

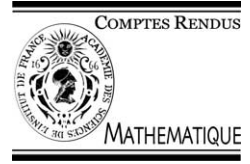


ELSEVIER

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

C. R. Acad. Sci. Paris, Ser. I 337 (2003) 451–456



Partial Differential Equations

A semilinear non-classical pseudodifferential boundary value problem in the Sobolev spaces

Youri V. Egorov^a, Nguyen Minh Chuong^b, Dang Anh Tuan^c

^a *Laboratoire des mathématiques pour l'industrie et la physique, UMR 5640, UFRMIG, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse, France*

^b *Institute of Mathematics, NCST, PO Box 631, BoHo, 10000 Hanoi, Viet Nam*

^c *Faculty of Mathematics–Mechanics–Informatics, Hanoi National University, 334 Nguyen Trai Str., Thanh Xuan, Hanoi, Viet Nam*

Received 31 July 2003; accepted 30 August 2003

Presented by Louis Nirenberg

Abstract

We study the solvability of a semilinear non-classical pseudodifferential boundary value problem in the Sobolev spaces $H_{l,p,q}$, $1 < p < \infty$, depending on a complex parameter q . **To cite this article:** *Y.V. Egorov et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).*

© 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Résumé

Un problème aux limites semilinéaire non-classique pour des opérateurs pseudodifférentiels dans les espaces de Sobolev. On étudie la résolution d'un problème aux limites semilinéaire non-classique pour des opérateurs pseudodifférentiels dans les espaces de Sobolev $H_{l,p,q}$, $1 < p < \infty$, munis de normes dépendant d'un paramètre complexe q . **Pour citer cet article :** *Y.V. Egorov et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).*

© 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Version française abrégée

Nous étudions des problèmes aux limites elliptiques non-linéaires dépendant d'un paramètre complexe q pour des opérateurs pseudodifférentiels dans les espaces de Sobolev $H_{l,p,q}$, $1 < p < \infty$, munis des normes dépendant de q . Nous considérons le cas quand dans les conditions au bord contient un champ de vecteurs qui peut être tangent au bord sur une sous-variété. En établissant des résultats pour le cas linéaire dans des classes nouvelles des opérateurs pseudodifférentiels et en utilisant le théorème des points fixés de Schauder on peut démontrer l'existence et l'unicité des solutions du problème. Cette Note est la continuation des articles publiés [7,8,12].

E-mail address: egorov@mip.ups-tlse.fr (Y.V. Egorov).

1. Oblique derivative problem

The oblique derivative problem for an elliptic differential equation of second order was investigated by many authors, in particular, Egorov and Kondratiev studied in [7] the problem in the case when the vector field may be tangent to the bounded smooth boundary Ω on a smooth submanifold Γ_0 . Trung discussed in [12] the problem for singular integro-differential operators of Agranovich type [3]. Egorov and Nguyen Minh Chuong studied in [8] some semilinear classical and non-classical boundary value problems for the mentioned above operators in the Sobolev spaces $H_{l,2}$.

Note that the assumptions and the proof of our main Theorem 5.2. here are new and quite different from the corresponding ones in [8].

2. Definitions

Let $p, l \in \mathbb{R}, 1 < p < \infty$. Let F, F' be the Fourier transforms with respect to $x = (x', x_n) \in \mathbb{R}^n, x'$, respectively. Denote by $H_{l,p}(\mathbb{R}^n)$ the completion of the space $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{l,p} = \|u\|_{l,p,\mathbb{R}^n} = \left(\int_{\mathbb{R}_\xi^n} (1 + |\xi|)^{pl} |Fu(\xi)|^p d\xi \right)^{1/p}.$$

The spaces $H_{l,p}(\mathbb{R}_+^n), H_{l,p}(\Omega), H_{l,p}(\partial\Omega), H_{l,p}(\Gamma_0)$ are defined in the standard way (see [1,10,11]). Set $Q = \{q \in \mathbb{C} \mid \alpha_0 \leq \arg q \leq \beta_0\}$. By $H_{l,p,q}$ we denote the above spaces depending on a complex parameter $q \in Q$: $\|u\|_{l,p} = (\|u\|_{l,p}^p + |q|^{lp} \|u\|_{0,p}^p)^{1/p}$. We use the functions $K_{\alpha\beta}(x, z) = H_{\alpha\beta}(z) + k_{\alpha\beta}(x, z)$, where

- (i) $H_{\alpha\beta}(z), k_{\alpha\beta}(x, z)$ are positively homogeneous of degree $(-n)$ with respect to z ;
- (ii) $k_{\alpha\beta}(x, z) \in C^\infty(\mathbb{R}^n)$ with respect to $x, D_x^\gamma k_{\alpha\beta}(x, z) \rightarrow 0$, as $|x| \rightarrow \infty$, for all γ ;
- (iii) $\int_{|z|=1} H_{\alpha\beta}(z) d\sigma(z) = 0, \int_{|z|=1} k_{\alpha\beta}(x, z) d\sigma(z) = 0$;
- (iv) $\int_{|z|=1} |H_{\alpha\beta}(z)|^2 d\sigma(z) < +\infty, \int_{\mathbb{R}_x^n} \int_{|z|=1} |k_{\alpha\beta}(x, z)|^2 d\sigma(z) dx < +\infty$, where $\sigma(z)$ is an area element of the sphere $\{z \in \mathbb{R}^n \mid |z| = 1\}$.

Let $s \in \mathbb{Z}_+$. Let us consider the following class of pseudodifferential operators in $C_0^\infty(\mathbb{R}^n)$ with symbols $\sigma_A(x, \xi, q)$:

(a) $Au(x, q) = (2\pi)^{-n/2} \int_{\mathbb{R}_\xi^n} e^{i(x,\xi)} \sigma_A(x, \xi, q) Fu(\xi) d\xi,$

where

- (b) $\sigma_A(x, \xi, q) = \sum_{|\alpha|+\beta \leq s} q^\beta g_{\alpha\beta}(x, \xi) \xi^\alpha$;
- (c) $g_{\alpha\beta}(x, \xi) = g_{\alpha\beta}^0(x, \xi) + g_{\alpha\beta}^1(\xi),$

with

- (d) $g_{\alpha\beta}^0(x, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}_z^n} e^{-i(z,\xi)} k_{\alpha\beta}(x, z) dz$;
- (e) $g_{\alpha\beta}^1(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}_z^n} e^{-i(z,\xi)} H_{\alpha\beta}(z) dz.$

Theorem 2.1. *The operator A can be extended to a bounded linear operator from $H_{1,p,q}(\mathbb{R}^n)$ to $H_{l-s,p,q}(\mathbb{R}^n)$ and the estimate*

$$\|Au\|_{l-s,p} \leq C \|u\|_{l,p}, \quad u \in H_{1,p,q}(\mathbb{R}^n) \tag{1}$$

holds true, where C is a constant not depending on u, q .

Theorem 2.2. *Let B_0, B, B' be the Banach spaces with the norms $\|\cdot\|_0, \|\cdot\|, \|\cdot\|'$, respectively. Assume that the embedding $B \hookrightarrow B_0$ is compact and $A : B \rightarrow B'$ is a bounded linear operator. Then $\dim(\text{Ker } A) < +\infty$ and $\text{Im } A$ is closed in B' if and only if $\|u\| \leq C(\|Au\|' + \|u\|_0)$, $u \in B$, where C is a constant not depending on u .*

Proof. See [4]. \square

3. Linear boundary value problem

Now consider the linear boundary value problem

$$Au(x, q) = f(x, q), \quad \text{in } \Omega, \tag{2}$$

$$B_j u(x, q) = g_j(x, q), \quad \text{on } \partial\Omega, \quad j = 1, \dots, s, \tag{3}$$

where A, B_j are pseudodifferential operators of orders $2s, m_j$, respectively. It is assumed that the problem defined by (A, B_j) is elliptic that is A is an elliptic operator and the Shapiro–Lopatinski condition is satisfied (see [2,9]). Moreover they are *admissible*, that is, for instance, the symbol of the principal part A_0 of the operator A has the form: $\sigma_{A_0}(x', 0, \eta', \eta_n) = \sum_{k=0}^{2s} \sigma_{A_{0,k}}(x', \eta') \eta_n^k$, $\sigma_{A_{0,k}}$ is positively homogeneous of degree $2s - k$ in η' , $\sigma_{A_{0,2s}}$ is independent of η' . Here the direction x_n is normal.

Let $1 < p < \infty, \ell \geq \max\{2s, m_j + 1\}$, and put $\mathcal{U} = (A, B_j|_{\partial\Omega}), H_{\ell,p,q}(\Omega, \partial\Omega) = H_{\ell-2s,p,q}(\Omega) \times \prod_{j=1}^s H_{\ell-m_j-1/p,p,q}(\partial\Omega)$ with the norm

$$\|(f, g)\|_{\ell,p} = \|(f, g_1, \dots, g_s)\|_{\ell,p} = \|f\|_{\ell-2s,p} + \sum_{j=1}^s \|g_j\|_{\ell-m_j-1/p,p}.$$

If the problem (2), (3) is elliptic, the operator \mathcal{U} is called elliptic.

The following theorem (see [4,5]) is used to prove theorems in the next sections.

Theorem 3.1. *If \mathcal{U} is elliptic then*

- (i) \mathcal{U} is a bounded linear operator from $H_{\ell,p,q}(\Omega)$ to $H_{\ell,p,q}(\Omega, \partial\Omega)$ and has a parametrix;
- (ii) \mathcal{U} is a Noether operator from $H_{\ell,p,q}(\Omega)$ to $H_{\ell,p,q}(\Omega, \partial\Omega)$;
- (iii) The estimate $\|u\|_{\ell,p,q} \leq C(\|\mathcal{U}u\|_{\ell,p,q} + \|u\|_{0,p,q})$, $u \in H_{\ell,p,q}(\Omega)$, holds true, where C is a constant not depending on u, q .

4. Linear non-classical pseudodifferential boundary value problem

Consider a linear non-classical pseudodifferential boundary value problem

$$Au(x, q) = f(x, q), \quad \text{in } \Omega, \tag{4}$$

$$B_j(D_\nu u(x, q)) = g_j(x, q), \quad \text{on } \partial\Omega, \quad j = 1, 2, \dots, s, \tag{5}$$

where A, B_j are admissible pseudodifferential operators of orders $2s, m_j - 1$, respectively.

If D_ν is a vector field that can be tangent to $\partial\Omega$ on a smooth manifold $\Gamma_0 \subset \partial\Omega$ of dimension $n - 2$ we suppose that it is not tangent to Γ_0 . We use here the classification of Γ_0 given in [7]. If Γ_0 belongs to the first class, then we add the following conditions

$$D_n^k u(x, q) = u_{0k}(x, q), \quad \text{on } \Gamma_0, \quad k = 0, 1, \dots, s - 1. \tag{6}$$

Let $1 < p < \infty$, $\ell \geq \max\{2s, m_j + 1\}$. Denote by $\Pi_{l,p,q}(\Omega)$, $G_{l,p,q}$, $\Lambda_{l,p,q}$ the spaces introduced in [7], with norms depending on complex parameter $q \in Q$.

Theorem 4.1. *If Γ_0 belongs to the second or third class (or Γ_0 belongs to the first class), $u \in H_{l,p,q}(\Omega)$, and u is a solution of problem (4), (5) (or (4)–(6)) and $Au \in H_{l-2s+2,p,q}(\Omega)$, $B_j D_\nu u|_{\partial\Omega} \in H_{l-m_j+2-1/p,p,q}(\partial\Omega)$ (or additionally $D_n^k u|_{\Gamma_0} \in H_{l-k+1-1/p,p,q}(\Gamma_0)$), then $u \in H_{l+1,p,q}(\Omega)$.*

To prove Theorem 4.1. we need the following lemmas.

Lemma 4.2. *Let Γ_0 belong to the first class. For each $\varepsilon > 0$, there exists a small enough neighborhood Q_p of $P \in \Gamma_0$ such that for $u \in H_{\ell,p,q}(\Omega)$ and $u(x, q) = 0$ outside of Q_p the following estimate holds $\|\chi(z) \int_0^{z_1} u(\eta, z_2, \dots, z_n, q) d\eta\|_{\ell,p} < \varepsilon \|u\|_{\ell,p}$, where χ is the characteristic function of the support of u , supposing that Γ_0 is situated in the plane $x_1 = 0$.*

Proof. See [5,7]. \square

In the sequel we denote $Su(z, q) = \chi(z) \int_0^{z_1} u(\eta, z_2, \dots, z_n, q) d\eta$.

Lemma 4.3. *Let Γ_0 belong to the first class. Let U_p be a neighborhood of $P \in \Gamma_0$. Let R be the parametrix of the elliptic problem: $Au = f$ in U_p , $B_j u = g_j$ on ∂U_p , $j = 1, \dots, s$. Set $L_{2s} = AD_\nu - D_\nu A$, $L_0 w = R(L_{2s} w, 0)$. Then, for the neighborhood U_p with small enough diameter we have the following representation: $L_0 S w = L_1 w + L'_1 w$, $S L_0 w = L_2 w + L'_2 w$ where L_1, L_2 are operators from $H_{l,p,q}(U_p)$ to $H_{l,p,q}(U_p)$ having norms $< \frac{1}{2}$, and L'_1, L'_2 are bounded operators from $H_{l,p,q}(U_p)$ to $H_{l+1,p,q}(U_p)$.*

Using Lemmas 4.2, 4.3 and Theorem 3.1 we get Theorem 4.1.

Using now theory of classical boundary value problems, theory of Fredholm operators in Banach spaces, and the Sobolev embedding theorems in $H_{l,p,q}$ one can prove the following theorems on parametrix.

Theorem 4.4. *If Γ_0 belongs to the first class, then the operator*

$$V = (A, B_j D_\nu|_{\partial\Omega}, D_n^{j-1}|_{\Gamma_0}) : \Pi_{l,p,q} \rightarrow \Lambda_{l,p,q},$$

defined by (4)–(6), possesses a right parametrix $R : \Lambda_{l,p,q} \rightarrow \Pi_{l,p,q}$, that is $VR = I + T$, $T : \Lambda_{l,p,q} \rightarrow \Lambda_{l,p,q}$ is a compact operator.

Theorem 4.5. *If Γ_0 belongs to the third class, then the operator $V = (A, B_j D_\nu|_{\partial\Omega}) : \Pi_{l,p,q}(\Omega) \rightarrow \Pi_{l-2s,p,q}(\Omega) \times \prod_{j=1}^s G_{l-m_j-1/p,p,q}(\partial\Omega)$, defined by (4)–(5) possesses a right parametrix*

$$R : \Pi_{l-2s,p,q}(\Omega) \times \prod_{j=1}^s G_{l-m_j-1/p,p,q}(\partial\Omega) \rightarrow \Pi_{l,p,q}(\Omega),$$

that is $VR = I + T$,

$$T : \prod_{l=2s,p,q}(\Omega) \times \prod_{j=1}^s G_{l-m_j-1/p,p,q}(\partial\Omega) \rightarrow \prod_{l=2s,p,q}(\Omega) \times \prod_{j=1}^s G_{l-m_j-1/p,p,q}(\partial\Omega)$$

is a compact operator.

5. Semilinear non-classical pseudodifferential boundary value problem

Now we consider a semilinear non-classical pseudodifferential boundary value problem. Using the Sobolev spaces depending on a complex parameter $q \in Q$, for the above linear non-classical pseudodifferential boundary value problem, from the estimates in smoothness theorems and theorems on parametrix we get immediately uniqueness and existence theorems (with large enough $|q|$) as below.

Theorem 5.1. *Let Γ_0 be a manifold of the first or of the third class. Assume that the operator (A, B_j) is elliptic. Let $f \in \Pi_{l=2s,p,q}(\Omega)$, $g_j \in G_{l-m_j-1/p,p,q}(\partial\Omega)$, and $u_{0k} \in H_{l-k-1/p,p,q}(\Gamma_0)$, if Γ_0 belongs to the first class. Then for large enough $|q|$, there exists a unique solution of the problem (4), (5) or (4)–(6) belonging to $\Pi_{l,p,q}(\Omega)$.*

Let us now consider the semilinear non-classical pseudodifferential boundary value problem

$$Au(x, q) = f(x, q, u, \dots, D^{2s-1}), \quad \text{in } \Omega, \tag{7}$$

$$B_j(D_\nu u(x, q)) = g_j(x, q, u, \dots, D^{m_j-1}u), \quad \text{on } \partial\Omega, j = 1, \dots, s, \tag{8}$$

where A, B_j are pseudodifferential operators in (4), (5). If Γ_0 belongs to the first class then we add the conditions

$$D_n^k u(x, q) = u_{0k}(x, q), \quad k = 0, \dots, s - 1, \text{ on } \Gamma_0. \tag{9}$$

Using Theorem 5.1 and the Schauder’s fixed point theorem, one can prove the following main theorem.

Theorem 5.2. *Let Γ_0 be the manifold of first class. Assume that the operator (A, B_j) is elliptic, the functions f, g_j are measurable and satisfy (in a local coordinate system at x)*

- (i) $|E(u, \xi, q)| \leq M(h_1(\xi, q) + \sum_{k=0}^{2s-1} |q|^k |\xi|^{2s-1-k} |Fu(\xi, q)|)$,
- (ii) $|E(u, \xi, q) - E(v, \xi, q)| \leq M \sum_{k=0}^{2s-1} |q|^k |\xi|^{2s-1-k} |Fu(\xi, q) - Fv(\xi, q)|$,
- (iii) $|G_j(u, \xi', q)| \leq M(h_{2j}(\xi', q) + \sum_{k=0}^{m_j-1} |q|^k |\xi'|^{m_j-1-k} |F'u(\xi', q)|)$,
- (iv) $|G_j(u, \xi', q) - G_j(v, \xi', q)| \leq M \sum_{k=0}^{m_j-1} |q|^k |\xi'|^{m_j-1-k} |F'u(\xi', q) - F'v(\xi', q)|$, with

$$E(u, \xi, q) = F(f(x, q, u(x, q), \dots, D^{2s-1}u(x, q)))(\xi, q),$$

$$G_j(u, \xi', q) = F'(g_j(x', q, u(x', 0, q), \dots, D^{m_j-1}u(x', 0, q)))(\xi', q),$$

where M is a constant, $h_1(\xi, q) \geq 0, h_{2j}(\xi', q) \geq 0$,

$$\int_{\mathbb{R}^n} (1 + |\xi| + |q|)^{(l-2s)p} |h_1(\xi, q)|^p d\xi < L^p, \quad \int_{\mathbb{R}^{n-1}} (1 + |\xi'| + |q|)^{(l-m_j)p} |h_{2j}(\xi', q)|^p d\xi' < L^p.$$

Then, if $u_{0k} \in H_{l-k-1/p,p,q}(\Gamma_0)$, $k = 0, 1, \dots, s - 1$, the problem (7)–(9) has a unique solution $u \in \Pi_{l,p,q}$, for sufficiently large $|q|$.

Proof. For each $\omega \in \Pi_{l,p,q}$, the problem $Au(x, q) = f(x, q, \omega, \dots, D^{2s-1}\omega)$, $x \in \Omega$, $B_j D_\nu u(x, q) = g_j(x, q, \omega, \dots, D^{m_j-1}\omega)$, $x \in \partial\Omega$, $j = 1, \dots, s$, $D_n^j u(x, q) = u_{0j}(x, q)$, $x \in \Gamma_0$, $j = 0, 1, \dots, s-1$, possesses a solution $J\omega \in \Pi_{l,p,q}$, for sufficiently large $|q|$. We obtain

$$\begin{aligned} \|J\omega\|_{\Pi_{l,p,q}} &\leq C_1 \left(\|f\|_{\Pi_{l-2s,p,q}} + \sum_{j=1}^s \|g_j\|_{G_{l-m_j-1/p,p,q}} + \sum_{j=0}^{s-1} \|u_{0j}\|_{l-j-1/p,p,\Gamma_0} \right) \\ &\leq C_2 \left(L + \|\omega\|_{\Pi_{l-1,p,q}} + \sum_{j=1}^s (L + \|\omega\|_{\Pi_{l-1,p,q}}) + \sum_{j=0}^{s-1} \|u_{0j}\|_{l-j-1/p,p,\Gamma_0} \right) \\ &\leq C_3 + C_4 \|\omega\|_{\Pi_{l-1,p,q}} \leq C_3 + C_4 |q|^{-1} \|\omega\|_{\Pi_{l,p,q}}. \end{aligned} \quad (10)$$

Note that

$$\begin{aligned} \|J\omega_1 - J\omega_2\|_{\Pi_{l,p,q}} &\leq C_5 \left(\|f(x, q, \omega_1, \dots, D^{2s-1}\omega_1) - f(x, q, \omega_2, \dots, D^{2s-1}\omega_2)\|_{\Pi_{l-2s,p,q}} \right. \\ &\quad \left. + \sum_{j=1}^s \|g_j(x', q, \omega_1, \dots, D^{m_j-1}\omega_1) - g_j(x', q, \omega_2, \dots, D^{m_j-1}\omega_2)\|_{G_{l-m_j-1/p,p,q}} \right) \\ &\leq C_6 \|\omega_1 - \omega_2\|_{\Pi_{l-1,p,q}}. \end{aligned} \quad (11)$$

From (10), there exists a positive number R such that $J : S \rightarrow S$, $S = \{\omega \in \Pi_{l,p,q} \mid \|\omega\|_{\Pi_{l,p,q}} \leq R\}$, and by (11), $J : S \rightarrow S$ is continuous.

Assume that $\{\omega_n\} \subset S$. Because the embedding $S \hookrightarrow \Pi_{l-1,p,q}$ is compact, $\{\omega_n\}$ is relatively compact in $\Pi_{l-1,p,q}$. Consequently by (11), $\{J\omega_n\}$ is relatively compact in $\Pi_{l,p,q}$. It follows that $J : S \rightarrow S$ is compact, while S is convex, closed, bounded, so by Schauder's fixed point theorem (see [6], p. 60), J possesses a fixed point $v \in S$, which is the solution of (7)–(9). By (11) the fixed point of J is unique, i.e., $v \in S$ is thus a unique solution of (7)–(9). \square

In the same way one can study the case when Γ_0 belongs to the third class.

References

- [1] R.A. Adams, Sobolev Spaces, Academic Press, 1975.
- [2] S. Agmon, A. Douglis, L. Nirenberg, Comm. Pure Appl. Math. (1959) 624–727.
- [3] M.S. Agranovich, Uspekhi Mat. Nauk 20 (5) (1965) 3–120.
- [4] Nguyen Minh Chuong, Dang Anh Tuan, Preprint 2002/28, Institute of Mathematics, Hanoi, 2002.
- [5] Nguyen Minh Chuong, Tran Tri Kiet, Preprint 2003/04, Institute of Mathematics, Hanoi, 2003, Uspekhi Mat. Nauk (in Russian), submitted for publication.
- [6] K. Deimling, Nonlinear Analysis, Springer-Verlag, 1985.
- [7] Yu.V. Egorov, V.A. Kondratiev, Mat. Sb. 78(120) (1) (1969) 148–176.
- [8] Yu.V. Egorov, Nguyen Minh Chuong, Uspekhi Mat. Nauk 53 324 (6) (1998) 249–250.
- [9] Yu.V. Egorov, Mat. Sb. 73 (1967) 356–374.
- [10] Tran Tri Kiet, Preprint 2003/03, Institute of Mathematics, Hanoi, 2003, Differentsial'nye Uravneniya (in Russian), submitted for publication.
- [11] F. Trèves, Introduction to Pseudodifferential and Fourier Integral Operators, Plenum Press, New York, 1982.
- [12] Le Quang Trung, Uspekhi Mat. Nauk 44 (5) (1989) 169–170.