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Jensen's inequality for g-expectation, Part 2

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Abstract

Chen et al. (C. R. Acad. Sci. Paris, Ser. I 337 (11) (2003)) studied a Jensen's inequality for g-expectation under the assumption that g does not depend on (t, y). In this Note we consider some applications of this inequality. ***To cite this article: Z. Chen et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***
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Résumé

L'inégalité de Jensen pour la g-espérance, 2ème partie. Chen et al. (C. R. Acad. Sci. Paris, Ser. I 337 (11) (2003)) a étudié l'inégalité de Jensen pour la g-espérance sous la prétention que g n'est pas fonction de (t, y). Comme suite a cette étude, nous considérons les applications de l'inégalité de Jensen dans cet article. ***Pour citer cet article : Z. Chen et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***
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Version française abrégée

Dans [3], nous démontrons une inégalité de Jensen sous la prétention que $g:\mathbf{R}^d \to \mathbf{R}$ est une fonction convexe. En fait, si on suppose que g a les mêmes propriétés que dans Briand et al. [2], l'inégalité de Jensen se tiendrait toujours. Dans cet article, nous considérons quelques applications de l'inégalité de Jensen.

1. Introduction

In [3], we show a Jensens's inequality under the assumption $g:\mathbf{R}^d \to \mathbf{R}$ is a convex function. In fact, if g is of the same form as in Briand et al. [2], i.e., $g:\Omega \times [0, T] \times \mathbf{R}^d \to \mathbf{R}$ is continuous in t and for each $z \in \mathbf{R}^d$, $\{g(\cdot, z)\}$

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is a progressively measurable process and $g(\omega, t, z)$ is convex with respect to z, then Jensens's inequality still is true. In this Note, we consider some applications of Jensen's inequality.

2. Main result

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(B_t)_{t\geqslant 0}$ be a d-dimensional standard Brownian motion with respect to filtration $(\mathcal{F}_t)_{t\geqslant 0}$ generated by the Brownian motion and all P-null subsets, i.e., $\mathcal{F}_t = \sigma\{B_s,\ s \in [0, t]\} \vee \mathcal{N}$, $t \in [0, T]$, where $\mathcal{N}$ is the set of all P-null subsets. Fix a real number $T > 0$. We assume that $\mathcal{F}_T = \mathcal{F}$. Denote $L^2(\Omega, \mathcal{F}, P) := \{\xi\colon \xi \text{ is a } \mathcal{F}_T\text{-measurable and } E\xi^2 < \infty\}$.

Let $g(\omega, t, zt) : \Omega \times [0, T] \times \mathbf{R}^d \mapsto \mathbf{R}$ satisfy the following conditions:

(A1) (Lipschitz Condition) There exists a constant $K > 0$, such that $\forall z_1, z_2 \in \mathbf{R}^d$

$$\left|g(\omega, t, z_1) - g(\omega, t, z_2)\right| \leqslant K|z_1 - z_2|, \quad t \geqslant 0,$$

where $|z|$ is the norm of $z \in \mathbf{R}^d$.

(A2) g is continuous in t and $\forall t \in [0, T]$, $g(t, 0) \equiv 0$.

Let $\mathcal{E}_g(\cdot)$ be the g-expectations in [3] or [5]. The remainder of the notation in this Note is from [3]. Under the above assumptions, Chen et al. [3] studied a Jensen's inequality for g-expectation under the assumption that g does not depend on t. This Note is a continuation of [3], and considers applications of Jensen's inequality for g-expectations. In fact, Theorem 3.1 in [3] is still true when g depends on t. The function g is said to be positively homogeneous if for each $z \in \mathbf{R}^d$ and any positive real number $\lambda \geqslant 0$, then $g(t, \lambda z) = \lambda g(t, z)$, $\forall t \in [0, T]$.

Theorem 2.1. *Let g satisfy* (A1) *and* (A2). *Suppose $\xi \in L^2(\Omega, \mathcal{F}, P)$ and $\varphi : \mathbf{R} \to \mathbf{R}$ is a convex function such that $\varphi(\xi) \in L^2(\Omega, \mathcal{F}, P)$. Then*

(i) *Jensen's inequality*

$$\varphi\big(\mathcal{E}_g[\xi|\mathcal{F}_t]\big) \leqslant \mathcal{E}_g\big[\varphi(\xi)|\mathcal{F}_t)\big], \quad 0 \leqslant t \leqslant T, \tag{1}$$

holds if and only if g is uniformly positively homogeneous with respect to z;

(ii) *If $d = 1$, the necessary and sufficient condition of Jensen's inequality* (1) *to hold is that there exist two bounded adapted processes $a \geqslant 0$ and b such that $g(t, z) = a_t|z| + b_t z$ a.e., $t \in [0, T]$.*

Proof. Applying the measurable maximum principle (see Aliprantis and Boder [1, Theorem 14.91], Revuz and Yor [6, p. 44]), for each t and z there exists $b_t(z)$ which is $\mathcal{F}_t$-measurable, such that $b_t(z) \cdot z = g(t, z)$. The rest can be proved in a similar manner as Theorem 3.1 in [3].

Example 1. Let z^i be the i-th component of $z \in \mathbf{R}^d$, $i = 1, \ldots, d$. It is easy to check the following functions are convex and positively homogeneous:

$$g(z) = \sum_{i=1}^{d} \left|z^i\right|; \qquad g(z) = \sqrt{\sum_{i=1}^{d} \left|z^i\right|^2}.$$

Thus Jensen's inequality (1) is true for the related g-expectations.

Remark 1. Note that for $g \equiv 0$, the g-expectation is the classical mathematical expectation. Moreover, if g is nonlinear, then the g-expectation usually is nonlinear. In this regard, Theorem 2.1 extends the classical Jensen's inequality.

Corollary 2.2. *Suppose H is a bounded, convex and closed subset of $\mathbf{R}^d$ and $D :=$ the set of $\mathbf{R}^d$-valued progressively measurable processes $\{v_t\}$ such that for each $t \geqslant 0$, $v_t \in H$ a.s. Define the probability measure Q^v by*

$$\frac{\mathrm{d}Q^v}{\mathrm{d}P} := \exp\left\{-\frac{1}{2}\int_0^T |v_s|^2\,\mathrm{d}s + \int_0^T v_s \cdot \mathrm{d}B_s\right\}. \tag{2}$$

Thus for any convex function φ

$$\varphi\Big(\sup_{v\in D} E_{Q^v}[\xi]\Big) \leqslant \sup_{v\in D} E_{Q^v}\big[\varphi(\xi)\big]$$

whenever $\xi, \varphi(\xi) \in L^2(\Omega, \mathcal{F}, P)$.

Proof. $g(z) = \sup_{v\in D} v \cdot z$ is convex and positively homogeneous. From [4], $\sup_{v\in D} E_{Q^v}[\xi] = \mathcal{E}_g[\xi]$ is a g-expectation. Applying Theorem 3.1 in [1] yields the result.

The proof is now complete.

Definition 2.3 (*Peng* [5]). A square integrable adapted process $\{X_t\}$ is called a g-martingale (sub-martingale), if for any $s \geqslant t$

$$\mathcal{E}_g[X_s|\mathcal{F}_t] = (\geqslant) X_t.$$

Applying Jensen's inequality, immediately we have

Corollary 2.4. *Let g satisfy the conditions of Theorem* 2.1. *If $\{X_t\}$ is a g-martingale and φ is a convex function such that $\varphi(X_t) \in L^2(\Omega, \mathcal{F}, P)$, then $\{\varphi(X_t)\}$ is a g-sub-martingale.*

Many results in mathematical finance depend on the notion of complete market models. In an incomplete market, sometimes we need to calculate the upper (lower) bound of the pricing of contingent claim for European option (see [4] for details), for example, if the price of a stock is a geometric Brownian motion with initial value x:

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t. \tag{3}$$

We sometimes need to calculate

$$\sup_{Q\in\mathcal{P}} E_Q\big[(S_T - k)^+\big],$$

where $\mathcal{P}$ is a subset of all equivalent martingale measures with respect to P. In a complete market model $\mathcal{P}$ is a set of size 1. In an incomplete market model $\mathcal{P}$ is an infinite set. The calculation usually is very complicated, but using Corollary 2.2 we have the following property:

Corollary 2.5. *Let μ and $\sigma > 0$ in* (3) *be two constants and $\mathcal{P}$ be the set of all probability measures Q^v defined in* (2) *and with $v \in D$, where D is the set of all adapted bounded processes such that $|v_t| \leqslant |\mu|/\sigma$. Then*

$$\operatorname{ess\,sup}_{v\in D} E_{Q^v}\big[(S_T - k)^+|\mathcal{F}_t\big] \geqslant (S_t - k)^+,$$

where k is called the strike price. In particular when $t = 0$ and $S_0 = x$

$$\operatorname{ess\,sup}_{v\in D} E_{Q^v}\big[(S_T - k)^+\big] \geqslant (x - k)^+. \tag{4}$$

Proof. Let $y_t := \operatorname{ess\,sup}_{v \in D} E_{Q^v}[(S_T - k)^+ | \mathcal{F}_t]$. Then y_t is the solution of BSDE:

$$y_t = (S_T - k)^+ + \int_t^T \frac{|\mu|}{\sigma} |z_s| \, \mathrm{d}s - \int_t^T z_s \, \mathrm{d}B_s.$$

If we choose $g(z) := \frac{|\mu|}{\sigma}|z|$, then from [4]

$$\operatorname{ess\,sup}_{v \in D} E_{Q^v}\big[(S_T - k)^+ | \mathcal{F}_t\big] = \mathcal{E}_g\big[(S_T - k)^+ | \mathcal{F}_t\big].$$

On the other hand, let (y_t, z_t) be the solution of the following BSDE:

$$y_t = S_T - k + \int_t^T \frac{|\mu|}{\sigma} |z_s| \, \mathrm{d}s - \int_t^T z_s \, \mathrm{d}B_s.$$

It is easy to check that if $\mu > 0$, then $(y_t, z_t) = (S_t - k, \sigma S_t)$ and if $\mu < 0$ then $y_t \geqslant S_t - k$. Applying Jensen's inequality yields

$$\operatorname{ess\,sup}_{v \in D} E_{Q^v}\big[(S_T - k)^+ | \mathcal{F}_t\big] = \mathcal{E}_g\big[(S_T - k)^+ | \mathcal{F}_t\big] \geqslant \big(\mathcal{E}_g\big[(S_T - k) | \mathcal{F}_t\big]\big)^+ \geqslant (S_t - k)^+.$$

Taking $t = 0$ gives (4).

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