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Algebra

Koszul duality for PROPs

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Abstract

The notion of PROP models the operations with multiple inputs and multiple outputs, acting on some algebraic structures like the bialgebras or the Lie bialgebras. We prove a Koszul duality theory for PROPs generalizing the one for associative algebras and for operads. ***To cite this article: B. Vallette, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***
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Résumé

Dualité de Koszul des PROPs. La notion de PROP modélise les opérations à plusieurs entrées et plusieurs sorties, agissant sur certaines structures algébriques comme les bigèbres et les bigèbres de Lie. Nous montrons une théorie de dualité de Koszul pour les PROPs qui généralise celle des algèbres associatives et des opérades. ***Pour citer cet article : B. Vallette, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***
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Version française abrégée

On travaille sur un corps de caractéristique nulle.

Suivant J.-P. Serre dans [12], on regroupe sous le terme de *gèbre* différentes structures algébriques comme les algèbres, les cogèbres et les bigèbres.

L'ensemble $\mathcal{P}(m,\ n)$ des opérations à n entrées et m sorties agissant sur un certain type de gèbres est un module à gauche sur le groupe symétrique $\mathbb{S}_m$ et à droite sur $\mathbb{S}_n$. Ces deux actions sont compatibles.

On appelle $\mathbb{S}$-*bimodule* toute collection $(\mathcal{P}(m,n))_{m,\,n\in\mathbb{N}^*}$ de tels modules. Nous définissons un produit $\boxtimes$ dans la catégorie des $\mathbb{S}$-bimodules qui représente les compositions d'opérations à plusieurs entrées et plusieurs sorties. Ce produit est basé sur les graphes dirigés (cf. Fig. 1).

On définit un *PROP* comme un $\mathbb{S}$-bimodule muni d'une composition $\mathcal{P}\boxtimes\mathcal{P}\xrightarrow{\mu}\mathcal{P}$ associative. On donne les exemples du PROP $\mathcal{B}i\mathcal{L}ie$ des bigèbres de Lie (cf. [3]), du PROP $\mathcal{B}i\mathcal{L}ie_0$ des bigèbres de Lie combinatoires (cf. [2]) et du PROP $\mathcal{I}nf\mathcal{B}i$ des bigèbres de Hopf infinitésimales (cf. [1]). On appelle $\mathcal{P}$-*gèbre*, tout module sur le PROP $\mathcal{P}$. On retrouve les définitions des gèbres classiques. Par exemple, une $\mathcal{B}i\mathcal{L}ie$-gèbre est exactement une bigèbre de Lie.

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Nous étendons les définitions de bar et cobar constructions des algèbres et des opérades aux PROPs, et nous généralisons les lemmes de comparaison de B. Fresse [4] aux PROPs. Remarquons que les démonstrations opéradiques ne sont pas reconductibles ici, car ces dernières reposent sur les propriétés combinatoires des arbres.

A partir d'un PROP gradué par un poids, par exemple quadratique c'est-à-dire défini par des générateurs et des relations quadratiques, on construit une *coPROP* dual $\mathcal{P}^{\text{¡}}$ et un *complexe de Koszul* $(\mathcal{P}^{\text{¡}} \boxtimes \mathcal{P},\ d_K)$.

Le principal théorème de cette théorie est le suivant :

Théorème 0.1. *Soit $\mathcal{P}$ un PROP différentiel augmenté gradué par un poids* (*par exemple, un PROP quadratique*), *les propositions suivantes sont équivalentes*

(1) *le complexe de Koszul $(\mathcal{P}^{\text{¡}} \boxtimes \mathcal{P}, d_K)$ est acyclique,*
(2) *la cobar construction sur le coPROP dual $\mathcal{P}^{\text{¡}}$ fournit une résolution du PROP $\mathcal{P}$* :

$$\overline{\mathcal{B}}^c\left(\mathcal{P}^{\text{¡}}\right) \longrightarrow \mathcal{P}.$$

Dans ce cas, la résolution obtenue est le *modèle minimal* de $\mathcal{P}$ et elle permet de définir la notion de *$\mathcal{P}$-gèbre à homotopie près*.

Nous montrons que le PROP $\mathcal{B}i\mathcal{L}ie$ des bigèbres de Lie (cf. [3]), le PROP $\mathcal{B}i\mathcal{L}ie_0$ des bigèbres de Lie combinatoires (cf. [2]) et le PROP $\mathcal{I}nf\mathcal{B}i$ des bigèbres de Hopf infinitésimales (cf. [1]) sont des PROPs de Koszul. Ce qui permet de donner les définitions de bigèbres de Lie, bigèbres de Lie combinatoires et bigèbres de Hopf infinitésimales à homotopie près.

1. Introduction

The Koszul duality theory for algebras, proved by S. Priddy in [11] has been generalized to the operads by V. Ginzburg and M.M. Kapranov in [6].

An operad models the operations acting on a certain type of algebras (associative, commutative and Lie algebras for instance). Since these operations have multiple inputs but only one output, their compositions can be represented by trees. This theory has many applications. It gives the minimal model of an operad $\mathcal{P}$, the notion of $\mathcal{P}$-algebras up to homotopy and a natural homology theory for the $\mathcal{P}$-algebras.

To study algebraic structures defined by operations with multiple inputs and multiple outputs, like bialgebras or Lie bialgebras for instance, one needs to generalize the notion of operad and introduce the notion of PROP.

It is natural to try to generalize the Koszul duality for PROPs. A first result in the direction is due to W.L. Gan in [5]; see also M. Markl and A.A. Voronov in [10].

We work over a field k of characteristic 0. The symmetric group on n elements is denoted by $\mathbb{S}_n$.

2. PROPs and $\mathcal{P}$-gebras

Over a vector space, various algebraic structures can be considered like algebras, coalgebras, bialgebras. Following J.-P. Serre in [12], we call *gebra* any one of these structures. The set $\mathcal{P}(m,n)$ of the operations of n inputs and m outputs acting on a gebra A is a module over $\mathbb{S}_m$ on the left and over $\mathbb{S}_n$ on the right. We have the following morphisms of $\mathbb{S}_m$-modules:

$$\mathcal{P}(m,n) \otimes_{\mathbb{S}_n} A^{\otimes n} \longrightarrow A^{\otimes m}.$$

Definition 2.1 (*$\mathbb{S}$-bimodule*). An *$\mathbb{S}$-bimodule* $(\mathcal{P}(m,n))_{m,n\in\mathbb{N}^*}$ is a collection of $\mathbb{S}_m \times \mathbb{S}_n^{\text{op}}$-modules.

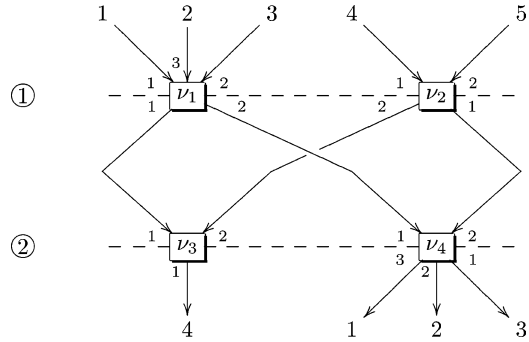


Fig. 1. Example of a 2-levels graph.

2.1. The composition product $\boxtimes$

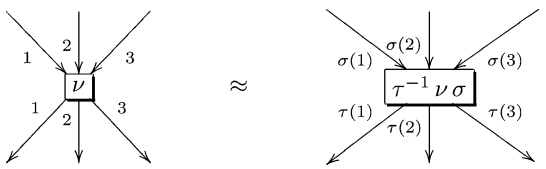
We introduce a product on $\mathbb{S}$-bimodules which describes the composition of operations.

A *graph with a flow* is a graph where the orientations of the edges are given by a global flow (from the top to the bottom, for instance). Let $\mathcal{G}$ be the set of finite graphs with a flow. We suppose that the inputs and the outputs of each vertex are labeled by integers. When the vertices of a graph g can be dispatched on two levels, we denote $\mathcal{N}_i$ $(i = 1, 2)$ the set of vertices belonging to the ith level. We denote by $\mathcal{G}^2$ the set of such graphs (cf. Fig. 1).

Definition 2.2 (*Product* $\boxtimes$). Given two $\mathbb{S}$-bimodules $\mathcal{P}$ and $\mathcal{Q}$, we define their product by the formula

$$\mathcal{P} \boxtimes \mathcal{Q} = \Big(\bigoplus_{g \in \mathcal{G}^2} \bigotimes_{v \in \mathcal{N}_2} \mathcal{P}\big(|Out(v)|, |In(v)|\big) \otimes \bigotimes_{v \in \mathcal{N}_1} \mathcal{Q}\big(|Out(v)|, |In(v)|\big) \Big) \Big/ \approx,$$

where the relation $\approx$ is generated by



Here $|Out(v)|$ and $|In(v)|$ are the numbers of the outgoing and the incoming edges of the vertex v.

This product has an algebraic writing using the symmetric groups (cf. [13]).

2.2. PROPs

The notion of PROP models the operations acting on a certain type of gebras and their compositions.

We denote by $\tilde{I}$ the identity $\mathbb{S}$-bimodule defined by the formula

$$\begin{cases} \tilde{I}(n,n) = k[\mathbb{S}_n], \\ \tilde{I}(m,n) = 0 \quad \text{elsewhere.} \end{cases}$$

Definition 2.3 (PROP). A structure of *PROP* over an $\mathbb{S}$-bimodule $\mathcal{P}$ is given by the following data

- an associative *composition* $\mathcal{P} \boxtimes \mathcal{P} \xrightarrow{\mu} \mathcal{P}$,
- a *unit* $\tilde{I} \xrightarrow{\eta} \mathcal{P}$.

Remark 1. This definition of a PROP is equivalent to the definition given by S. Mac Lane (cf. [9]).

Example 1. For any vector space V, the sets $(\mathrm{Hom}(V^{\otimes n}, V^{\otimes m}))_{m,n\in\mathbb{N}^*}$ of morphisms from $V^{\otimes n}$ to $V^{\otimes m}$ with the composition of morphisms as μ form a PROP, denoted $\mathrm{End}(V)$.

The associative algebras and the operads are examples of PROPs.

Dually, we define the notion of *coPROP*, which is a PROP in the opposite category of the category of $\mathbb{S}$-bimodules equipped with the product $\boxtimes$.

2.3. Quadratic PROPs

We give a categorical construction of the free monoid in [13], which applied here, gives the free PROP.

Proposition 2.4. *The free PROP over an $\mathbb{S}$-bimodule V, denoted $\mathcal{F}(V)$, is given by the direct sum on the set of graphs* (*without level*)*, where each vertex is indexed by an element of V*:

$$\mathcal{F}(V) = \left(\bigoplus_{g\in\mathcal{G}} \bigotimes_{v\in\mathcal{N}} V\big(|Out(v)|, |In(v)|\big) \right) \Big/ \approx .$$

Remark 2. This construction is analytic in V. The part of weight n of $\mathcal{F}(V)$, denoted $\mathcal{F}_{(n)}(V)$, is the direct summand generated by the finite graphs with n vertices.

Dually, we define the *cofree connected coPROP $\mathcal{F}^c(V)$ on V* with the same underlying $\mathbb{S}$-bimodule $\mathcal{F}(V)$.

Definition 2.5 (*Quadratic* PROP). A *quadratic PROP* $\mathcal{P}$ is a *PROP* $\mathcal{P} = \mathcal{F}(V)/(R)$ generated by an $\mathbb{S}$-bimodule V and a space of relations $R \subset \mathcal{F}_{c(2)}(V)$, where $\mathcal{F}_{c(2)}(V)$ is the direct summand of $\mathcal{F}(V)$ generated by the connected graphs with 2 vertices.

Since the relations R of a quadratic PROP are homogenous, a quadratic PROP is weight-graded.

Example 2. The *PROP* of Lie bialgebras, denoted $\mathcal{B}i\mathcal{L}ie$ (cf. V. Drinfeld [3]), the *PROP* of combinatorial Lie bialgebras, denoted $\mathcal{B}i\mathcal{L}ie_0$ (cf. M. Chas [2]) and the *PROP* of infinitesimal Hopf bialgebras, denoted $\mathcal{I}nf\mathcal{B}i$ (cf. M. Aguiar [1]), are examples of quadratic *PROP*s.

2.4. $\mathcal{P}$-gebras

The notion of a gebra over a PROP $\mathcal{P}$ is the generalisation of the notion of an algebra over an operad.

Definition 2.6 (*$\mathcal{P}$-gebra*). A structure of *$\mathcal{P}$-gebra* over a vector space A is given by a morphism of *PROP*s $: \mathcal{P} \to \mathrm{End}(A)$.

Example 3. A $\mathcal{B}i\mathcal{L}ie$-gebra is exactly a Lie bialgebra, a $\mathcal{B}i\mathcal{L}ie_0$-gebra is a combinatorial Lie bialgebra and an $\mathcal{I}nf\mathcal{B}i$-gebra is an infinitesimal Hopf bialgebra.

3. Koszul duality

We generalize the Koszul duality theory of associative algebras and operads to PROPs.

3.1. Bar and cobar constructions

We generalize the bar and the cobar constructions of the algebras and operads to PROPs.

Definition 3.1 (*Partial product*). For an augmented *PROP* $(\mathcal{P} = I \oplus \overline{\mathcal{P}},\ \mu,\ \eta)$, we define the *partial product* as the restriction of the composition μ to the sub-module of $\mathcal{P} \boxtimes \mathcal{P}$ made of connected graphs with only one vertex on each level indexed by an element of $\overline{\mathcal{P}}$ (and the other vertices indexed by I, the unit).

We denote by Σ the homological *suspension*. For instance, we have $(\Sigma\mathcal{P})_{n+1} = \mathcal{P}_n$, where n is the homological degree.

Proposition 3.2. *There exists a unique coderivation d_θ on the coPROP $\mathcal{F}^c(\Sigma\overline{\mathcal{P}})$ whose restriction on $\mathcal{F}^c_{(2)}(\Sigma\overline{\mathcal{P}})$ is the partial product.*

Remark 3. This notion generalizes the *edge contraction*, given by M. Kontsevich [7], which defines the boundary map in *graph homology*.

Definition 3.3 (*Bar construction*). Let $(\mathcal{P}, \delta)$ be a differential augmented PROP. The *bar construction* of $\mathcal{P}$ is the following chain complex:

$$\mathcal{B}(\mathcal{P}) = \big(\mathcal{F}^c(\Sigma\overline{\mathcal{P}}), \delta + d_\theta\big).$$

Dually, we define the *cobar construction* of a differential co-augmented coPROP $(\mathcal{C}, \delta)$ and we denote this differential graded PROP by $\mathcal{B}^c(\mathcal{C})$.

3.2. Koszul dual and Koszul complex

We give the basic definitions of the objects involved in the Koszul duality theory for PROPs.

A PROP in the category of weight-graded vector spaces is called a *weight-graded PROP*. Quadratic PROPs are examples of weight-graded PROPS.

Definition 3.4 (*Koszul dual*). To a weight-graded augmented *PROP* $\mathcal{P}$, we associate a dual *coPROP*, denoted $\mathcal{P}^{\text{¡}}$, which is a sub-*coPROP* of the bar construction $\mathcal{P}^{\text{¡}} \hookrightarrow \mathcal{B}(\mathcal{P})$.

Remark 4. Under finite dimensional hypothesis, the linear dual of $\mathcal{P}^{\text{¡}}$ gives a *PROP* which corresponds, in the cases of associative algebras and operads, to the classical Koszul dual $\mathcal{P}^!$ (cf. S. Priddy [11], V. Ginzburg and M.M. Kapranov [6] and J.-L. Loday [8]).

On the $\mathbb{S}$-bimodule $\mathcal{P}^{\text{¡}} \boxtimes \mathcal{P}$, we define a map d_K by the following compositions:

$$d_K : \mathcal{P}^{\text{¡}} \boxtimes \mathcal{P} \xrightarrow{\Delta\boxtimes\mathcal{P}} \mathcal{P}^{\text{¡}} \boxtimes \mathcal{P}^{\text{¡}} \boxtimes \mathcal{P} \twoheadrightarrow \mathcal{P}^{\text{¡}} \boxtimes \underbrace{\mathcal{P}^{\text{¡}}}_{(1)} \boxtimes \mathcal{P} \xrightarrow{\sim} \mathcal{P}^{\text{¡}} \boxtimes \underbrace{\mathcal{P}}_{(1)} \boxtimes \mathcal{P} \xrightarrow{\mathcal{P}^{\text{¡}}\boxtimes\mu} \mathcal{P}^{\text{¡}} \boxtimes \mathcal{P}.$$

Lemma 3.5. *For any differential weight-graded PROP $\mathcal{P}$, we have ${d_K}^2 = 0$.*

Definition 3.6 (*Koszul complex*). The chain complex $\mathcal{P}^{\text{¡}} \boxtimes \mathcal{P}$ with the boundary map d_K is called the *Koszul complex* of $\mathcal{P}$.

3.3. Koszul criterion

We give a criterion that determines whether the cobar construction on the dual coPROP gives a resolution of $\mathcal{P}$ or not.

Theorem 3.7 (Koszul criterion). *Let $\mathcal{P}$ be a differential weight-graded augmented PROP* (*for instance a quadratic PROP*)*, the following assertions are equivalent*

(1) *The Koszul complex $(\mathcal{P}^{\text{¡}} \boxtimes \mathcal{P}, d_K)$ is acyclic.*
(1′) *The Koszul complex $(\mathcal{P} \boxtimes \mathcal{P}^{\text{¡}}, d'_K)$ is acyclic.*
(2) *The inclusion $\mathcal{P}^{\text{¡}} \hookrightarrow \mathcal{B}(\mathcal{P})$ is a quasi-isomorphism of differential weight-graded PROPs.*
(2′) *The projection $\mathcal{B}^c(\mathcal{P}^{\text{¡}}) \twoheadrightarrow \mathcal{P}$ is a quasi-isomorphism of differential weight-graded PROPs.*

Sketch of the proof.

- We remark that the product $\boxtimes$ induces functors $A \boxtimes -$ and $- \boxtimes A$ which are analytic functors. With the graduations given by these functors and the weight graduation of the PROP $\mathcal{P}$, we generalize homological lemmas, called *comparison lemmas*, proved by B. Fresse in [4] for operads, to PROPs.
- We prove that the augmented bar construction $\mathcal{B}(\mathcal{P}) \boxtimes \mathcal{P}$ of a PROP $\mathcal{P}$ and the co-augmented cobar construction $\mathcal{C} \boxtimes B^c(\mathcal{C})$ are acyclic.
- We define a natural morphism of differential weight-graded PROPs from the bar-cobar construction $\mathcal{B}^c(\mathcal{B}(\mathcal{P}))$ of a differential weight-graded PROP to $\mathcal{P}$:

$$\mathcal{B}^c\big(\mathcal{B}(\mathcal{P})\big) \longrightarrow \mathcal{P}.$$

  We apply the comparison lemmas to show that this bar-cobar construction gives a resolution of $\mathcal{P}$.
- We simplify the bar-cobar construction with the comparison lemmas to conclude. □

Remark 5. This theorem includes the cases of associative algebras and operads.

A PROP $\mathcal{P}$ that verifies these assertions is called a *Koszul PROP*. In this case, the cobar construction on the dual $\mathcal{P}^!$ is a resolution of $\mathcal{P}$. Since it is a resolution built on a free $\mathbb{S}$-bimodule with a decomposable boundary map, we call it the *minimal model* of $\mathcal{P}$.

Example 4. We prove that the *PROP*s $\mathcal{B}i\mathcal{L}ie$, $\mathcal{B}i\mathcal{L}ie_0$ and $\mathcal{I}nf\mathcal{B}i$ are Koszul *PROP*s. This result can be interpreted in terms of graph homology as in [10].

3.4. $\mathcal{P}$-gebras up to homotopy

One of the main application of the Koszul duality for PROPs is the definition of a $\mathcal{P}$-gebra up to homotopy.

Definition 3.8 (*$\mathcal{P}$-gebra up to homotopy*). Let $\mathcal{P}$ be a Koszul PROP. A gebra over the cobar construction $\mathcal{B}(\mathcal{P})$ of $\mathcal{P}$ is called a *$\mathcal{P}$-gebra up to homotopy* and denoted a *$\mathcal{P}_\infty$-gebra*.

Example 5. Applied to the examples given above this defines the notions of Lie bialgebras, combinatorial Lie bialgebras and infinitesimal Hopf bialgebras up to homotopy.

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