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Behavior of random walks on $\mathbb{Z}$ in Gibbsian medium

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Abstract

We consider random walks on $\mathbb{Z}$ in a random medium with $\{-L, \ldots, -1, 0, +1\}$ as possible jumps, where $L \geqslant 1$ is fixed. When the environment is defined by a Gibbs measure on a subshift of finite type, we show a dichotomy in the recurrent case between the pointwise functional CLT and the slow behavior described by Sinaï. In the transient cases and under natural integrability conditions, we prove the validity of the averaged CLT. ***To cite this article: J. Bremont, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Comportement des marches aléatoires sur $\mathbb{Z}$ en milieu Gibbsien. Nous étudions des marches aléatoires sur $\mathbb{Z}$ en milieu aléatoire avec $\{-L, \ldots, -1, 0, +1\}$ comme sauts possibles, où $L \geqslant 1$ est fixé. Pour un environnement défini par une mesure de Gibbs sur un sous-shift de type fini, nous montrons dans le cas récurrent une dichotomie entre le TCL fonctionnel ponctuel et le comportement lent décrit par Sinaï. Dans les cas transients et sous des conditions d'intégrabilité naturelles, nous montrons la validité du TCL en moyenne. ***Pour citer cet article : J. Bremont, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Presentation

Random walks in random media appear in the modelisation of the statistical homogeneity properties of a solid where a diffusion takes place (see [2]). We consider a model on $\mathbb{Z}$ with a stationary and ergodic environment and when the possible jumps form a set of consecutive integers $\Lambda = \{-L, \ldots, -1, 0, +1\}$, where $L \geqslant 1$ is fixed. Let $(\Omega, \mathcal{F}, \mu, T)$ be an ergodic and invertible dynamical system, where $(\Omega, \mathcal{F}, \mu)$ is a probability space and T is an invertible transformation, measurable as well as T^{-1}. Let then $(p_z)_{z\in\Lambda}$ be a family of positive random variables on $(\Omega, \mathcal{F})$ verifying $\sum_{z\in\Lambda} p_z = 1$, μ-as, and such that there exists $\varepsilon > 0$ with: $\forall z \in \Lambda$, $z \neq 0$, $p_z \geqslant \varepsilon$.

Fixing $\omega \in \Omega$, we introduce the Markov chain $(\xi_n(\omega))_{n\geqslant 0}$ on $\mathbb{Z}$ with $\xi_0(\omega) = 0$ and the transition laws: $\mathcal{P}_0^\omega(\xi_{n+1}(\omega) = x + z \mid \xi_n(\omega) = x) := p_z(T^x\omega)$, $\forall x \in \mathbb{Z}$, $\forall z \in \Lambda$. The "quenched problem" is to describe the behavior of $(\xi_n(\omega))_{n\geqslant 0}$ with $\mathcal{P}_0^\omega$-probability one, for μ-almost every ω.

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This model has been widely studied when $L = 1$. See [12,4] or [5] for a detailed review. The general case $L \geqslant 1$ was considered in [4] and the aim of this paper is to bring some completeness to this study by considering environments $(\Omega, \mathcal{F}, \mu, T)$ of Gibbs type. We first introduce some definitions and recall a few results.

For $1 \leqslant i \leqslant L$, set $a_i = (p_{-i} + \cdots + p_{-L})/p_1$ and define the random matrix M of size $L \times L$:

$$M := \begin{pmatrix} a_1 & \cdots & a_{L-1} & a_L \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The main Lyapunov exponent $\gamma(M, T)$ of M with respect to T is defined by:

$$\gamma(M, T) = \lim_{n \to +\infty} \frac{1}{n} \log \left\| T^{n-1} M \cdots M \right\|, \quad \mu\text{-as.}$$

Denote also by V and λ respectively the unique positive random vector with $\|V\|_1 = 1$ and the unique positive random scalar such that $MV = \lambda TV$ (see [4]). The minoration condition on $(p_z)_{z \in \Lambda}$ implies that $\log \lambda$ is a bounded map. We also have $\int \log \lambda \, \mathrm{d}\mu = \gamma(M, T)$. The next result is detailed in [4] and can be seen as a particular case of Key's Theorem [10].

Theorem 1.1. *The asymptotic behavior of the Random Walk is the following*:

(i) *If* $\gamma(M, T) < 0$, *then*: $\xi_n(\omega) \to +\infty$, $\mathcal{P}_0^\omega$*-as*, μ*-as.*
(ii) *If* $\gamma(M, T) = 0$, *then*: $-\infty = \liminf \xi_n(\omega) < \limsup \xi_n(\omega) = +\infty$, $\mathcal{P}_0^\omega$*-as*, μ*-as.*
(iii) *If* $\gamma(M, T) > 0$, *then*: $\xi_n(\omega) \to -\infty$, $\mathcal{P}_0^\omega$*-as*, μ*-as.*

We now present an extension of Sinaï's Theorem [11] on the existence of a slow behavior in the recurrent case $\gamma(M, T) = 0$. A proof is given in [5], adapting the method of [12] when $L = 1$. The limit law $\mathcal{L}$ appearing in the statement of the theorem was computed independently by Golosov [7] and Kesten [9]. We introduce the ergodic sums $S_n(\log \lambda) = \sum_{k=0}^{n-1} T^k \log \lambda$.

Theorem 1.2. *Let* $\mathcal{L}$ *be the law on* $\mathbb{R}$ *with density* $d(x) = \frac{2}{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} \exp\{-\frac{(2k+1)^2}{8} \pi |x|\}$, $x \in \mathbb{R}$.

Assume that $\gamma(M, T) = 0$ *and that* $n^{-1/2} S_n(\log \lambda)$ *converges in law to a Brownian Motion with a diffusion coefficient* $\sigma^2 > 0$. *Then there exists a sequence of random variables* $(m_n(\omega))_{n \geqslant 1}$ *on* $(\Omega, \mathcal{F})$ *such that* $(\sigma^2 m_n(\omega))_{n \geqslant 1}$ *converges in law to* $\mathcal{L}$ *and the sequence of random variables* $(\xi_n(\omega)/(\log n)^2 - m_n(\omega))$ *converges to* 0 *in probability under the probability* $\int_\Omega \mathcal{P}_0^\omega \, d\mu(\omega)$.

2. Gibbsian environments

We introduce the environments that we consider in the sequel. Fix an integer $d \geqslant 2$ and let $(\Omega, \mathcal{F}, T, A)$ be a bilateral subshift of finite type, where T is the left shift and $\Omega \subset \{1, \ldots, d\}^{\mathbb{Z}}$ is described by a transition matrix A of zero's and one's. An infinite word $\omega = (\omega_i)_{i \in \mathbb{Z}}$ is admissible if and only if $A(\omega_i, \omega_{i+1}) = 1$ for all $i \in \mathbb{Z}$. The σ-algebra $\mathcal{F}$ is induced on Ω by the one generated by the cylinder sets on $\{1, \ldots, d\}^{\mathbb{Z}}$. We suppose that $(\Omega, \mathcal{F}, T, A)$ is topologically mixing, that is some power of A has positive components.

We now consider classes of regular functions on Ω. Let δ be the distance on $\{1, \ldots, d\}^{\mathbb{Z}}$ defined by $\delta(x, y) = 2^{-N}$, where N is the largest integer such that $x_i = y_i$ for $0 \leqslant |i| < N$. For $k \geqslant 0$ and $\varphi : \Omega \to \mathbb{R}$, set $\mathrm{Var}_k(\varphi) = \sup\{|\varphi(x) - \varphi(y)|,\ (x, y) \in \Omega \times \Omega,\ \delta(x, y) \leqslant 2^{-k}\}$. For $p \geqslant 0$, let $\mathcal{H}_p$ be the set of real functions φ defined on Ω such that $\sum_{k \geqslant 0} k^p \,\mathrm{Var}_k(\varphi) < +\infty$.

For $\varphi \in \mathcal{H}_3$, the associated Gibbs measure ν_φ is the unique T-invariant probability measure on $(\Omega, \mathcal{F})$ realizing the maximum in the Variational Principle: $P(\varphi) = \sup\{h(\mu) + \int \varphi \, d\mu \mid \mu = T\mu\}$. For a general presentation of Gibbs measures in the traditional Hölderian context, we refer to [3].

A classical frame of construction of ν_φ consists in considering the unilateral subshift Ω' associated to Ω. A direct adaptation of the Bowen Lemma (Lemma 1.6 of [3]) gives that if $\varphi \in \mathcal{H}_3$, then there is $\psi \in \mathcal{H}_2$ such that $\varphi - \psi + T\psi$ depends only on the positive coordinates. Therefore supposing that $\varphi \in \mathcal{H}_2$ on Ω', one introduces the transfer operator R_φ on the space of continuous functions on Ω': $R_\varphi(g)(x) = \sum_{y \in \Omega',\, Ty=x} e^{\varphi(y)} g(y)$. Next, R_φ admits a unique positive eigenfunction h. Moreover h is regular if φ is regular and the corresponding eigenvalue is $e^{P(\varphi)}$. The restriction of ν_φ to Ω' is then the unique invariant probability measure of $g \mapsto Q(g) = e^{-P(\varphi)} h^{-1} R_\varphi(hg)$ which is a Markovian operator. We now present the central result of this paper.

Theorem 2.1. *Let $(\Omega, \mathcal{B}, T, A)$ be a topologically mixing subshift of finite type with ν_φ the Gibbs measure associated to some $\varphi \in \mathcal{H}_3$. Assume that $(p_{-i}/p_1) \in \mathcal{H}_2$ for $1 \leqslant i \leqslant L$.*

(i) *If $\gamma(M, T) = 0$, then the following dichotomy holds*:
- *either ν_φ-as ω, under the measure $\mathcal{P}_0^\omega$ a non-degenerated CLT is valid,*
- *or the hypotheses of Sinaï's Theorem are verified.*

In particular, if $\gamma(M, T) = 0$ and if for a p-periodic word $\omega \in \Omega$ the matrix $(T^{p-1}M \cdots M)(\omega)$ does not have 1 as eigenvalue, then $\log(\lambda)$ is not a coboundary and we are in Sinaï's situation.

(ii) *If $\gamma(M, T) < 0$ and $\int (\sum_{n=1}^{+\infty} (T^{-n}\lambda \cdots T^{-1}\lambda))^2 (\sum_{p=0}^{+\infty} (T^{p-1}\lambda \cdots \lambda)) \, d\mu < +\infty$, then the Random Walk verifies a pointwise Law of Large Numbers with a non-zero drift in the right direction and a non-degenerated annealed CLT.*

(iii) *If $\gamma(M, T) > 0$ and $\int (\sum_{n=0}^{+\infty} (T^n\lambda \cdots \lambda)^{-1})^2 (\sum_{p=1}^{+\infty} (T^{-p}\lambda \cdots T^{-1}\lambda)^{-1}) \, d\mu < +\infty$, then the Random Walk verifies a pointwise Law of Large Numbers with a non-zero drift in the left direction and a non-degenerated annealed CLT.*

3. Proof of Theorem 2.1

A simple adaptation of the proof of Theorem 5.2 of [4] gives that if $(p_{-i}/p_1) \in \mathcal{H}_2$ for $1 \leqslant i \leqslant L$, then $\log \lambda \in \mathcal{H}_2$. Then the Bowen Lemma gives the existence of functions f and g in $\mathcal{H}_1$ with f depending only of the positive coordinates such that $\log \lambda = f + g - Tg$.

Assume now that $\gamma(M, T) = \int \log \lambda \, d\nu_\varphi = 0$. As u has zero mean, we use the relativized operator Q defined in the introduction of the previous section. There exists a function h (cf. [1] and [6]) on the unilateral subshift Ω' which is at least continuous (and therefore bounded) such that $f = h - Qh$. Consequently:

$$\log \lambda = h - TQh + (g - Qh) - T(g - Qh).$$

Introducing the σ-algebra $\mathcal{F}'$ generated by the cylinder sets on Ω', we note that TQh is a version of the conditional expectation of h under ν_φ with respect to the sub-σ-algebra $T^{-1}\mathcal{F}'$. Therefore $(T^n(h - TQh))_{n \geqslant 0}$ is a sequence of reversed martingale differences with respect to $(T^{-n}\mathcal{F}')_{n \geqslant 0}$. Thus the following discussion holds:

- If $\nu_\varphi\{h - TQh \neq 0\} > 0$, then one checks that the invariance principle holds with a positive variance for the sequence $(T^n(h - TQh))_{n \geqslant 0}$, see Hall and Heyde [8]. Thus the hypotheses of Sinaï's Theorem are verified.
- If $h - TQh = 0$, then $\log \lambda = (g - Qh) - T(g - Qh)$ is a bounded coboundary and one applies Theorem 4.5 of [4]. The pointwise functional CLT holds.

When ω is p-periodic, that is if $T^p\omega = \omega$, and if $\lambda = T\psi/\psi$, then $(T^{p-1}M \cdots MV)(\omega) = V(\omega)$, which proves the last assertion of (i).

The transient cases $\gamma(M,T) \neq 0$ with the integrability conditions of Theorem 2.1 are treated with similar methods, using the existence of the absolutely continuous invariant measure for the random walk of the environments viewed from the particle and the existence of harmonic coordinates. A presentation of these notions and a proof are detailed in [5]. □

Remark 1. Both cases of Theorem 2.1(i) are easily seen to be non-empty, for example taking $L = 1$. We also mention that the dichotomy is not valid in general and requires hypotheses on the environment. In [5], we provide an example with $L = 1$ of a Sinaï type behavior in the recurrent case but with a scaling in $\log n(\log\log\log n)^{3/2}$.

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