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Mathematical Problems in Mechanics

An example of nonuniqueness for the continuous static unilateral contact model with Coulomb friction

Patrick Hild

Laboratoire de mathématiques de Besançon, Université de Franche-Comté/CNRS UMR 6623, 16, route de Gray, 25030 Besançon, France

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Abstract

The aim of this Note is to propose an example of nonuniqueness for the continuous static unilateral contact model with Coulomb friction in linear elasticity. ***To cite this article: P. Hild, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Un exemple de non-unicité pour le modèle continu statique de contact unilatéral avec frottement de Coulomb. Le but de cette Note est de proposer un exemple de non-unicité pour le modèle continu statique de contact unilatéral avec frottement de Coulomb en élasticité linéaire. ***Pour citer cet article : P. Hild, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

1. Introduction

The Coulomb model [2] is the most common law of friction used in solid mechanics in order to describe slipping or sticking bodies on a contact surface. This law is very often coupled with the unilateral contact model which takes into account the possible separation of the body from the surface. In the simple case of elastostatics, the variational formulation of the unilateral contact problem with Coulomb friction (see [3,4]) was followed some years later by an existence result in the case of an infinitely long strip with small friction (see [8]). These results were generalized with more classical geometries and the bounds ensuring existence were improved, particularly in references [7] and [5]. Nevertheless, this (simple) model shows numerous mathematical difficulties so that there does not exist, to our knowledge neither uniqueness results nor nonuniqueness or nonexistence examples.

The aim of this Note is to propose a simple example of nonuniqueness for the continuous unilateral contact model with Coulomb friction for a linear elastic body lying on a rigid foundation. This example admits at least two solutions provided the friction coefficient is greater than a critical value. Moreover these two solutions (one which

E-mail address: patrick.hild@math.univ-fcomte.fr (P. Hild).

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separates from the foundation, and another corresponding to adherence on the foundation) do not depend on the friction coefficient.

2. Problem set-up

Let us consider the deformation of an elastic body occupying, in the initial unconstrained configuration, a domain Ω in $\mathbb{R}^2$. The boundary $\partial\Omega$ of Ω consists of Γ_D, Γ_N and Γ_C where the measure of Γ_D does not vanish. The body Ω is submitted to given displacements $\mathbf{U}$ on Γ_D, it is subjected to surface traction forces $\mathbf{F}$ on Γ_N and the body forces are denoted $\mathbf{f}$. In the initial configuration, the part Γ_C is considered as the candidate contact surface on a rigid foundation which means that the contact zone cannot enlarge during the deformation process. The contact is assumed to be frictional and the stick, slip and separation zones on Γ_C are not known in advance. We denote by $\mu > 0$ the given friction coefficient on Γ_C. The unit outward normal and tangent vectors on $\partial\Omega$ are $\mathbf{n} = (n_x, n_y)$ and $\mathbf{t} = (-n_y, n_x)$ respectively.

The unilateral contact problem with Coulomb's friction law consists of finding the displacement field $\mathbf{u} : \Omega \to \mathbb{R}^2$ satisfying (1)–(6):

$$\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} \quad \text{in } \Omega, \tag{1}$$

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{U} \quad \text{on } \Gamma_D, \tag{3}$$

$$\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{F} \quad \text{on } \Gamma_N. \tag{4}$$

The notation $\boldsymbol{\sigma}(\mathbf{u}) : \Omega \to S_2$ represents the stress tensor field lying in S_2, the space of second order symmetric tensors on $\mathbb{R}^2$. The linearized strain tensor field is $\boldsymbol{\varepsilon}(\mathbf{u}) = (\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}^T\mathbf{u})/2$ and $\mathbf{C}$ is the fourth order symmetric and elliptic tensor of linear elasticity.

We now choose the following notation for any displacement field $\mathbf{u}$ and for any density of surface forces $\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}$ defined on $\partial\Omega$:

$$\mathbf{u} = u_n\mathbf{n} + u_t\mathbf{t} \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \sigma_n(\mathbf{u})\mathbf{n} + \sigma_t(\mathbf{u})\mathbf{t}.$$

On Γ_C, the three conditions representing unilateral contact are as follows:

$$u_n \leqslant 0, \qquad \sigma_n(\mathbf{u}) \leqslant 0, \qquad \sigma_n(\mathbf{u})u_n = 0, \tag{5}$$

and the Coulomb friction law on Γ_C is described by the following conditions:

$$\begin{cases} u_t = 0 \Longrightarrow \left|\sigma_t(\mathbf{u})\right| \leqslant \mu\left|\sigma_n(\mathbf{u})\right|, \\ u_t \neq 0 \Longrightarrow \sigma_t(\mathbf{u}) = -\mu\left|\sigma_n(\mathbf{u})\right|\dfrac{u_t}{|u_t|}. \end{cases} \tag{6}$$

Remark 1. Let us mention that the true Coulomb friction law involves the tangential contact velocities and not the tangential displacements. However, a problem analogous to the one discussed here is obtained by time discretization of the quasi-static frictional contact evolution problem. In this case (see [1]) $\mathbf{u}$, $\mathbf{f}$ and $\mathbf{F}$ stand for $\mathbf{u}((i+1)\Delta t)$, $\mathbf{f}((i+1)\Delta t)$ and $\mathbf{F}((i+1)\Delta t)$ respectively and u_t has to be replaced by $u_t((i+1)\Delta t) - u_t(i\Delta t)$, where Δt denotes the time step. For simplicity, and without any loss of generality only the static case described above will be considered in the following.

The variational formulation of problem (1)–(6) consists of finding $\mathbf{u} \in \mathbf{K}$ satisfying (see [3,4]):

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - \int_{\Gamma_C} \mu\sigma_n(\mathbf{u})\big(|v_t| - |u_t|\big)\,\mathrm{d}\Gamma \geqslant L(\mathbf{v} - \mathbf{u}), \quad \forall \mathbf{v} \in \mathbf{K}, \tag{7}$$

where

$$a(\mathbf{u},\mathbf{v}) = \int_{\Omega} \big(\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})\big) : \boldsymbol{\varepsilon}(\mathbf{v})\, \mathrm{d}\Omega, \qquad L(\mathbf{v}) = \int_{\Omega} \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{F}\cdot\mathbf{v}\,\mathrm{d}\Gamma,$$

for any $\mathbf{u}$ and $\mathbf{v}$ in the Sobolev space $(H^1(\Omega))^2$. In these definitions the notations $\cdot$ and $:$ represent the canonical inner products in $\mathbb{R}^2$ and S_2 respectively.

In (7), the set $\mathbf{K}$ stands for the convex of admissible displacement fields:

$$\mathbf{K} = \big\{\mathbf{v} \in \big(H^1(\Omega)\big)^2;\ \mathbf{v} = \mathbf{U} \text{ on } \Gamma_D,\ v_n \leqslant 0 \text{ on } \Gamma_C\big\}.$$

As far as we know there only exist existence results in the case of small friction coefficients (see [8,7,5]) for problem (1)–(6) and there are neither uniqueness results (unless the loads $\mathbf{f}$, $\mathbf{F}$ and $\mathbf{U}$ are equal to zero) nor nonuniqueness or nonexistence examples available.

3. A nonuniqueness example

We consider the triangle Ω of vertexes $A = (0,0)$, $B = (1,0)$ and $C = (3/4, 1/4)$ and we define $\Gamma_D = [B, C]$, $\Gamma_N = [A, C]$, $\Gamma_C = [A, B]$. The body Ω lies on the rigid foundation, the half-space delimited by the straight line (A, B). We suppose that the body Ω is governed by Hooke's law concerning homogeneous isotropic materials so that (2) becomes

$$\boldsymbol{\sigma}(\mathbf{u}) = \frac{E\nu}{(1-2\nu)(1+\nu)} \operatorname{tr}\big(\boldsymbol{\varepsilon}(\mathbf{u})\big)\mathbf{I} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}(\mathbf{u}),$$

where $\mathbf{I}$ represents the identity matrix, tr is the trace operator, E and ν denote Young's modulus and Poisson ratio, respectively. The chosen material characteristics are $\nu = 1/5$ and $E = 1$ (the choice of E is only made for the sake of simplicity and any choice of a postive E would lead to the same kind of nonuniqueness example). Let $(x = (1,0), y = (0,1))$ stand for the canonical basis of $\mathbb{R}^2$. We suppose that the volume forces $\mathbf{f} = (f_x, f_y) = (0,0)$ are absent and that the surface forces denoted $\mathbf{F} = (F_x, F_y)$ are such that

$$F_x = -\frac{35\sqrt{10}}{48}\alpha, \qquad F_y = 0,$$

where $\alpha > 0$. On Γ_D, the presribed displacements $\mathbf{U} = (U_x, U_y)$ are given by

$$U_x = 6\alpha(x-1), \qquad U_y = \frac{3}{4}\alpha(x-1),$$

where $\alpha < 4/3$ (to avoid some penetration of Γ_D in the rigid foundation).

Set $0 < \alpha < 4/3$ and introduce two linear displacement fields $\mathbf{u} = (u_x, u_y)$ and $\bar{\mathbf{u}} = (\bar{u}_x, \bar{u}_y)$ in Ω:

$$u_x = (7x + y - 7)\alpha, \qquad u_y = \left(-x - \frac{7}{4}y + 1\right)\alpha, \tag{8}$$

$$\bar{u}_x = -6y\alpha, \qquad \bar{u}_y = -\frac{3}{4}y\alpha. \tag{9}$$

The displacement field $\mathbf{u}$ moves points A and C to $(-7\alpha, \alpha)$ and $(3/4 - 3\alpha/2, 1/4 - 3\alpha/16)$ respectively whereas position of point B remains unchanged. When considering $\bar{\mathbf{u}}$ the points A and B are stuck and the new position of point C becomes $(3/4 - 3\alpha/2, 1/4 - 3\alpha/16)$.

In the next proposition we show that the two displacement fields $\mathbf{u}$ and $\bar{\mathbf{u}}$ are solutions of the frictional contact problem (1)–(6) if the friction coefficient μ is large enough.

Proposition 3.1. *Let be given Ω, Γ_D, Γ_N, Γ_C, E, ν,* $\mathbf{f}$, $\mathbf{F}$, $\mathbf{U}$, α *as previously. For any $\mu \geqslant 3$ there exist at least two solutions* (*given by* (8) *and* (9)) *of the Coulomb frictional contact problem* (1)–(6).

Proof. Using the constitutive relation (2), one easily obtains:

$$\boldsymbol{\sigma}(\mathbf{u}) = \alpha \begin{pmatrix} \frac{175}{24} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}(\bar{\mathbf{u}}) = \alpha \begin{pmatrix} -\frac{5}{24} & -\frac{5}{2} \\ -\frac{5}{2} & -\frac{5}{6} \end{pmatrix}.$$

As a result the equilibrium equations $\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{0}$ and $\mathbf{div}\,\boldsymbol{\sigma}(\bar{\mathbf{u}}) = \mathbf{0}$ are satisfied in Ω. On Γ_D, the two fields $\mathbf{u}$ and $\bar{\mathbf{u}}$ coincide with the prescribed displacement field $\mathbf{U}$ and the stress vectors $\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}$ and $\boldsymbol{\sigma}(\bar{\mathbf{u}})\mathbf{n}$ are equal to $\mathbf{F}$ on Γ_N (since $\mathbf{n} = (-1/\sqrt{10}, 3/\sqrt{10})$). It remains to verify the fulfillment of the frictional contact conditions for both fields. We begin with $\mathbf{u}$. On Γ_C, $\mathbf{n} = (0, -1)$, $\mathbf{t} = (1, 0)$ and we get

$$\sigma_n(\mathbf{u}) = 0, \quad \sigma_t(\mathbf{u}) = 0, \quad u_n = \alpha(x-1), \quad u_t = 7\alpha(x-1).$$

Since $\alpha > 0$ it follows that $u_n \leqslant 0$. This together $\sigma_n(\mathbf{u}) = \sigma_t(\mathbf{u}) = 0$ implies that $\mathbf{u}$ satisfies conditions (5), (6) for any positive μ. In fact $\mathbf{u}$ is a solution which separates from the rigid foundation except at point B. We now consider the displacement field $\bar{\mathbf{u}}$. The displacements, the normal and tangential stresses on Γ_C are:

$$\sigma_n(\bar{\mathbf{u}}) = -\frac{5}{6}\alpha, \quad \sigma_t(\bar{\mathbf{u}}) = \frac{5}{2}\alpha, \quad \bar{u}_n = 0, \quad \bar{u}_t = 0.$$

Obviously $\sigma_n(\bar{\mathbf{u}}) < 0$ and $|\sigma_t(\bar{\mathbf{u}})| \leqslant \mu|\sigma_n(\bar{\mathbf{u}})|$ when $\mu \geqslant 3$. As a consequence, the displacement field $\bar{\mathbf{u}}$ satisfies conditions (5), (6) for any $\mu \geqslant 3$. Note that this solution is stuck on the rigid foundation. This concludes the proof of the proposition.

Remark 2. The values $\mu \geqslant 3$ in the proposition correspond to quite large friction coefficients from an practical point of view. Nevertheless, such a choice allows one to exhibit a simple non-uniqueness example. In fact, it is possible to obtain examples of non-uniqueness with $\mu \geqslant 1 + \varepsilon$ for any $\varepsilon > 0$. This result is obtained together with the theoretical framework considering the general setting in which such nonunique isolated solutions occur. This work is in preparation.

Remark 3. The are probably some other types of nonuniqueness cases for problem (1)–(6): a different approach from the one presented in this Note consists of searching sufficient conditions of nonuniqueness involving infinitely many solutions (which all remain in slipping contact) for critical (eigen)values of the friction coefficient (see [6]).

Remark 4. The existence of examples with nonunique solutions to (1)–(6) for arbitrary small friction coefficients is an open question.

References

[1] M. Cocu, E. Pratt, M. Raous, Formulation and approximation of quasistatic frictional contact, Int. J. Engrg. Sci. 22 (1996) 783–798.

[2] C.A. de Coulomb, Théorie des machines simples, en ayant égard au frottement de leurs parties et la raideur des cordages. Pièce qui a remporté le Prix double de L'Académie des Sciences pour l'année 1781, Mémoire des Savants Etrangers, X (1785) 163–332. Reprinted by Bachelier, Paris, 1809.

[3] G. Duvaut, Problèmes unilatéraux en mécanique des milieux continus, in: Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 3, Gauthier-Villars, Paris, 1971, pp. 71–77.

[4] G. Duvaut, J.-L. Lions, Les inéquations en mécanique et en physique, Dunod, Paris, 1972.

[5] C. Eck, J. Jarušek, Existence results for the static contact problem with Coulomb friction, Math. Models Methods Appl. Sci. 8 (1998) 445–468.

[6] R. Hassani, P. Hild, I. Ionescu, Sufficient condition of non-uniqueness for the Coulomb friction problem, Math. Methods Appl. Sci. 27 (3) (2004), in press.

[7] J. Jarušek, Contact problems with bounded friction. Coercive case, Czechoslovak. Math. J. 33 (1983) 237–261.

[8] J. Nečas, J. Jarušek, J. Haslinger, On the solution of the variational inequality to the Signorini problem with small friction, Boll. Un. Mat. Ital. B(5) 17 (1980) 796–811.