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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

On the stability of radial solutions of semilinear elliptic equations in all of $\mathbb{R}^n$

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Abstract

We establish that every nonconstant bounded radial solution u of $-\Delta u = f(u)$ in all of $\mathbb{R}^n$ is unstable if $n \leqslant 10$. The result applies to every C^1 nonlinearity f satisfying a generic nondegeneracy condition. In particular, it applies to every analytic and every power-like nonlinearity. We also give an example of a nonconstant bounded radial solution u which is stable for every $n \geqslant 11$, and where f is a polynomial. ***To cite this article: X. Cabré, A. Capella, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Sur la stabilité des solutions radiales des équations elliptiques semi-linéaires dans tout $\mathbb{R}^n$. On montre que toute solution u non constante, bornée et radiale de l'équation $-\Delta u = f(u)$ dans tout $\mathbb{R}^n$ est instable si $n \leqslant 10$. Ce résultat s'applique à toute nonlinéarité f de classe C^1 qui satisfait une condition générique de non dégénérescence. Il s'applique, en particulier, à toute nonlinéarité analytique et à toute nonlinéarité de type puissance. On donne aussi un exemple de solution u non constante, bornée et radiale qui est stable pour tout $n \geqslant 11$, et où f est un polynôme. ***Pour citer cet article : X. Cabré, A. Capella, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

On étudie les propriétés de stabilité des solutions bornées de l'équation elliptique

$$-\Delta u = f(u) \quad \text{dans } \mathbb{R}^n, \tag{1}$$

où $f \in C^1(\mathbb{R})$. La forme quadratique associée au problème linéarisé de (1) en u est donnée par $Q(\xi) = \int_{\mathbb{R}^n} \{|\nabla \xi|^2 - f'(u)\xi^2\}\,\mathrm{d}x$ où $\xi \in C_c^\infty(\mathbb{R}^n)$, c'est-à-dire, ξ est C^∞ avec support compact dans $\mathbb{R}^n$.

On dit qu'une solution bornée u de (1) est stable si $Q(\xi) \geqslant 0$ pour toute $\xi \in C_c^\infty(\mathbb{R}^n)$. Dans le cas contraire, on dit que u est instable.

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Dans un premier résultat de cette Note, on utilise des méthodes récemment développées dans [2,1] pour établir que toute solution de (1) non constante, bornée et dans $H^1(\mathbb{R}^n)$ est nécessairement instable, pour toute f.

Les méthodes de [2,1] montrent aussi que, pour $n \leqslant 2$, une solution u non constante et bornée de (1) est stable si et seulement si u ne dépend que d'une seule variable et u est croissante ou décroissante. En particulier, les solutions non constantes, bornées et radiales sont toujours instables pour $n \leqslant 2$. On rappelle qu'on dit que u est radiale si u est de la forme $u = u(r)$, où $r = |x|$ et $x \in \mathbb{R}^n$.

Dans cette Note, on étudie les propriétés de stabilité des solutions radiales en dimensions n supérieures et pour toute nonlinéarité f. Le théorème suivant est notre résultat principal.

Théorème 0.1. (a) *Soient $n \leqslant 10$, $f \in C^1(\mathbb{R})$ et u une solution non constante, bornée et radiale de* (1). *Si $9 \leqslant n \leqslant 10$, on suppose que pour tout $s_0 \in \mathbb{R}$ il existe des nombres réels $q \geqslant 0$ et $a > 0$ (qui peuvent dépendre de s_0) tels que $\lim_{s \to s_0} |f'(s)||s - s_0|^{-q} = a \in (0, \infty)$. Alors, u est instable.*

(b) *Pour $n \geqslant 11$, il existe un polynôme f qui admet une solution stable, non constante, bornée et radiale de* (1).

Toute nonlinearité f analytique et toute f de la forme $f(s) = |s|^p$ où $f(s) = |s|^{p-1}s$ avec $p > 1$, satisfait l'hypothèse du Théorème 0.1(a) quand $9 \leqslant n \leqslant 10$. L'hypothèse est aussi satisfaite par toute $f \in C^\infty(\mathbb{R})$ telle que pour chaque $s_0 \in \mathbb{R}$ il existe un nombre entier $k = k(s_0) \geqslant 1$ avec $f^{(k)}(s_0) \neq 0$.

Exemple 1. Pour établir la partie (b) du Théorème 0.1, on considère $u(r) = (1 + r^2)^{-1/8}$, qui est une solution bornée et C^∞ de $-\Delta u = ((4n - 9)u^9 + 9u^{17})/16 =: f(u)$. Pour $n \geqslant 11$ on peut vérifier que $f'(u) = (9(4n - 9)r^2 + 36(n + 2))/(16(1 + r^2)^2) \leqslant (n - 2)^2/(4r^2)$ pour tout $r > 0$. En conséquence, u est stable pour $n \geqslant 11$, grâce à l'inegalité de Hardy : $\int_{\mathbb{R}^n} \{(n - 2)^2/(4r^2)\}\xi^2 \leqslant \int_{\mathbb{R}^n} |\nabla\xi|^2$ pour toute $\xi \in C_c^\infty(\mathbb{R}^n)$.

1. Introduction

We study the stability properties of bounded solutions of the elliptic equation

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n, \tag{2}$$

where $f \in C^1(\mathbb{R})$. The energy functional associated to (2) in a bounded domain $\Omega \subset \mathbb{R}^n$ is defined by $E_\Omega(u) = \int_\Omega \{|\nabla u|^2/2 - F(u)\}\,\mathrm{d}x$, where $F' = f$. The second variation of energy is given by

$$Q(\xi) = \int_{\mathbb{R}^n} \left\{|\nabla\xi|^2 - f'(u)\xi^2\right\} \mathrm{d}x \tag{3}$$

for $\xi \in C_c^\infty(\mathbb{R}^n)$, that is, ξ is C^∞ with compact support in $\mathbb{R}^n$.

Definition 1.1. We say that a bounded solution u of (2) is stable if the second variation of energy Q satisfies $Q(\xi) \geqslant 0$ for all $\xi \in C_c^\infty(\mathbb{R}^n)$. Otherwise we say that u is unstable.

The following is our first result. It originated from methods recently developed in [2,1] in connection with a conjecture of De Giorgi.

Proposition 1.2. *Let $f \in C^1(\mathbb{R})$ and u be a nonconstant bounded solution of* (2).

(a) *Assume that $u \in H^1(\mathbb{R}^n)$ (for $n \geqslant 2$ it suffices to assume that $|\nabla u| \in L^2(\mathbb{R}^n)$). Then, u is unstable.*
(b) *Assume that $n \leqslant 2$. Then, u is stable if and only if u is of the form $u = u(\langle e, x\rangle)$ and satisfies $\partial_e u \neq 0$ in all of $\mathbb{R}^n$ for some $e \in \mathbb{R}^n \setminus \{0\}$. In particular, if u is radial then it is unstable.*

Proposition 1.2 is proven below. Note that it applies to every bounded solution, not necessarily radial. We recall that u is said to be radial if it is of the form $u = u(r)$, where $r = |x|$ and $x \in \mathbb{R}^n$.

Proposition 1.2(b) characterizes all stable solutions when $n \leqslant 2$, a difficult open task in higher dimensions. Its last statement, that stable nonconstant bounded solutions are never radial for $n \leqslant 2$, is a very particular consequence of it, that we study here in higher dimensions and still for every f. The following theorem is the main result of this Note.

2. Main result

Theorem 2.1.

(a) *Let $n \leqslant 10$, $f \in C^1(\mathbb{R})$, and u be a nonconstant bounded radial solution of* (2). *If* $9 \leqslant n \leqslant 10$, *assume also that for every $s_0 \in \mathbb{R}$ there exist real numbers $q \geqslant 0$ and $a > 0$ (which may depend on s_0) such that* $\lim_{s \to s_0} |f'(s)||s - s_0|^{-q} = a \in (0, \infty)$. *Then, u is unstable.*

(b) *For $n \geqslant 11$, there exists a polynomial f which admits a stable nonconstant bounded radial solution u of* (2).

Note that every analytic nonlinearity f, and every f of the form $f(s) = |s|^p$ or $f(s) = |s|^{p-1}s$ with $p > 1$, satisfies the hypothesis of Theorem 2.1(a) for $9 \leqslant n \leqslant 10$. The same holds for every $f \in C^\infty(\mathbb{R})$ such that for each $s_0 \in \mathbb{R}$ there exists an integer $k = k(s_0) \geqslant 1$ with $f^{(k)}(s_0) \neq 0$.

Example 1. To establish Theorem 2.1(b), consider $u(r) = (1 + r^2)^{-1/8}$, a bounded C^∞ solution of $-\Delta u = ((4n - 9)u^9 + 9u^{17})/16 =: f(u)$. For $n \geqslant 11$ it can be shown that

$$f'(u) = \frac{9(4n-9)r^2 + 36(n+2)}{16(1+r^2)^2} \leqslant \frac{(n-2)^2}{4r^2} \quad \text{for all } r > 0.$$

Hence u is stable for $n \geqslant 11$, by Hardy inequality: $\int_{\mathbb{R}^n} \{(n-2)^2/(4r^2)\}\xi^2 \leqslant \int_{\mathbb{R}^n} |\nabla \xi|^2$, $\xi \in C_c^\infty(\mathbb{R}^n)$.

Some geometric criteria to determine the stability or unstability of radial solutions for $n \geqslant 11$ will be given in [5]. They are related to recent developments from [1] that establish relations between minimality and monotonicity properties of solutions.

Berestycki et al. [3,4] proved the existence and the unstability (also under the flow of the parabolic equation) of a radial solution $u \in H^1(\mathbb{R}^n)$ of (2) under the assumptions $n \geqslant 3$, $f(0) = 0$, $f'(0) < 0$, $F(\zeta) > 0$ for some $\zeta > 0$, and f subcritical at infinity. Proposition 1.2(a) extends part of this result by establishing the unstability of every $H^1(\mathbb{R}^n)$ solution for general f.

The cutting dimension $n = 10$ appears in the 1992 paper by Gui et al. [8], which studies positive solutions of $u_t = \Delta u + u^p$ for $p > 1$. Among other things, they prove that for $n \leqslant 10$ every stationary radial solution is unstable, while for $n \geqslant 11$ there exists an exponent $p_c \in (0, \infty)$ such that for $p \geqslant p_c$ there exists a stable stationary radial solution. Theorem 2.1(a) above extends the first of these results to the case of general f.

Existence of solutions for equations of mean curvature and p-Laplacian type are studied by Franchi et al. [6]. Corresponding stability results will be given in [5].

In a forthcoming paper, we use methods developed in the present Note to study the boundedness of weak stable solutions, and in particular of extremal solutions, for semilinear problems in a ball.

To prove Theorem 2.1(a) we need two preliminary results. The first one, Lemma 2.2 below, was inspired by the proof of Simons theorem on the nonexistence of singular minimal cones in $\mathbb{R}^n$ for $n \leqslant 7$ (see the proof of Theorem 10.10 of [7]).

Lemma 2.2. *Let u be a bounded radial solution of* (2). *Then, for every $\eta \in (H^1 \cap L^\infty)(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n \setminus \{0\}$, we have that $u_r \eta \in H^1(\mathbb{R}^n)$ has compact support in $\mathbb{R}^n \setminus \{0\}$ and*

$$Q(u_r\eta) = \int_{\mathbb{R}^n} u_r^2 \left\{ |\nabla \eta|^2 - \frac{n-1}{r^2}\eta^2 \right\} \mathrm{d}x,$$

where $Q(\xi)$ is defined by (3) *for $\xi \in H^1(\mathbb{R}^n)$.*

Proof. Let $\eta \in (H^1 \cap L^\infty)(\mathbb{R}^n)$ have compact support in $\mathbb{R}^n \setminus \{0\}$, and $c \in (H^2_{\mathrm{loc}} \cap L^\infty)(\mathbb{R}^n \setminus \{0\})$. Take $\xi = c\eta$ in (3). We obtain that $Q(c\eta) = \int_{\mathbb{R}^n} c^2|\nabla\eta|^2 + \nabla\eta^2 \cdot c\nabla c + \eta^2|\nabla c|^2 - f'(u)c^2\eta^2 = \int_{\mathbb{R}^n} c^2|\nabla\eta|^2 - \eta^2\nabla\cdot(c\nabla c) + \eta^2|\nabla c|^2 - f'(u)c^2\eta^2 = \int_{\mathbb{R}^n} c^2|\nabla\eta|^2 - \eta^2(c\Delta c + f'(u)c^2)$.

Differentiating (2) with respect to r, we have

$$-\Delta u_r + \frac{n-1}{r^2}u_r = f'(u)u_r \quad \text{for } r > 0. \tag{4}$$

By local $W^{2,p}$ estimates for (2) and (4), we have that $c := u_r \in (H^2_{\mathrm{loc}} \cap L^\infty)(\mathbb{R}^n \setminus \{0\})$. Using (4) in the last expression for $Q(c\eta)$, we conclude Lemma 2.2. □

Lemma 2.3. *Let $f \in C^1(\mathbb{R})$, and u be a stable nonconstant bounded radial solution of* (2). *Then*:

(a) *u_r has constant sign in $(0,\infty)$. In particular, $\int_0^\infty u_r^2\,\mathrm{d}r \leqslant C\int_0^\infty |u_r|\,\mathrm{d}r < \infty$.*
(b) *$f(u_\infty) = 0$ and $f'(u_\infty) \leqslant 0$, where $u_\infty = \lim_{r\to\infty} u(r)$.*
(c) *If f satisfies the hypothesis of Theorem* 2.1(a) *for $s_0 = u_\infty$, then*

$$|u_r(r)| \leqslant C/r \quad \text{for all } r > 0, \tag{5}$$

for some constant C. In particular,

$$\int_0^\infty u_r^2 r\,\mathrm{d}r < \infty. \tag{6}$$

Open problem. Does (6) hold for every $f \in C^1$ and every stable bounded radial solution u? If the answer were yes, then Theorem 2.1(a) would hold for every $f \in C^1$ even when $9 \leqslant n \leqslant 10$.

Proof of Proposition 1.2. Assume that u is a stable nonconstant bounded solution. By Proposition 4.2 of [1], there exists a continuous function $\varphi \in H^2_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\varphi > 0$ and $-\Delta\varphi = f'(u)\varphi$ in $\mathbb{R}^n$. Assume either that $|\nabla u| \in L^2(\mathbb{R}^n)$, that $n = 1$, or that $n = 2$. In the three cases we have that $\int_{B_R} |\nabla u|^2 \leqslant CR^2$ for $R > 1$. Hence, the Liouville property of Theorem 3.1 in [1] can be applied to the equation satisfied by $(\partial_{x_i} u)/\varphi$. One concludes that, for every $i \in \{1,\ldots,n\}$, $\partial_{x_i} u = c_i\varphi$ for some constant c_i. This easily implies (see [2] or [1]) that u is of the form $u = u(\langle e, x\rangle)$ and satisfies either $\partial_e u > 0$ in $\mathbb{R}^n$, or $\partial_e u < 0$ in $\mathbb{R}^n$, for some $e \in \mathbb{R}^n \setminus \{0\}$.

(a) To prove part (a), we argue by contradiction and assume that u is stable. The previous argument gives that u must be of the form above, that is, a 1D solution either increasing or decreasing. But then $u \notin L^2(\mathbb{R}^n)$, and hence $u \notin H^1(\mathbb{R}^n)$. Moreover, ∇u is constant in parallel hyperplanes, and hence $\int_{B_R} |\nabla u|^2 \geqslant cR^{n-1}$ for all $R > 1$. In particular, $|\nabla u| \notin L^2(\mathbb{R}^n)$ if $n \geqslant 2$, a contradiction.

(b) One implication is already proven in the argument above. The other (i.e., that 1D monotone solutions are stable) is trivial (see the proof of Corollary 4.3 of [1]). The last statement of part (b) (that u cannot be radial) follows from $\partial_e u \neq 0$ in $\mathbb{R}^n$ and the fact that $\nabla u(0) = 0$ if u is radial. □

Using Lemma 2.3, that we prove later, we can give the

Proof of Theorem 2.1. Part (b) is established by the Example 1 above. To prove part (a), we may assume $3 \leqslant n \leqslant 10$, since the cases $n = 1$ and $n = 2$ are already covered by Proposition 1.2(b).

We argue by contradiction and assume that u is a stable nonconstant bounded radial solution. By approximation, the stability of u implies that $Q(\xi) \geqslant 0$ for all $\xi \in H^1(\mathbb{R}^n)$ with compact support. Hence, Lemma 2.2 leads to

$$(n-1) \int_{\mathbb{R}^n} \frac{u_r^2 \eta^2}{r^2} \, \mathrm{d}x \leqslant \int_{\mathbb{R}^n} u_r^2 |\nabla \eta|^2 \, \mathrm{d}x, \tag{7}$$

for every $\eta \in (H^1 \cap L^\infty)(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n \setminus \{0\}$. Let now $\eta \in (H^1 \cap L^\infty)(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$ (now not necessarily vanishing around 0), and take $\zeta \in C^\infty$ such that $\zeta \equiv 0$ in B_1 and $\zeta \equiv 1$ in $\mathbb{R}^n \setminus B_2$. By local $W^{2,p}$ estimates for (2), ∇u (and hence also u_r) are bounded in $\mathbb{R}^n$. Applying (7) to $\eta(\cdot)\zeta(\cdot/\varepsilon)$, letting $\varepsilon \to 0$, and using $u_r \in L^\infty(\mathbb{R}^n)$ and $n \geqslant 3$, we see that (7) also holds for every $\eta \in (H^1 \cap L^\infty)(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$.

For $\alpha > 0$, choose

$$\eta(r) = \begin{cases} 1 & \text{if } r < 1, \\ r^{-\alpha} & \text{if } r \geqslant 1, \end{cases} \tag{8}$$

and apply (7) to $\eta - R^{-\alpha}$ extended by zero outside $B_R(0)$. Letting $R \to \infty$ and using monotone convergence, we see that (7) also holds for η given by (8). Now, we take $\alpha > 0$ such that the right-hand side of (7) is finite, i.e.,

$$\int_1^\infty u_r^2 r^{n-2\alpha-3} \, \mathrm{d}r < \infty. \tag{9}$$

If (9) holds, then (7) applied to η given by (8) leads to

$$0 < \left\{\alpha^2 - (n-1)\right\} \int_1^\infty u_r^2 r^{n-2\alpha-3} \, \mathrm{d}r < \infty, \tag{10}$$

where the first strict inequality is a consequence of (9) and of having dropped the contribution from 0 to 1 in the integral of the left-hand side of (7) (together with $n - 1 > 0$).

By Lemma 2.3(a), (9) will hold if we can choose $\alpha > 0$ such that $n - 2\alpha - 3 \leqslant 0$. For $3 \leqslant n \leqslant 8$, we can take $\alpha > 0$ such that $(n-3)/2 \leqslant \alpha \leqslant \sqrt{n-1}$. Now the last inequality, $\alpha^2 \leqslant n - 1$, gives a contradiction with (10).

Finally, for $9 \leqslant n \leqslant 10$ we use (6) of Lemma 2.3(c) to ensure (9) whenever $n - 2\alpha - 3 \leqslant 1$. Now, since $n \leqslant 10$, we can take $\alpha > 0$ such that $(n-4)/2 \leqslant \alpha \leqslant \sqrt{n-1}$. The last inequality, $\alpha^2 \leqslant n - 1$, gives a contradiction with (10). □

Proof of Lemma 2.3. We can assume that $n \geqslant 3$ since, by Proposition 1.2(b), there are no stable nonconstant bounded radial solutions for $n = 1$ and $n = 2$.

(a) Arguing by contradiction, assume $u_r(R) = 0$ for some $R > 0$. By $W^{2,p}$ estimates for (2), $\nabla u \in (H^1 \cap L^\infty)(B_R(0))$. Hence $u_r = \langle \nabla u, x/r \rangle \in (H_0^1 \cap L^\infty)(B_R(0))$, since $n \geqslant 3$. Multiply (4) by $\zeta(\cdot/\varepsilon) u_r$ (with ζ vanishing around 0 as in the proof of Theorem 2.1), integrate by parts, let $\varepsilon \to 0$ and use $n \geqslant 3$, to obtain $Q(u_r \chi_{B_R(0)}) = \int_{B_R(0)} \{|\nabla u_r|^2 - f'(u) u_r^2\} \, \mathrm{d}x = -(n-1) \int_{B_R(0)} \frac{u_r^2}{r^2} \, \mathrm{d}x < 0$, a contradiction with the stability of u.

Now, since ∇u is bounded, $u_r^2 \leqslant C|u_r|$. Moreover, $\int_0^\infty |u_r| \, \mathrm{d}r < \infty$ since u_r has constant sign. Indeed, say that $u_r < 0$ for $r > 0$. This implies that the limit of u at infinity, u_∞, exists. In addition, $\int_0^\infty |u_r| \, \mathrm{d}r = -\int_0^\infty u_r \, \mathrm{d}r = u(0) - u_\infty < \infty$.

(b) From (a) we have that u_∞ exists. Choose a function $0 \not\equiv \zeta \in C_c^\infty(B_1(0))$. For $y \in \mathbb{R}^n$, let $\zeta^y(\cdot) := \zeta(\cdot - y)$. Multiply $-\Delta(u - u_\infty) = f(u)$ by ζ^y and integrate by parts twice on $B_1(y)$. Letting $|y| \to \infty$, we conclude $f(u_\infty) = 0$.

For the second statement, we argue by contradiction. Assume $f'(u_\infty) > 0$. Then, for large r, $f'(u(r)) \geqslant \varepsilon > 0$. Taking ξ supported on a ring centered at the origin and of large inner radius R, from (3) we get $\varepsilon \int \xi^2\,\mathrm{d}x \leqslant \int |\nabla\xi|^2\,\mathrm{d}x$. Choosing $\xi(x) = \tilde{\xi}(r/R)$, where $\tilde{\xi} \equiv 0$ in $(0,1) \cup (4,\infty)$ and $\tilde{\xi} \equiv 1$ in $(2,3)$, we obtain $\varepsilon R^n \leqslant CR^{n-2}$, a contradiction for R large enough.

(c) By adding a constant to u, and perhaps changing u by $-u$, we may assume $u_\infty = 0$, $u > 0$, and $u_r < 0$. Our hypothesis on f implies that the limit

$$\lim_{s\to 0^+} f'(s)s^{-q} = b \in \mathbb{R} \setminus \{0\} \tag{11}$$

also exists and it is nonzero, for some $q \geqslant 0$.

Case 1. $b < 0$. In this case (11) leads to $f'(s) < 0$ for small $s > 0$. Note also that since u has a limit at infinity, there exist $r_k \to +\infty$ such that $u_r(r_k) \to 0$. We have $f'(u)u_r \geqslant 0$ for large r, and hence equation (4) leads to $r^{1-n}\partial_r(r^{n-1}\partial_r u_r) \leqslant 0$ for large r. That is, $r^{n-1}\partial_r u_r$ is a nonincreasing function for large r, and therefore $\partial_r u_r \leqslant Cr^{1-n}$ for large r, where throughout the proof C denotes positive constants that may differ in each occurrence. Integrating on r from t to r_k (here and in similar situations later in the proof, we use $n \geqslant 3$) and letting $k \to \infty$, we get (5).

Case 2. $b > 0$. From (b) we know that $f(0) = 0$ and $f'(0) \leqslant 0$. Hence, $q = 0$ is impossible by (11), since $b > 0$. Therefore $q > 0$, and we deduce $f(s) \geqslant Cs^{q+1}$ for small $s > 0$. This implies that $-\partial_r(r^{n-1}u_r) \geqslant Cu^{q+1}r^{n-1}$ for large r. Integrating on r from s to t, we get $-u_r(t)t^{n-1} \geqslant C\int_s^t u^{q+1}(r)r^{n-1}\,\mathrm{d}r - u_r(s)s^{n-1} \geqslant C\int_s^t u^{q+1}(r)r^{n-1}\,\mathrm{d}r$ for large $s < t$. Since $u^{q+1}(r) > u^{q+1}(t)$ for $r < t$, we deduce $-u_r(t)u^{-(q+1)}(t) \geqslant C(t - s^n/t^{n-1})$ for large $s < t$. Integrating on t from s to r, using $q > 0$, and choosing a value of s large enough, we get $u^{-q}(r) \geqslant Cr^2 - D$ for large r, where $D > 0$ is a constant. Considering both large and small values of r, we conclude

$$u^q(r) \leqslant Cr^{-2} \quad \text{for } r > 0. \tag{12}$$

By (11), we also have that $f(s) \leqslant Cs^{q+1}$ for all $s \in [0, \max\{u\}]$. Using Eq. (2), we deduce $-\partial_r(r^{n-1}u_r) \leqslant Cu^{q+1}r^{n-1} \leqslant Cur^{n-3}$ for $r > 0$, where we have used estimate (12) in the last inequality. Now, we integrate on r from 0 to t and obtain $-t^{n-1}u_r(t) \leqslant C\|u\|_{L^\infty}t^{n-2}$, which gives estimate (5).

Finally, (6) follows from (5) and $\int_0^\infty |u_r|\,\mathrm{d}r < \infty$. □

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