

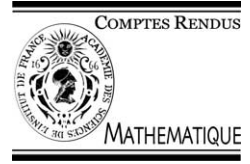


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Geometry

Construction of pseudo-isometries for treelike hyperbolic 3-manifolds of infinite volume

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Abstract

We introduce a family of *rigid* hyperbolic 3-manifolds of infinite volume with possibly infinitely many ends: the *treelike manifolds*. These manifolds generalize a family of constructive non compact surfaces – the equational surfaces – for which the homeomorphism problem is decidable. The proof of rigidity relies firstly on Thurston’s theorem of compactness of the Teichmüller space of acylindrical compact 3-manifolds, and secondly, on Sullivan’s rigidity theorem. **To cite this article:** O. Ly, *C. R. Acad. Sci. Paris, Ser. I 337 (2003)*.

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Résumé

Construction de pseudo-isométries pour les 3-variétés hyperboliques de volume infini arborescentes. Nous introduisons une famille de 3-variétés hyperboliques *rigides* de volume infini à nombre de bouts infini : les *variétés arborescentes*. Ces variétés généralisent une famille de surfaces non compactes constructives – les surfaces équationnelles – pour lesquelles le problème de l’homéomorphisme est décidable. La démonstration de rigidité s’appuie sur, premièrement, le théorème de Thurston de compacité de l’espace de Teichmüller des 3-variétés compactes acylindriques, et deuxièmement, le théorème de rigidité de Sullivan. **Pour citer cet article :** O. Ly, *C. R. Acad. Sci. Paris, Ser. I 337 (2003)*.

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1. Introduction

Sullivan showed in [6] that if the action of a discrete group of hyperbolic motions Γ on the sphere at infinity is conservative then the quotient manifold \mathbb{H}^n/Γ is Mostov-rigid, i.e., any pseudo-isometry between \mathbb{H}^n/Γ and another hyperbolic manifold is homotopic to an isometry. McMullen gave in [3] a sufficient condition for the conservativity of this action: the action of Γ on the sphere at infinity is conservative if the injectivity radius of the quotient manifold is uniformly bounded.

Here we present a method of constructions of pseudo-isometries for a family of complete hyperbolic 3-manifolds of infinite volume – the *treelike manifolds* – satisfying this criterion. These manifolds are defined as follows:

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Definition 1.1. Let M be a complete hyperbolic 3-manifold without boundary. M is said to be *treelike* if there exists a geodesic triangulation \mathcal{T} of M such that

- the lengths of the edges of \mathcal{T} vary in a fixed compact interval not containing zero,
- there exists along \mathcal{T} a system of pairwise disjoint separating incompressible surfaces $\{S_u\}_u$ which cuts M into simple pieces in the sense that there exists a finite number of bordered compact triangulated 3-manifolds M_1, \dots, M_n such that for each component C of $M \setminus \bigcup_u S_u$, there exists a simplicial homeomorphism $\varphi_c : \bar{C} \rightarrow M_{\iota(c)}$ where $\iota(c) \in \{1, 2, \dots, n\}$,
- for each component C of $M \setminus \bigcup_u S_u$, \bar{C} is acylindrical.

The main result of this paper is the following:

Theorem 1.2. *Let M, M' be two treelike complete hyperbolic 3-manifolds. Then every isomorphism between $\pi_1(M)$ and $\pi_1(M')$ is induced by an isometry.*

The main step of the proof consists in Lemma 3.3: given an isomorphism between the fundamental groups of a pair of treelike hyperbolic manifolds, one can construct a homotopy equivalence inducing it which is k -Lipschitz for some constant $k > 0$. The main ingredient to get this map is Thurston's compactness theorem for Teichmüller spaces of acylindrical compact 3-manifolds [8] which allows us to control deformations of treelike manifolds. This result allows us to use a standard construction of pseudo-isometry that we shall not describe here (see [4]). The conclusion comes with the use of theorem of Sullivan and the criterion of McMullen.

2. Examples of treelike manifolds

The following lemma, whose proof is omitted, allows us to construct examples of treelike manifolds (see [7]).

Lemma 2.1. *Let N be a prime acylindrical compact 3-manifold whose boundary components are incompressible surfaces of genus greater than two, then N can be endowed with a complete hyperbolic structure with totally geodesic boundary.*

Taking this result into account, given such a prime acylindrical compact 3-manifold N with $m > 1$ incompressible boundary components of genus greater than 2, we construct a treelike complete hyperbolic manifold as follows: by Lemma 2.1, N can be endowed with a complete hyperbolic metric with totally geodesic boundary. For each boundary component B of N , let us consider a new copy of N which we glue on N itself along B according to the identity mapping of B . We get a manifold with $m(m - 1)$ boundary components. We repeat this operation to the new manifold obtained. Iterating in this way leads to a treelike complete hyperbolic manifold of infinite volume. Let us note that this manifold may have infinitely many ends, and its fundamental group may be not finitely generated.

Treelike manifolds arises as generalizations of equational surfaces (see [1]). Indeed, it can be checked that the manifold constructed previously is constructive in the sense that it has a triangulation generated by a graph grammar.

3. Proof of Theorem 1.2

Notations. First, we fix some notations. Let M be a treelike complete hyperbolic 3-manifold. Let $\{S_u\}_u, M_i$ and φ_c be like in Definition 1.1. Let $\{B_{ij}\}_{1 \leq j \leq b_i}$ be the components of the boundary of M_i . Let us consider $S_u \subset \bar{C}$; $\varphi_c|_{S_u}$ defines a simplicial homeomorphism between S_u and a component of the boundary of $M_{\iota(c)}$. But there is only finitely many B_{ij} . So there is finitely many S_u up to triangulation isomorphism. Let $\{T_k\}_{1 \leq k \leq m}$ be a family of representatives of the S_u , given with a family of simplicial homeomorphisms $\{\psi_u : S_u \rightarrow T_{\mu(u)} \mid u\}$.

Let i, j, u and C be such that $\varphi_c(S_u) = B_{ij}$. Then $\psi_u \circ \varphi_c^{-1}|_{B_{ij}} : B_{ij} \rightarrow T_{\mu(u)}$ is a simplicial homeomorphism. Let us note that the number of simplicial homeomorphism between two triangulated surfaces is finite up to

homotopy. Therefore, even if it means adding new representative of the components of $M \setminus \bigcup_u S_u$, namely new M_i , we can suppose that $\psi_u \circ \varphi_c^{-1}|_{B_{ij}}$ neither depends on u nor on C ; let it be denoted by g_{ij} .

For each $k \in \{1, 2, \dots, m\}$, let us choose a vertex x_k of the triangulation of T_k . For each pair i, j , let $k \in [1, m]$ be such that B_{ij} is mapped onto T_k by g_{ij} , we then define $x_{ij} = g_{ij}^{-1}(x_k)$. Let t_i be a simplicial tree in M_i of root x_{i0} connecting x_{i0} to all the x_{ij} for $j \neq 0$. Let us assume that t_i is a star contained in the interior of M_i , except for the x_{ij} . Let us consider a spanning tree of the 1-skeleton of T_k . It induces via g_{ij}^{-1} a spanning tree of the 1-skeleton of each B_{ij} . Let s_i be a spanning tree of the 1-skeleton of M_i which contains these trees together with t_i . By using φ_c^{-1} for each component C of $M \setminus \bigcup_u S_u$, the family $\{s_i\}_i$ defines a spanning tree of the 1-skeleton of M , let us denote it by s . For each C , let $x_c = \varphi_c^{-1}(x_{i(c)0})$. Let us choose one component C_0 of $M \setminus \bigcup_u S_u$, let us call it the *root component* of M . The union of the family $\{\varphi_c^{-1}(t_i)\}_c$, denoted by t , is called a *spine* of M and for any component C , x_c is called a *node* of t . Even if it means adding new representatives of component, we can assume that if $C \neq C'$ then $x_c \neq x_{c'}$.

Since t is a tree containing x_{c_0} , $\pi_1(M, t)$ is canonically isomorphic to $\pi_1(M, x_{c_0})$. On the other hand, for each component C , there is a canonical homomorphism induced by φ_c^{-1} from $\pi_1(M_{i(c)}, x_{i(c)0})$ into $\pi_1(M, t)$. Let Γ_i denote $\pi_1(M_i, x_{i0})$. Then the resulting homomorphism $j_c: \Gamma_{i(c)} \rightarrow \pi_1(M, x_{c_0})$ is injective because S_u is incompressible.

Step 1. Let Γ, Γ' be two discrete subgroups of hyperbolic elements of $\text{Isom} \mathbb{H}^3$ such that $M = \mathbb{H}^3/\Gamma$ and $M' = \mathbb{H}^3/\Gamma'$ are treelike complete hyperbolic manifolds. Let $\Phi: \Gamma \rightarrow \Gamma'$ be an isomorphism. We keep the above notations concerning the treelike decomposition of M . Let A_i denote the generating subset of Γ_i defined using s_i and the edges of the 1-skeleton of M_i not belonging to s_i . Let us consider the function m_i which associates to each faithful representation ρ of Γ_i the point $m_i(\rho) \in \mathbb{H}^3$ which minimizes the function $d_\rho(x) = \sum_{\alpha \in A_i} d_{\mathbb{H}^3}(x, \rho(\alpha)(x))$. It can be checked that m_i is well defined and continuous regarding the algebraic topology of the space of faithful representations of Γ_i .

Now, let us pick an element $\tilde{x}_{c_0} \in \mathbb{H}^3$ in the fiber of x_{c_0} . Together with the covering map $\mathbb{H}^3 \rightarrow M$, this defines a holonomy map $H: \pi_1(M, x_{c_0}) \rightarrow \Gamma$. Let C be a component of $M \setminus \bigcup_u S_u$, let $\rho_c = \Phi \circ H \circ j_c$, this is a faithful representation of $\Gamma_{i(c)}$ in $\text{Isom} \mathbb{H}^3$. Let $\tilde{x}'_c = m_{i(c)}(\rho_c)$.

Lemma 3.1. *There exists $k_1 > 0$ such that $\forall C_1, C_2$ such that $\overline{C}_1 \cap \overline{C}_2 \neq \emptyset$, $d_{\mathbb{H}^3}(\tilde{x}'_{c_1}, \tilde{x}'_{c_2}) \leq k_1$.*

Proof. Let us consider two components C_1 and C_2 such that $\overline{C}_1 \cap \overline{C}_2 \neq \emptyset$. Let $K = \pi_1(\overline{C}_1 \cup \overline{C}_2, x_{c_1})$. As above, there is a natural injection $j_K: K \rightarrow \pi_1(M, x_{c_0})$. For $i \in \{1, 2\}$, $\varphi_{c_i}^{-1}$ induces an injection j'_{c_i} of $\Gamma_{i(c_i)}$ into K . We have $j_{c_i} = j_K \circ j'_{c_i}$. The function defined by $\rho \mapsto d_{\mathbb{H}^3}(m_{i(c_1)}(\rho \circ j'_{c_1}), m_{i(c_2)}(\rho \circ j'_{c_2}))$ from the space of faithful representations of K to \mathbb{R}_+ is continuous because m_i is continuous. To other extends, it is invariant by conjugation and then, it defines a continuous function $\tilde{d}: \text{AH}(K) \rightarrow \mathbb{R}_+$, where $\text{AH}(K)$ denotes the Teichmüller space of K endowed with the algebraic topology. Let us note that $\overline{C}_1 \cup \overline{C}_2$ is acylindrical. By Thurston's theorem (see [8]) $\text{AH}(K)$ is compact and thus, \tilde{d} is bounded. Finally, the number of such triple $(K, \Gamma_{i(c_1)}, \Gamma_{i(c_2)})$ arising as above is finite up to isomorphism and for each pair of intercepting components C_1 and C_2 , $\Phi \circ H$ gives rise in a canonical way to a faithful representation of $K = \pi_1(\overline{C}_1 \cup \overline{C}_2, x_{c_1})$. The conclusion follows. \square

Lemma 3.2. *There exists $k_2 > 0$ such that $\forall C, \forall \alpha \in A_{i(c)}: d_{\mathbb{H}^3}(\tilde{x}'_C, \rho_c(\alpha)(\tilde{x}'_C)) \leq k_2$.*

Proof. The mapping $\rho \mapsto d_{\mathbb{H}^3}(m_{i(c)}(\rho), \rho(\alpha)(m_{i(c)}(\rho)))$ is continuous on the space of faithful representation of $\Gamma_{i(c)}$ and invariant by conjugation. We use again Thurston's theorem. \square

We now construct an homotopy equivalence f between M and M' which is k -Lipschitz for some constant k . First, we define f on the 1-skeleton $M^{(1)}$ of M .

Let $p': \mathbb{H}^3 \rightarrow M'$ be a covering map associated to the hyperbolic structure of M' . For all component C , let $f(x_c) = p'(\tilde{x}'_c)$.

Let e be an simple path of t connecting x_{c_1} and x_{c_2} where C_1 and C_2 are components such that $C_1 \cap C_2 \neq \emptyset$. The length of e has a lower bound, which is independent of e . By Lemma 3.1 the distance between \tilde{x}'_{c_1} and \tilde{x}'_{c_2} is bounded in \mathbb{H}^3 . We then choose $p'([\tilde{x}'_{c_1}, \tilde{x}'_{c_2}])$ for the image of e , where $[\tilde{x}'_{c_1}, \tilde{x}'_{c_2}]$ denotes the geodesic segment between \tilde{x}'_{c_1} and \tilde{x}'_{c_2} . Clearly, $f|_e$ can be defined to be Lipschitz for a constant not depending on e .

Let x be a point of s . As s is a tree containing t which is also a tree, x is connected to t within s by a unique path. We define $f(x)$ to be the image (as defined previously) of the other extremity of this path. Actually, we just retract s into t . Again, the restriction of f on each edge of s can be construct to be Lipschitz.

Let us now consider an edge e of $M^{(1)}$ not belonging to s . Let x_1 and x_2 be its extremities. Let C be a component such that $e \subset \overline{C}$. Let $\alpha \in A_{i(C)}$ be the element of $\Gamma_{i(C)}$ associated to $\varphi_C(e)$. Let \tilde{e}' denote the geodesic segment $[\tilde{x}'_C, \rho_C(\alpha)(\tilde{x}'_C)]$. The image of e by f is defined to be the concatenation $[f(x_1), f(x_c)]p'(\tilde{e}')[f(x_c), f(x_2)]$. The fact that f can be constructed to be Lipschitz follows from Lemma 3.2.

Now, it can be checked that $\Phi \circ H \circ (i_{M^{(1)}})_* = H' \circ f_*$, making us sure that f is going to be an homotopy equivalence inducing Φ . The construction of f on $M^{(2)}$, keeping the Lipschitz property, is now easy. Each 2-simplex σ comes from an M_i for some i , and its boundary γ_σ is mapped by f to a piecewise geodesic loop γ'_σ of uniformly bounded length. As this loop is null homotopic, it is lifted in \mathbb{H}^3 by a piecewise geodesic loop $\tilde{\gamma}'_\sigma$ of the same length. This last loop bounds a disk D of bounded “size” in \mathbb{H}^3 . We define $f(\sigma)$ to be the image $p'(D)$ of this disk in M' . The “size” of σ is bounded bellow in the sense that the lengths of all its edges are bounded below. So, $f|_\sigma$ can be constructed to be Lipschitz. The construction of f on $M^{(3)}$ follows the same ideas. It can be checked that f is k -Lipschitz for some constant $k > 0$. Eventually, we get the following result:

Lemma 3.3. *Let M and M' be complete hyperbolic 3-manifolds without boundary. Let us assume that M is treelike. Then every isomorphism between $\pi_1(M)$ and $\pi_1(M')$ is induced by a k -Lipschitz homotopy equivalence from M to M' for some $k > 0$.*

Step 2. This step consists in proving that f lifts to a pseudo-isometry of \mathbb{H}^3 . This does not really differ from the compact case which is standard and can be found in, e.g., [5, Theorem 11.6.1].

Step 4. By Sullivan’s theorem [6], if the action of Γ on the sphere at infinity is conservative then every Γ -equivariant pseudo-isometry is homotopic to a Γ -equivariant isometry by a Γ -equivariant homotopy. On the other hand, by the result of McMullen (see [3] or [2, Theorem 5.11]), the action of Γ on the sphere at infinity is conservative if and only if the injectivity radius of \mathbb{H}^3/Γ is uniformly bounded. This last condition is fulfilled by treelike manifolds. Theorem 1.2 is proved. \square

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