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A posteriori residual error estimation of a cell-centered finite volume method

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Abstract

We present an a posteriori residual error estimator for the Laplace equation using a cell-centered finite volume method in the plane. For that purpose we associate to the approximated solution a kind of Morley interpolant. The error is then the difference between the exact solution and this Morley interpolant. The residual error estimator is based on the jump of normal and tangential derivatives of the Morley interpolant. The equivalence between the discrete H^1-seminorm of the error and the residual error estimator is proved. The proof of the upper error bound uses the Helmholtz decomposition of the broken gradient of the error and some quasi-orthogonality relations. ***To cite this article: S. Nicaise, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Estimateur d'erreur a posteriori du type résiduel pour une méthode de volumes finis centrés par mailles. Nous présentons un estimateur d'erreur a posteriori du type résiduel pour le problème de Dirichlet dans le plan approché par une méthode de volumes finis centrés par mailles. Dans ce but nous associons à la solution approchée un interpolant du type Morley. L'erreur est alors la différence entre la solution exacte et cet interpolant de Morley. L'estimateur d'erreur résiduel est basé sur le saut des dérivées normale et tangentielle de l'interpolant de Morley. Nous démontrons l'équivalence entre la seminorme H^1 discrète de l'erreur et l'estimateur d'erreur résiduel. La preuve de la borne supérieure utilise une décomposition de Helmholtz du gradient par morceaux de l'erreur et des relations de quasi-orthogonalité. ***Pour citer cet article : S. Nicaise, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Introduction

Les estimations a posteriori sont devenues un outil indispensable pour l'approximation des problèmes aux limites. Contrairement aux méthodes d'éléments finis [7], des estimations a posteriori pour des méthodes de volumes finis sont peu developpées, citons par exemple [1,2,6]. Notre but est donc de considérer un estimateur

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d'erreur a posteriori du type résiduel pour le problème de Dirichlet dans le plan approché par une méthode de volumes finis centrés par mailles.

Soit Ω un ouvert du plan à bord polygonal Γ. Considérons le problème de Dirichlet : étant donné $f \in L^2(\Omega)$ soit $u \in H_0^1(\Omega)$ la solution variationelle de $-\Delta u = f$ dans Ω, autrement dit solution de (1).

Discrétisation du problème

Fixons une famille régulière (au sens de Ciarlet [3]) de triangulations T_h, $h > 0$, de Ω admissibles au sens de [5, Définition 9.1]. On considère maintenant le schéma centré par mailles suivant [5] : trouver $u_h := (u_K)_{K \in T_h}$ (u_K sera l'approximation de $u(x_K)$, pour $K \in T_h$, x_K étant le « centre » de la boite K) solution de (2).

Nous associons à $u_h = (u_K)_{K \in T_h}$ son interpolé de Morley $I_M u_h$ comme l'unique $v_h \in L^2(\Omega)$ tel que $v_{h|K} \in \mathbb{P}_2(K)$, pour tout $K \in T_h$ et satisfaisant (4)–(6). La propriété principale de cet interpolé est que si u_h est solution de (2), alors $I_M u_h$ satisfait l'identité (7).

Estimations d'erreur

La preuve de la borne supérieure sera basée sur les relations de quasi-orthogonalité suivantes :

Lemme 0.1. *Si u est solution de* (1) *et u_h solution de* (2)*, alors avec les notations* (8)*, les identités* (9) *et* (11) *ainsi que les estimées* (10) *et* (12) *sont satisfaites pour tout $\phi \in H_0^1(\Omega)$ et tout $\chi \in H^1(\Omega)$, lorsque $\mathcal{M}_K \phi$ (resp. $\mathcal{M}_E \phi$) est la moyenne de ϕ sur K (resp. E), et dorénavant la notation $a \lesssim b$ signifie qu'il existe une constante positive C indépendante de a et b et de la taille de la triangulation tel que $a \leqslant Cb$.*

Ces estimées (10) et (12) et une décomposition de Helmholtz du gradient par morceaux de l'erreur $e := u - I_M u_h$ (comme [4, Théorème 3.1]) permettent de prouver la borne supérieure. La preuve de la borne inférieure est plus classique et est basée sur des inégalités inverses et applications de formule de Green [7]. En résumé nous obtenons les estimées suivantes :

Théorème 0.2. *Si u est solution de* (1) *et u_h solution de* (2)*, alors*

$$|e|_{1,h} \lesssim \eta + \zeta \quad \textit{et} \quad \eta \lesssim |e|_{1,h} + \zeta,$$

où on rappelle que $|\cdot|_{1,h} := (\sum_{K \in T_h} |\cdot|_{1,K}^2)^{1/2}$, le résidu et le terme d'approximation étant définis par (15).

Remarque 1. Les résultats précédents s'étendent au cas d'une triangulation constituée de rectangles (voir Section 4).

Remarque 2. Le nouvel estimateur d'erreur introduit est à la fois efficace et fiable, de plus des premiers tests confirment ces faits et montrent la convergence de l'interpolant de Morley vers la solution exacte.

1. Introduction

A posteriori error estimates become in our days an essentiel tool for the approximation of boundary value problems. Contrary to the finite element methods [7], a posteriori error estimates for finite volume methods are not well developed and only a few numbers of results are obtained in that direction [1,2,6]. The goal of our paper is then to consider a posteriori error estimations of residual type for the Laplace equation using a cell-centered finite volume method.

Let Ω be an open subset of $\mathbb{R}^2$ with a polygonal boundary Γ. We consider the standard elliptic problem: for $f \in L^2(\Omega)$ let $u \in H_0^1(\Omega)$ be the variational solution of $-\Delta u = f$ in Ω, which means that u satisfies

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x, \quad \forall v \in H_0^1(\Omega). \tag{1}$$

2. Discretization of the Laplace equation

Let us fix a regular family (in Ciarlet's sense [3]) of triangular meshes T_h, $h > 0$, admissible in the sense of [5, Definition 9.1], i.e., meshes made of triangles and satisfying standard orthogonality conditions. Let us define E_h as the set of edges of the triangulation and set $E_h^{\text{int}} = \{E \in E_h \mid E \subset \Omega\}$ the set of interior edges of T_h, while $E_h^{\text{ext}} = E_h \setminus E_h^{\text{int}}$ is the set of exterior edges of T_h.

For an edge E of K introduce $n_{K,E} = (n_x, n_y)$ the unit outward normal vector to K along E. Furthermore for each edge E we fix one of the two normal vectors and denote it by n_E. Introduce additionally the tangent vector $t_{K,E} = n_{K,E}^{\perp} := (-n_y, n_x)^{\top}$ such that it is oriented positively (with respect to K). Similarly set $t_E := n_E^{\perp}$.

The jump of some (scalar or vector valued) function v across an edge E at a point $y \in E$ is defined as

$$[\![v(y)]\!]_E := \begin{cases} \lim_{\alpha\to+0} v(y+\alpha n_E) - v(y-\alpha n_E) & \forall E \in E_h^{\text{int}}, \\ v(y) & \forall E \in E_h^{\text{ext}}. \end{cases}$$

Finally we will need local patches, like ω_K, the union of all elements having a common edge with K and ω_E (resp. ω_a), the union of all elements having E as edge (resp. a as vertex).

We now consider the following cell-centered method [5]: find $u_h := (u_K)_{K\in T_h}$ (u_K being the approximation of $u(x_K)$, for $K \in T_h$, x_K being the "center" of the box K) solution of

$$-\sum_{E\in E_K} F_{K,E}(u_h) = \int_K f(x)\,\mathrm{d}x, \quad \forall K \in T_h, \tag{2}$$

where E_K is the set of edges of K and

$$F_{K,E}(u_h) := \begin{cases} \dfrac{|E|}{d(x_K, x_L)}(u_L - u_K) & \text{if } E = K \cap L, \\ -\dfrac{|E|}{d(x_K, \Gamma)} u_K & \text{if } E \subset K \cap \Gamma. \end{cases} \tag{3}$$

We recall that this system is well defined as proved for instance in [5].

For any $K \in T_h$ we denote by a_i^K, $i = 1, 2, 3$, the three vertices of K. For any vertex a of the triangulation we fix $(w_K(a))_{K\subset\omega_a}$ suitable weights of interpolation around K (for shorthness we do not describe them but they may be obtained using a discrete projection of piecewise constant functions over affine functions on ω_a, a standard technique to get a recovered gradient at the vertex a). Furthermore for any edge E we denote by m_E its midpoint.

Definition 2.1. For $u_h = (u_K)_{K\in T_h}$ we define its Morley interpolant $I_M u_h$ as the unique element $v_h \in L^2(\Omega)$ such that $v_{h|K} \in \mathbb{P}_2(K)$, for all $K \in T_h$ and satisfying:

$$v_{h|K}(a_i^K) = \sum_{L\in T_h: a_i^K \in L} w_L(a_i^K) u_L, \quad \forall K \in T_h,\ i \in \{1,2,3\}\colon a_i^K \in \Omega, \tag{4}$$

$$v_{h|K}(a_i^K) = 0, \quad \forall K \in T_h,\ i \in \{1,2,3\}\colon a_i^K \in \Gamma, \tag{5}$$

$$\frac{\partial v_{h|K}}{\partial n_{K,E}}(m_E) = \frac{1}{|E|} F_{K,E}(u_h), \quad \forall E \in E_K,\ K \in T_h. \tag{6}$$

We now prove a useful property of the Morley interpolant.

Lemma 2.2. *If u_h is solution of* (2)*, then $I_M u_h$ satisfies*

$$\int_K \Delta(I_M u_h)\,\mathrm{d}x = -\int_K f(x)\,\mathrm{d}x, \quad \forall K \in T_h. \tag{7}$$

Proof. Apply Green's formula on K, the property (6) and the identity (2). □

3. Error estimators

We first define the gradient jump of $I_M u_h$ in normal and tangentiel direction by

$$J_{E,n}(u_h) = \left[\!\left[\frac{\partial}{\partial n_E}(I_M u_h) \right]\!\right]_E, \quad \forall E \in E_h^{\mathrm{int}}, \qquad J_{E,t}(u_h) = \left[\!\left[\frac{\partial}{\partial t_E}(I_M u_h) \right]\!\right]_E, \quad \forall E \in E_h. \tag{8}$$

We now prove some quasi-orthogonality relations that will be used for the upper error bound.

Lemma 3.1. *If u is solution of (1) and u_h is solution of (2), then for all $\phi \in H_0^1(\Omega)$ we have*

$$\sum_{K\in T_h} \int_K \nabla(u - I_M u_h) \cdot \nabla\phi \, \mathrm{d}x$$
$$= \sum_{K\in T_h} \int_K (f - \mathcal{M}_K f)(\phi - \mathcal{M}_K \phi)\, \mathrm{d}x - \sum_{E\in E_h^{\mathrm{int}}} \int_E J_{E,n}(u_h)(\phi - \mathcal{M}_E \phi)\, \mathrm{d}s, \tag{9}$$

$$\left| \sum_{K\in T_h} \int_K \nabla(u - I_M u_h) \cdot \nabla\phi \, \mathrm{d}x \right| \lesssim (\eta + \zeta)|\phi|_{1,\Omega}, \tag{10}$$

where $\mathcal{M}_K \phi$ and $\mathcal{M}_E \phi$ are the mean of ϕ on K and E respectively, $|\cdot|_{1,\Omega}$ means the standard $H^1(\Omega)$-seminorm and from now on, $a \lesssim b$ means that there exists a positive constant C independent of a and b and of the meshsize of the triangulation such that $a \leqslant Cb$.

Proof. For the identity (9) we use the identity (1), Green's formula on each triangle K, the identity (7), the continuity of ϕ through the edges and the fact that $\phi = 0$ on Γ, as well as the property

$$\int_E J_{E,n}(u_h)\, \mathrm{d}s = |E| J_{E,n}(u_h)(m_E) = 0, \quad \forall E \in E_h^{\mathrm{int}},$$

consequence of (6) and the principle of conservation of flux.

The estimate (10) follows from (9) and the use of a Bramble–Hilbert argument and Cauchy–Schwarz's inequalities. □

Similarly we can prove the

Lemma 3.2. *If u is solution of (1) and u_h is solution of (2), then for all $\chi \in H^1(\Omega)$*

$$\sum_{K\in T_h} \int_K \nabla(u - I_M u_h) \cdot \operatorname{curl}\chi \, \mathrm{d}x = -\sum_{E\in E_h} \int_E J_{E,t}(u_h)(\chi - \mathcal{M}_E \chi)\, \mathrm{d}s, \tag{11}$$

$$\left| \sum_{K\in T_h} \int_K \nabla(u - I_M u_h) \cdot \operatorname{curl}\chi \, \mathrm{d}x \right| \lesssim \eta |\chi|_{1,\Omega}. \tag{12}$$

Our local and global residual error estimators and the approximation terms are now defined by

$$\eta_K^2 := h_K \sum_{E\in E_K \cap E_h^{\mathrm{int}}} \|J_{E,n}(u_h)\|_E^2 + h_K \sum_{E\subset\partial K} \|J_{E,t}(u_h)\|_E^2, \tag{13}$$

$$\zeta_K^2 := h_K^2 \sum_{K' \subset \omega_K} \|f - \mathcal{M}_{K'} f\|_{K'}^2, \tag{14}$$

$$\eta^2 := \sum_{T \in T_h} \eta_T^2, \qquad \zeta^2 := \sum_{T \in T_h} \zeta_T^2, \tag{15}$$

where $\|\cdot\|_\omega$ means the $L^2(\omega)$-norm.

Now we are able to show our upper error bound.

Theorem 3.3. *Let u be a solution of* (1) *and u_h a solution of* (2). *Then the error $e := u - I_M u_h$ is bounded by*

$$|e|_{1,h} \lesssim \eta + \zeta, \tag{16}$$

where we recall that $|\cdot|_{1,h} := (\sum_{K \in T_h} |\cdot|_{1,K}^2)^{1/2}$.

Proof. Denote by $\nabla_T e$ the brokent gradient of e, namely

$$(\nabla_T e)_{|K} = \nabla e_{|K} \quad \text{on } K,\ \forall K \in T_h.$$

As in Theorem 3.1 of [4] we use its Helmholtz decomposition:

$$\nabla_T e = \nabla w + \operatorname{curl} \psi, \tag{17}$$

with $w \in H_0^1(\Omega)$ and $\psi \in H^1(\Omega)$ such that

$$|w|_{1,\Omega} + |\psi|_{1,\Omega} \lesssim |e|_{1,h}. \tag{18}$$

Owing to the identity (17) we may write

$$|e|_{1,h}^2 = \int_\Omega |\nabla_T e|^2 \, \mathrm{d}x = \sum_{K \in T_h} \int_K \nabla e \cdot \nabla w \, \mathrm{d}x + \sum_{K \in T_h} \int_K \nabla e \cdot \operatorname{curl} \psi \, \mathrm{d}x.$$

We conclude using the estimates (10), (12) and (18). □

We now pass to the lower error bound proved in a quite standard way [7].

Theorem 3.4. *For all elements K, the following local lower error bound holds*:

$$\eta_K \lesssim \|\nabla_T e\|_{\omega_K} + \zeta_K. \tag{19}$$

Proof. For shortness let us estimate the normal jump, the estimate of the tangential jump being obtained in a similar manner. Fix an arbitrary edge $E \in E_h^{\mathrm{int}}$ and set

$$w_E := \mathrm{F}_{\mathrm{ext}}\big(J_{E,n}(u_h)\big) b_E \in H_0^1(\omega_E),$$

where $\mathrm{F}_{\mathrm{ext}}(\cdot)$ is the trivial extension to ω_E and b_E is the edge bubble associated with E [7]. By elementwise partial integration we get

$$\begin{aligned}\int_E J_{E,n}(u_h) w_E &= -\sum_{K \subset \omega_E} \int_{\partial K} \frac{\partial e}{\partial n_K} w_E = -\sum_{K \subset \omega_E} \int_K (\nabla e \cdot \nabla w_E + \Delta e w_E) \, \mathrm{d}x \\ &\lesssim \|\nabla_T e\|_{\omega_E} \|\nabla w_E\|_{\omega_E} + \sum_{K \subset \omega_E} \|\Delta e\|_K \|w_E\|_{\omega_E}.\end{aligned}$$

Therefore by the identity (7) we get

$$\int_E J_{E,n}(u_h) w_E \lesssim \|\nabla_T e\|_{\omega_E} \|\nabla w_E\|_{\omega_E} + \sum_{K \subset \omega_E} \|f - \mathcal{M}_K f\|_K \|w_E\|_{\omega_E}.$$

Standard inverse inequalities [3] in the previous estimate lead to

$$h_E \left\| J_{E,n}(u_h) \right\|_E^2 \lesssim \|\nabla_T e\|_{\omega_E}^2 + \sum_{K \subset \omega_E} \zeta_K^2. \qquad \square \tag{20}$$

4. Rectangular meshes

The above results extend to the case of a regular triangulation made of rectangles. In that case we use the following "Morley element": For a rectangle K of vertices a_i^K and edges E_i^K, $i = 1, \ldots, 4$, set $P_K = \mathbb{P}_2(K) \oplus \mathrm{Span}\{x^3 - 3xy^2, y^3 - 3yx^2\}$, and take as degree of freedom $\Sigma_K := \{q(a_i^K), \frac{1}{|E_i^K|} \int_{E_i^K} \frac{\partial q}{\partial n_K}\, \mathrm{d}s\}_{i=1,\ldots,4}$. We readily check that the triple (K, P_K, Σ_K) is a finite element.

The above choice is motivated by the fact that Δq is constant on K for any $q \in P_K$, since $x^3 - 3xy^2$ and $y^3 - 3yx^2$ are the unique polynomials of degree $\leqslant 3$ which are harmonic.

As in Section 2 we may take the

Definition 4.1. For $u_h = (u_K)_{K \in T_h}$ we define its Morley interpolant $I_M u_h$ as the unique element $v_h \in L^2(\Omega)$ such that $v_{h|K} \in P_K$, for all $K \in T_h$ and satisfying:

$$v_{h|K}\left(a_i^K\right) = \frac{1}{4} \sum_{L \in T_h : a_i^K \in L} u_L, \quad \forall K \in T_h,\ i \in \{1,2,3,4\}:\ a_i^K \in \Omega, \tag{21}$$

$$v_{h|K}\left(a_i^K\right) = 0, \quad \forall K \in T_h,\ i \in \{1,2,3\}:\ a_i^K \in \Gamma, \tag{22}$$

$$\int_E \frac{\partial v_{h|K}}{\partial n_{K,E}}\, \mathrm{d}s = F_{K,E}(u_h), \quad \forall E \in E_K,\ K \in T_h. \tag{23}$$

Using the definition of $I_M u_h$ one directly sees that Lemmas 2.2, 3.1 and 3.2 are still valid. As a consequence the upper error bound (16) from Theorem 3.3 holds. On the other hand the lower error bound (19) is also valid since its proof only uses Green's formula and inverse inequalities that also hold for rectangular meshes.

5. Conclusion

We have presented and analysed a new efficient and reliable a posteriori error estimator of residual type for a cell-centered finite volume method. First numerical tests show the convergence of the Morley interpolant to the exact solution as well as the efficiency and reliability of the method.

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