Partial Differential Equations

Stabilization for 1D radiative and reactive viscous gas flows

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Abstract

We consider the compressible 1D-Navier–Stokes system for a viscous heat-conducting gas, with a pressure law $p(\rho, \theta) = R\rho\theta + \frac{a}{3}\theta^4$ and a one-order kinetics to include radiative and reactive processes. The mass force and the ignition phenomenon are also taken into account. For large data and under general assumptions on the heat conductivity, we establish global in time bounds and exponential stabilization for solutions in L^q and H^1-norms, by using new Lyapunov functionals. ***To cite this article: B. Ducomet, A. Zlotnik, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

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Résumé

Stabilisation pour un écoulement 1D de fluide compressible radiatif et réactif. Nous étudions le système de Navier–Stokes 1D décrivant un fluide compressible conducteur avec contribution radiative, couplé à une cinétique chimique du premier ordre. On tient compte d'un champ de force externe ainsi que d'une température d'ignition. Pour de grandes données et sous des conditions générales sur la conductivité, nous prouvons l'existence globale d'une solution et sa stabilisation en normes L^q et H^1, en introduisant de nouvelles fonctionnelles de Lyapunov. ***Pour citer cet article : B. Ducomet, A. Zlotnik, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

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Version française abrégée

Nous étudions un modèle monodimensionnel de gaz réactif décrivant de manière simplifiée l'évolution d'une étoile gazeuse radiative. Le milieu est décrit par le système de Navier–Stokes compressible couplé à une cinétique chimique du premier ordre. On suppose que le gaz et le rayonnement sont à l'équilibre thermodynamique local (modèle du gaz de Rosseland) : la pression, l'énergie et la conductivité thermique ont une composante radiative de type Stefan–Boltzmann. On tient également compte d'un champ de force externe qui sert à rendre compte de l'autogravitation ainsi que d'une température d'ignition positive pour la cinétique, qui modélise l'allumage des réactions thermonucléaires.

On considère un problème à frontière libre associé, où les bords de l'étoile sont libres de se déplacer sous l'action de la dynamique interne, du champ de forces et de la combustion du gaz.

Pour de grandes données et sous des conditions générales sur la conductivité, nous prouvons l'existence globale d'une solution et sa stabilisation en normes L^q et H^1, à l'aide de nouvelles fonctionnelles de Lyapunov.

Précisons que, contrairement aux travaux précédents sur des sujets reliés [2–5], la prise en compte d'un champ externe et d'une température d'ignition complique notablement l'analyse du comportement asymptotique de la solution aux grands temps.

1. Statement of the problem and main results

We consider the compressible Navier–Stokes system describing 1D-flows of a viscous heat-conducting radiative and reactive gas

$$\eta_t = v_x, \quad v_t = \sigma_x + g, \quad \hat{e}_t = \pi_x + \sigma v_x + \lambda f(\theta)z, \quad z_t = s_x - f(\theta)z, \tag{1}$$

$$\sigma = \nu \rho v_x - \hat{p}, \quad \rho = \eta^{-1}, \quad \pi = \hat{\kappa}\rho\theta_x, \quad s = d\rho^2 z_x, \tag{2}$$

for $(x, t) \in Q \equiv \Omega \times \mathbf{R}^+ = (0, M) \times (0, \infty)$, together with the equations of state

$$p(\eta, \theta) = R\eta^{-1}\theta + \frac{a}{3}\theta^4, \qquad e(\eta, \theta) = c_V\theta + a\eta\theta^4. \tag{3}$$

Hereafter the abbreviation for the compositions $\hat{b}(x, t) = b(\eta(x, t), \theta(x, t))$ is used, for $b = e, p, \kappa$, etc.; x, t are the Lagrangian mass coordinates. We exploit the following boundary and initial conditions:

$$v|_{x=0} = 0, \quad \sigma|_{x=M} = -p_\Gamma, \quad \pi|_{x=0} = 0, \quad \theta|_{x=M} = \theta_\Gamma, \quad s|_{x=0,M} = 0, \tag{4}$$

$$(\eta, v, \theta, z)|_{t=0} = \left(\eta^0(x), v^0(x), \theta^0(x), z^0(x)\right), \quad \text{for } x \in \Omega, \tag{5}$$

with given outer pressure $p_\Gamma = \text{const} > 0$ and temperature $\theta_\Gamma = \text{const} > 0$.

The unknown quantities $\eta > 0$, v, $\theta > 0$ and $0 \leqslant z \leqslant 1$ are the specific volume, the velocity, the absolute temperature and the concentration of unburnt gas. The other quantities ρ, σ, p, e, $-\pi$ and $-s$ are the density, the stress, the pressure, the internal energy, the heat flux and the diffusion flux. Let also $S(\eta, \theta) := R\log\eta + c_V\log\theta + \frac{4}{3}a\eta\theta^3$ be the entropy. Finally $\kappa(\eta, \theta)$ and $g(x)$ are the heat conductivity and the mass force, and the quantities ν, R, a, c_V, λ, d and M are positive physical constants.

The linear θ-terms in (3) correspond to the perfect polytropic contribution whereas the θ^4-terms correspond to the Stefan–Boltzmann radiative law. The last terms complicate the problem significantly, even for $z \equiv 0$.

For the non-radiative ($a = 0$) reactive case, global solutions and their stabilization were analyzed in [1–3,11,13], etc.; see also a local in time 3D result for the radiative non-reactive case in [10]. The general case was studied in [5] but for highly growing (in θ) $\kappa(\eta, \theta)$ only (not covering the most physically interesting case).

In this paper, for large data and under general assumptions on $\kappa(\eta, \theta)$, we present various global bounds for the strong solution and establish its global exponential decay as $t \to \infty$ in L^q and H^1-norms. Moreover, we show that the dynamics of solutions is described by (new) Lyapunov functionals.

In contrast to previous works, notice that we take into account large mass force $g = g(x)$ as a simplified description of self-gravitation. Thus the stationary specific volume is no more a constant, which complicates the study. Moreover the condition we impose on the data g, p_Γ and θ_Γ to get stabilization is sharp. Finally, we treat the case of a chemical reaction with an ignition temperature, which strongly modifies the dynamics.

We use the classical Lebesgue and Sobolev spaces $L^q(G)$ and $W^{1,q}(G)$, for $q \in [1, \infty]$, with $H^1(G) = W^{1,2}(G)$ and $H^0(G) = L^2(G)$. We also need the Lebesgue and Sobolev spaces $L^{q,r}(Q_T)$ and $H^{2,1}(Q_T) \equiv W_2^{2,1}(Q_T)$ (see [8]), for $Q_T := \Omega \times (0, T)$ with $0 < T \leqslant \infty$, endowed with the norms

$$\|w\|_{L^{q,r}(Q_T)} = \left\| \|w\|_{L^q(\Omega)} \right\|_{L^r(0,T)}, \qquad \|w\|_{H^{2,1}(Q_T)} = \|w\|_{V_2(Q_T)} + \|w_x\|_{V_2(Q_T)} + \|w_t\|_{L^2(Q_T)},$$

where $V_2(Q_T)$ is the standard energy space with the norm $\|w\|_{V_2(Q_T)} = \|w\|_{L^{2,\infty}(Q_T)} + \|w_x\|_{L^2(Q_T)}$. We denote $Iw(x) = \int_0^x w(\xi)\,d\xi$, $I^*w(x) = \int_x^M w(\xi)\,d\xi$, $\langle w \rangle = \frac{1}{M}\int_\Omega w(x)\,dx$ and $D_t = \frac{d}{dt}$.

Assume first that $\kappa(\eta, \theta)$ is continuously differentiable and satisfies $0 < \underline{\kappa} \leqslant \kappa(\eta, \theta)$ on $\mathbf{R}^+ \times \overline{\mathbf{R}}^+$ only. Let also f be a (globally) Lipschitz continuous function on $\overline{\mathbf{R}}^+$, such that $f(\theta) = 0$, for $0 \leqslant \theta < \theta_I$, and $0 < f(\theta) \leqslant \bar{f}$, for $\theta > \theta_I$, where $\theta_I \geqslant 0$ is the ignition temperature (the case $\theta_I = 0$, without ignition, is accepted, too). Finally let us limit ourselves to the case of global strong solutions such that

$$\eta, \eta_x, \eta_t \in L^{2,\infty}(Q_T), \quad \min_{\overline{Q}_T} \eta > 0; \quad v, \theta, z \in H^{2,1}(Q_T), \quad \theta > 0 \quad \text{in } \overline{\Omega} \times (0, T], \ \forall T < \infty. \tag{6}$$

To insure the existence of a solution, the following requirements on the data are necessary

$$\eta^0, v^0, \theta^0, z^0 \in H^1(\Omega), \ g \in L^2(\Omega), \quad \min_{\overline{\Omega}} \eta^0 > 0, \quad \theta^0 \geqslant 0, \quad v^0(0) = 0, \quad \theta^0(M) = \theta_\Gamma. \tag{7}$$

We begin with an elementary result on the stationary problem, i.e., the system

$$v_x = 0, \quad \sigma_x + g = 0, \quad \pi_x + \lambda f(\theta)z = 0, \quad s_x - f(\theta)z = 0,$$

together with (2), for $x \in \Omega$, and also with the first relation (3), under the boundary conditions (4).

Proposition 1.1. *Let* $g \in L^1(\Omega)$. *The weak solution* $\eta_S, v_S, \theta_S, z_S \in W^{1,1}(\Omega)$ *of the stationary problem such that* $\min_{\overline{\Omega}} \eta_S > 0$ *and* $\theta_S \geqslant 0$ *exists if and only if the condition*

$$\frac{a}{3}\theta_\Gamma^4 < p_{S\min} := \min_{\overline{\Omega}} p_S, \quad \text{with } p_S := p_\Gamma - I^*g, \tag{8}$$

is satisfied. Under this condition, the solution is such that

$$v_S(x) \equiv 0, \quad p(\eta_S(x), \theta_S(x)) = p_S(x), \quad \theta_S(x) \equiv \theta_\Gamma, \quad z_S(x) \equiv const \quad on \ \overline{\Omega}; \tag{9}$$

consequently $\eta_S = R\theta_\Gamma/(p_S - \frac{a}{3}\theta_\Gamma^4)$. *Moreover the component* $z_S = 0$ *is unique provided that* $\theta_\Gamma > \theta_I$.

Due to condition (8), for the stationary pressure p_S both the gas pressure and radiation pressure can be identified (see (3) and (9)). For $g = 0$, condition (8) takes the simple form $\frac{a}{3}\theta_\Gamma^4 < p_\Gamma$.

Let $N > 1$ be an arbitrarily large parameter and $K = K(N)$, $K_1 = K_1(N)$ be positive nondecreasing functions of N which may also depend on the physical constants of the problem, $\underline{\kappa}$, $\bar{f}$, $\underline{\eta}$ (see (12) below), etc. We also define the function $\Psi(\zeta) := \zeta - \log \zeta - 1$, for $\zeta > 0$, and set $\tilde{\eta} = \eta/\eta_S$, $\tilde{\theta} = \theta/\theta_\Gamma$, $\tilde{\theta}^0 = \theta^0/\theta_\Gamma$.

The first theorem describes the $L^q(\Omega)$-behaviour of the solution together with the $H^1(\Omega)$-behaviour of z.

Theorem 1.2. *Assume that the data are such that*

$$N^{-1} \leqslant \eta^0 \leqslant N, \quad \|v^0\|_{L^2(\Omega)} + \|\theta^0\|_{L^4(\Omega)} + \|\log \theta^0\|_{L^1(\Omega)} \leqslant N, \quad 0 \leqslant z^0 \leqslant 1, \quad j\|z_x^0\|_{L^2(\Omega)} \leqslant N, \tag{10}$$

$$\|g\|_{L^1(\Omega)} \leqslant N, \qquad (1 + N^{-1})\frac{a}{3}\theta_\Gamma^4 \leqslant p_{S\min}, \tag{11}$$

where the parameter j *may be equal to* 0 *or* 1.

(1) *The following bounds hold in* Q

$$\begin{aligned}
&0 < \underline{\eta} \leqslant \eta(x, t) \leqslant \bar{\eta} = K \quad in \ \overline{Q}, \\
&\|\eta - \eta_S\|_{L^2(Q)} + \|v\|_{V_2(Q)} + \|\theta^{-1/2}v_x\|_{L^2(Q)} + \|\sigma + p_S\|_{L^2(Q)} \leqslant K, \\
&\|\theta\|_{L^{4,\infty}(Q)} + \|\log \theta\|_{L^{1,\infty}(Q)} + \|\sqrt{\kappa}\theta^{-1}\theta_x\|_{L^2(Q)} \leqslant K, \\
&0 \leqslant z(x, t) \leqslant 1 \quad in \ \overline{Q}, \quad \|f(\theta)z\|_{L^1(Q)} \leqslant M, \quad \|z_x\|_{L^2(Q)} + j(\|z_t\|_{L^2(Q)} + \|s\|_{V_2(Q)}) \leqslant K.
\end{aligned} \tag{12}$$

Moreover, the following stabilization properties of the solution hold as $t \to \infty$

$$\mathbf{V}^{(1)}(t) := \left\langle \left(\eta(\cdot, t) - \eta_S(\cdot) \right)^2 + v^2(\cdot, t) + \left(1 + \theta(\cdot, t) \right)^2 \left(\theta(\cdot, t) - \theta_\Gamma \right)^2 + \Psi \left(\bar{\theta}(\cdot, t) \right) \right\rangle \to 0, \tag{13}$$

$$\left\| z(\cdot, t) - z_S \right\|_{H^j(\Omega)} \to 0, \tag{14}$$

for some constant z_S (depending on the data) such that $0 \leqslant z_S \leqslant \langle z^0 \rangle$.

(2) *In the case $\theta_\Gamma > \theta_I$, one has $z_S = 0$, and the behaviour of the solution with respect to time is described by a Lyapunov functional $\mathcal{W}^{(1)}$, the explicit form of which is given below, such that*

$$D_t \mathcal{W}^{(1)} + K^{-1} \mathcal{W}^{(1)} \leqslant 0, \quad K^{-1} \mathbf{W}^{(1)} \leqslant \mathcal{W}^{(1)} \leqslant K \mathbf{W}^{(1)}, \quad \text{with } \mathbf{W}^{(1)} := \mathbf{V}^{(1)} + \langle z + j z_x^2 \rangle,$$

on $\mathbf{R}^+$. Consequently one has $\mathbf{W}^{(1)}(t) \leqslant K \, e^{-\alpha_1 t} \mathbf{W}^{(1),0}$, for $t \geqslant 0$, where $\alpha_1 = \frac{1}{K}$ and $\mathbf{W}^{(1),0} := \langle (\eta^0 - \eta_S)^2 + (v^0)^2 + (1 + \theta^0)^2 (\theta^0 - \theta_\Gamma)^2 + \Psi(\bar{\theta}^0) + z^0 + j(z_x^0)^2 \rangle$.

(3) *In the case $\theta_\Gamma < \theta_I$, let us define $t_R := \min\{T \geqslant 0; \ \theta(x, t) \leqslant \theta_I, \ \forall x \in \overline{\Omega}, \ t \geqslant T\}$ and suppose that $t_R < \infty$. Then, for $t \geqslant t_R$, the dynamics of the solution with respect to time is essentially decoupled*

$$D_t \mathcal{V}^{(1)} + K^{-1} \left(\mathcal{V}^{(1)} + \langle v_x^2 \rangle \right) \leqslant 0, \quad D_t \mathcal{Z} + K^{-1} \mathcal{Z} \leqslant j \, K \langle v_x^2 \rangle,$$

$$K^{-1} \mathbf{V}^{(1)} \leqslant \mathcal{V}^{(1)} \leqslant K \mathbf{V}^{(1)}, \quad K^{-1} \| z - z_S \|_{H^j(\Omega)}^2 \leqslant \mathcal{Z} \leqslant K \| z - z_S \|_{H^j(\Omega)}^2,$$

together with the property $\langle z(\cdot, t) \rangle \equiv z_S$; moreover, the lower bound holds $z_S \geqslant e^{-\bar{f} t_R} \langle z^0 \rangle$. Clearly one has

$$\mathbf{V}^{(1)}(t) \leqslant K \, e^{-\alpha_1 (t - t_R)} \mathbf{V}^{(1)}(t_R), \quad \| z(\cdot, t) - z_S \|_{H^j(\Omega)}^2 \leqslant K \, e^{-\bar{\alpha}_1 (t - t_R)} \left(\| z(\cdot, t_R) - z_S \|_{H^j(\Omega)}^2 + j \mathbf{V}^{(1)}(t_R) \right),$$

for $t \geqslant t_R$, with $\alpha_1 := 1/K$ and $\bar{\alpha}_1 := 1/K_1$. Hereafter, K, K_1 do not depend on t_R.

Remark 1. The following inequality holds $\langle |\hat{p} - p_S| + |\hat{e} - e(\eta_S, \theta_\Gamma)| + |\hat{S} - S(\eta_S, \theta_\Gamma)| \rangle \leqslant K \sqrt{\mathbf{V}^{(1)}}$ on $\overline{\mathbf{R}}^+$ describing the $L^1(\Omega)$-behaviour in time for the pressure, the internal energy and the entropy.

Let us give some comments concerning this theorem.

(1) The second condition (10) can be replaced equivalently by more physical ones $\langle \frac{1}{2}(v^0)^2 + e(\eta^0, \theta^0) + \lambda z^0 \rangle \leqslant N$, $\langle S(\eta^0, \theta^0) \rangle \geqslant -N$, involving the total initial energy and entropy in the left-hand sides.
(2) Condition (8) implies the second condition (11), for sufficiently large N.
(3) In (12), $\bar{\eta}$ and $\underline{\eta}$ do not depend on each other.
(4) In fact, we have $L^4(\Omega)$-decay of θ along with $L^q(\Omega)$-decay for η and z (even for $j = 0$), since

$$\| \theta - \theta_\Gamma \|_{L^4(\Omega)}^4 + \| \eta - \eta_S \|_{L^q(\Omega)}^q \leqslant K(q) \mathbf{V}^{(1)}, \quad \forall q \in [2, \infty); \qquad \| z \|_{L^q(\Omega)}^q \leqslant M \langle z \rangle, \quad \forall q \in [1, \infty).$$

(5) Both θ-terms in (13) are essential to garantee the validity of the important Remark 1.
(6) In the case $\theta_\Gamma > \theta_I$, the property $z_S = 0$ means that all the fuel is burnt as $t \to \infty$. In the case $\theta_\Gamma < \theta_I$, the quantity t_R is the duration of the chemical reaction. In the latter case (see Theorem 1.3 below) under some additional assumptions on the data, we show that $t_R < \infty$ in spite of the presence of unburnt fuel (since $z_S > 0$ for $z^0 \not\equiv 0$).
(7) In the case $\theta_\Gamma \leqslant \theta_I$, the stationary solution (see Proposition 1.1) serves as an example of the nonstationary one, for any $0 \leqslant z_S \leqslant 1$. Thus any value of $z_S = \langle z^0 \rangle \in [0, 1]$ can be implemented in (14).

Our main result describes now the H^1-behaviour of the solution. Let us suppose as above that $j = 0$ or 1.

Theorem 1.3. *Assume that the data are such that*

$$N^{-1} \leqslant \eta^0, \quad \left\|\left(\eta^0, v^0, \theta^0\right)\right\|_{H^1(\Omega)} + \left\|\log \theta^0\right\|_{L^1(\Omega)} \leqslant N, \quad 0 \leqslant z^0 \leqslant 1, \quad j\left\|z_x^0\right\|_{L^2(\Omega)} \leqslant N,$$

$$\|g\|_{L^2(\Omega)} \leqslant N, \qquad \left(1 + N^{-1}\right)\frac{a}{3}\theta_\Gamma^4 \leqslant p_{S\min},$$

$$0 < \kappa_1(\delta)(1+\theta)^q \leqslant \kappa(\eta, \theta),$$

$$\left(\kappa + |\kappa_\eta| + |\kappa_\theta| + |\kappa_{\eta\eta}|\right)(\eta, \theta) \leqslant \kappa_2(\delta)(1+\theta)^q, \quad \forall \eta \in \left[\delta^{-1}, \delta\right], \ \theta \geqslant 0, \ \forall \delta > 1$$

for some $q \geqslant 2$, where the derivative $\kappa_{\eta\eta}$ is continuous on $\mathbf{R}^+ \times \overline{\mathbf{R}}^+$.

(1) *The following bounds in Q together with the $H^1(\Omega)$-stabilization of the solution hold*

$$\|\eta_x\|_{L^{2,\infty}(Q)} \leqslant K \quad and, \quad for \ j = 1, \quad \|z\|_{H^{2,1}(Q)} \leqslant K, \tag{15}$$

$$\sup_{\overline{Q}} \theta \leqslant K, \quad \|v\|_{H^{2,1}(Q)} + \|\theta\|_{H^{2,1}(Q)} \leqslant K, \quad \|\sigma + p_S\|_{V_2(Q)} + \|\pi\|_{V_2(Q)} \leqslant K, \tag{16}$$

$$\mathbf{V}^{(2)}(t) := \left\|\left(\eta(\cdot, t) - \eta_S(\cdot), v(\cdot, t), \theta(\cdot, t) - \theta_\Gamma\right)\right\|_{H^1(\Omega)}^2 + \left\langle\Psi\left(\bar{\theta}(\cdot, t)\right)\right\rangle \to 0 \quad as \ t \to \infty.$$

See also (14). Moreover, if in addition $N^{-1} \leqslant \theta^0$, then $0 < \underline{\theta} \leqslant \theta(x, t)$ in $\overline{Q}$.

(2) *In the case $\theta_\Gamma > \theta_I$, a Lyapunov functional $\mathcal{W}^{(2)}$ can be constructed such that*

$$D_t \mathcal{W}^{(2)} + K^{-1}\mathcal{W}^{(2)} \leqslant 0, \quad K^{-1}\mathbf{W}^{(2)} \leqslant \mathcal{W}^{(2)} \leqslant K\mathbf{W}^{(2)}, \quad with \ \mathbf{W}^{(2)} := \mathbf{V}^{(2)} + \left\langle z + jz_x^2\right\rangle,$$

on $\mathbf{R}^+$. Clearly $\mathbf{W}^{(2)}(t) \leqslant K\,\mathrm{e}^{-\alpha_2 t}\mathbf{W}^{(2),0}$, for $t \geqslant 0$, where $\alpha_2 = \frac{1}{K}$, and $\mathbf{W}^{(2),0} := \|(\eta^0 - \eta_S, v^0, \theta^0 - \theta_\Gamma)\|_{H^1(\Omega)}^2 + \left\langle\Psi(\bar{\theta}^0) + z^0 + j(z_x^0)^2\right\rangle$. Moreover, in the case $N^{-1} \leqslant \theta^0$, one has $\underline{\theta} = \frac{1}{K}$.

(3) *In the case $\theta_\Gamma < \theta_I$, one has $t_R < \infty$. Moreover, the functional $\mathcal{V}^{(2)} := \mathcal{W}^{(2)}|_{z \equiv 0}$ satisfies, for $t \geqslant t_R$*

$$D_t \mathcal{V}^{(2)} + K^{-1}\mathcal{V}^{(2)} \leqslant 0, \qquad K^{-1}\mathbf{V}^{(2)} \leqslant \mathcal{V}^{(2)} \leqslant K\mathbf{V}^{(2)}.$$

Clearly $\mathbf{V}^{(2)}(t) \leqslant K\,\mathrm{e}^{-\alpha_2(t-t_R)}\mathbf{V}^{(2)}(t_R)$, for $t \geqslant t_R$, with $\alpha_2 := \frac{1}{K}$. For the behaviour of z, see the claim (3) of Theorem 1.2.

2. Sketch of the proofs

To prove Theorem 1.2, we begin with a combined energy and entropy conservation law together with the equality $D_t\langle z\rangle + \langle f(\theta)z\rangle = 0$, which give starting bounds in Q. Then we prove bounds (12) by applying a technique from [6]. Next we derive the key relation

$$D_t \mathcal{V}^{(1)} + \left\langle\beta_0 R\rho_S\theta_\Gamma\rho(\eta - \eta_S)^2 + \left(\tilde{\theta}^{-1} + \beta_1\right)v\rho v_x^2 - \beta_0 v^2 + \theta_\Gamma\rho\hat{\kappa}\theta^{-2}\theta_x^2\right\rangle$$
$$= \left\langle\left(\hat{p} - p(\eta, \theta_\Gamma)\right)\left(\beta_0(\eta - \eta_S) + \beta_1 v_x\right) + \lambda\left(1 - \tilde{\theta}^{-1}\right)f(\theta)z\right\rangle,$$

with $\mathcal{V}^{(1)} := \langle[(1+\beta_1)R\theta_\Gamma + \beta_0\nu\eta_S]\Psi(\tilde{\eta}) + \frac{1+\beta_1}{2}v^2 + \beta_0 vI(\eta - \eta_S) + c_V\theta_\Gamma\Psi(\tilde{\theta}) + a\theta_\Gamma^4\eta(\tilde{\theta}^4 - \frac{4}{3}\tilde{\theta}^3 + \frac{1}{3})\rangle$, where $\beta_0, \ \beta_1$ are parameters.

By studying this equality first for $\beta_1 = 0$ then for $\beta_1 > 0$, using the inequalities $\frac{1}{2}D_t\langle(z - \langle z\rangle)^2\rangle + d\,\bar{\eta}^{-2}\langle z_x^2\rangle \leqslant \langle f(\theta)z\rangle$ and, for $j = 1$, $\frac{1}{d}D_t\langle(\eta s)^2\rangle + \langle s_x^2\rangle \leqslant K\langle v_x^2\rangle + 2\bar{f}\langle f(\theta)z\rangle$, we obtain the remaining part of Theorem 1.2, for the Lyapunov functionals $\mathcal{W}^{(1)} := \mathcal{V}^{(1)} + \langle(\lambda + \tilde{\beta}_1)z + j\bar{\beta}_1(\rho z_x)^2\rangle$ and $\mathcal{Z} := \langle(1 - j)(z - z_S)^2 + j(\rho z_x)^2\rangle$, where $\beta_0, \ \beta_1, \ \tilde{\beta}_1, \ \bar{\beta}_1$ are suitably chosen constants of "$\frac{1}{K}$ type".

To prove Theorem 1.3 we establish sequentially bounds (15), $\sup_{\overline{Q}} \theta \leqslant K$ and the remaining bounds (16). For this purpose we first derive energy-type inequalities for the quantities $r := v - \nu(\log \tilde{\eta})_x$, $\sigma + p_S$ and π, applying techniques borrowed from [12,6]. For the crucial bound $\sup_{\overline{Q}} \theta \leqslant K$, we modify essentially an approach from [9] by involving $\sigma + p_S$ and π. After bounds (15) and (16), we revisit those energy-type inequalities, combine them

with the Lyapunov-type inequalities of Theorem 1.2 and obtain all the required stabilization properties, for the Lyapunov functional $\mathcal{W}^{(2)} := \mathcal{W}^{(1)} + \langle \beta_2 r^2 + \beta_3(\eta(\sigma + p_S)^2 + \pi^2) \rangle$, where β_2, β_3 are constants of "$\frac{1}{K}$ type".

Remarks.

(1) As in [6], our bounds enable us to prove that under conditions (7), $\log \theta^0 \in L^1(\Omega)$, $0 \leqslant z^0 \leqslant 1$ and (8) there exists a unique strong solution to the problem (1)–(5) with the properties (6) and $0 \leqslant z \leqslant 1$.
(2) Our Lyapunov functionals generalize those suggested in [12]; they belong to $W^{1,1}(0, T)$ for any $T < \infty$.
(3) The approach described above can also be applied in the much more involved case of spherically symmetric flows, which is of interest in the astrophysical framework of stellar evolution [4].

The full proofs of the above results may be found in [7], for various boundary conditions.

Acknowledgements

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