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On the contact boundaries of normal surface singularities

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Abstract

The abstract boundary M of a normal complex-analytic surface singularity is canonically equipped with a contact structure. We show that if M is a rational homology sphere, then this contact structure is uniquely determined by the topological type of M. An essential tool is the notion of open book carrying a contact structure, defined by E. Giroux. ***To cite this article: C. Caubel, P. Popescu-Pampu, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Sur les bords de contact des singularités de surfaces normales. Le bord abstrait M d'une singularité analytique complexe de surface normale est canoniquement muni d'une structure de contact. Nous montrons que si M est une sphère d'homologie rationnelle, alors cette structure de contact est uniquement déterminée par le type topologique de M. Un outil essentiel est la notion de livre ouvert portant une structure de contact, définie par E. Giroux. ***Pour citer cet article : C. Caubel, P. Popescu-Pampu, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Soit $(\mathcal{S}, 0)$ un germe d'espace analytique complexe normal à singularité isolée. Le bord abstrait $M(\mathcal{S})$ de $(\mathcal{S}, 0)$ est muni d'une structure de contact $\xi(\mathcal{S})$ définie canoniquement à contactomorphisme près par la structure complexe du germe (Section 1). Nous nous intéressons au problème suivant :

Une variété fermée orientée M étant donnée, trouver le nombre de structures de contact distinctes (à contactomorphisme près) sur M provenant d'un germe dont le bord abstrait est difféomorphe à M.

Notre résultat principal (Théorème 1.2) concerne les sphères d'homologie rationnelles de dimension 3 :

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Si M est une sphère d'homologie rationnelle de dimension 3*, alors M peut être munie à contactomorphisme près d'au plus une structure de contact provenant d'un germe de surface complexe normale.*

Notre outil essentiel est celui de *livre ouvert portant* une structure de contact, introduit par E. Giroux (Définitions 2.2 et 2.3). Giroux montre que deux structures de contact de même orientation et portées par le même livre ouvert sont isotopes. D'où notre idée de prouver le Théorème 1.2 en montrant que *si la sphère d'homologie rationnelle M est le bord abstrait d'un germe de surface normale, alors la structure de contact associée est portée par un livre ouvert déterminé à isomorphisme près par la topologie de M.*

Esquissons la preuve de cette affirmation. Supposons que $(\mathcal{S}, 0)$ soit un germe de surface normale. On montre d'abord que toute fonction holomorphe $f : (\mathcal{S}, 0) \to (\mathbf{C}, 0)$ à singularité isolée définit un *livre ouvert de Milnor* $(N(f), \theta)$ sur $M(\mathcal{S})$ qui porte la structure de contact $\xi(\mathcal{S})$ (Proposition 2.4). Ensuite, on construit une fonction f dont l'entrelacs $(M(\mathcal{S}), N(f))$ a un type topologique déterminé par celui de $M(\mathcal{S})$ (Proposition 3.1). On conclut en remarquant que sur une sphère d'homologie rationnelle, la classe d'isotopie de tout livre ouvert est déterminée par celle de l'entrelacs orienté associé (Proposition 2.2).

Si M n'est pas une sphère d'homologie rationnelle, notre méthode permet seulement de dire que *M peut être munie à contactomorphisme près d'au plus un nombre fini de structures de contact provenant d'un germe de surface complexe normale* (Proposition 4.1). Dans la Section 4 nous présentons quelques idées qui permettraient de faire des progrès dans cette direction.

1. Introduction

A *contact structure* on a $(2n+1)$-dimensional manifold M is a completely non-integrable smooth field ξ of hyperplanes on M. This means that, locally, ξ is the field of kernels of a smooth contact form α, i.e. a 1-form such that $\alpha \wedge (\mathrm{d}\alpha)^{\wedge n}$ is nowhere vanishing. This shows that a contact structure on a 3-manifold automatically induces an orientation.

Let $(\mathcal{S}, 0)$ be a germ of normal complex analytic space with isolated singularity. If $z_1, \ldots, z_s$ is a system of generators of the ideal $\mathcal{M}_{\mathcal{S},0}$ of germs of holomorphic functions on $(\mathcal{S}, 0)$ vanishing at 0, define the function $\rho := \sum_{k=1}^{s} |z_k|^2$. Then there exists an $\varepsilon_0 > 0$ such that the level set $M_\varepsilon := \rho^{-1}(\varepsilon)$ is smooth for any positive $\varepsilon < \varepsilon_0$. Furthermore, its diffeomorphism type does not depend on the choice of functions $z_1, \ldots, z_s$. We call it *the abstract boundary* of the germ $(\mathcal{S}, 0)$ and we denote it by $M(\mathcal{S})$. It is supposed oriented with the canonical orientation obtained as the boundary of a neighborhood of 0 in $\mathcal{S}$.

If $f \in \mathcal{M}_{\mathcal{S},0}$ has an isolated singularity at 0, then for $\varepsilon > 0$ sufficiently small $\{f = 0\}$ cuts M_ε transversally and the oriented diffeomorphism type of the pair $(M_\varepsilon, M_\varepsilon \cap \{f = 0\})$ does not depend on the choices. We call it *the abstract link* of the function f and we denote it by $(M(\mathcal{S}), N(f))$.

Notice now that the function ρ is strictly plurisubharmonic (abridged: *spsh*) on $\mathcal{S} \setminus \{0\}$, for any representative of $\mathcal{S}$. This implies that for any positive $\varepsilon < \varepsilon_0$, the field ξ_ε of tangent hyperplanes to M_ε which are kept invariant by the complex multiplication is a contact structure on M_ε. Varchenko showed in [11] that this contact structure is also independent on the choices, and so we get a well-defined oriented contact manifold $(M(\mathcal{S}), \xi(\mathcal{S}))$. We call it *the contact boundary* of $\mathcal{S}$.

Definition 1.1. Let (M, ξ) be a connected closed oriented contact manifold. If there exists a germ $(\mathcal{S}, 0)$ of normal complex analytic space with isolated singularity such that (M, ξ) is isomorphic to the contact boundary $(M(\mathcal{S}), \xi(\mathcal{S}))$ of $\mathcal{S}$, we say that (M, ξ) *admits a Milnor filling* (or is *Milnor fillable*).

The name is motivated by the fact that in the case of isolated hypersurface singularities (studied by Milnor in [7]), the contact boundaries are obtained by intersecting the hypersurface with so-called Milnor spheres. In [3], the first author and M. Tibăr surveyed results on contact boundaries of isolated singularities, as well as some open problems. Here we deal with the following one:

Problem. An oriented closed manifold M being given, find the number of distinct (up to contactomorphism) contact structures on M which admit a Milnor filling.

In fact, for the moment this problem only seems tractable in dimension 3:

- the class of 3-dimensional manifolds which arise as abstract boundaries of singularities is known (see Neumann [8]). It is precisely the class of graph-manifolds which are obtained by plumbing along a weighted graph which has a negative definite intersection form.
- in higher dimension, the situation is much more complicated. For instance, the infinite number of different contact structures on the spheres S^{4n+1}, $n \geqslant 1$, discovered by Ustilovsky [10], all admit a Milnor filling.

Our main result, proved at the end of Section 3, concerns 3-dimensional *rational homology spheres*, i.e. oriented closed connected real 3-manifolds M such that $H_1(M, \mathbf{Q}) = 0$:

Theorem 1.2. *If the 3-manifold M is a rational homology sphere, then there is at most one isomorphism class of contact structures on it which admits a Milnor filling.*

2. Carrying open books for contact boundaries

We use as the main tool for studying contact manifolds the notion of *open book carrying a contact structure*, defined by E. Giroux in [4]. We will give its definition after having reminded the classical definition of an open book. This was introduced in the context of singularity theory by Milnor [7]. The name seems to appear slightly later.

Definition 2.1. An *open book* with *binding* N in a manifold M is a couple (N, θ), where N is a (not necessarily connected) codimension-2 closed submanifold of M and $\theta : M \setminus N \to \mathbf{S}^1$ is a smooth fibration which in a neighborhood $N \times \mathbf{D}^2$ of N coincides with the angular coordinate.

In the previous definition, the map $U(N) \to N \times \mathbf{D}^2$ which trivializes a tubular neighborhood of N in M is supposed to be differentiable. If M and N are oriented, the co-orientation of a fiber of θ given by $d\theta$ induces an orientation of this fiber. If this orientation induces the given orientation of its boundary N, we say that (N, θ) is *compatible with the orientations*. The open books (N, θ) and (N', θ') on the manifolds M, respectively M' are said *isomorphic* if there exists an orientation-preserving diffeomorphism $\phi : (M, N) \to (M', N')$ which carries the fibers of θ in fibers of θ'.

Main example. Let $(M_\varepsilon, M_\varepsilon \cap \{f = 0\})$ be a representative of the abstract link of a function f on the singularity $(\mathcal{S}, 0)$, as defined in the introduction. Then the function $\theta := \arg(f) : M_\varepsilon \setminus (M_\varepsilon \cap \{f = 0\}) \to S^1$ defines, for $\varepsilon > 0$ sufficiently small, an open book compatible with the orientations of $(M_\varepsilon, M_\varepsilon \cap \{f = 0\}) \simeq (M(\mathcal{S}), N(f))$. We call *Milnor open book* associated to f any open book on the abstract boundary of $(\mathcal{S}, 0)$ isomorphic to such one.

The following proposition shows that, in 3-dimensional rational homology spheres, any open book is determined up to an isotopy by its binding alone.

Proposition 2.2. *Let M be a* 3*-dimensional rational homology sphere and let N be an oriented link in M. If N is the binding of an open book compatible with the orientations of (M, N), then this open book is unique up to isotopy.*

Sketch of proof. By Alexander duality with coefficients in $\mathbf{Q}$ (see Massey [6] for the version we use), $H_1(M \setminus N, \mathbf{Q}) \simeq H^1(N, \mathbf{Q})$ and so $H_1(M \setminus N, \mathbf{Q})$ is generated by loops linking positively each component of N. If F is the closure of a fiber of θ, its intersection number with any of these loops is $+1$. By applying Lefschetz duality on $M \setminus U(N)$, where $U(N)$ is an open regular neighborhood of N, we see that the class of F in $H_2(M \setminus U(N), \partial U(N), \mathbf{Q})$ is determined by the couple (M, N). By applying results of Stallings and Waldhausen (see Cantwell, Conlon [2] and the references therein), we get that the isotopy type of the foliation on $M \setminus U(N)$ formed by the fibers of θ is determined by (M, N). □

Now let us come to the relation between contact structures and open books.

Definition 2.3. A contact structure ξ on M is *carried by an open book* (N, θ) if it admits a defining contact form α which verifies the following:

- α induces a contact structure on N;
- $d\alpha$ induces a symplectic structure on each fiber of θ;
- the orientation of N as defined by α coincides with its orientation as the boundary of the symplectic fibers of θ.

Giroux proved (see [4] and [5]) that *on a 3-manifold, two contact structures inducing the same orientation and carried by the same open book are isotopic*. So, in order to show that two contact structures on a given 3-manifold are isomorphic, it is enough to show that they are carried by isomorphic open books.

However, we have the following:

Proposition 2.4. *Let $(\mathcal{S}, 0)$ be a normal complex analytic germ with an isolated singularity and $f \in \mathcal{M}_{\mathcal{S},0}$ be a germ which has an isolated singularity at 0. Then the natural contact structure $\xi(\mathcal{S})$ on the abstract boundary $M(\mathcal{S})$ is carried by a Milnor open book associated to f.*

Sketch of proof. We follow the method given by Giroux in [5] in the case where $(\mathcal{S}, 0)$ is smooth. Let $z_1, \ldots, z_s$ and ρ be defined as in section 1. Let $N \in \mathbf{N}^*$ and $c : (0, +\infty) \to (0, +\infty)$, $c(\varepsilon) = \varepsilon^{-N}$. For $\varepsilon > 0$, let C_ε be the subspace of $\mathcal{S}$ defined by the equation $\rho + c(\varepsilon)|f|^2 = \varepsilon^2$. Then we show that for N large enough and $\varepsilon > 0$ small enough:

– the levels C_ε are contact isotopic to the contact boundary $M(\mathcal{S})$;
– $\theta = \arg(f)$ defines an open book on them which is a Milnor open book of f;
– this open book carries the natural contact structure on C_ε. □

This gave us the idea to prove Theorem 1.2 by showing that *there exists on any Milnor filling of M a function whose Milnor open book is determined by the topology of M*. Proposition 2.2 shows that, for M a 3-dimensional rational homology sphere, it is enough to find a function whose abstract link, which is the binding of its Milnor open book, is so determined. This is what we do in the next section.

3. Construction of a Milnor open book with prescribed binding

Let us restrict now to the case where $\dim_{\mathbf{C}} \mathcal{S} = 2$. Let $p : (\Sigma, E) \to (\mathcal{S}, 0)$ be the *good* minimal resolution of $(\mathcal{S}, 0)$. Namely, Σ is smooth, p is proper and realises an isomorphism over $\mathcal{S} \setminus \{0\}$, the set-theoretical fibre $E = \sum_{i=1}^{r} E_i$ over 0 is a divisor with normal crossings having smooth irreducible components E_i and p is minimal with these properties. If D is a divisor on Σ, we denote by $|D|$ its support. There exists a unique decomposition $D = D_e + D_s$ such that $|D_e| \subset |E|$ and no component of $|D_s|$ is included in $|E|$. We call D_e *the exceptional part* of D and D_s *the strict part* of D. We say also that D_e is *purely exceptional*.

The following proposition was proved in collaboration with A. Némethi:

Proposition 3.1. *Let D be a purely exceptional effective divisor on Σ, such that $(D + E + K_\Sigma) \cdot E_i + 3 \leqslant 0$ for any $i \in \{1, \dots, r\}$. Denote by $q : \widetilde{\Sigma} \to \Sigma$ the morphism obtained by blowing-up all the singular points of E. Let Q denote the exceptional divisor of q. Then there exists a function $f \in \mathcal{M}_{\mathcal{S},0}$ with isolated singularity at 0 such that $\operatorname{div}(f \circ p \circ q)$ is a divisor with normal crossings whose exceptional part is $q^*(D) + Q$. In particular, the topology of $\operatorname{div}(f \circ p \circ q)$ only depends on D.*

Sketch of proof. We use the following vanishing theorem, proved in Bădescu [1], page 53, and attributed there to Laufer and Ramanujam:

If L is a divisor on Σ such that $L \cdot E_i \geqslant K_\Sigma \cdot E_i$, $\forall i \in \{1, \dots, n\}$, then $H^1(\mathcal{O}_\Sigma(L)) = 0$.

We apply the theorem to $L = -D - E$. So, our hypothesis implies that $H^1(\mathcal{O}_\Sigma(-D - E)) = 0$. Then, from the exact cohomology sequence associated to the short exact sequence of sheaves of $\mathcal{O}_\Sigma$-modules $0 \to \mathcal{O}_\Sigma(-D - E) \to \mathcal{O}_\Sigma(-D - \sum_{j \neq i} E_j) \xrightarrow{\psi_i} \mathcal{O}_{E_i}(-D - \sum_{j \neq i} E_j) \to 0$ we deduce the *surjectivity* of the restriction map $\psi_{i*} : H^0(\mathcal{O}_\Sigma(-D - \sum_{j \neq i} E_j)) \to H^0(\mathcal{O}_{E_i}(-D - \sum_{j \neq i} E_j))$.

At this point we have only used the fact that $(D + E + K_\Sigma) \cdot E_i \leqslant 0$. Now we use the stronger inequality $(D + E + K_\Sigma) \cdot E_i + 3 \leqslant 0$. This implies, by the adjunction formula, that $(-D - \sum_{j \neq i} E_j) \cdot E_i \geqslant 2g(E_i) + 1$, which shows that the line bundle $\mathcal{O}_{E_i}(-D - \sum_{j \neq i} E_j))$ is very ample. So, there exists a non-identically zero section $s_i \in H^0(\mathcal{O}_{E_i}(-D - \sum_{j \neq i} E_j))$, which cuts E_i transversely and only in points of $E_i \setminus \bigcup_{j \neq i} E_j$. The surjectivity of ψ_{i*} implies that there exists $\sigma_i \in H^0(\mathcal{O}_\Sigma(-D - \sum_{j \neq i} E_j))$ such that $\psi_{i*}(\sigma_i) = s_i$. As $(\mathcal{S}, 0)$ is normal, there exists $f_i \in \mathcal{M}_{\mathcal{S},0}$ with $\sigma_i = f_i \circ p$. Our hypothesis on s_i implies that f_i has an isolated singularity at 0 and that the divisor $\operatorname{div}(f_i \circ p)$ has normal crossings in a neighborhood of E_i. Then one shows by analyzing what happens in local coordinates, that any linear combination $f = \sum_{i=1}^r \lambda_i f_i$ of the functions so constructed, with $\lambda_i \neq 0$, $\forall i \in \{1, \dots, r\}$, verifies $F_e = q^*(D) + Q$, where $F = \operatorname{div}(f \circ p \circ q)$.

Let $\widetilde{E}$ be the exceptional divisor of $p \circ q$. Then $F \cdot \widetilde{E}_k = 0$ for any component $\widetilde{E}_k$ of $\widetilde{E}$, which shows that $F_s \cdot \widetilde{E}_k = -(q^*(D) + Q) \cdot \widetilde{E}_k$. But since by construction all the components of F_s are smooth and cut $\widetilde{E}_k$ transversely outside $\widetilde{E}_k \cap (\cup_{l \neq k} \widetilde{E}_l)$, the divisor D determines the topology of F_s near $\widetilde{E}_k$. It follows that the topology of F only depends on D. □

Remark. A divisor D which verifies the hypothesis of the previous proposition always exists. Moreover, one can also assume that D is fixed by the automorphism group of the dual graph of E, weighted by the self-intersection numbers E_i^2. Indeed, one can take a sufficiently high multiple of the sum $\sum_\gamma \gamma(D_0)$, where D_0 is any effective divisor which verifies $D_0 \cdot E_i < 0$, $\forall i \in \{1, \dots, r\}$ and γ varies among the elements of the automorphism group.

Proof of Theorem 1.2. Suppose M is a 3-dimensional rational homology sphere which admits a Milnor filling. Let $(\mathcal{S}, 0)$ be a germ of normal surface which satisfies $M(\mathcal{S}) \simeq M$. By W. Neumann's work [8], the topology of the minimal good resolution of $(\mathcal{S}, 0)$ is determined by M. Choose a divisor D which verifies the hypothesis of Proposition 3.1. Then this choice only depends on the topology of M.

Now let $f \in \mathcal{M}_{\mathcal{S},0}$ be a function associated to D by this proposition. Then D determines the topological type of the divisor $\operatorname{div}(f \circ p \circ q)$ and so, by plumbing, of the link $(M(\mathcal{S}), N(f))$. By Proposition 2.2, $M(\mathcal{S})$ being a rational homology sphere, the isomorphism type of the Milnor open book $(N(f), \theta)$ is thus also determined by D. Since by Proposition 2.4, this open book carries the contact structure $\xi(\mathcal{S})$, we are done. Moreover, the previous remark shows that $\xi(\mathcal{S})$ is invariant by a big subgroup of the mapping class group of $M(\mathcal{S})$. □

4. Perspectives

When the closed oriented 3-manifold M is not a rational homology sphere, the conclusion of proposition 2.2 is no longer true, even if (M, N) is the abstract boundary of a pair $(\mathcal{S}, \{f = 0\})$. Indeed, Pichon gave in [9] an example of two topologically equivalent such pairs, but which determine non-isomorphic Milnor open books. Moreover, she gives an algorithm which determines, for each fixed topological type of the pair (M, N), the topological types of the associated Milnor open books. As this algorithm shows that there is only a finite number of such topological types, we get:

Proposition 4.1. *Any closed oriented 3-manifold admits at most a finite number of Milnor fillable contact structures up to contactomorphism.*

In [4], Giroux gave a topological criterion to determine if two open books on the same 3-manifold carry isotopic contact structures: *in a closed 3-dimensional manifold, two open books which carry isotopic contact structures have isotopic stabilizations* (for details, see [4] and [5]).

The Milnor open books have geometric monodromies which are quasi-finite. Pichon gave in [9] a coding of quasi-finite monodromies, following ideas of Nielsen. In order to extend our method to manifolds which are not rational homology spheres, one should first *give an algorithm to decide if two open books having quasi-finite monodromies have isotopic stabilizations*.

In the opposite direction, if one proves that for some 3-manifold there is only one isomorphism class of Milnor fillable contact structures, one gets an infinite family of open books which have isotopic stabilizations. This is for example the case for the class of rational homology spheres we considered in this note. To our knowledge, presently there is no other method to prove that they are stably equivalent.

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