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Abelian fibrations on $S^{[n]}$

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Abstract

Let $S \xrightarrow{\phi} \mathbb{P}^1$ be an elliptic fibration on a $K3$ surface S. Then the composition $S^{[n]} \xrightarrow{\pi} S^{(n)} \xrightarrow{\mathrm{sym}^n\phi} \mathbb{P}^n$ gives an Abelian fibration on $S^{[n]}$. Let E be the exceptional divisor of π, then $\mathrm{sym}^n\phi \circ \pi(E)$ is of dimension $n-1$. We prove the inverse in this Note. ***To cite this article: B. Fu, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Fibrations abéliennes sur $S^{[n]}$. Soit $S \xrightarrow{\phi} \mathbb{P}^1$ une fibration elliptique sur une surface S, $K3$. Alors la composition $S^{[n]} \xrightarrow{\pi} S^{(n)} \xrightarrow{\mathrm{sym}^n\phi} \mathbb{P}^n$ donne une fibration abélienne sur $S^{[n]}$. Soit E le diviseur exceptionel de π, alors $\mathrm{sym}^n\phi \circ \pi(E)$ est de dimension $n-1$. Dans cette Note, nous démontrons la réciproque. ***Pour citer cet article : B. Fu, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

1. Introduction

Let X be a $2n$-dimensional irreducible symplectic manifold. Recall that an *Abelian fibration* on X is a proper surjective morphism $X \to \mathbb{P}^n$ whose generic fiber is a smooth Abelian variety. This is more or less the only non trivial fibration structure that could exist on X, owing to a result of Matsushita [5]. To understand Abelian fibrations on holomorphic symplectic manifolds is one of three-part programme to understand the mysteries of holomorphic symplectic manifolds (see, for example, [7]).

As remarked by Hassett and Tschinkel (Remark 5.6, [2]), the existence of an Abelian fibration on the Hilbert scheme $S^{[2]}$ of a $K3$ surface S does not imply that S admits an Abelian fibration, i.e., it does not imply that S is an elliptic $K3$ (compare [4]). A classical example is the following (communicated to the author by A. Beauville): let $S \subset \mathbb{P}^5$ be the intersection of three quadrics $Q_1 = 0$, $Q_2 = 0$ and $Q_3 = 0$, which does not contain any line. If we take a general such S, then $\mathrm{Pic}(S)$ has rank 1, thus it contains no non-trivial divisor with zero self-intersection, i.e., S is not elliptic. An Abelian fibration on $S^{[2]}$ can be constructed as follows: any point $I \in S^{[2]}$ defines a line

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in $\mathbb{P}^5$, which is not contained in S. Then there exists a unique plan in $V = \mathbb{C}\langle Q_1, Q_2, Q_3\rangle$ which vanishes on this line. This gives an Abelian fibration: $S^{[2]} \to \mathbb{P}(V^*) \simeq \mathbb{P}^2$.

The purpose of this Note is to study Abelian fibrations on $S^{[n]}$. If S is elliptic, i.e., there exists an elliptic fibration $\phi : S \to \mathbb{P}^1$, then the composition,

$$S^{[n]} \xrightarrow{\pi} S^{(n)} \xrightarrow{\mathrm{sym}^n\phi} \mathrm{Sym}^n\mathbb{P}^1 \simeq \mathbb{P}^n,$$

gives an Abelian fibration on $S^{[n]}$. If we denote by $E \subset S^{[n]}$ the exceptional divisor of π, then the image of E by $\mathrm{sym}^n(\phi) \circ \pi$ is of dimension $n-1$ in $\mathbb{P}^n$. Our aim of this Note is to prove the inverse.

Theorem 1.1. *Let $S^{[n]} \xrightarrow{f} \mathbb{P}^n$ be an Abelian fibration on $S^{[n]}$. Suppose that $\dim(f(E)) \leqslant n-1$, then S is elliptic and f is isomorphic to an Abelian fibration coming from an elliptic fibration $S \xrightarrow{\phi} \mathbb{P}^1$.*

2. Abelian varieties contained in products of $K3$ surfaces

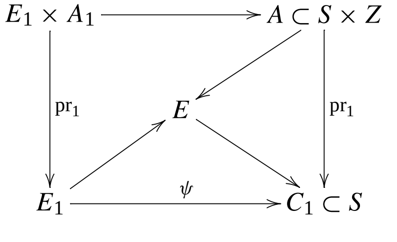
Lemma 2.1. *Let A be an Abelian variety and S a $K3$ surface. Then there exists no surjective morphism from A to S.*

Proof. Suppose we had a surjective morphism $f : A \to S$. By Stein factorization, there exist a normal surface B, a finite morphism $f_1 : B \to S$ and a morphism with connected fibers $f_2 : A \to B$ such that $f = f_1 \circ f_2$. Notice that f_2 has connected fibers, so by the rigidity lemma [6], there exists an Abelian subvariety A_0 of A, such that $f_2^{-1}(f_2(a)) = a + A_0$ for any $a \in A$. This implies that B is isomorphic to A/A_0, thus it is an Abelian surface.

Now consider the finite morphism $f_1 : B \to S$, which induces $f_{1*}f_1^* = \deg(f_1)\,\mathrm{Id}$ in cohomology. Thus $f_1^* : H^2(S, \mathbb{R}) \to H^2(B, \mathbb{R})$ is injective, this gives $b_2(S) \leqslant b_2(B)$, which is absurd since $b_2(S) = 22$ and $b_2(B) = 6$. □

Lemma 2.2. *Let A be an Abelian variety contained in the product $S \times Z$, where S is a $K3$ surface and Z an algebraic variety. Then either A is contained in $\{p\} \times Z$ for some point $p \in S$ or A is isogeny to $E_1 \times A_1$, where E_1 is an elliptic curve and A_1 is an Abelian variety contained in $\{p'\} \times Z$ for some point $p' \in S$.*

Proof. Consider the projection $\mathrm{pr}_1 : A \subset S \times Z \to S$. If $\mathrm{Img}(\mathrm{pr}_1)$ is just a point $p \in S$, then A is contained in $\{p\} \times Z$. If $\mathrm{Img}(\mathrm{pr}_1)$ is not a point, it is a curve $C_1 \subset S$, by the above lemma. Now by the Stein factorization, there exist a normal curve E, a finite morphism $E \to C_1$ and a morphism with connected fibers $A \to E$. The argument in the proof of Lemma 2.1 shows that E is an elliptic curve. Now by Poincaré's theorem on complete reducibility, there exist an elliptic curve E_1, a finite morphism $E_1 \to E$ and an Abelian variety A_1, such that A is isogeny to $E_1 \times A_1$ and the following diagram commutes:



If we take the identity point $e \in E_1$, then $\{e\} \times A_1$ is contained in $\mathrm{pr}_1^{-1}(\psi(e)) \subset \{\psi(e)\} \times Z$, thus we can chose A_1 such that A_1 is contained in $\{p'\} \times Z$ for some point $p' \in S$. □

Theorem 2.3. *Let A be a k-dimensional Abelian variety contained in a product of $K3$ surfaces $S_1 \times \cdots \times S_n$. Then A is isomorphic to a product of elliptic curves $E_1 \times \cdots \times E_k$, with $E_i \subset S_{k_i}$.*

Proof. Applying the above lemma, an induction argument shows that A is isogeny to a product of elliptic curves $E_1 \times \cdots \times E_k$. Re-ordering the index if necessary, we can suppose that E_i projects onto a curve C_i on S_i. The above lemma also shows that E_k can be chosen on S_k. Now we show that E_{k-1} can also be chosen to be an elliptic curve on S_{k-1}.

Let B be the Abelian surface contained in $S_{k-1} \times S_k$, which is the image of $E_{k-1} \times E_k$. Applying Lemma 2.2 with $S = S_k$ and $Z = S_{k-1}$, then we can chose E_{k-1} to be a curve on S_{k-1}. Now the projection curve C_{k-1} (resp. C_k) should be E_{k-1} (resp. E_k), thus B is isomorphic to $E_{k-1} \times E_k$.

An induction with the above arguments concludes the proof. □

Remark 1. It is proved by Hwang and Mok (see [3]) that if $B \to S$ is a finite morphism from an Abelian surface to a projective surface S, then S is either an Abelian surface, a $\mathbb{P}^1$-bundle over a curve or $\mathbb{P}^2$. Using this result and above arguments, the theorem still holds if we replace $K3$ surfaces S_i by surfaces which is neither an Abelian surface, a $\mathbb{P}^1$-bundle over a curve nor $\mathbb{P}^2$.

3. Proof of Theorem 1.1

Let A be a general fiber of f, which is an Abelian variety. By hypothesis, $\dim(f(E)) \leqslant n-1$, A is contained in $S^{[n]} - E$, and the latter can be identified with $S^{(n)} - \delta$, where δ is the big diagonal. Notice that $q : S^n - \Delta \to S^{(n)} - \delta$ is an unbranched covering of order n, where Δ is the preimage of δ in S^n. Then $q^{-1}(A)$ is an unbranched covering of A of order n. Let A_1 be any connected component of $q^{-1}(A)$, which is still an Abelian variety, contained in S^n.

Now by our Theosrem 2.3, A_1 is isomorphic to products $E_1 \times \cdots \times E_n$, where E_i are elliptic curves on S. Notice that $A \cap \delta = \emptyset$, thus E_i and E_j have no common points if $i \neq j$, thus $E_i,\ i = 1, \ldots, n$, are fibers of an elliptic fibration $\phi : S \to \mathbb{P}^1$. In particular, S is elliptic.

Take another general fiber of f, then the arguments above give another elliptic pencil on S with fibers $E'_i,\ i = 1, \ldots, n$. If $E'_{i_0} \cdot E_{j_0} \neq 0$, then $E'_i \cdot E_j \neq 0$ for any i, j, thus there exists a point $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n \cap E'_1 \times \cdots \times E'_n$, which contradicts to the fact that the intersection of two fibers is empty. Thus $E'_i,\ i = 1, \ldots, n$, are all fibers of $\phi : S \to \mathbb{P}^1$, i.e., all elliptic fibrations defined in this way on S are the same.

This implies that there exists an open set $U = S - \phi(W)$ for some closed subvariety $W \subset \mathbb{P}^1$ such that every fiber of the composition $U^n - \Delta_U \to U^{(n)} - \delta_U \xrightarrow{f} \mathbb{P}^n$ is an Abelian variety, thus it is of the form $\phi^{-1}(x_1) \times \cdots \times \phi^{-1}(x_n)$. This gives a birational automorphism $\beta : \mathbb{P}^n - \to \mathbb{P}^n$, such that the following diagram commutes:

$$\begin{array}{ccc} & U^{(n)} - \delta_U \subset S^{[n]} & \\ \swarrow^{f} & & \searrow^{\mathrm{sym}^n(\phi)} \\ \mathbb{P}^n & - - \beta \to & \mathbb{P}^n \end{array}$$

Thus the two birational morphisms $\mathrm{sym}^n(\phi) : S^{[n]} \to \mathbb{P}^n$ and $\beta \circ f : S^{[n]} - - \to \mathbb{P}^n$ agree over an open set of $S^{[n]}$, which shows $\beta \circ f = \mathrm{sym}^n(\phi)$ over the whole of $S^{[n]}$. In particular, $\beta : \mathbb{P}^n \to \mathbb{P}^n$ is a birational morphism, thus an isomorphism, which concludes the theorem. □

4. Another proof

Here we want to give a quick proof of the following part of Theorem 1.1, which is communicated to the author by A. Beauville.

Theorem 4.1. *Let* $S^{[n]} \xrightarrow{f} \mathbb{P}^n$ *be an Abelian fibration on* $S^{[n]}$*. Suppose that* $\dim(f(E)) \leqslant n-1$*, then* S *is elliptic.*

Proof. Let $q_S(-)$ be the Beauville–Bogomolov form on holomorphic symplectic varieties. Then we have $\mathrm{Pic}(S^{[n]}) \simeq \mathrm{Pic}(S) \oplus \mathbb{Z} \cdot [E/2]$ which is also orthogonal with respect to the quadric form $q_S(-)$ (see [1]). Take a hyperplane class h on $\mathbb{P}^n$. By Fujiki's formula, we have $[q_S(E + tf^*h)]^n = c_n(E + tf^*h)^{2n}$ for some constant c_n. Notice that $q_S(E+tf^*h) = q_S(E)+2tq_S(E, f^*h)$, then by comparing the coefficient of t^n, we have $q_S(f^*h, E) = cE^n \cdot (f^*h)^n$ for some constant c. Notice that $(f^*h)^n$ is nothing but fibers of the fibration f. By hypothesis $\dim(f(E)) \leqslant n-1$, the general fibers have empty intersection with E, thus $q_S(f^*h, E) = cE^n \cdot (f^*h)^n = 0$. This implies that $f^*h \in \mathrm{Pic}(S^{[n]})$ comes from some divisor D on S. Furthermore, $D \cdot D = q_S(f^*h) = 0$, thus S is elliptic. □

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