Available online at www.sciencedirect.com
SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 338 (2004) 347–352

Comptes Rendus Mathematique

Number Theory

Weyl's law for the cuspidal spectrum of SL_n

Werner Müller

Universität Bonn, Mathematisches Institut, Beringstrasse 1, 53115 Bonn, Germany

Received 21 October 2003; accepted after revision 5 January 2004

Presented by Jean-Michel Bismut

Abstract

Let Γ be a principal congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$ and let σ be an irreducible unitary representation of $\mathrm{SO}(n)$. Let $N_{\mathrm{cus}}^{\Gamma}(\lambda,\sigma)$ be the counting function of the eigenvalues of the Casimir operator acting in the space of cusp forms for Γ which transform under $\mathrm{SO}(n)$ according to σ. In this Note we prove that the counting function $N_{\mathrm{cus}}^{\Gamma}(\lambda,\sigma)$ satisfies Weyl's law. In particular, this implies that there exist infinitely many cusp forms for the full modular group $\mathrm{SL}_n(\mathbb{Z})$. ***To cite this article: W. Müller, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une formule de Weyl pour le spectre cuspidal de SL_n**.** Soit Γ un sous-groupe de congruence principal de $\mathrm{SL}_n(\mathbb{Z})$ et soit σ une représentation irréductible unitaire de $\mathrm{SO}(n)$. Soit $N_{\mathrm{cus}}^{\Gamma}(\lambda,\sigma)$ la fonction de dénombrement des valeurs propres de l'opérateur de Casimir, agissant sur l'espace des formes automorphes cuspidales pour Γ qui se transforment sous $\mathrm{SO}(n)$ par σ. Dans cette Note, nous prouvons une formule de Weyl pour le comportement asymptotique de la fonction de comptage $N_{\mathrm{cus}}^{\Gamma}(\lambda,\sigma)$. ***Pour citer cet article : W. Müller, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Soit G un groupe algébrique réductif connexe défini sur $\mathbb{Q}$ et soit $\Gamma\subset G(\mathbb{Q})$ un sous-groupe arithmétique de G. Un problème important dans la théorie des formes automorphes est la question de l'existence et de la constuction de formes cuspidales pour Γ.

Dans cette Note, nous étudions le problème d'existence pour le groupe $G=\mathrm{SL}_n$, $n\geqslant 2$. Soit Γ un sous-groupe de congruence de $\mathrm{SL}_n(\mathbb{Z})$. Soit $L^2_{\mathrm{cus}}(\Gamma\setminus\mathrm{SL}_n(\mathbb{R}))$ la fermeture hilbertienne de l'espace engendré par les formes automorphes cuspidales. Soit (σ,V_σ) une représentation irréductible unitaire de $\mathrm{SO}(n)$. On pose

$$L^2\big(\Gamma\setminus\mathrm{SL}_n(\mathbb{R}),\sigma\big)=\big(L^2\big(\Gamma\setminus\mathrm{SL}_n(\mathbb{R})\big)\otimes V_\sigma\big)^{\mathrm{SO}(n)},$$

E-mail address: mueller@math.uni-bonn.de (W. Müller).

1631-073X/$ – see front matter © 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crma.2004.01.003

et on définit $L^2_{\mathrm{cus}}(\Gamma\setminus \mathrm{SL}_n(\mathbb{R}),\sigma)$ de manière similaire. Soit $\Omega\in\mathcal{Z}(\mathfrak{sl}(n,\mathbb{C}))$ l'élément de Casimir de $\mathrm{SL}_n(\mathbb{R})$. Alors $-\Omega\otimes\mathrm{Id}$ induit un opérateur auto-adjoint Δ_σ, agissant sur l'espace de Hilbert $L^2(\Gamma\setminus \mathrm{SL}_n(\mathbb{R}),\sigma)$. Cet opérateur est borné inférieurement et la restriction de Δ_σ au sous-espace $L^2_{\mathrm{cus}}(\Gamma\setminus \mathrm{SL}_n(\mathbb{R}),\sigma)$ est un opérateur à spectre ponctuel, formé de valeurs propres $\lambda_0(\sigma)<\lambda_1(\sigma)<\cdots$ de multiplicité finie. Soit $\mathcal{E}(\lambda_i(\sigma))$ l'espace propre associé à la valeur propre $\lambda_i(\sigma)$. Pour $\lambda\geqslant 0$ on pose

$$N^{\Gamma}_{\mathrm{cus}}(\lambda,\sigma)=\sum_{\lambda_i(\sigma)\leqslant\lambda}\dim\mathcal{E}\big(\lambda_i(\sigma)\big).$$

Alors notre résultat principal est le théorème suivant.

Théorème 0.1. *Pour $n\geqslant 2$, soit $X_n=\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. Soit $d_n=\dim X_n$. Alors pour tout sous-groupe de congruence principal Γ de $\mathrm{SL}_n(\mathbb{Z})$ et pour toute représentation irréductible unitaire σ de $\mathrm{SO}(n)$ tels que $\sigma|_{Z_\Gamma}=\mathrm{Id}$, on a*

$$N^{\Gamma}_{\mathrm{cus}}(\lambda,\sigma)\sim\dim(\sigma)\frac{\mathrm{vol}(\Gamma\setminus X_n)}{(4\pi)^{d_n/2}\Gamma(d_n/2+1)}\lambda^{d_n/2}$$

pour $\lambda\to\infty$.

La démonstration de ce théorème utilise la formule des traces d'Arthur combinée avec la méthode de l'équation de la chaleur.

1. Introduction

Let G be a connected reductive algebraic group over $\mathbb{Q}$ and let $\Gamma\subset G(\mathbb{Q})$ be an arithmetic subgroup. An important problem in the theory of automorphic forms is the question of existence and the construction of cusp forms for Γ.

In this paper we address the problem of existence for $G=\mathrm{SL}_n$, $n\geqslant 2$. Let Γ be a congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$. Let $L^2_{\mathrm{cus}}(\Gamma\setminus \mathrm{SL}_n(\mathbb{R}))$ be the closure of the span of cusp forms for Γ. Let (σ,V_σ) be an irreducible unitary representation of $\mathrm{SO}(n)$. Set

$$L^2\big(\Gamma\setminus \mathrm{SL}_n(\mathbb{R}),\sigma\big)=\big(L^2\big(\Gamma\setminus \mathrm{SL}_n(\mathbb{R})\big)\otimes V_\sigma\big)^{\mathrm{SO}(n)},$$

and define $L^2_{\mathrm{cus}}(\Gamma\setminus \mathrm{SL}_n(\mathbb{R}),\sigma)$ similarly. Let $\Omega\in\mathcal{Z}(\mathfrak{sl}(n,\mathbb{C}))$ be the Casimir element of $\mathrm{SL}_n(\mathbb{R})$. Then $-\Omega\otimes\mathrm{Id}$ induces a selfadjoint operator Δ_σ in the Hilbert space $L^2(\Gamma\setminus \mathrm{SL}_n(\mathbb{R}),\sigma)$ which is bounded from below. The restriction of Δ_σ to the subspace $L^2_{\mathrm{cus}}(\Gamma\setminus \mathrm{SL}_n(\mathbb{R}),\sigma)$ has a pure point spectrum consisting of eigenvalues $\lambda_0(\sigma)<\lambda_1(\sigma)<\cdots$ of finite multiplicity. Let $\mathcal{E}(\lambda_i(\sigma))$ be the eigenspace corresponding to the eigenvalue $\lambda_i(\sigma)$. For $\lambda\geqslant 0$ set

$$N^{\Gamma}_{\mathrm{cus}}(\lambda,\sigma)=\sum_{\lambda_i(\sigma)\leqslant\lambda}\dim\mathcal{E}\big(\lambda_i(\sigma)\big).$$

Then our main result is the following theorem.

Theorem 1.1. *For $n\geqslant 2$ let $X_n=\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. Let $d_n=\dim X_n$. For every principal congruence subgroup Γ of $\mathrm{SL}_n(\mathbb{Z})$ and every irreducible unitary representation σ of $\mathrm{SO}(n)$ such that $\sigma|_{Z_\Gamma}=\mathrm{Id}$ we have*

$$N^{\Gamma}_{\mathrm{cus}}(\lambda,\sigma)\sim\dim(\sigma)\frac{\mathrm{vol}(\Gamma\setminus X_n)}{(4\pi)^{d_n/2}\Gamma(d_n/2+1)}\lambda^{d_n/2}$$

as $\lambda\to\infty$.

This is Weyl's law for principal congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$. For $n = 2$ this was proved by Selberg [13]. For $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ and σ the trivial representation, Weyl's law was proved by Miller [7]. It has been conjectured by Sarnak [12] and also by Müller [9] that Weyl's law holds for every arithmetic subgroup of a reductive group G.

2. The adèlic version of Weyl's law

Let $G = \mathrm{GL}_n$ regarded as algebraic group over $\mathbb{Q}$ and let A_G be the split component of the center of G. Let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$. Denote by ξ_0 the trivial character of $A_G(\mathbb{R})^0$. Let $\Pi(G(\mathbb{A}), \xi_0)$ be the set of all irreducible unitary representations of $G(\mathbb{A})$ whose central character is trivial on $A_G(\mathbb{R})^0$ and let $\Pi_{\mathrm{cus}}(G(\mathbb{A}), \xi_0)$ be the subset of cuspidal automorphic representations in $\Pi(G(\mathbb{A}), \xi_0)$. Let $\mathbb{A}_f$ be the ring of finite adèles. Given an irreducible unitary representation of $G(\mathbb{A})$, write $\pi = \pi_\infty \otimes \pi_f$, where π_∞ and π_f are irreducible unitary representations of $G(\mathbb{R})$ and $G(\mathbb{A}_f)$, respectively. Let $\mathcal{H}_{\pi_\infty}$ and $\mathcal{H}_{\pi_f}$ be the Hilbert spaces of the representations π_∞ and π_f, respectively. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$. Denote by $\mathcal{H}_{\pi_f}^{K_f}$ the subspace of K_f-invariant vectors in $\mathcal{H}_{K_f}$. Let $G(\mathbb{R})^1$ be the subgroup of all $g \in G(\mathbb{R})$ with $|\det(g)| = 1$. Given $\pi \in \Pi(G(\mathbb{A}), \xi_0)$, denote by λ_π the Casimir eigenvalue of the restriction of π_∞ to $G(\mathbb{R})^1$. For $\lambda \geqslant 0$ let $\Pi_{\mathrm{cus}}(G(\mathbb{A}), \xi_0)_\lambda$ be the space of all $\pi \in \Pi_{\mathrm{cus}}(G(\mathbb{A}), \xi_0)$ which satisfy $|\lambda_\pi| \leqslant \lambda$. Set $\varepsilon_{K_f} = 1$, if $-1 \in K_f$ and $\varepsilon_{K_f} = 0$ otherwise. Then we have

Theorem 2.1. *Let $G = \mathrm{GL}_n$ and let $d_n = \dim \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and let (τ, V_τ) be an irreducible unitary representation of $\mathrm{O}(n)$ such that $\sigma(-1) = \mathrm{Id}$ if $-1 \in K_f$. Then*

$$\sum_{\pi \in \Pi_{\mathrm{cus}}(G(\mathbb{A}),\xi_0)_\lambda} \dim\bigl(\mathcal{H}_{\pi_f}^{K_f}\bigr) \dim(\mathcal{H}_{\pi_\infty} \otimes V_\tau)^{\mathrm{O}(n)} \sim \dim(\tau) \frac{\mathrm{vol}(G(\mathbb{Q})A_G(\mathbb{R})^0 \setminus G(\mathbb{A})/K_f)}{(4\pi)^{d_n/2}\Gamma(d_n/2+1)}(1+\varepsilon_{K_f})\lambda^{d_n/2} \tag{1}$$

as $\lambda \to \infty$.

Let $N \in \mathbb{N}$ and let $N = \prod_p p^{r_p}$, $r_p \geqslant 0$, be the prime factor decomposition of N. Put $K_p(N) = \{k \in \mathrm{GL}_n(\mathbb{Z}_p) \mid k \equiv 1 \bmod p^{r_p}\mathbb{Z}_p\}$ and $K(N) = \prod_{p<\infty} K_p(N)$. Then $K(N)$ is an open compact subgroup of $G(\mathbb{A}_f)$ and as an $\mathrm{SL}_n(\mathbb{R})$-module, $L^2(G(\mathbb{Q})A_G(\mathbb{R})^0 \setminus G(\mathbb{A})/K(N))$ is isomorphic to the direct sum of $\#[(\mathbb{Z}/NZ)^*]$ copies of $L^2(\Gamma(N) \setminus \mathrm{SL}_n(\mathbb{R}))$, where $\Gamma(N)$ is the principal congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$ of level N. Using this fact, Theorem 1.1 is an immediate consequence of Theorem 2.1.

The proof of Theorem 2.1 is based on Arthur's trace formula combined with the heat equation method. Let $G(\mathbb{A})^1$ be the subgroup of all $g \in G(\mathbb{A})$ satisfying $|\det(g)| = 1$. The noninvariant trace formula of Arthur [1] is an identity

$$\sum_{\chi \in \mathfrak{X}} J_\chi(f) = \sum_{\mathfrak{o} \in \mathfrak{O}} J_{\mathfrak{o}}(f), \quad f \in C_c^\infty\bigl(G(\mathbb{A})^1\bigr), \tag{2}$$

between distributions on $G(\mathbb{A})^1$. The left-hand side is the spectral side $J_{\mathrm{spec}}(f)$ and the right-hand side the geometric side $J_{\mathrm{geo}}(f)$ of the trace formula.

We construct a special family of test functions $\tilde{\phi}_t^1 \in C_c^\infty(G(\mathbb{A})^1)$, $t > 0$, as follows. Let τ be an irreducible unitary representation of $\mathrm{O}(n)$. Let $\widetilde{E}_\tau \to G(\mathbb{R})^1/\mathrm{O}(n)$ be the homogeneous vector bundle attached to τ and let $\tilde{\Delta}_\tau$ be the elliptic operator induced by $-\Omega \otimes \mathrm{Id}$ in $C^\infty(\widetilde{E}_\tau)$. Let $H_t^\tau : G(\mathbb{R})^1 \to \mathrm{End}(V_\tau)$ be the kernel of the heat operator $\mathrm{e}^{-t\tilde{\Delta}_\tau}$. Set $h_t^\tau = \mathrm{tr}\, H_t^\tau$. We extend h_t^τ to a smooth function on $G(\mathbb{R})$ by $h_t^\tau(zg) = h_t^\tau(g)$, $g \in G(\mathbb{R})^1$, $z \in Z_{G(\mathbb{R})}$, the center of $G(\mathbb{R})$. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and let χ_{K_f} be the normalized characteristic function of K_f in $G(\mathbb{A}_f)$. For $t > 0$ we define a smooth function ϕ_t on $G(\mathbb{A})$ by

$\phi_t(g) = h_t^\tau(g_\infty)\chi_{K_f}(g_f),\ g = g_\infty g_f$. Let $\varphi \in C^\infty(\mathbb{R})$ be such that $\varphi(u) = 1$, if $|u| \leqslant 1/2$, and $\varphi(u) = 0$, if $|u| \geqslant 1$. Given $g_\infty \in G(\mathbb{R})^1$, let $r(g_\infty)$ be the Riemannian distance of the cosets in $G(\mathbb{R})^1/\mathrm{O}(n)$ of g_∞ and e, respectively. Put $\varphi_t(g_\infty) = \varphi(r^2(g_\infty)/t^{1/4}),\ g_\infty \in G(\mathbb{R})^1$. Extend φ_t to a smooth function on $G(\mathbb{R})$ by $\varphi_t(zg) = \varphi_t(g),\ g \in G(\mathbb{R})^1,\ z \in Z_{G(\mathbb{R})}$, and then to a smooth function on $G(\mathbb{A})$ by multiplying φ_t by the characteristic function of K_f. Put $\tilde{\phi}_t(g) = \varphi_t(g)\phi_t(g),\ g \in G(\mathbb{A})$.

Let $\tilde{\phi}_t^1$ be the restriction of $\tilde{\phi}_t$ to $G(\mathbb{A})^1$. Then $\tilde{\phi}_t^1 \in C_c^\infty(G(\mathbb{A})^1)$. To prove Theorem 2.1 we insert $\tilde{\phi}_t^1$ in the trace formula and compare the asymptotic behaviour of the left and right-hand side of the trace formula as $t \to 0$.

3. The spectral side of the Arthur trace formula

In this section we determine the asymptotic behaviour of $J_{\mathrm{spec}}(\tilde{\phi}_t^1)$ as $t \to 0$. By a parabolic subgroup of G we will always mean a parabolic subgroup which is defined over $\mathbb{Q}$. Let M_0 be the Levi component of the standard minimal parabolic subgroup P_0 of G. By a Levi subgroup we will mean a subgroup of G which contains M_0 and which is the Levi component of a parabolic subgroup of G. Let $\mathcal{L}$ be the set of all Levi subgroups of G. Given $M \in \mathcal{L}$, let $\mathcal{L}(M)$ be the set of Levi subgroups containing M and denote by $\mathcal{P}(M)$ the set of parabolic subgroups with Levi component M.

Let $\mathcal{C}^1(G(\mathbb{A})^1)$ denote the space of integrable rapidly decreasing functions on $G(\mathbb{A})^1$ [10, §1.3]. By Theorem 0.1 of [11] the spectral side $J_{\mathrm{spec}}(f)$ of the trace formula is absolutely convergent for all $f \in \mathcal{C}^1(G(\mathbb{A})^1)$ and can be written as a finite linear combination

$$J_{\mathrm{spec}}(f) = \sum_{M\in\mathcal{L}} \sum_{L\in\mathcal{L}(M)} \sum_{P\in\mathcal{P}(M)} \sum_{s\in W^L(\mathfrak{a}_M)_{\mathrm{reg}}} a_{M,s} J_{M,P}^L(f,s), \tag{3}$$

of distributions $J_{M,P}^L(f,s)$, where $W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ is a certain set of Weyl group elements. The main ingredients of the distribution $J_{M,P}^L(f,s)$ are generalized logarithmic derivatives of intertwining operators $M_{Q|P}(\lambda):\mathcal{A}^2(P) \to \mathcal{A}^2(Q),\ P, Q \in \mathcal{P}(M),\ \lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, acting between spaces of square-integrable automorphic forms attached to P and Q, respectively. For a detailed description of $J_{M,P}^L(f,s)$ see [11].

Let ϕ_t^1 denote the restriction of ϕ_t to $G(\mathbb{A})^1$. Then ϕ_t^1 belongs to $\mathcal{C}^1(G(\mathbb{A})^1)$ and it follows from the proof of the absolute convergence of the spectral side [11], [10] that

$$\left|J_{\mathrm{spec}}(\tilde{\phi}_t^1) - J_{\mathrm{spec}}(\phi_t^1)\right| \leqslant C\,\mathrm{e}^{-c\sqrt{t}}$$

as $t \to 0$. Thus it suffices to determine the asymptotic behaviour of $J_{\mathrm{spec}}(\phi_t^1)$ as $t \to 0$.

Let ξ_0 be the trivial character of $A_G(\mathbb{R})^0$ and let $\Pi_{\mathrm{dis}}(G(\mathbb{A}),\xi_0)$ be the set of all irreducible unitary representations of $G(\mathbb{A})$ which are equivalent to a subrepresentation of the regular representation of $G(\mathbb{A})$ in $L^2(G(\mathbb{Q})A_G(\mathbb{R})^0 \setminus G(\mathbb{A}))$. Given $\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}),\xi_0)$, let $m(\pi)$ denote the multiplicity with which π occurs in $L^2(G(\mathbb{Q})A_G(\mathbb{R})^0 \setminus G(\mathbb{A}))$. Let $\tau \in \Pi(\mathrm{O}(n))$.

Theorem 3.1. *We have*

$$J_{\mathrm{spec}}(\phi_t^1) = \sum_{\pi\in\Pi_{\mathrm{dis}}(G(\mathbb{A}),\xi_0)} m(\pi)\dim\left(\mathcal{H}_{\pi_f}^{K_f}\right)\dim\left(\mathcal{H}_{\pi_\infty}\otimes V_\tau\right)^{\mathrm{O}(n)}\mathrm{e}^{t\lambda_\pi} + \mathrm{O}\left(t^{-(d_n-1)/2}\right), \tag{4}$$

as $t \to 0^+$, and the series on the right-hand side is convergent for all $t > 0$.

The proof of this theorem is based on (3). We evaluate the distributions $J_{M,P}^L$ at ϕ_t^1. If $M = L = G$, then $s = 1$ and $J_{G,G}^G(\phi_t^1, 1)$ equals the series on the right-hand side of (4). The proof is completed by showing that for all proper Levi subgroups $M \in \mathcal{L}$, all $L \in \mathcal{L}(M),\ P \in \mathcal{P}(M)$ and $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ we have

$$J_{M,P}^L\left(\phi_t^1, s\right) = \mathrm{O}\left(t^{-(d_n-1)/2}\right) \tag{5}$$

as $t \to 0$. This is the key result. The proof of (5) relies on estimations of generalized logarithmic derivatives of the intertwining operators $M_{Q|P}(\lambda)$, $P, Q \in \mathcal{P}(M)$, on $\lambda \in i\mathfrak{a}_M^*$. Given $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}), \xi_0)$, let $M_{Q|P}(\pi, \lambda)$ be the restriction of the intertwining operator $M_{Q|P}(\lambda)$ to the subspace $\mathcal{A}_\pi^2(P)$ of automorphic forms of type π. The intertwining operators can be normalized by certain meromorphic functions $r_{Q|P}(\pi, \lambda)$ [3]. Using Arthur's theory of (G, M)-families [2], our problem can be reduced to the estimation of derivatives of the normalized intertwining operators $N_{Q|P}(\pi, \lambda)$ and the normalizing factors $r_{Q|P}(\pi, \lambda)$ on $i\mathfrak{a}_M^*$. The derivatives of $N_{Q|P}(\pi, \lambda)$ can be estimated using Proposition 0.2 of [11]. The normalizing factors are defined in terms of the Rankin–Selberg L-functions $L(s, \pi_i \times \tilde{\pi}_j)$. So the problem is reduced to the estimation of the logarithmic derivatives of Rankin–Selberg L-functions on the line $\mathrm{Re}(s) = 1$. Estimates are derived using the analytic properties of the Rankin–Selberg L-functions together with standard methods of analytic number theory.

4. The geometric side of the Arthur trace formula

To study the asymptotic behaviour of the geometric side $J_{\mathrm{geo}}(\tilde{\phi}_t^1)$ of the trace formula, we use the fine $\mathfrak{o}$-expansion [4]

$$J_{\mathrm{geo}}(f) = \sum_{M \in \mathcal{L}} \sum_{\gamma \in (M(\mathbb{Q}_S))_{M,S}} a^M(S, \gamma) J_M(f, \gamma), \quad f \in C_c^\infty\big(G(\mathbb{A})^1\big), \tag{6}$$

which expresses the distribution $J_{\mathrm{geo}}(f)$ in terms of weighted orbital integrals $J_M(\gamma, f)$. Here S is a finite set of places of $\mathbb{Q}$, and $(M(\mathbb{Q}_S))_{M,S}$ is a certain set of equivalence classes in $M(\mathbb{Q}_S)$. This reduces our problem to the investigation of weighted orbital integrals. The key result is that

$$\lim_{t \to 0} t^{d_n/2} J_M\big(\tilde{\phi}_t^1, \gamma\big) = 0, \tag{7}$$

unless $M = G$ and $\gamma = \pm 1$. The contributions to (6) of the terms where $M = G$ and $\gamma = \pm 1$ are easy to determine. Set $\varepsilon_{K_f} = 1$, if $-1 \in K_f$ and $\varepsilon_{K_f} = 0$ otherwise. Using the behaviour of the heat kernel $h_t^\tau(\pm 1)$ as $t \to 0$, it follows that

$$J_{\mathrm{geo}}\big(\tilde{\phi}_t^1\big) \sim \dim(\tau) \frac{\mathrm{vol}(G(\mathbb{Q}) \setminus G(\mathbb{A})^1 / K_f)}{(4\pi)^{d_n/2}} (1 + \varepsilon_{K_f}) t^{-d_n/2} \tag{8}$$

as $t \to 0$.

5. Proof of the main theorem

By the trace formula (2) we have $J_{\mathrm{spec}}(\tilde{\phi}_t^1) = J_{\mathrm{geo}}(\tilde{\phi}_t^1)$, $t > 0$. Using (4) and (8), it follows that

$$\sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}), \xi_0)} m(\pi) \dim\big(\mathcal{H}_{\pi_f}^{K_f}\big) \dim(\mathcal{H}_{\pi_\infty} \otimes V_\tau)^{\mathrm{O}(n)} \mathrm{e}^{t\lambda_\pi}$$
$$\sim \dim(\tau) \frac{\mathrm{vol}(G(\mathbb{Q}) \setminus G(\mathbb{A})^1 / K_f)}{(4\pi)^{d/2}} (1 + \varepsilon_{K_f}) t^{-d_n/2} \tag{9}$$

as $t \to 0$. Using [5] and [8] it follows that in (9) one can replace $\Pi_{\mathrm{dis}}(G(\mathbb{A}), \xi_0)$ by $\Pi_{\mathrm{cus}}(G(\mathbb{A}), \xi_0)$ and the same asymptotic formula remains true. Then Theorem 2.1 is an immediate consequence of Karamata's theorem [6, p. 446]. As explained above, Theorem 2.1 implies Theorem 1.1.

References

[1] J. Arthur, A trace formula for reductive groups I: terms associated to classes in $G(\mathbb{Q})$, Duke. Math. J. 45 (1978) 911–952.
[2] J. Arthur, The trace formula in invariant form, Ann. Math. 114 (1981) 1–74.
[3] J. Arthur, Intertwining operators and residues. I. Weighted characters, J. Funct. Anal. 84 (1989) 19–84.
[4] J. Arthur, On a family of distributions obtained from orbits, Canad. J. Math. 38 (1986) 179–214.
[5] H. Donnelly, On the cuspidal spectrum for finite volume symmetric spaces, J. Differential Geom. 17 (1982) 239–253.
[6] W. Feller, An Introduction to Probability Theory and its Applications, vol. II, second ed., Wiley, New York, 1971.
[7] St. Miller, On the existence and temperedness of cusp forms for $\mathrm{SL}_3(\mathbb{Z})$, J. Reine Angew. Math. 533 (2001) 127–169.
[8] C. Moeglin, J.-L. Waldspurger, Le spectre résiduel de $\mathrm{GL}(n)$, Ann. Sci. École Norm. Sup. (4) 22 (1989) 605–674.
[9] W. Müller, Eigenvalue estimates for locally symmetric spaces of finite volume, in: Symposium Partial Differential Equations, Holzhau, 1988, in: Teubner-Texte Math., vol. 112, Teubner, Leipzig, 1989, pp. 179–196.
[10] W. Müller, On the spectral side of the Arthur trace formula, Geom. Funct. Anal. 12 (2002) 669–722.
[11] W. Müller, B. Speh, With appendix by E. Lapid, Absolute convergence of the spectral side of the Arthur trace formula for GL_n, Geom. Funct. Anal., in press.
[12] P. Sarnak, On cusp forms, in: D. Hejhal, et al. (Eds.), The Selberg Trace Formula and Related Topics, in: Contemp. Math., vol. 53, Amer. Math. Soc., 1984, pp. 393–407.
[13] A. Selberg, Harmonic analysis, in: Collected Papers, vol. I, Springer-Verlag, Berlin, 1989, pp. 626–674.