V(u(t)) \geq -C\|u(t)\|_{L^2}^{\beta-\theta} \|u(t)\|_{H^1}^\theta - C.$$

Therefore, the conservation laws (1.3) yield a bound independent with respect to t for $\|u(t)\|_{H^1(M)}$ which allows to reiterate the local existence argument and thus to achieve the existence of $u(t)$ on an arbitrary time interval. Finally, thanks to the Sobolev embedding $H^s(M) \subset L^\infty(M)$, $s > 3/2$, and the propagation of regularity, we easily obtain that if $u_0 \in H^s(M)$, $s > 3/2$, then the uniqueness holds in the class $C([-T, T]; H^s(M))$. This completes the proof of Proposition 3.3. \square

With Proposition 3.3 in hand the assertion of Theorem 1 follows from the following statement.

THEOREM 4. – *Let $M = S^3$ or $M = S_\rho^2 \times S^1$ endowed with the standard metrics. For every $1 < \alpha < 5$ and $s \in [1, 2)$ there exists $(b, b') \in \mathbb{R}^2$ satisfying (3.7) such that (3.8) and (3.9) hold.*

The next two sections are devoted to the proof of Theorem 4.

4. Bilinear Strichartz estimates and applications

In this section we prove Theorem 4 for $M = S^3$ with the standard metric.

4.1. Bilinear Strichartz estimates on S^3

In the case $M = S^3$ the eigenvalues of $-\Delta$ are $\lambda_k = k^2 - 1$, $k \geq 1$, and this fact plays a key role in the analysis. The starting point is the following bilinear improvement of the L^4 Strichartz inequality on S^3 established in [5].

PROPOSITION 4.1. – *For every interval $I \subset \mathbb{R}$, every $\varepsilon > 0$ there exists a constant C such that for every $N_1, N_2 \geq 1$, every $f_1, f_2 \in L^2(M)$,*

$$\left\| \prod_{j=1}^2 e^{it\Delta} (\Delta_{N_j} f_j) \right\|_{L^2(I \times M)} \leq C (\min(N_1, N_2))^{\frac{1}{2} + \varepsilon} \prod_{j=1}^2 \|\Delta_{N_j} f_j\|_{L^2(M)}.$$

Proof. – By a time translation we can suppose that $I = [0, T]$. Moreover, for $f \in L^2(M)$ the function $e^{it\Delta} f$ is periodic with respect to t and therefore it suffices to give the proof with $T = 2\pi$. Let us write

$$\prod_{j=1}^2 e^{it\Delta} (\Delta_{N_j} f_j) = \sum_{N_j \leq \langle \lambda_{k_j} \rangle^{1/2} < 2N_j} e^{-it(\lambda_{k_1} + \lambda_{k_2})} P_{k_1}(f_1) P_{k_2}(f_2).$$

Using the Parseval identity with respect to t we get

$$\left\| \prod_{j=1}^2 e^{it\Delta}(\Delta_{N_j} f_j) \right\|_{L^2([0,2\pi] \times M)}^2 = \sum_{\tau \in \mathbb{Z}} \left\| \sum_{\tau = \lambda_{k_1} + \lambda_{k_2}} P_{k_1}(f_1) P_{k_2}(f_2) \right\|_{L^2(M)}^2,$$

where the summation over (k_1, k_2) is restricted to $N_j \leq \langle \lambda_{k_j} \rangle^{1/2} < 2N_j, j = 1, 2$. Applying the triangle inequality for the $L^2(M)$ norm, the Cauchy–Schwarz inequality in the summation over (k_1, k_2) , and the bilinear estimate of Theorem 2 for $d = 3$ yields that for every $\varepsilon > 0$,

$$\begin{aligned} & \left\| \prod_{j=1}^2 e^{it\Delta}(\Delta_{N_j} f_j) \right\|_{L^2([0,2\pi] \times M)}^2 \\ & \leq C_\varepsilon (\min(N_1, N_2))^{1+\varepsilon} \sup_{\tau \in \mathbb{Z}} \alpha_{N_1, N_2}(\tau) \prod_{j=1}^2 \|\Delta_{N_j} f_j\|_{L^2(M)}^2, \end{aligned}$$

where

$$\alpha_{N_1, N_2}(\tau) = \#\{(k_1, k_2) \in \mathbb{N}^2: \tau + 2 = k_1^2 + k_2^2, N_j \leq \langle \lambda_{k_j} \rangle^{1/2} < 2N_j, j = 1, 2\}.$$

We claim that $\alpha_{N_1, N_2}(\tau) \leq C_\varepsilon N^\varepsilon$. Indeed this follows from the next lemma.

LEMMA 4.2. – *For every $\varepsilon > 0$ there exists $C > 0$ such that for every positive integers τ and N ,*

$$(4.1) \quad \#\{(k_1, k_2) \in \mathbb{N}^2: N \leq k_1 < 2N, k_1^2 + k_2^2 = \tau\} \leq CN^\varepsilon.$$

Proof. – This lemma already appeared in [8] (see [8, Lemma 3.2]). We recall the proof. For $\tau \leq 10N^4$ it follows from the divisor bound in the ring of Gaussian integers which is a Euclidean division domain. For $\tau \geq 10N^4$ there is at most one value of (k_1, k_2) satisfying the imposed restriction since in this case k_2 should range in an interval of size smaller than one. Hence for $\tau \geq 10N^4$ the left-hand side of (4.1) is bounded by 1. This completes the proof of Lemma 4.2. \square

Proposition 4.1 now readily follows from Lemma 4.2. \square

4.2. Using bilinear Strichartz estimates

From now on we simply assume that M is a three dimensional compact manifold satisfying Proposition 4.1. Proceeding as in [8, Section 3.2] one can show, for instance, that three dimensional Zoll manifolds have this property. As a consequence it can be remarked that in fact Theorem 1 holds for any such manifold.

First we deduce from Proposition 4.1 the following bilinear estimate in the $X^{s,b}$ context.

PROPOSITION 4.3. – *For every $\varepsilon > 0$ there exist $\beta < 1/2$ and $C > 0$ such that for every $N_1, N_2, L_1, L_2 \geq 1$, every $u_1, u_2 \in L^2(\mathbb{R} \times M)$,*

$$\left\| \prod_{j=1}^2 \Delta_{N_j L_j}(u_j) \right\|_{L^2(\mathbb{R} \times M)} \leq C(L_1 L_2)^\beta (\min(N_1, N_2))^{\frac{1}{2} + \varepsilon} \prod_{j=1}^2 \|\Delta_{N_j L_j}(u_j)\|_{L^2(\mathbb{R} \times M)}.$$

Proof. – Let us suppose that $N_1 \leq N_2$. Using Lemma 3.2 and the Hölder inequality we can write

$$\begin{aligned}
 (4.2) \quad \left\| \prod_{j=1}^2 \Delta_{N_j L_j}(u_j) \right\|_{L^2(\mathbb{R} \times M)} &\leq \left\| \Delta_{N_1 L_1}(u_1) \right\|_{L^4(\mathbb{R}; L^\infty(M))} \left\| \Delta_{N_2 L_2}(u_2) \right\|_{L^4(\mathbb{R}; L^2(M))} \\
 &\leq C N_1^{\frac{3}{2}} (L_1 L_2)^{\frac{1}{4}} \prod_{j=1}^2 \left\| \Delta_{N_j L_j}(u_j) \right\|_{L^2(\mathbb{R} \times M)}.
 \end{aligned}$$

Estimate (4.2) is better than the needed one with respect to the L_j localization but is far from the needed one with respect to the N_j localization.

We now estimate $\left\| \prod_{j=1}^2 \Delta_{N_j L_j}(u_j) \right\|_{L^2}$ by means of Proposition 4.1. It is indeed possible thanks to the following lemma.

LEMMA 4.4. – *For every $b \in]1/2, 1]$, every $\delta > 0$, there exists $C_{b,\delta}$ such that for every $u_1, u_2 \in X^{0,b}(\mathbb{R} \times M)$, every $1 \leq N_1 \leq N_2$,*

$$\left\| \prod_{j=1}^2 \Delta_{N_j}(u_j) \right\|_{L^2(\mathbb{R} \times M)} \leq C_{b,\delta} N_1^{\frac{1}{2} + \delta} \prod_{j=1}^2 \left\| \Delta_{N_j}(u_j) \right\|_{X^{0,b}(\mathbb{R} \times M)}.$$

Proof. – Let us set $v_j(t) := e^{it\Delta} \Delta_{N_j}(u_j)(t)$, $j = 1, 2$. Then we can write

$$\Delta_{N_j}(u_j)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\tau} e^{-it\Delta} \widehat{v}_j(\tau) d\tau.$$

Therefore

$$\prod_{j=1}^2 \Delta_{N_j}(u_j)(t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it(\tau_1 + \tau_2)} \left(\prod_{j=1}^2 e^{-it\Delta} \widehat{v}_j(\tau_j) \right) d\tau_1 d\tau_2.$$

Using the triangle inequality and Proposition 4.1 gives that for every unit interval $I \subset \mathbb{R}$, every $\delta > 0$ there exists C_δ such that

$$\left\| \prod_{j=1}^2 \Delta_{N_j}(u_j) \right\|_{L^2(I \times M)} \leq C_\delta N_1^{\frac{1}{2} + \delta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{j=1}^2 \left\| \widehat{v}_j(\tau_j) \right\|_{L^2(M)} d\tau_1 d\tau_2.$$

Hence using the Cauchy–Schwarz inequality in (τ_1, τ_2) gives for $b > 1/2$,

$$\begin{aligned}
 (4.3) \quad \left\| \prod_{j=1}^2 \Delta_{N_j}(u_j) \right\|_{L^2(I \times M)} &\leq C_{b,\delta} N_1^{\frac{1}{2} + \delta} \prod_{j=1}^2 \left\| \langle \tau \rangle^b \widehat{v}_j(\tau) \right\|_{L^2(\mathbb{R}_\tau \times M)} \\
 &= C_{b,\delta} N_1^{\frac{1}{2} + \delta} \prod_{j=1}^2 \left\| \Delta_{N_j}(u_j) \right\|_{X^{0,b}(\mathbb{R} \times M)}.
 \end{aligned}$$

Using a partition of unity, we can find a $\psi \in C_0^\infty(\mathbb{R})$, supported in $[0, 1]$ such that

$$(4.4) \quad \Delta_{N_j}(u_j)(t) = \sum_{n \in \mathbb{Z}} \psi\left(t - \frac{n}{2}\right) \Delta_{N_j}(u_j(t)) = \sum_{n \in \mathbb{Z}} \Delta_{N_j} \left(\psi\left(t - \frac{n}{2}\right) u_j(t) \right).$$

Notice that if for $u \in X^{0,b}(\mathbb{R} \times M)$, $b \in [0, 1]$, we set $u_n(t) = \psi(t - \frac{n}{2})u(t)$ then

$$(4.5) \quad \sum_{n \in \mathbb{Z}} \|u_n\|_{X^{0,b}(\mathbb{R} \times M)}^2 \leq C \|u\|_{X^{0,b}(\mathbb{R} \times M)}^2.$$

Indeed (4.5) is straightforward for $b = 0$ and $b = 1$ and it follows by complex interpolation for $b \in]0, 1[$. Using the almost disjointness of the supports of $\psi(t - \frac{n}{2})$, $n \in \mathbb{Z}$, the triangle inequality, estimates (4.3), (4.4) and (4.5) complete the proof of Lemma 4.4. \square

Next we apply Lemma 4.4 with $\Delta_{N_j L_j}(u_j)$, $j = 1, 2$, in the place of u_j and it follows from the definition of $X^{s,b}$ that, for any $b > 1/2$ and any $\delta > 0$,

$$(4.6) \quad \left\| \prod_{j=1}^2 \Delta_{N_j L_j}(u_j) \right\|_{L^2(\mathbb{R} \times M)} \leq C_{b,\delta} N_1^{\frac{1}{2} + \delta} (L_1 L_2)^b \prod_{j=1}^2 \|\Delta_{N_j L_j}(u_j)\|_{L^2(\mathbb{R} \times M)}.$$

It is now clear that the proof of Proposition 4.3 can be completed from (4.2) and (4.6) by a suitable Hölder inequality \square

Let us now turn to the proof of (3.8). Set $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$. Thanks to (1.2) and using that $F(0) = 0$, we obtain that the function

$$F(u) - (\partial F)(0)u - (\bar{\partial} F)(0)\bar{u}$$

is vanishing at least of order 3 at the origin. Therefore, in order to prove (3.8), it suffices to prove

$$(4.7) \quad \|F(u)\|_{X^{s,-b'}(\mathbb{R} \times M)} \leq C (\|u\|_{X^{1,b}(\mathbb{R} \times M)}^2 + \|u\|_{X^{1,b}(\mathbb{R} \times M)}^p) \|u\|_{X^{s,b}(\mathbb{R} \times M)}$$

assuming that $F(u)$ is vanishing to order 3 in zero. We can write

$$F(u) = \sum_{N_1} [F(S_{N_1}(u)) - F(S_{N_1/2}(u))],$$

where the sum is taken over all dyadic values of N_1 (recall that $S_{1/2}(u) = 0$). We have for $z, w \in \mathbb{C}$,

$$F(z) - F(w) = (z - w) \int_0^1 \partial F(tz + (1-t)w) dt + (\bar{z} - \bar{w}) \int_0^1 \bar{\partial} F(tz + (1-t)w) dt.$$

Therefore

$$\begin{aligned} & F(S_{N_1}(u)) - F(S_{N_1/2}(u)) \\ &= \Delta_{N_1}(u) G_1(\Delta_{N_1}(u), S_{N_1/2}(u)) + \overline{\Delta_{N_1}(u)} G_2(\Delta_{N_1}(u), S_{N_1/2}(u)), \end{aligned}$$

with

$$G_1(z_1, z_2) = \int_0^1 \partial F(tz_1 + z_2) dt, \quad G_2(z_1, z_2) = \int_0^1 \bar{\partial} F(tz_1 + z_2) dt.$$

We have thus the splitting $F(u) = F_1(u) + F_2(u)$, where

$$F_1(u) = \sum_{N_1} \Delta_{N_1}(u) G_1(\Delta_{N_1}(u), S_{N_1/2}(u)).$$

Thanks to the growth assumption on $V(z)$, we have the bound

$$|G_j(z_1, z_2)| \leq C(1 + |z_1| + |z_2|)^{\alpha-1}, \quad j = 1, 2.$$

We will provide a bound only for $F_1(u)$. The analysis for $F_2(u)$ is exactly the same.

We have for dyadic integers N_1, N_2

$$\Delta_{N_1} \Delta_{N_2} = \begin{cases} \Delta_{N_1} & \text{if } N_1 = N_2, \\ 0 & \text{otherwise} \end{cases}$$

and hence we can write that for dyadic integers N_1, N_2 ,

$$G_1(\Delta_{N_1} S_{N_2}(u), S_{N_1/2} S_{N_2}(u)) - G_1(\Delta_{N_1} S_{N_2/2}(u), S_{N_1/2} S_{N_2/2}(u))$$

is equal to

$$\begin{cases} G_1(0, S_{N_2}(u)) - G_1(0, S_{N_2/2}(u)) & \text{if } 2N_2 \leq N_1, \\ G_1(\Delta_{N_1}(u), S_{N_1/2}(u)) - G_1(0, S_{N_1/2}(u)) & \text{if } N_2 = N_1, \\ G_1(\Delta_{N_1}(u), S_{N_1/2}(u)) - G_1(\Delta_{N_1}(u), S_{N_1/2}(u)) = 0 & \text{if } N_2 \geq 2N_1. \end{cases}$$

Using the vanishing property of F at the origin allows us to write

$$\begin{aligned} G_1(\Delta_{N_1}(u), S_{N_1/2}(u)) &= \sum_{N_2: N_2 \leq N_1} \Delta_{N_2}(u) H_1^{N_2}(\Delta_{N_2}(u), S_{N_2/2}(u)) \\ &\quad + \sum_{N_2: N_2 \leq N_1} \overline{\Delta_{N_2}(u)} H_2^{N_2}(\Delta_{N_2}(u), S_{N_2/2}(u)), \end{aligned}$$

with

$$H_1^{N_2}(a, b) = \begin{cases} \int_0^1 \partial_2 G_1(0, ta + b) dt & \text{if } 2N_2 \leq N_1, \\ \int_0^1 \partial_1 G_1(ta, b) dt & \text{if } N_2 = N_1, \end{cases}$$

where (∂_1, ∂_2) are the derivatives of G_1 with respect to the first and the second arguments respectively. Moreover

$$H_2^{N_2}(a, b) = \begin{cases} \int_0^1 \overline{\partial_2} G_1(0, ta + b) dt & \text{if } 2N_2 \leq N_1, \\ \int_0^1 \overline{\partial_1} G_1(ta, b) dt & \text{if } N_2 = N_1. \end{cases}$$

Notice that

$$|H_j^{N_2}(a, b)| \leq C(1 + |a| + |b|)^{\max(\alpha-2, 0)}, \quad j = 1, 2.$$

We can write

$$F_1(u) = \sum_{N_2 \leq N_1} \Delta_{N_1}(u) \Delta_{N_2}(u) H_1^{N_2}(\Delta_{N_2}(u), S_{N_2/2}(u))$$

$$+ \sum_{N_2 \leq N_1} \Delta_{N_1}(u) \overline{\Delta_{N_2}(u)} H_2^{N_2}(\Delta_{N_2}(u), S_{N_2/2}(u)) := F_{11}(u) + F_{12}(u).$$

We will provide a bound only for $F_{11}(u)$. The analysis for $F_{12}(u)$ is exactly the same.

Similarly to the analysis for $G_1(\Delta_{N_1}(u), S_{N_1/2}(u))$, using once again the vanishing property of F at the origin, allows us to expand $H_1^{N_2}$ as follows

$$\begin{aligned} H_1^{N_2}(\Delta_{N_2}(u), S_{N_2/2}(u)) &= \sum_{N_3: N_3 \leq N_2} \Delta_{N_3}(u) H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)) \\ &\quad + \sum_{N_3: N_3 \leq N_2} \overline{\Delta_{N_3}(u)} H_{12}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)), \end{aligned}$$

where, due to the growth assumptions on V , $H_{1j}^{N_2, N_3}(a, b)$ satisfies

$$(4.8) \quad |\partial_{a,b}^\gamma H_{1j}^{N_2, N_3}(a, b)| \leq C(1 + |a| + |b|)^{\max(\alpha-3-|\gamma|, 0)}, \quad j = 1, 2.$$

Of course we can write explicit formulas for $H_{1j}^{N_2, N_3}(a, b)$ as we did for $H_1^{N_2}(a, b)$ but it will not be needed in the sequel. The only information for $H_{1j}^{N_2, N_3}(a, b)$ that we will use is the bound (4.8). Now, we can write

$$\begin{aligned} F_{11}(u) &= \sum_{N_3 \leq N_2 \leq N_1} \Delta_{N_1}(u) \Delta_{N_2}(u) \Delta_{N_3}(u) H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)) \\ &\quad + \sum_{N_3 \leq N_2 \leq N_1} \Delta_{N_1}(u) \Delta_{N_2}(u) \overline{\Delta_{N_3}(u)} H_{12}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)) \\ &:= F_{111}(u) + F_{112}(u). \end{aligned}$$

We will provide a bound only for $F_{111}(u)$. The analysis for $F_{112}(u)$ is exactly the same. Notice that

$$(4.9) \quad \Delta_N = \sum_L \Delta_{NL},$$

where the sum is taken over all dyadic values of L . For $w \in X^{-s, b'}(\mathbb{R} \times M)$, we set

$$I := \sum_{\substack{L_0, L_1, L_2, L_3, N_0 \\ N_3 \leq N_2 \leq N_1}} \int_{\mathbb{R} \times M} \Delta_{N_0 L_0}(w) \prod_{j=1}^3 \Delta_{N_j L_j}(u) H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)),$$

where the sum is taken over dyadic values of $N_j, L_j, j = 0, 1, 2, 3$. By duality, to prove (4.7) it suffices to establish the bound

$$|I| \leq C \|w\|_{X^{-s, b'}(\mathbb{R} \times M)} (\|u\|_{X^{1, b}(\mathbb{R} \times M)}^2 + \|u\|_{X^{1, b}(\mathbb{R} \times M)}^p) \|u\|_{X^{s, b}(\mathbb{R} \times M)}.$$

Set

$$I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} := \left| \int_{\mathbb{R} \times M} \Delta_{N_0 L_0}(w) \prod_{j=1}^3 \Delta_{N_j L_j}(u) H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)) \right|.$$

We split I as $|I| \leq I_1 + I_2$, where we define I_1 and I_2 to be the sums of the terms $I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3}$ associated to indexes such that $N_0 \leq \Lambda N_1$ and $N_0 > \Lambda N_1$ respectively, and $\Lambda > 1$ is a large constant to be determined later.

We first evaluate I_1 . Using Proposition 4.3, and the Hölder inequality, we get, that for every $\varepsilon > 0$ there exists $\beta < 1/2$ such that,

$$I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} \leq C_\varepsilon (N_2 N_3)^{\frac{1}{2} + \varepsilon} (L_0 L_1 L_2 L_3)^\beta \|H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u))\|_{L^\infty(\mathbb{R} \times M)} \\ \times \|\Delta_{N_0 L_0}(w)\|_{L^2(\mathbb{R} \times M)} \prod_{j=1}^3 \|\Delta_{N_j L_j}(u)\|_{L^2(\mathbb{R} \times M)}.$$

Thanks to (4.8) we can write

$$\|H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u))\|_{L^\infty} \leq C(1 + \|\Delta_{N_3}(u)\|_{L^\infty} + \|S_{N_3/2}(u)\|_{L^\infty})^{\max(\alpha-3, 0)}.$$

Using Lemma 3.2, (3.5) and the Cauchy–Schwarz inequality yields, for $b > 1/2$,

$$\|\Delta_{N_3}(u)\|_{L^\infty(\mathbb{R} \times M)} \leq \sum_L \|\Delta_{N_3 L}(u)\|_{L^\infty(\mathbb{R} \times M)} \\ \leq \sum_L C N_3^{\frac{3}{2}} L^{\frac{1}{2}} \|\Delta_{N_3 L}(u)\|_{L^2(\mathbb{R} \times M)} \\ \leq C N_3^{\frac{1}{2}} \sum_L L^{\frac{1}{2}-b} L^b N_3 \|\Delta_{N_3 L}(u)\|_{L^2(\mathbb{R} \times M)} \\ \leq C N_3^{\frac{1}{2}} \left(\sum_L L^{1-2b}\right)^{\frac{1}{2}} \left(\sum_L L^{2b} N_3^2 \|\Delta_{N_3 L}(u)\|_{L^2(\mathbb{R} \times M)}^2\right)^{\frac{1}{2}} \\ \leq C N_3^{\frac{1}{2}} \|u\|_{X^{1,b}}.$$

We next estimate $\|S_{N_3/2}(u)\|_{L^\infty}$.

$$\|S_{N_3/2}(u)\|_{L^\infty(\mathbb{R} \times M)} \\ \leq \sum_{N_4: N_4 \leq N_3/2} \|\Delta_{N_4}(u)\|_{L^\infty(\mathbb{R} \times M)} \leq C \sum_{N_4: N_4 \leq N_3/2} \sum_L N_4^{\frac{3}{2}} L^{\frac{1}{2}} \|\Delta_{N_4 L}(u)\|_{L^2(\mathbb{R} \times M)} \\ \leq C \left(\sum_{N_4: N_4 \leq N_3/2} \sum_L (N_4^{\frac{1}{2}})^2 L^{1-2b}\right)^{\frac{1}{2}} \left(\sum_{N_4: N_4 \leq N_3/2} \sum_L L^{2b} N_4^2 \|\Delta_{N_4 L}(u)\|_{L^2}^2\right)^{\frac{1}{2}} \\ \leq C N_3^{\frac{1}{2}} \|u\|_{X^{1,b}},$$

provided $b > 1/2$. Using the last two estimates, we obtain the bound

$$(4.10) \quad \|H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u))\|_{L^\infty(\mathbb{R} \times M)} \leq 1 + C(N_3^{\frac{1}{2}} \|u\|_{X^{1,b}})^{\max(\alpha-3, 0)}.$$

With (4.10) in hand, we estimate I_1 . Let us recall a discrete Schur lemma.

LEMMA 4.5. – *For every $\Lambda > 0$, every $s > 0$ there exists $C > 0$ such that if (c_{N_0}) and (d_{N_1}) are two sequences of nonnegative numbers indexed by the dyadic integers, then,*

$$\sum_{N_0 \leq \Lambda N_1} \frac{N_0^s}{N_1^s} c_{N_0} d_{N_1} \leq C \left(\sum_{N_0} c_{N_0}^2\right)^{\frac{1}{2}} \left(\sum_{N_1} d_{N_1}^2\right)^{\frac{1}{2}}.$$

Proof. – Let us set

$$K(N_0, N_1) := \mathbb{1}_{N_0 \leq \Lambda N_1} \frac{N_0^s}{N_1^s}.$$

Summing geometric series implies that there exists $C > 0$ such that

$$\sup_{N_0} \sum_{N_1} K(N_0, N_1) + \sup_{N_1} \sum_{N_0} K(N_0, N_1) \leq C.$$

Therefore the Schur lemma implies the boundedness on $l_{N_0}^2 \times l_{N_1}^2$ of the bilinear form with kernel $K(N_0, N_1)$. This completes the proof of Lemma 4.5. \square

In estimating I_1 , we first sum with respect to L_0, L_1, N_0, N_1 . Writing

$$1 = \frac{N_0^s}{N_1^s} N_0^{-s} N_1^s, \quad (L_0 L_1)^\beta = L_0^{b'} L_1^b L_0^{\beta-b'} L_1^{\beta-b},$$

using Lemma 4.5 and (4.10), after summing geometric series in L_0, L_1 , we can write for $b > 1/2$ and $1/2 > b' > \beta$,

$$\begin{aligned} I_1 &\leq C_\varepsilon \|u\|_{X^{s,b}(\mathbb{R} \times M)} \|w\|_{X^{-s,b'}(\mathbb{R} \times M)} \left(1 + \|u\|_{X^{1,b}(\mathbb{R} \times M)}^{\max(\alpha-3,0)}\right) \\ &\quad \times \sum_{L_2, L_3} \sum_{N_3 \leq N_2} (N_2 N_3)^{\frac{1}{2} + \varepsilon} (L_2 L_3)^\beta N_3^{\frac{\max(\alpha-3,0)}{2}} \prod_{j=2}^3 \|\Delta_{N_j L_j}(u)\|_{L^2(\mathbb{R} \times M)}. \end{aligned}$$

Since $\alpha < 5$ and $N_3 \leq N_2$, we have, choosing $\varepsilon > 0$ small enough,

$$(N_2 N_3)^{\frac{1}{2} + \varepsilon} N_3^{\frac{\max(\alpha-3,0)}{2}} \leq N_2 N_3 (N_2 N_3)^{-\varepsilon}.$$

Therefore, by summing geometric series in N_2, N_3, L_2, L_3 , we get the bound

$$I_1 \leq C \|w\|_{X^{-s,b'}(\mathbb{R} \times M)} \left(\|u\|_{X^{1,b}(\mathbb{R} \times M)}^2 + \|u\|_{X^{1,b}(\mathbb{R} \times M)}^{\alpha-1}\right) \|u\|_{X^{s,b}(\mathbb{R} \times M)}.$$

It remains to estimate I_2 . This is performed by using the following proposition and summing geometric series.

PROPOSITION 4.6. – *Let $s \in [1, 2)$. Then there exists $\Lambda > 0$, b, b' satisfying (3.7), $\gamma > 0$ and p, C such that for every $w \in X^{-s,b'}(\mathbb{R} \times M)$, $u \in X^{1,b}(\mathbb{R} \times M)$, if N_0, N_1, N_2, N_3 satisfy*

$$N_0 \geq \Lambda N_1, \quad N_3 \leq N_2 \leq N_1,$$

then

$$(4.11) \quad I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} \leq C (N_0 L_0 L_1 L_2 L_3)^{-\gamma} \|w\|_{X^{-s,b'}} \left(\|u\|_{X^{1,b}}^2 + \|u\|_{X^{1,b}}^p\right) \|u\|_{X^{s,b}}.$$

Proof. – To prove Proposition 4.6, we consider three regimes:

4.2.1. Case 1: $N_0^{1-\delta} \leq N_3$, $\delta > 0$, **small enough**

We use Proposition 4.3, the Hölder inequality and (4.10). For every $\varepsilon > 0$, there exists $\beta < 1/2$ such that,

$$I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} \leq C_\varepsilon (N_2 N_3)^{\frac{1}{2} + \varepsilon} (L_0 L_1 L_2 L_3)^\beta N_3^{\frac{\max(\alpha-3, 0)}{2}} \|\Delta_{N_0 L_0}(w)\|_{L^2(\mathbb{R} \times M)} \\ \times \prod_{j=1}^3 \|\Delta_{N_j L_j}(u)\|_{L^2(\mathbb{R} \times M)} (1 + \|u\|_{X^{1, b}(\mathbb{R} \times M)}^{\max(\alpha-3, 0)}).$$

Therefore

$$I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} \leq C_\varepsilon \frac{(L_0 L_1 L_2 L_3)^\beta}{L_0^{b'} (L_1 L_2 L_3)^b} \frac{N_0^s}{N_1^s (N_2 N_3)} (N_2 N_3)^{\frac{1}{2} + \varepsilon} N_3^{\frac{\max(\alpha-3, 0)}{2}} \|w\|_{X^{-s, b'}(\mathbb{R} \times M)} \\ \times (\|u\|_{X^{1, b}(\mathbb{R} \times M)}^2 + \|u\|_{X^{1, b}(\mathbb{R} \times M)}^{\alpha-1}) \|u\|_{X^{s, b}(\mathbb{R} \times M)}.$$

Since $\alpha < 5$, we observe that there exist $\varepsilon > 0$, $\delta > 0$ and $\gamma > 0$ such that

$$\frac{N_0^s}{N_1^s (N_2 N_3)} (N_2 N_3)^{\frac{1}{2} + \varepsilon} N_3^{\frac{\max(\alpha-3, 0)}{2}} \leq N_0^{-\gamma}.$$

The parameter $\varepsilon > 0$ being fixed, we choose β as imposed by Proposition 4.3. Finally we chose $(b, b') \in \mathbb{R}^2$ satisfying (3.7) such that $b' > \beta$.

4.2.2. Case 2: $N_3 \leq N_0^{1/2}$

We start with a rough bound for $H_{11}^{N_2, N_3}$. By a repetitive use of Leibniz rule, the Sobolev embeddings and (4.8), we obtain the following statement.

LEMMA 4.7. – *There exists $A > 0$ such that for every coordinate patch*

$$\kappa : U \subset \mathbb{R}^3 \longmapsto M,$$

every $\gamma \in \mathbb{N}^3$, there exists $C_\gamma > 0$ such that for every $u \in H^1(M)$,

$$\|\partial_x^\gamma H_{11}^{N_2, N_3}(\Delta_{N_3}(u(\kappa(x))), S_{N_3/2}(u(\kappa(x))))\|_{L^\infty(U)} \leq C_\gamma N_3^{\frac{3|\gamma|}{2} + A} (1 + \|u\|_{H^1(M)}^{|\gamma| + A}).$$

We next state a bound for products of eigenfunctions.

LEMMA 4.8. – *Let $2/3 > \delta > 0$. There exists $\Lambda > 0$ such that if*

$$(4.12) \quad N_0 \geq \Lambda N_1, \quad N_3 \leq N_2 \leq N_1, \quad N_3 \leq N_0^{\frac{2}{3} - \delta},$$

then for every $\gamma > 0$ there exists C and p such that for every $u, w \in L^2(M)$,

$$\left| \int_M H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)) P_{k_0} w \prod_{j=1}^3 P_{k_j} u \right| \leq \frac{C}{N_0^\gamma} \|u\|_{L^2}^3 \|w\|_{L^2} (1 + \|u\|_{H^1(M)}^p),$$

provided $\langle \lambda_{k_j} \rangle^{1/2} \in [N_j, 2N_j]$, $j = 0, 1, 2, 3$.

Proof. – A similar argument already appeared in Lemma 2.6 of our previous paper [8]. The new point here is the presence of $H_{11}^{N_2, N_3}$. Working in local coordinates, due to Lemma 2.3, we can substitute $P_{k_0} w$ with the oscillatory integral

$$(4.13) \quad \int e^{i\lambda_{k_0}^{1/2} \varphi(x, y_0)} a_0(x, y_0, \lambda_{k_0}^{1/2}) w(y_0) dy_0.$$

Indeed the remainder term can be estimated thanks to the Sobolev embeddings and Lemma 4.7. We consider three cases.

- Case 1. Suppose first that $N_1 \leq N_0^{1-\delta}$. Using Lemma 2.5, we integrate by parts in the variable x by means of the oscillating factor

$$e^{i\lambda_{k_0}^{1/2}\varphi(x,y_0)},$$

and after q integrations, we gain a factor N_0^{-q} . On the other hand, due to Lemma 4.7, the assumption $N_1 \leq N_0^{1-\delta}$ and the Sobolev inequality, we obtain that the derivation of the amplitude is causing at most a factor

$$\max(N_1^{q+A}, N_3^{3q/2+A}) \leq N_0^{q(1-\delta)+A}.$$

By taking $q \gg 1$, this completes the proof in the case $N_1 \leq N_0^{1-\delta}$.

- Case 2. Suppose next that $N_1 \geq N_0^{1-\delta}$ but $N_2 \leq N_0^{1-\delta}$. In this case we can substitute $P_{k_0}w$ with (4.13) and $P_{k_1}u$ with

$$\int e^{i\lambda_{k_1}^{1/2}\varphi(x,y_1)} a_1(x, y_1, \lambda_{k_1}^{1/2}) u(y_1) dy_1.$$

Indeed in the considered case the remainders in the approximation for $P_{k_0}w$ and $P_{k_1}u$ given by Lemma 2.3 are both $\mathcal{O}(N_0^{-\infty})$ as operators from L^2 to the Sobolev spaces. Thanks to Lemma 2.5, if we take $\Lambda \gg 1$, we can again integrate by parts in x with the slightly modified oscillatory factor

$$e^{i\lambda_{k_0}^{1/2}\Phi(x,y_0,y_1)},$$

where

$$\Phi(x, y_0, y_1) = \varphi(x, y_0) + \lambda_{k_0}^{-1/2} \lambda_{k_1}^{1/2} \varphi(x, y_1).$$

- Case 3. Suppose finally that $N_1 \geq N_0^{1-\delta}$ and $N_2 \geq N_0^{1-\delta}$. Then we can substitute $P_{k_0}w$, $P_{k_1}u$ and $P_{k_2}u$ with the corresponding oscillatory integrals and we can then argue as in case 2.

This completes the proof of Lemma 4.8. \square

Lemma 4.8 (with $\delta = 1/6$) is now used to prove Proposition 4.6 for space time functions in this regime.

Define $\Pi_{k,L}$ as follows

$$\Pi_{k,L}(u) := \frac{1}{2\pi} \int_{L \leq \langle \tau + \lambda_k \rangle \leq 2L} \widehat{P_k u}(\tau) e^{it\tau} d\tau.$$

Further we set

$$\Lambda(N_0, N_1, N_2, N_3) := \{(k_0, k_1, k_2, k_3) : N_j \leq \langle \lambda_{k_j} \rangle^{1/2} \leq 2N_j, j = 0, 1, 2, 3\}.$$

Since

$$\Delta_{NL} = \sum_{k: N \leq \langle \lambda_k \rangle^{1/2} < 2N} \Pi_{k,L}$$

we get the bound

$$I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} \leq \sum_{\Lambda(N_0, N_1, N_2, N_3)} \left| \int_{\mathbb{R} \times M} H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)) \Pi_{k_0, L_0}(w) \prod_{j=1}^3 \Pi_{k_j, L_j}(u) \right|.$$

Since $P_k \Pi_{k, L} = \Pi_{k, L}$, under the assumption (4.12) a use of Lemma 4.8 yields,

$$I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} \leq C_\gamma N_0^{-\gamma} \sup_{t \in \mathbb{R}} (1 + \|u(t)\|_{H^1(M)}^p) \times \sum_{\Lambda(N_0, N_1, N_2, N_3)} \int_{-\infty}^{\infty} \|\Pi_{k_0, L_0} w(t)\|_{L^2(M)} \prod_{j=1}^3 \|\Pi_{k_j, L_j} u(t)\|_{L^2(M)} dt.$$

For $b > 1/2$, a use of (3.2) and the Hölder inequality implies that

$$I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} \leq C_\gamma N_0^{-\gamma} (1 + \|u\|_{X^{1,b}(\mathbb{R} \times M)}^p) \times \sum_{\Lambda(N_0, N_1, N_2, N_3)} \|\Pi_{k_0, L_0} w\|_{L^2(\mathbb{R} \times M)} \|\Pi_{k_1, L_1} u\|_{L^2(\mathbb{R} \times M)} \times \prod_{j=2}^3 \|\Pi_{k_j, L_j} u\|_{L^\infty(\mathbb{R}; L^2(M))}.$$

Since $\Delta_{N_j L_j} \Pi_{k_j, L_j} = \Pi_{k_j, L_j}$, for k_j such that $N_j \leq \langle \lambda_{k_j} \rangle^{1/2} < 2N_j$, using (3.4), we get

$$(4.14) \quad \|\Pi_{k_0, L_0} w\|_{L^2(\mathbb{R} \times M)} \leq C N_0^s L_0^{-b'} \|w\|_{X^{-s, b'}(\mathbb{R} \times M)},$$

$$(4.15) \quad \|\Pi_{k_1, L_1} u\|_{L^2(\mathbb{R} \times M)} \leq C N_1^{-s} L_1^{-b} \|u\|_{X^{s, b}(\mathbb{R} \times M)} \leq C L_1^{-b} \|u\|_{X^{s, b}(\mathbb{R} \times M)},$$

and using Lemma 3.2, for $j = 2, 3$,

$$(4.16) \quad \|\Pi_{k_j, L_j} u\|_{L^\infty(\mathbb{R}; L^2(M))} \leq L_j^{1/2} \|\Pi_{k_j, L_j} u\|_{L^2(\mathbb{R} \times M)} \leq C L_j^{\frac{1}{2}-b} \|u\|_{X^{0, b}(\mathbb{R} \times M)} \leq C L_j^{\frac{1}{2}-b} \|u\|_{X^{1, b}(\mathbb{R} \times M)}.$$

Using a crude form of Weyl asymptotics, we get a bound

$$(4.17) \quad |\Lambda(N_0, N_1, N_2, N_3)| \leq C \left(\prod_{j=0}^3 N_j \right)^c.$$

Estimate (4.11) in this regime follows in view of (4.14), (4.15), (4.16) and (4.17).

4.2.3. Case 3: $N_0^{1/2} \leq N_3 \leq N_0^{1-\delta}$ where δ is the small number fixed in case 1

We shall denote by $\mathcal{O}(1)$ any quantity bounded by

$$(L_0 L_1 L_2 L_3)^{-\gamma} \|w\|_{X^{-s, b'}} (\|u\|_{X^{1, b}}^2 + \|u\|_{X^{1, b}}^p) \|u\|_{X^{s, b}}$$

for some $\gamma > 0, p \in \mathbb{N}$. Let

$$v = \Delta_{N_0 L_0}(w) \Delta_{N_1 L_1}(u) \Delta_{N_2 L_2}(u) \Delta_{N_3 L_3}(u).$$

LEMMA 4.9. – *There exists $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ such that*

$$\|(1 - \varphi(N_0^{-2} \Delta))v\|_{L^1(\mathbb{R} \times M)} \leq \mathcal{O}(1)N_0^{-k}$$

for any k .

Proof. – Indeed, working in local coordinates, according to [5, Proposition 2.1], there exists $\chi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ such that for any k

$$\left\| v - \prod_{j=0}^3 \chi(N_j^{-1} D)(u_j) \right\|_{L^1(\mathbb{R} \times M)} \leq \mathcal{O}(1)N_0^{-k}.$$

Therefore, modulo negligible terms, the Fourier spectrum of v lies in a ring of size N_0 and consequently, applying again [5, Proposition 2.1] in the other way round, this proves the lemma. \square

Next we take advantage of this spectral localization to perform integrations by parts: we have (for some function $\psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$)

$$\varphi(N_0^{-2} \Delta) = N_0^{-2} \Delta \circ \psi(N_0^{-2} \Delta)$$

and modulo negligible terms

$$I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} = N_0^{-2} \left| \int_{\mathbb{R} \times M} v \psi(N_0^{-2} \Delta) [\Delta(H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)))] dt dx \right|.$$

Applying Proposition 4.3 we obtain

$$I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} \leq \mathcal{O}(1)N_0^{s-2}N_1^{-s}N_2^{-\frac{1}{2}+\varepsilon}N_3^{-\frac{1}{2}+\varepsilon} \|\Delta(H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)))\|_{L^\infty}.$$

But, by Sobolev embedding, we have

$$\|\Delta(H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)))\|_{L^\infty} \leq CN_3^3(1 + \|u\|_{X^{1,b}}^2)$$

and thus, since $N_3 \leq N_2 \leq N_1$ and $N_3 \leq N_0^{1-\delta}$,

$$(4.18) \quad \begin{aligned} I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} &\leq \mathcal{O}(1)N_0^{s-2}N_1^{-s}N_2^{-\frac{1}{2}+\varepsilon}N_3^{\frac{5}{2}+\varepsilon} \\ &\leq \mathcal{O}(1)N_0^{(s-2)}N_3^{2-s+2\varepsilon} \leq \mathcal{O}(1)N_0^{(s-2)\delta+2\varepsilon} \end{aligned}$$

and we can choose $\varepsilon > 0$ small enough such that $(s - 2)\delta + 2\varepsilon < 0$.

This completes the proof of Proposition 4.6 (and thus of (3.8)) in the case of a three dimensional compact manifold satisfying Proposition 4.1. \square

Remark 4.10. – Let us notice that the estimate (3.8) holds for any sub-quintic nonlinearity, not necessarily satisfying the gauge condition (1.2). We used (1.2) in the reduction to F vanishing of order three at zero performed above because it simplifies a bit the analysis. More precisely for an arbitrary F in the expansions of $G_1(\Delta_{N_1}(u), S_{N_1/2}(u))$ and $H_1^{N_2}(\Delta_{N_2}(u), S_{N_2/2}(u))$ above one should add a constant. This would force one to analyze quadratic nonlinearities separately which can be done with our methods.

Thanks to the multilinear nature of our arguments, the proof of (3.9) is essentially the same as for (3.8). Indeed for suitable F_1, F_2 one writes

$$F(u) - F(v) = (u - v)F_1(u, v) + (\bar{u} - \bar{v})F_2(u, v).$$

Then we expand

$$u - v = \sum_{N_1} \Delta_{N_1}(u - v)$$

and for $j = 1, 2$,

$$F_j(u, v) = \sum_{N_2} [F_j(S_{N_2}(u), S_{N_2}(v)) - F_j(S_{N_2/2}(u), S_{N_2/2}(v))].$$

One then further expands the difference and after a duality argument the proof of (3.9) is reduced to a bound for a 4-linear expression multiplied with a factor similar to

$$H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u))$$

which appeared in the proof of (3.8). We omit the details.

5. Trilinear Strichartz estimates and applications

In this section we prove Theorem 4 for $M = S_\rho^2 \times S^1$ with the standard metric.

5.1. Trilinear Strichartz estimates on $M = S_\rho^2 \times S^1$

We do not know whether Proposition 4.1 holds in this case. Instead, we shall prove a trilinear Strichartz-type estimate. Let us first introduce some notation. As usual we identify S^1 with $\mathbb{R}/(2\pi\mathbb{Z})$. The eigenfunctions of Δ in the considered case are

$$\lambda_{m,n} = m^2 + \kappa(n^2 + n), \quad m \geq 0, n \geq 0, \kappa = \frac{1}{\rho^2}.$$

Let us denote by Π_n the spectral projector on spherical harmonics of degree $n \geq 0$ on S_ρ^2 . For $f(\omega, \theta) \in L^2(S_\rho^2 \times S^1)$, we set

$$\Theta_m f(\omega) := \frac{1}{2\pi} \int_0^{2\pi} f(\omega, \theta) e^{-im\theta} d\theta.$$

The crucial estimate is the following.

PROPOSITION 5.1. – *For every interval $I \subset \mathbb{R}$, every $\varepsilon > 0$ there exists a constant C such that for every $N_1 \geq N_2 \geq N_3 \geq 1$, every $f_1, f_2, f_3 \in L^2(M)$,*

$$\left\| \prod_{j=1}^3 e^{it\Delta}(\Delta_{N_j} f_j) \right\|_{L^2(I \times M)} \leq C N_3^{\frac{5}{3}} N_2^{\frac{3}{4} + \varepsilon} \prod_{j=1}^3 \|\Delta_{N_j} f_j\|_{L^2(M)}.$$

Proof. – By a time translation we can suppose that $I = [0, T]$. Since κ is not necessarily integer, we cannot employ the argument of Proposition 4.1 which reduces the analysis to the case $I = [0, 2\pi]$. We shall instead use the following lemma, already used in a similar context in [4].

LEMMA 5.2. – *Let Λ be a countable set of real numbers. Then for every $T > 0$ there exists C_T such that for every sequence (a_λ) indexed by Λ one has*

$$\left\| \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda t} \right\|_{L^2(0, T)} \leq C_T \left(\sum_{l \in \mathbb{Z}} \left(\sum_{\lambda: |\lambda-l| \leq 1/2} |a_\lambda| \right)^2 \right)^{\frac{1}{2}}.$$

Proof. – Let $\psi_T \in C_0^\infty(\mathbb{R})$ be such that $\psi_T = 1$ on the interval $[0, T]$. Set

$$f(t) := \sum_{\lambda \in \Lambda} \psi_T(t) a_\lambda e^{i\lambda t}.$$

Then

$$\widehat{f}(\tau) = \sum_{\lambda \in \Lambda} \widehat{\psi_T}(\tau - \lambda) a_\lambda$$

and the problem is to show that

$$\|\widehat{f}\|_{L^2(\mathbb{R})} \leq C_T \left(\sum_{l \in \mathbb{Z}} \left(\sum_{\lambda: |\lambda-l| \leq 1/2} |a_\lambda| \right)^2 \right)^{\frac{1}{2}}.$$

Next, we write

$$|\widehat{f}(\tau)| \leq \sum_{l \in \mathbb{Z}} \sum_{\lambda: |\lambda-l| \leq 1/2} |\widehat{\psi_T}(\tau - \lambda)| |a_\lambda| \leq \sum_{l \in \mathbb{Z}} K(l, \tau) h(l),$$

where

$$h(l) = \sum_{\lambda: |\lambda-l| \leq 1/2} |a_\lambda|, \quad K(l, \tau) = \sup_{\lambda: |\lambda-l| \leq 1/2} |\widehat{\psi_T}(\tau - \lambda)|.$$

It is clear that $|\lambda - l| \leq 1/2$ implies

$$\frac{1}{1 + |\tau - \lambda|} \leq \frac{C}{1 + |\tau - l|}$$

and therefore, using that $\psi_T \in C_0^\infty(\mathbb{R})$, we deduce that for every $N \in \mathbb{N}$ there exists $C_{T, N}$ such that

$$|K(l, \tau)| \leq \frac{C_{T, N}}{(1 + |\tau - l|)^N}.$$

A use of the Schur lemma completes the proof of Lemma 5.2. \square

We expand

$$\left(\prod_{j=1}^3 e^{it\Delta} (\Delta_{N_j} f_j) \right) (\omega, \theta)$$

$$= \sum e^{-i(\lambda_{m_1, n_1} + \lambda_{m_2, n_2} + \lambda_{m_3, n_3})t} e^{i(m_1 + m_2 + m_3)\theta} \prod_{j=1}^3 (\Pi_{n_j} \Theta_{m_j} f_j)(\omega),$$

where the sum is taken over (m_j, n_j) , $j = 1, 2, 3$, such that $N_j \leq \langle \lambda_{m_j, n_j} \rangle^{1/2} < 2N_j$. Using the Parseval identity with respect to θ and Lemma 5.2, we obtain

$$\begin{aligned} & \left\| \prod_{j=1}^3 e^{it\Delta}(\Delta_{N_j} f_j) \right\|_{L^2([0, T] \times M)}^2 \\ & \leq C_T \sum_{(l, \xi) \in \mathbb{Z}^2} \left\| \sum_{\substack{|l - \lambda_{m_1, n_1} - \lambda_{m_2, n_2} - \lambda_{m_3, n_3}| \leq 1/2 \\ \xi = m_1 + m_2 + m_3}} \prod_{j=1}^3 |\Pi_{n_j} \Theta_{m_j} f_j| \right\|_{L^2(S_\rho^2)}^2, \end{aligned}$$

where the summation over $(m_1, m_2, m_3, n_1, n_2, n_3)$ is restricted to (m_j, n_j) such that $N_j \leq \langle \lambda_{m_j, n_j} \rangle^{1/2} < 2N_j$, $j = 1, 2, 3$. Applying the triangle inequality for the $L^2(S_\rho^2)$ norm, the Cauchy–Schwarz inequality in the summation over $(m_1, m_2, m_3, n_1, n_2, n_3)$, and the trilinear estimate (1.6) of Theorem 2 yields

$$\begin{aligned} & \left\| \prod_{j=1}^3 e^{it\Delta}(\Delta_{N_j} f_j) \right\|_{L^2([0, T] \times M)}^2 \\ & \leq \sum_{(l, \xi) \in \mathbb{Z}^2} \sum_{\substack{|l - \lambda_{m_1, n_1} - \lambda_{m_2, n_2} - \lambda_{m_3, n_3}| \leq 1/2 \\ \xi = m_1 + m_2 + m_3}} |\Lambda(l, \xi)| (N_2 N_3)^{\frac{1}{2}} \prod_{j=1}^3 \|\Pi_{n_j} \Theta_{m_j} f_j\|_{L^2(S_\rho^2)}^2 \\ & \leq (N_2 N_3)^{\frac{1}{2}} \sup_{(l, \xi) \in \mathbb{Z}^2} |\Lambda(l, \xi)| \prod_{j=1}^3 \|f_j\|_{L^2(M)}^2, \end{aligned}$$

where

$$\begin{aligned} \Lambda(l, \xi) := & \left\{ (m_1, m_2, m_3, n_1, n_2, n_3) \in \mathbb{N}^6 : \left| l - \sum_{j=1}^3 (m_j^2 + \kappa(n_j^2 + n_j)) \right| \leq \frac{1}{2}, \right. \\ & \left. \xi = m_1 + m_2 + m_3, N_j \leq \langle \lambda_{m_j, n_j} \rangle^{1/2} < 2N_j, j = 1, 2, 3 \right\}. \end{aligned}$$

It remains to bound the size of $\Lambda(l, \xi)$. The number of possible (m_3, n_3) is bounded by CN_3^2 . The number of possible m_2 is bounded by CN_2 . Thus the number of possible (m_2, m_3, n_3) is bounded by $CN_2 N_3^2$. Let us now **fix** a possible triple (m_2, m_3, n_3) . Our goal is evaluate the number of possible (m_1, n_1, n_2) such that $(m_1, m_2, m_3, n_1, n_2, n_3) \in \Lambda(l, \xi)$. In view of the imposed restrictions, we can eliminate m_1 by concluding that (n_1, n_2) should satisfy

$$\left| l - (\xi - m_2 - m_3)^2 - m_2^2 - m_3^2 - \kappa[n_1^2 + n_2^2 + n_3^2 + n_1 + n_2 + n_3] \right| \leq \frac{1}{2}$$

or equivalently

$$(5.1) \quad \left| (2n_1 + 1)^2 + (2n_2 + 1)^2 - R \right| \leq \frac{2}{\kappa},$$

where

$$R = -4(n_3^2 + n_3) + 2 + \frac{4}{\kappa} [l - (\xi - m_2 - m_3)^2 - m_2^2 - m_3^2].$$

Using Lemma 4.2, uniformly with respect to R , the number of integer solutions $(n_1, n_2) \in [0, CN_1] \times [0, CN_2]$ of the inequality (5.1) is bounded by $C_\varepsilon N_2^\varepsilon$ which implies the estimate

$$|\Lambda(l, \xi)| \leq C_\varepsilon N_3^2 N_2^{1+\varepsilon}.$$

The proof of Proposition 5.1 is now completed. \square

5.2. Using trilinear Strichartz estimates

From now on we simply assume that M is a three dimensional compact manifold satisfying Proposition 5.1. Proceeding as in [8, Section 3.2] one can show, for instance, that the product of any Zoll surface with S^1 has this property. As a consequence it can be remarked that in fact Theorem 1 holds for any such manifold.

For our purpose in this section, we will first use the following weaker form of Proposition 4.1 which is a consequence of Proposition 5.1.

PROPOSITION 5.3. – *For every interval $I \subset \mathbb{R}$, every $\varepsilon > 0$ there exists a constant C such that for every $N_1, N_2 \geq 1$, every $f_1, f_2 \in L^2(M)$,*

$$\left\| \prod_{j=1}^2 e^{it\Delta}(\Delta_{N_j} f_j) \right\|_{L^2(I \times M)} \leq C(\min(N_1, N_2))^{\frac{3}{4}+\varepsilon} \prod_{j=1}^2 \|\Delta_{N_j} f_j\|_{L^2(M)}.$$

Proof. – It suffices to apply Proposition 5.1 with $f_3 = 1$. \square

Propositions 5.1 and 5.3 now imply the following statement.

PROPOSITION 5.4. – *For every $\varepsilon > 0$ there exist $\beta < 1/2$ and $C > 0$ such that for every $N_1 \geq N_2 \geq N_3 \geq 1$, $L_1, L_2, L_3 \geq 1$, every $u_1, u_2, u_3 \in L^2(\mathbb{R} \times M)$,*

$$(5.2) \quad \left\| \prod_{j=1}^2 \Delta_{N_j L_j}(u_j) \right\|_{L^2} \leq C(L_1 L_2)^\beta N_2^{\frac{3}{4}+\varepsilon} \prod_{j=1}^2 \|\Delta_{N_j L_j}(u_j)\|_{L^2}$$

and

$$(5.3) \quad \left\| \prod_{j=1}^3 \Delta_{N_j L_j}(u_j) \right\|_{L^2} \leq C(L_1 L_2 L_3)^\beta N_3^{\frac{5}{4}+\varepsilon} N_2^{\frac{3}{4}+\varepsilon} \prod_{j=1}^3 \|\Delta_{N_j L_j}(u_j)\|_{L^2}.$$

Proof. – One can show that Proposition 5.3 implies (5.2) exactly as we did in the proof of Proposition 4.3. The proof of (5.3) follows similar lines. First, using Lemma 3.2 and the Hölder inequality we get

$$(5.4) \quad \left\| \prod_{j=1}^3 \Delta_{N_j L_j}(u_j) \right\|_{L^2} \leq \|\Delta_{N_1 L_1}(u_1)\|_{L^6(\mathbb{R}; L^2(M))} \prod_{j=2}^3 \|\Delta_{N_j L_j}(u_j)\|_{L^6(\mathbb{R}; L^\infty(M))} \\ \leq C(N_2 N_3)^{\frac{3}{2}} (L_1 L_2 L_3)^{\frac{1}{3}} \prod_{j=1}^3 \|\Delta_{N_j L_j}(u_j)\|_{L^2}.$$

Next, exactly as in the proof of Proposition 4.3, we obtain that for every unit interval $I \subset \mathbb{R}$, every $b > 1/2$, every $\delta > 0$ there exists $C_{b,\delta}$ such that for every $N_1 \geq N_2 \geq N_3 \geq 1$, every

$u_1, u_2, u_3 \in X^{0,b}(\mathbb{R} \times M)$,

$$(5.5) \quad \left\| \prod_{j=1}^3 \Delta_{N_j}(u_j) \right\|_{L^2(I \times M)} \leq C_{b,\delta} N_2^{\frac{3}{4}+\delta} N_3^{\frac{5}{4}} \prod_{j=1}^3 \|\Delta_{N_j}(u_j)\|_{X^{0,b}}.$$

Using the partition of unity (4.4), we get the bound

$$(5.6) \quad \left\| \prod_{j=1}^3 \Delta_{N_j L_j}(u_j) \right\|_{L^2(\mathbb{R} \times M)} \leq C_{b,\delta} N_2^{\frac{3}{4}+\delta} N_3^{\frac{5}{4}} (L_1 L_2 L_3)^b \prod_{j=1}^3 \|\Delta_{N_j L_j}(u_j)\|_{L^2(\mathbb{R} \times M)}.$$

Finally, a suitable interpolation between (5.4) and (5.6) completes the proof of Proposition 5.4. \square

Let us now turn to the proof of Theorem 4 in the case $M = S_\rho^2 \times S^1$ (or more generally any manifold satisfying Proposition 5.1). We can again suppose that $F(u)$ is vanishing at least at order three at zero. We expand $F(u)$ as we did in Section 4 and are led to estimating terms of the form

$$I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} := \left| \int_{\mathbb{R} \times M} \Delta_{N_0 L_0}(w) \prod_{j=1}^3 \Delta_{N_j L_j}(u) H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)) \right|.$$

As in Section 4, we consider two cases and denote by I_1 the contribution corresponding to $N_0 \leq \Lambda N_1$, where $\Lambda \gg 1$ is a large constant and by I_2 the contribution corresponding to $N_0 > \Lambda N_1$. To study I_1 , we even make one more expansion of the terms $H_{1j}^{N_2, N_3}$ and it results that estimate (3.8) is a consequence of the bounds

$$(5.7) \quad J \leq C \|w\|_{X^{-s,b'}(\mathbb{R} \times M)} \|u\|_{X^{1,b}(\mathbb{R} \times M)} \|u\|_{X^{s,b}(\mathbb{R} \times M)}^2$$

and

$$(5.8) \quad I_1 \leq C \|w\|_{X^{-s,b'}(\mathbb{R} \times M)} (\|u\|_{X^{1,b}(\mathbb{R} \times M)}^3 + \|u\|_{X^{1,b}(\mathbb{R} \times M)}^{\alpha-1}) \|u\|_{X^{s,b}(\mathbb{R} \times M)},$$

$$(5.9) \quad I_2 \leq C \|w\|_{X^{-s,b'}(\mathbb{R} \times M)} (\|u\|_{X^{1,b}(\mathbb{R} \times M)}^2 + \|u\|_{X^{1,b}(\mathbb{R} \times M)}^p) \|u\|_{X^{s,b}(\mathbb{R} \times M)},$$

where

$$J = \sum_{L_0, L_1, L_2, L_3} \sum_{N_0} \sum_{N_3 \leq N_2 \leq N_1} \left| \int_{\mathbb{R} \times M} \Delta_{N_0 L_0}(w) \prod_{j=1}^3 \Delta_{N_j L_j}(u) \right|$$

and

$$I_1 = \sum_{L_0, L_1, L_2, L_3, L_4} \sum_{N_0 \leq \Lambda N_1} \sum_{N_4 \leq N_3 \leq N_2 \leq N_1} \left| \int_{\mathbb{R} \times M} \Delta_{N_0 L_0}(w) \left(\prod_{j=1}^4 \Delta_{N_j L_j}(u) \right) H^{N_2, N_3, N_4}(\Delta_{N_4}(u), S_{N_4/2}(u)) \right|,$$

$$I_2 = \sum_{L_0, L_1, L_2, L_3} \sum_{N_0 > \Lambda N_1} \sum_{N_3 \leq N_2 \leq N_4} \left| \int_{\mathbb{R} \times M} \Delta_{N_0 L_0}(w) \left(\prod_{j=1}^3 \Delta_{N_j L_j}(u) \right) H^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3/2}(u)) \right|,$$

with sums taken over the dyadic values of N_j and $L_j, j = 0, 1, 2, 3, 4$. Moreover $H^{N_2, N_3, N_4}(a, b)$ enjoys the bound

$$|H^{N_2, N_3, N_4}(a, b)| \leq C(1 + |a| + |b|)^{\max(\alpha-4, 0)}.$$

In addition for $4 < \alpha < 5$, we can further expand H^{N_2, N_3, N_4} and we can get the bound

$$(5.10) \quad |H^{N_2, N_3, N_4}(\Delta_{N_4}(u), S_{N_4/2}(u))|^{\frac{1}{\alpha-4}} \leq C \sum_{N_5: N_5 \leq N_4} |\Delta_{N_5}(u)|.$$

The proof of (5.7) is a consequence of the bilinear estimate (5.2). More precisely, using (5.2) and the Hölder inequality, we obtain that for every $\varepsilon > 0$ there exists $\beta < 1/2$ such that

$$\left| \int \Delta_{N_0 L_0}(w) \prod_{j=1}^3 \Delta_{N_j L_j}(u) \right| \leq C(N_2 N_3)^{\frac{3}{4} + \varepsilon} L_0^\beta \|\Delta_{N_0 L_0}(w)\|_{L^2} \prod_{j=1}^3 L_j^\beta \|\Delta_{N_j L_j}(u)\|_{L^2}.$$

Since for $\varepsilon < 1/4$ we have $3/4 + \varepsilon < 1$, we can complete the proof of (5.7) as we did in Section 4. A similar argument (using both (5.2) and (5.3)) is valid for (5.8), if $\alpha \leq 4$.

To prove (5.8) if $4 < \alpha < 5$, we use Proposition 5.4 in its full strength. Set

$$I_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4} := \left| \int_{\mathbb{R} \times M} \Delta_{N_0 L_0}(w) \left(\prod_{j=1}^4 \Delta_{N_j L_j}(u) \right) H^{N_2, N_3, N_4}(\Delta_{N_4}(u), S_{N_4/2}(u)) \right|.$$

In order to estimate $I_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4}$, we use the following form of Hölder's inequality:

$$(5.11) \quad \forall \gamma \in]0, 1[, \quad \left| \int_{\mathbb{R} \times M} fg \right| \leq \left(\int_{\mathbb{R} \times M} |f| \right)^{1-\gamma} \left(\int_{\mathbb{R} \times M} |f||g|^{\frac{1}{\gamma}} \right)^\gamma.$$

Since $4 < \alpha < 5, \gamma = \alpha - 4 \in]0, 1[$. Using (5.11), we can write,

$$(5.12) \quad I_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4} \leq [J_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4}]^{1-\gamma} [K_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4}]^\gamma,$$

where

$$J_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4} = \int_{\mathbb{R} \times M} \left| \Delta_{N_0 L_0}(w) \left(\prod_{j=1}^4 \Delta_{N_j L_j}(u) \right) \right|.$$

Thanks to (5.10), the second factor $K_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4}$ in (5.12) enjoys the bound

$$K_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4} \leq C \sum_{N_5: N_5 \leq N_4} \int_{\mathbb{R} \times M} \left| \Delta_{N_0 L_0}(w) \left(\prod_{j=1}^4 \Delta_{N_j L_j}(u) \right) \right| |\Delta_{N_5}(u)|.$$

Let us now bound $J_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4}$. Using Hölder’s inequality and Proposition 5.4 (both (5.2) and (5.3)), we obtain that for every $\varepsilon > 0$ there exists $\beta < 1/2$ such that

$$J_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4} \leq C(N_2 N_3)^{\frac{3}{4}+\varepsilon} N_4^{\frac{5}{4}+\varepsilon} L_0^\beta \|\Delta_{N_0 L_0}(w)\|_{L^2} \prod_{j=1}^4 L_j^\beta \|\Delta_{N_j L_j}(u)\|_{L^2}.$$

Next we estimate $K_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4}$. By writing $\Delta_{N_5} = \sum_{L_5} \Delta_{N_5 L_5}$, using Hölder’s inequality and Proposition 5.4 (twice (5.3)), we obtain that for every $\varepsilon > 0$ there exists $\beta < 1/2$ such that $K_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4}$ is bounded by

$$C \sum_{N_5: N_5 \leq N_4} \sum_{L_5} (N_2 N_3)^{\frac{3}{4}+\varepsilon} (N_4 N_5)^{\frac{5}{4}+\varepsilon} L_0^\beta \|\Delta_{N_0 L_0}(w)\|_{L^2} \prod_{j=1}^5 L_j^\beta \|\Delta_{N_j L_j}(u)\|_{L^2}.$$

Writing $N_5^{\frac{5}{4}+\varepsilon} = N_5 N_5^{\frac{1}{4}+\varepsilon}$, using the Cauchy–Schwarz inequality, we get for $b > \beta$,

$$\begin{aligned} & \sum_{N_5: N_5 \leq N_4} \sum_{L_5} N_5^{\frac{5}{4}+\varepsilon} L_5^\beta \|\Delta_{N_5 L_5}(u)\|_{L^2} \\ & \leq \left(\sum_{N_5: N_5 \leq N_4} \sum_{L_5} [L_5^{\beta-b} N_5^{\frac{1}{4}+\varepsilon}]^2 \right)^{\frac{1}{2}} \|u\|_{X^{1,b}} \leq C N_4^{\frac{1}{4}+\varepsilon} \|u\|_{X^{1,b}}. \end{aligned}$$

Therefore, we have the estimate,

$$\begin{aligned} K_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4} & \leq C(N_2 N_3)^{\frac{3}{4}+\varepsilon} N_4^{\frac{5}{4}+\varepsilon} L_0^\beta \|\Delta_{N_0 L_0}(w)\|_{L^2} \\ & \quad \times \left(\prod_{j=1}^4 L_j^\beta \|\Delta_{N_j L_j}(u)\|_{L^2} \right) N_4^{\frac{1}{4}+\varepsilon} \|u\|_{X^{1,b}}. \end{aligned}$$

Coming back to (5.12), we obtain the following estimate

$$\begin{aligned} (5.13) \quad I_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4} & \leq C \frac{N_0^s (N_2 N_3)^{\frac{3}{4}+\varepsilon} N_4^{\frac{5}{4}+\varepsilon} N_4^{\gamma(\frac{1}{4}+\varepsilon)}}{N_1^s N_2 N_3 N_4} L_0^{\beta-b'} (L_1 L_2 L_3 L_4)^{\beta-b} \\ & \quad \times (N_0^{-s} L_0^{b'} \|\Delta_{N_0 L_0}(w)\|_{L^2}) (N_1^s L_1^b \|\Delta_{N_1 L_1}(u)\|_{L^2}) \\ & \quad \times \left(\prod_{j=2}^4 N_j L_j^b \|\Delta_{N_j L_j}(u)\|_{L^2} \right) \|u\|_{X^{1,b}}^\gamma. \end{aligned}$$

Let us take $\varepsilon > 0$ such that $(\frac{3}{2} + 2\varepsilon) + (\frac{5}{4} + \varepsilon) + \gamma(\frac{1}{4} + \varepsilon) < 3$ or equivalently,

$$0 < \varepsilon < \frac{1 - \gamma}{4(3 + \gamma)} = \frac{5 - \alpha}{4(\alpha - 1)}.$$

Note that a proper choice of ε is possible thanks to the subcritical assumption $\alpha < 5$. Therefore there exists $\theta > 0$ such that for $N_4 \leq N_3 \leq N_2$,

$$(5.14) \quad \frac{(N_2 N_3)^{\frac{3}{4}+\varepsilon} N_4^{\frac{5}{4}+\varepsilon} N_4^{\gamma(\frac{1}{4}+\varepsilon)}}{N_2 N_3 N_4} \leq \frac{C}{(N_2 N_3 N_4)^\theta}.$$

Thanks to (5.14) and (5.13), we obtain

$$(5.15) \quad I_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4} \leq C \frac{N_0^s}{N_1^s} \frac{1}{(N_2 N_3 N_4)^\theta} L_0^{\beta-b'} (L_1 L_2 L_3 L_4)^{\beta-b} \\ \times (N_0^{-s} L_0^{b'} \|\Delta_{N_0 L_0}(w)\|_{L_2}) (N_1^s L_1^b \|\Delta_{N_1 L_1}(u)\|_{L_2}) \\ \times \left(\prod_{j=2}^4 N_j L_j^b \|\Delta_{N_j L_j}(u)\|_{L_2} \right) \|u\|_{X^{1,b}}^\gamma.$$

Estimate (5.8) follows by summing geometric series in $L_0, L_1, L_2, L_3, L_4, N_2, N_3, N_4$ while the sum over (N_0, N_1) is performed by invoking Lemma 4.5.

We now turn to the proof of (5.9). As in Section 4, we shall denote by $\mathcal{O}(1)$ any quantity bounded (for some $\gamma > 0, p \in \mathbb{N}$) by

$$(L_0 L_1 L_2 L_3)^{-\gamma} \|w\|_{X^{-s,b'}} (\|u\|_{X^{1,b}}^2 + \|u\|_{X^{1,b}}^p) \|u\|_{X^{s,b}}.$$

We have three regimes:

1. $N_0^{1-\delta} \leq N_3, \delta > 0$ small enough,
2. $N_3 \leq N_0^{1/2}$,
3. $N_0^{1/2} \leq N_3 \leq N_0^{1-\delta}$.

In the first regime, we expand once more the nonlinear term, writing

$$I_{L_0 L_1 L_2 L_3}^{N_0 N_1 N_2 N_3} \leq \sum_{L_4, N_4} I_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4}$$

and apply the same strategy as when $N_0 \leq \Lambda N_1$. Indeed, in this regime, $N_0/N_1 \leq N_0^\delta$ and we obtain with $\eta > 0$

$$I_{L_0 L_1 L_2 L_3 L_4}^{N_0 N_1 N_2 N_3 N_4} \leq \mathcal{O}(1) \frac{N_0^s}{N_1^s} \frac{1}{(N_2 N_3 N_4)^\theta} \leq \mathcal{O}(1) N_0^{2\delta} N_3^{-\theta} \leq \mathcal{O}(1) N_0^{2\delta - \theta(1-\delta)} \leq \mathcal{O}(1) N_0^{-\eta}$$

which gives the summability in $N_4 \leq N_3 \leq N_2 \leq N_1 \leq \Lambda^{-1} N_0$.

The second regime can be dealt with in the same way as in the previous section (in this regime, we gain arbitrary powers of N_0^{-1}). Finally we concentrate on the last regime. Let

$$v = \Delta_{N_0 L_0}(w) \Delta_{N_2 L_2}(u) \Delta_{N_3 L_3}(u).$$

By the same argument as in Lemma 4.9, there exists $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ such that

$$\|(1 - \varphi(N_0^{-2} \Delta))v\|_{L^1(\mathbb{R} \times M)} \leq \mathcal{O}(1) N_0^{-k}$$

for any k .

Using that for some function $\Psi \in C_0^\infty$ we have

$$\varphi(N_0^{-2} \Delta) = N_0^{-2} \Delta \circ \Psi(N_0^{-2} \Delta)$$

we can integrate by parts in the integral defining $I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3}$:

$$(5.16) \quad I_{L_0, L_1, L_2, L_3}^{N_0, N_1, N_2, N_3} = N_0^{-2} \left| \int_M \Psi(N_0^{-2} \Delta)(v) \Delta(\Delta_{N_1 L_1}(u) H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3}(u))) \right|$$

$$\begin{aligned}
 &= N_0^{-2} \left| \int_M \Psi(N_0^{-2} \Delta)(v) \times [\Delta(\Delta_{N_1 L_1}(u)) H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3}(u)) \right. \\
 &\quad + \nabla(\Delta_{N_1 L_1}(u)) \cdot \nabla(H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3}(u))) \\
 &\quad \left. + \Delta_{N_1 L_1}(u) \Delta(H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3}(u))) \right] \Big|.
 \end{aligned}$$

Now to estimate the first term in the right-hand side of (5.16) we simply apply the strategy already used in the case $N_0 \leq \Lambda N_1$, the only difference being the additional factor $(N_1/N_0)^2$ (the factor N_1^2 coming from the action of the Laplace operator on $\Delta_{N_1 L_1}(u)$), which allows, since $s < 2$, to exchange the roles of N_0 and N_1 and gain summability. Remark here that the additional operator $\Psi(N_0^{-2} \Delta)$ applied to $\Delta_{N_0 L_0}(w) \Delta_{N_2 L_2}(u) \Delta_{N_3 L_3}(u)$ plays no role since it disappears when one takes L^2 norms.

Next we estimate the two other terms. We expand the derivatives and observe that

$$|\nabla(H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3}(u)))|(x) \leq C w_{N_3,1}(x)$$

with

$$\begin{aligned}
 w_{N_3,1}(x) &= (|\nabla \Delta_{N_3}(u)| + |\nabla S_{N_3}(u)|)(1 + |\Delta_{N_3}(u)| + |S_{N_3}(u)|)(x) \\
 &= w_{N_3,1,q} + (|\nabla \Delta_{N_3}(u)| + |\nabla S_{N_3}(u)|), \\
 &\quad |\Delta(H_{11}^{N_2, N_3}(\Delta_{N_3}(u), S_{N_3}(u)))|(x) \leq w_{N_3,2}(x)
 \end{aligned}$$

with

$$\begin{aligned}
 w_{N_3,2}(x) &= (|\Delta \Delta_{N_3}(u)| + |\Delta S_{N_3}(u)|)(1 + |\Delta_{N_3}(u)| + |S_{N_3}(u)|)(x) \\
 &\quad + (|\nabla \Delta_{N_3}(u)| + |\nabla S_{N_3}(u)|)^2(x) \\
 &= w_{N_3,2,q}(x) + (|\Delta \Delta_{N_3}(u)| + |\Delta S_{N_3}(u)|).
 \end{aligned}$$

Next we use that for $\phi \in C_0^\infty(\mathbb{R})$, $\phi(N^{-2} \Delta)$ is an N^{-1} -semi-classical operator and consequently the gradient of a spectrally localized function is essentially spectrally localized. This allows to apply the trilinear estimate (5.3) to v on one hand and to

$$(5.17) \quad |\nabla(\Delta_{N_1 L_1}(u))| \times w_{N_3,1,q} \quad \text{or} \quad \Delta_{N_1 L_1}(u) \times w_{N_3,2,q}$$

on the other hand (remark that the moduli in (5.17) do not spoil the estimate since we take L^2 norms). Similarly, we apply the bilinear estimate (5.2) to v on one hand and to

$$\begin{aligned}
 (5.18) \quad &|\nabla(\Delta_{N_1 L_1}(u))| \times (|\nabla \Delta_{N_3}(u)| + |\nabla S_{N_3}(u)|) \\
 &\text{or} \quad \Delta_{N_1 L_1}(u) \times (|\Delta \Delta_{N_3}(u)| + |\Delta S_{N_3}(u)|)
 \end{aligned}$$

on the other hand.

We obtain that the contribution of these terms is bounded (for any $\varepsilon > 0$) by

$$\begin{aligned}
 (5.19) \quad &C_\varepsilon \mathcal{O}(1) N_0^{-2} \left(\frac{N_0}{N_1}\right)^s N_2^{-1/4+\varepsilon} N_3^{1/4+\varepsilon} (N_1 N_3 + N_3^2) N_3^{-1/4+\varepsilon} N_3^{1/4+\varepsilon} \\
 &\leq C_\varepsilon \mathcal{O}(1) N_0^{s-2} N_1^{1-s} N_3^{4\varepsilon+1} \leq C_\varepsilon \mathcal{O}(1) N_0^{s-2} N_3^{2-s+4\varepsilon}.
 \end{aligned}$$

Using that $\delta > 0$ and $N_3 \leq N_0^{1-\delta}$, for $\varepsilon > 0$ small enough, this term can be bounded by $N_0^{-\eta}$, $\eta > 0$ giving the required summability in $\Lambda^{-1} N_0 \geq N_1 \geq N_2 \geq N_3$.

Remark 5.5. – A careful examination of the proof above shows that Theorem 4 still holds for a three dimensional manifold M satisfying the more general trilinear Strichartz estimate,

$$\begin{aligned} \exists a > 0: \forall T > 0, \forall \varepsilon > 0, \exists C > 0: \forall N_3 \leq N_2 \leq N_1, \forall f_1, f_2, f_3 \in L^2(M), \\ \left\| \prod_{j=1}^3 e^{it\Delta}(\Delta_{N_j} f_j) \right\|_{L^2([0,T] \times M)} \leq CN_3^{1+a} N_2^{1-a+\varepsilon} \prod_{j=1}^3 \|\Delta_{N_j} f_j\|_{L^2(M)}. \end{aligned}$$

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Appendix A

This appendix is devoted to the optimality of the assumption $\alpha < 5$ in Theorem 1. Let us again consider a 3d-manifold M endowed with a Riemannian metric g and Δ the Laplace–Beltrami operator acting on functions of M . We consider the following nonlinear Schrödinger equation on M

$$(A.1) \quad (i\partial_t + \Delta)u = F(u), \quad u|_{t=0} = u_0 \in H^1(M),$$

where $F(z) = (1 + |z|^2)^{(\alpha-1)/2}z$ and $\alpha > 5$.

Let us fix $s > 3/2$. Eq. (A.1) is well-posed for data in $H^s(M)$ by the energy method. In particular, for every bounded set $B \subset H^s(M)$ there exists T_s such that for every $u_0 \in H^s(M)$ the Cauchy problem (A.1) has a unique solution on the interval $[-T_s, T_s]$ in the class $C([-T_s, T_s]; H^s(M))$. Moreover the flow map

$$\Phi: u_0 \longrightarrow u$$

is continuous (and even Lipschitz continuous) from B , endowed with the $H^s(M)$ metric, to $C([-T_s, T_s]; H^s(M))$. The next statement shows that Φ cannot be extended as a continuous map on bounded sets of $H^1(M)$.

THEOREM 5. – *Let B be a bounded set of $H^1(M)$. There is no $T > 0$ such that the map Φ can be extended as a continuous map from B to $C([-T, T]; H^1(M))$.*

The result of Theorem 5 readily follows from the following statement.

THEOREM 6. – *There exist a sequence $(t_n)_{n \in \mathbb{N}}$ of positive numbers tending to zero and a sequence $(u_n(t))_{n \gg 1}$ of $C^\infty(M)$ functions defined for $t \in [0, t_n]$, such that*

$$(i\partial_t + \Delta)u_n = (1 + |u_n|^2)^{\frac{\alpha-1}{2}} u_n$$

with

$$\lim_{n \rightarrow \infty} \|u_n(0)\|_{H^1(M)} = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} \|u_n(t_n)\|_{H^1(M)} = \infty.$$

Remark A.1. – The result of Theorem 6, in the particular case $M = \mathbb{R}^3$, endowed with the standard metric, can be found in [11]. We also refer to [11] for more ill-posedness results for NLS on \mathbb{R}^d , $d \geq 1$, with power-like nonlinearities and data in H^s , $s > 0$.

Remark A.2. – The proof of Theorem 6 is strongly inspired by [11]. The only observation we make here is that the dilation arguments involved in the proof in [11] are not essential. It is clear from the proof we present that the discussed phenomenon is completely local, i.e. the whole analysis is close to a point of M for very small times.

Proof. – We work in a local coordinate patch around 0 and consider as initial data the sequence $u_n(0) = \kappa_n n^{1/2} \varphi(nx)$, $n \gg 1$, where φ is a fixed nonnegative smooth compactly supported $\kappa_n = \log^{-\delta}(n)$ with $\delta > 0$ to be fixed later. Remark that

$$\|u_n(0)\|_{H^1(M)} \sim \kappa_n.$$

Let us set $f(z) := (1 + |z|^2)^{\frac{\alpha-1}{2}}$. Then

$$v_n(t) = \kappa_n n^{1/2} \varphi(nx) e^{-itf(\kappa_n n^{1/2} \varphi(nx))}$$

is the solution of the equation

$$(A.2) \quad i\partial_t v_n = F(v_n), \quad v_n|_{t=0} = u_n(0).$$

Let us give a basic bound for $v_n(t)$.

LEMMA A.3. – *There exist $c > 0$ and $C > 0$ such that for any $t \geq 0$,*

$$\|\nabla_x v_n(t)\|_{L^2} \geq \kappa_n (ct\kappa_n^{\alpha-1} n^{(\alpha-1)/2} - C).$$

Proof. – The change of variable $y = nx$ gives

$$(A.3) \quad \begin{aligned} \|\nabla_x v_n(t)\|_{L^2} &= \kappa_n \|\nabla_y [\varphi(y) e^{-itf(\kappa_n n^{1/2} \varphi(y))}]\|_{L^2} \\ &\geq \kappa_n (2t\kappa_n n^{1/2} \|\varphi(y) \partial_z f(\kappa_n n^{1/2} \varphi(y)) \cdot \nabla_y \varphi(y)\|_{L^2} - \|\nabla_y \varphi\|_{L^2}) \\ &\geq ct\kappa_n^{\alpha} n^{(\alpha-1)/2} - C\kappa_n \end{aligned}$$

which implies the lemma. \square

For functions u on M , we define the quantity,

$$E_n(u) := [n^2 \|u\|_{L^2}^2 + n^{-2} \|\Delta u\|_{L^2}^2]^{\frac{1}{2}}.$$

The key point in the proof of Theorem 6 is the next statement.

LEMMA A.4. – *The solution u_n of (A.1) with initial data $u_0 = \kappa_n n^{1/2} \varphi(nx) \in C^\infty(M)$ exists for $0 \leq t \leq t_n$, with $t_n = \log^{1/8}(n) n^{-(\alpha-1)/2}$. Moreover, there exists $\epsilon > 0$ such that for $t \in [0, t_n]$,*

$$E_n(u_n(t) - v_n(t)) \leq Cn^{-\epsilon}.$$

Proof. – Since the initial data are in H^s , $s > 3/2$, we know that $u_n(t)$ exists on a (nonempty) maximal time interval $[0, \tilde{t}_n[$. Consequently, to prove Lemma A.4, we simply prove the *a priori* estimates which ensure, by a classical bootstrap argument, both the existence and the control on $E_n(u_n(t) - v_n(t))$ for $t \in [0, t_n] \cap [0, \tilde{t}_n[$. Let us set $w_n := u_n - v_n$. For the sake of conciseness, in the rest of the proof of Lemma A.4, we drop the subscript n of u_n , v_n and w_n . The *a priori* estimates involved in the proof are simply energy inequalities in the equations

$$\begin{aligned} (i\partial_t + \Delta)w &= F(u) - F(v) - \Delta v = \mathcal{O}(1 + |v|^{\alpha-1} + |w|^{\alpha-1})w - \Delta v, \\ (i\partial_t + \Delta)\Delta w &= \Delta(F(u) - F(v)) - \Delta^2 v = -\Delta^2 v + \Lambda, \end{aligned}$$

where

$$\begin{aligned} \Lambda &:= \mathcal{O}(1 + |w|^{\alpha-1} + |v|^{\alpha-1})\Delta w \\ &\quad + \mathcal{O}((1 + |w|^{\alpha-2} + |v|^{\alpha-2})(1 + |w| + |v| + |\nabla w| + |\nabla v|))\nabla w \\ &\quad + \mathcal{O}((1 + |w|^{\alpha-3} + |v|^{\alpha-3})(1 + |\nabla v|^2 + (|v| + |w|)|\nabla^2 v|))w. \end{aligned}$$

From the explicit formula for v , we have for $0 \leq t \leq t_n$, $k = 0, 1, 2, \dots$,

$$\|\nabla^k v\|_{L^\infty} \leq Cn^{1/2+k} \log^{k/8}(n).$$

According to the Gagliardo–Nirenberg inequalities,

$$(A.4) \quad \|f\|_{L^\infty} \leq C\|f\|_{H^2}^{3/4}\|f\|_{L^2}^{1/4} \leq Cn^{1/2}E_n(f)$$

we deduce

$$(A.5) \quad n\|\mathcal{O}(1 + |v|^{\alpha-1} + |w|^{\alpha-1})w\|_{L^2} \leq C(1 + \|v\|_{L^\infty}^{\alpha-1} + \|w\|_{L^\infty}^{\alpha-1})n\|w\|_{L^2} \leq Cn^{(\alpha-1)/2}(E_n(w) + E_n^\alpha(w)).$$

To estimate Λ , we proceed similarly. More precisely, thanks to (A.4), we estimate systematically the terms involving v or w below the \mathcal{O} sign in L^∞ . The only term which cannot be estimated by invoking (A.4) is

$$(A.6) \quad \mathcal{O}((1 + |w|^{\alpha-2} + |v|^{\alpha-2})(|\nabla w|))\nabla w.$$

In order to evaluate (A.6), we use the bound

$$\|\nabla w\|_{L^4} \leq C\|\nabla w\|_{H^{3/4}} \leq Cn^{3/4}E_n(w)$$

and we obtain

$$(A.7) \quad \|(A.6)\|_{L^2} \leq Cn^{(\alpha+1)/2}(E_n(w) + E_n^\alpha(w)).$$

We are therefore conducted to the following estimate for Λ

$$(A.8) \quad n^{-1}\|\Lambda\|_{L^2} \leq Cn^{(\alpha-1)/2} \log^{1/4}(n)(E_n(w) + E_n^\alpha(w)).$$

Next, thanks to the formula for v , for $0 \leq t \leq t_n$, we estimate the source terms,

$$(A.9) \quad n\|\Delta v\|_{L^2} + n^{-1}\|\Delta^2 v\|_{L^2} \leq Cn^2 \log^{1/2}(n).$$

According to (A.5), (A.7) and (A.9), we obtain

$$\frac{d}{dt}E_n^2(w) \leq Cn^{(\alpha-1)/2} \log^{1/4}(n)(E_n^2(w) + E_n^{\alpha+1}(w)) + Cn^2 \log^{1/2}(n)E_n(w).$$

Suppose first that $E_n(w) \leq 1$ which is clearly the case at least for $t \ll 1$ since $w|_{t=0} = 0$. Notice that

$$2n^2 \log^{1/2}(n)E_n(w) \leq n^{(\alpha-1)/2} \log^{1/4}(n)E_n^2(w) + \frac{n^4 \log(n)}{n^{(\alpha-1)/2} \log^{1/4}(n)}.$$

Therefore

$$\frac{d}{dt} [e^{-Ctn^{(\alpha-1)/2} \log^{1/4}(n)} E_n^2(w)] \leq Cn^{4-\frac{\alpha-1}{2}} \log^{3/4}(n) e^{-Ctn^{(\alpha-1)/2} \log^{1/4}(n)}.$$

Integrating the last inequality between 0 and t gives the estimate

$$E_n(w) \leq Cn^{2-\frac{\alpha-1}{2}} \log^{1/4}(n) e^{Ctn^{(\alpha-1)/2} \log^{1/4}(n)}.$$

For every $\gamma > 0$ there exists C_γ such that for $t \in [0, t_n]$,

$$Ctn^{(\alpha-1)/2} \log^{1/4}(n) \leq C \log^{3/8}(n) \leq \gamma \log n + C_\gamma.$$

Since $\alpha > 5$, by taking $\gamma > 0$ small enough, we obtain that there exists $\varepsilon > 0$ such that for $t \in [0, t_n]$, we have

$$(A.10) \quad E_n(w) \leq Cn^{-\varepsilon}.$$

Finally the usual bootstrap argument allows to drop the assumption $E_n(w) \leq 1$. This completes the proof of Lemma A.4. \square

It is clear that by interpolation, the quantity $E_n(u)$ controls uniformly with respect to n the H^1 norm of u . Consequently, it follows from Lemmas A.3 and A.4 that for $\delta < \frac{1}{8\alpha}$ and $n \gg 1$,

$$\|u_n(t_n)\|_{H^1} \geq C \log^{\frac{1}{8}-\alpha\delta}(n).$$

This completes the proof of Theorem 6. \square

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