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Differential Geometry

Secondary characteristic classes of super-foliations

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Abstract

We construct secondary classes for super-foliations of codimension $0 + \epsilon 1$ and $1 + \epsilon 1$. We indicate how to generalize this construction for any regular super-foliations on super-manifolds. We interpret the secondary classes as classes of foliated flat connections. ***To cite this article: C. Laurent-Gengoux, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Classes caractéristiques secondaires des super-feuilletages. Nous déterminons des classes caractéristiques secondaires de super-feuilletages de codimension $0+\epsilon 1$ et $1+\epsilon 1$. Nous indiquons comment généraliser cette construction pour les feuilletages réguliers de codimension quelconque sur des super-variétés. Nous interprètons ensuite les classes ainsi construites comme des classes caractéristiques associées à des connexions feuilletées plates. ***Pour citer cet article : C. Laurent-Gengoux, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

La classe de Godbillon–Vey est un élément de $H^3(M,\mathbb{R})$ associée à un feuilletage régulier de codimension 1 sur une variété M. De plus depuis les travaux de Fuks, Bernshtein ou Rozenfel'd [2,3], on sait associer à tout feuilletage de codimension n avec trivialisation du fibré normal sur une variété M un homomorphisme $H^*(\mathrm{Vect}(n)) \to H^*(M,\mathbb{R})$, ou, en d'autres mots, on sait construire autant de classes caractéristiques qu'il y a de classes de cohomologie dans la cohomologie de Gelfand–Fuks de l'algèbre de Lie $\mathrm{Vect}(n)$ des champs de vecteurs formels. On appelle ces classes caractéristiques des classes secondaires du feuilletage.

Considérons $\mathcal{M}$ une super-variété et M sa variété de base. Nous associons dans la présente Note un élément de $H^1(M,\mathbb{R})$ à des super-feuilletages $\mathcal{F}$ de codimension $0 + \epsilon 1$ par le théorème suivant :

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Théorème 0.1. *Soit un super-feuilletage $\mathcal{F}$ de codimension $0+\epsilon 1$ déterminé par une 1-forme paire a et soit $\mathcal{X}$ un super-champ de vecteur tel que $a(\mathcal{X})=1$. La classe de cohomologie de $p(L_{\mathcal{X}}a)\in\Omega^1(M)$, où $p:\Omega(\mathcal{M})\to\Omega(M)$ est la projection canonique, ne dépend pas des choix de a et $\mathcal{X}$.*

Nous donnons aussi une méthode pratique pour construire des classes secondaires dans le cas de la codimension $1+\epsilon 1$. Plus généralement, nous affirmons que l'on peut associer à un super-feuilletage au complémentaire trivial une application de $H^*(\mathrm{Vect}(n,m)_0)$ dans $H^*(M,\mathbb{R})$, où $H^*(\mathrm{Vect}(n,m)_0)$ est la cohomologie de Gelfand–Fuks de la partie paire $\mathrm{Vect}(n,m)_0$ de la super-algèbre de Lie $\mathrm{Vect}(n,m)$ des super-champs de vecteurs formels à n paramètres pairs et m paramètres impairs. En d'autres mots, étant donné un $C\in H^*(\mathrm{Vect}(n,m)_0)$, nous construisons une classe secondaire $\phi_{\mathcal{M},\mathcal{F}}(C)\in H^*(M,\mathbb{R})$.

Théorème 0.2. *Soit un super-feuillatege $\mathcal{F}$ de codimension $n+\epsilon m$ admettant un fibré normal trivialisé. Il existe un homomorphisme $\phi_{\mathcal{M},\mathcal{F}}:H^*(\mathrm{Vect}(n,m)_0)\to H^*(M)$ tel que* (i) *si $f^*:\mathcal{N}\to\mathcal{M}$ est une submersion, alors $\phi_{\mathcal{N},f^*(\mathcal{F})}=\tilde{f}^*\circ\phi_{\mathcal{M},\mathcal{F}}$, où $f^*(\mathcal{F})$ est le tiré en arrière du super-feuilletage $\mathcal{F}$ et où $\tilde{f}:N\to M$ est l'application lisse induite par la restriction de f aux variétés de bases,* (ii) *si $\mathcal{M}$ est une variété différentielle ordinaire, $\phi_{\mathcal{M},\mathcal{F}}$ est l'application de Bernshtein, Fuks et Rozenfel'd,* [2,3].

Il peut sembler surprenant que ce soit la cohomologie $H^*(M)$ de la variété de base qui joue un rôle ici : cela est dû à un résultat de Batchelor [7,9] selon lequel cette cohomologie est isomorphe à celle de la super-variété $\mathcal{M}$. Enfin, remarquons que c'est la cohomologie de la partie paire $\mathrm{Vect}(n,m)_0$ et non celle de $\mathrm{Vect}(n,m)$ qui généralise le résultat classique. Il est en fait heureux qu'il en soit ainsi puisque l'on peut montrer [1,4] que lorsque $n<m$ l'espace vectoriel $H^*(\mathrm{Vect}(n,m)_0)$ n'est que de dimension 2. Nous montrons néanmoins (Proposition 4.2) que l'on pourrait utiliser la cohomologie de $\mathrm{Vect}(n,m)$ mais que l'on obtiendrait alors moins de classes secondaires. On retrouve toutefois dans certains cas particuliers les classes de Godbillon–Vey construites en [5].

Nous expliquerons enfin pourquoi à partir de tout super-feuilletage sur une super-variété $\mathcal{M}$, on peut construire ce que l'on appelle des connexions feuilletées plates sur la variété de base M ; c'est-à-dire un feuilletage sur la variété de base, un fibré vectoriel et une connexion plate de ce fibré vectoriel définie uniquement le long des feuilles du feuilletage. On obtient alors le résultat suivant.

Théorème 0.3. *Si à deux super-feuilletages $\mathcal{F}$ et $\mathcal{F}'$ sont associées des connnexions feuilletées plates isomorphes, alors $\phi_{\mathcal{M},\mathcal{F}}=\phi_{\mathcal{M},\mathcal{F}'}$.*

1. Introduction

The Godbillon–Vey class is an element of $H^3(M,\mathbb{R})$ associated to a regular foliation of codimension 1 on a manifold M. More generally, to any regular foliation F with a trivialized normal bundle on a manifold M is uniquely associated a (non-trivial in general) homomorphism from $H^*(\mathrm{Vect}(n))$ to $H^*(M,\mathbb{R})$, see [2] or [3] (where $H^*(\mathrm{Vect}(n))$ is the Gelfand–Fuks cohomology of the Lie algebra $\mathrm{Vect}(n)$ of formal vector fields with n parameters). These classes are called secondary classes of the foliation.

Let $\mathcal{M}$ be a super-manifold and M its basis manifold. We construct in this article secondary classes of foliated super-manifold $\mathcal{M}$ endowed with a super-foliation of codimension $0+\epsilon 1$ or $1+\epsilon 1$. For super-foliation of codimension $0+\epsilon 1$, we construct a secondary class as an element of $H^1(M,\mathbb{R})$. For super-foliation of codimension $1+\epsilon 1$, we construct three secondary classes as elements of $H^3(M,\mathbb{R})$. More generally, we will show how to associate to any super-foliation $\mathcal{F}$ of codimension $n+\epsilon m$ with a trivialized normal bundle, a homomorphism $\phi_{\mathcal{M},\mathcal{F}}:H^*(\mathrm{Vect}(n,m)_0)\to H^*(M,\mathbb{R})$ where $H^*(\mathrm{Vect}(n,m)_0)$ is the Gelfand–Fuks cohomology of the even part $\mathrm{Vect}(n,m)_0$ of the super-Lie algebra $\mathrm{Vect}(n,m)$ of formal vector fields with n even parameters and m odd parameters. In other words, we define for any $C\in H^*(\mathrm{Vect}(n,m)_0)$ a secondary class $\phi_{\mathcal{M},\mathcal{F}}(C)\in H^*(M,\mathbb{R})$ of super-foliation with a trivialized normal bundle. The reader must not be surprised by the fact that the de Rham cohomology of $\mathcal{M}$ is not used: by a result of Batchelor [7], this cohomology is indeed isomorphic to the cohomology of the basis manifold. Note that only the even part $\mathrm{Vect}(n,m)_0$ appears: it is indeed better like this, since $H^*(\mathrm{Vect}(n,m))$ has only two non-trivial generators for $n<m$, see [3].

2. Foliations on super-manifolds

We denote by $\mathcal{O}(\mathcal{M})$ the super-functions on a super-manifold $\mathcal{M}$, by $\mathrm{Vect}(\mathcal{M})$ the super-Lie algebra of super-vector fields and by $(\Omega(\mathcal{M}),d)$ the super-algebra of differential forms on $\mathcal{M}$ endowed with its de Rham differential.

On the super-manifold $\mathbb{R}^{p,q}$, we denote by $\mathcal{B}$ the $\mathcal{O}(\mathbb{R}^{p,q})$-sub-module of $\mathrm{Vect}(\mathbb{R}^{p,q})$ generated by the super-vector fields $(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_{p-n}})$ and $(\frac{\partial}{\partial\theta_1},\ldots,\frac{\partial}{\partial\theta_{q-m}})$ where $x_1,\ldots,x_p,\theta_1,\ldots,\theta_q$ are the natural parametrisation of $\mathbb{R}^{p,q}$. A *regular super-foliation $\mathcal{F}$ of codimension $n+\epsilon m$* is a sub-super-Lie algebra $\mathrm{Vect}(\mathcal{F})$ of $\mathrm{Vect}(\mathcal{M})$ such that for any x in the basis manifold M there exists a neighborhood $\mathcal{U}$ of x and a diffeomorphism from $\mathcal{U}$ and $\mathbb{R}^{p,q}$ that provides an isomorphism between $\mathcal{B}$ and the restriction of $\mathrm{Vect}(\mathcal{F})$ to $\mathcal{U}$.

We say that a super-foliation $\mathcal{F}$ admits a trivialized normal bundle when there exists odd 1-forms $\omega_1,\ldots,\omega_n$ and even 1-forms $a_1,\ldots,a_m$ generating a free $\mathcal{O}(\mathcal{M})$-module of super-dimension $n+\epsilon m$ such that $\mathrm{Vect}(\mathcal{F})=\{\mathcal{X}\in\mathrm{Vect}(\mathcal{M})\mid\omega_1(\mathcal{X})=\cdots=\omega_n(\mathcal{X})=a_1(\mathcal{X})=\cdots=a_m\mathcal{X}=0\}$.

An important point is that to any super-foliation $\mathcal{F}$ of codimension $n+\epsilon m$ on a super-manifold $\mathcal{M}$ is associated by a canonical way a super-foliation with a trivialized normal bundle on a $\mathrm{Gl}(n,m)$-principal bundle over $\mathcal{M}$, see [6]. This $\mathrm{Gl}(n,m)$-principal bundle is the super-manifold of free $n+\epsilon m$ families $T^*(M)$ that vanishes on $\mathrm{Vect}(\mathcal{F})$. Therefore, we can always assume that the super-foliation admits a trivialized normal bundle by replacing the initial super-foliation by this new one if necessary.

3. Secondary classes for super-foliations of codimension $0+\epsilon 1$ and $1+\epsilon 1$

Let $\mathcal{F}$ be a regular super-foliation of codimension $0+\epsilon 1$ with a trivialized normal bundle given by an even 1-form a and Θ an unique odd super-vector field satisfying $a(\Theta)=1$. Let us consider the 1-form on M given by $A=p(L_\Theta a)$ where $p:\Omega(\mathcal{M})\to\Omega(M)$ is the natural projection. This theorem allows us to consider $[A]\in H^1(M,\mathbb{R})$ as a secondary class of a super-foliation of codimension $0+\epsilon 1$.

Theorem 3.1. *The* 1*-form A is closed. Its cohomology class in $H^1(M,\mathbb{R})$ does not depend upon the choice of a and Θ.*

Proof. Since the 1-form a defines a super-foliation of codimension $0+\epsilon 1$, there exists an odd 1-form b such that $\mathrm{d}a=b\wedge a$, see [8]. Since b is odd, we have $b\wedge b=0$ and the identity $0=\mathrm{d}^2a=a\wedge b\wedge b+a\wedge\mathrm{d}b=a\wedge\mathrm{d}b$ holds. This implies that $\mathrm{d}b=0$ because, a being a regular even 1-form, we have the surprising property that $a\wedge c=0$ implies $c=0$ for any $c\in\Omega(\mathcal{M})$.

Now from $L_\Theta a=\imath_\Theta da=\imath_\Theta a\wedge b=b+a\wedge\imath_\Theta b$, we deduce immediately that $A=p(L_\Theta a)=p(b)$. The 1-form A is thus a closed 1-form on M that does not depend on the choice of Θ.

To show that $[A]$ does not depend on the choice of the 1-form a chosen, we just have to see that if we replace a by fa for f an even invertible super-function on $\mathcal{M}$, we have $\mathrm{d}(fa)=c\wedge fa$ with $c=b+\frac{\mathrm{d}f}{f}$ and, as a consequence, $p(c)=p(b)+p(\frac{\mathrm{d}f}{f})=p(b)+\frac{\mathrm{d}p(f)}{p(f)}$. The super-function f being invertible, $p(f)$ never vanishes, so the 1-form $p(\frac{\mathrm{d}f}{f})=\mathrm{d}(ln(|p(f)|))$ is exact on M. □

Example 1. On the super-circle $S^{1|1}$, let us denote $x\in S^1$ the even parameter and θ the odd parameter. For any $t\in\mathbb{R}$, let us consider the super-foliation of codimension $0+\epsilon 1$ defined by the 1-form $\mathrm{d}\theta+t\theta\,\mathrm{d}x$. Let us choose

$\Theta = \frac{\mathrm{d}}{\mathrm{d}\theta}$. We have $A = p(L_\Theta a) = t\,\mathrm{d}x$ and $[A] = t \in H^1(S^1, \mathbb{R}) \simeq \mathbb{R}$. The secondary class that we obtain in this case is thus non-zero if $t \neq 0$.

For super-foliation of codimension $1+\epsilon 1$, the following theorem (proved in [6]) defines three secondary classes.

Theorem 3.2. *Let $\mathcal{F}$ be a super-foliation of codimension $1+\epsilon 1$ defined by the odd 1-form ω and the even 1-form a. Let b be an odd n 1-form on $\mathcal{M}$ and c an even 1-form on $\mathcal{M}$ satisfying $\mathrm{d}_{\mathcal{M}} a = \omega \wedge c - a \wedge b$. Then*:

(i) *There exist $\alpha, \beta, \xi \in \Omega^1(M)$ on the base manifold M such that $\mathrm{d}_M p(\omega) = p(\omega) \wedge \alpha$, $\mathrm{d}_M \alpha = p(\omega) \wedge \beta$, $\mathrm{d}_M p(b) = p(\omega) \wedge \xi$.*
(ii) *The three 3-forms $[\beta \wedge \alpha \wedge p(\omega)]$, $[\xi \wedge \alpha \wedge p(\omega)]$ and $[\xi \wedge p(b) \wedge p(\omega)]$ are closed and their cohomology class do not depend on the choice of ξ, α, β.*

Example 2. Consider the super-manifold given by the trivial 1-vector bundle E over the 3-dimensional Torus $T^3 \simeq (S^1)^3$. Let $x, y, z \in S^1$ the coordinates of T^3 and let θ be the odd-parameter corresponding to the constant section of E. Set $f(x), g(x)$ two smooth functions on S^1 such that $\int_{S^1} W(f,g) \neq 0$ where W is the Wronskian.

We leave the reader to check that a super-foliation of codimension $1+\epsilon 1$ is defined by the odd 1-form $\omega = \mathrm{d}x$ and the even 1-form $a = \mathrm{d}\theta + \theta(f(x)\,\mathrm{d}y + g(x)\,\mathrm{d}z)$. In this case, we can chose $b = (f(x)\,\mathrm{d}y + g(x)\,\mathrm{d}z)$, $\xi = -\frac{f(x)}{\mathrm{d}x}\,\mathrm{d}y - \frac{g(x)}{\mathrm{d}x}\,\mathrm{d}z$ and it is routine to check that $[\xi \wedge p(b) \wedge p(\omega)] = [W(f,g)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z]$. Since $[W(f,g)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z] = (\int_{S^1} W(f,g))\,[\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z]$, $[\xi \wedge p(b) \wedge p(\omega)]$ is a non-zero class of the de Rham cohomology.

Example 3. Let M be a manifold and F a foliation of codimension 1 on M defined by $\omega \in \Omega^1(M)$ such that $\mathrm{d}_M \omega = \omega \wedge \alpha$. Assume moreover that the Godbillon–Vey class $-[\alpha \wedge \mathrm{d}_M \alpha] \in H^3(M)$ of F is not zero.

Let $E \to M$ be the trivial 1-dimensional bundle $E = \mathbb{R} \times M$. Consider the super-manifold $\mathcal{M}$ with $\mathcal{O}(\mathcal{M}) = \Gamma(\wedge E)$. We denote by θ the unique odd parameter corresponding to some constant section of E.

We define a super-foliation of codimension $1+\epsilon 1$ by the odd 1-form ω and by the even 1-form $a = \mathrm{d}\theta + \theta\alpha$. In this case, one can check (see [6]) that all three secondary classes constructed in Theorem 3.2(ii) are equal to the Godbillon–Vey class.

4. Generalization

The Lie super-algebra $\mathrm{Vect}(n,m)$ of super-vector fields on $\mathbb{R}^{n,m}$ with polynomial coefficients is defined for example in [1]: it is the Lie super-algebra of super-derivations of $\mathbb{R}[x_1, \ldots, x_n] \otimes \bigwedge \mathbb{R}^m$ where $\mathbb{R}[x_1, \ldots, x_n]$ is the algebra polynomial functions on $\mathbb{R}^n$. We denote by $H^*(\mathrm{Vect}(n,m)_0)$ the cohomology of the even part of $\mathrm{Vect}(n,m)$. The following theorem defines *secondary classes of a super-foliation with a trivialized normal bundle*. The proof of this statement turns out to be far from being easy and will appear in [6].

Theorem 4.1. *To any super-foliation with a trivialized normal bundle of codimension $n+\epsilon m$, we can associate a natural and non-trivial in general homomorphism $\phi_{\mathcal{M},\mathcal{F}}$ from $H^*(\mathrm{Vect}(n,m)_0)$ to $H^*(M,\mathbb{R})$ such that*

(i) *if $f : \mathcal{N} \to \mathcal{M}$ is a submersion of super-manifolds, then $\phi_{\mathcal{N}, f^*\mathcal{F}} = \tilde{f}^* \phi_{\mathcal{M},\mathcal{F}}$ where the super-foliation $f^*\mathcal{F}$ is the pull-back of $\mathcal{F}$ by f and $\tilde{f} : N \to M$ is the smooth map induced by f on basis manifolds*;
(ii) *if $\mathcal{M}$ is an (ordinary) smooth manifold, then $m = 0$ and $\phi_{\mathcal{M},\mathcal{F}}$ reduces to the well-known map of Fuks, Bernshtein and Rozenfel'd* [2,4].

Example 4. It is not difficult to see (see [6]) that $H^1(\mathrm{Vect}(0,1)_0) \simeq \mathbb{R}$ and $H^i(\mathrm{Vect}(0,1)) = 0$ for $i \geqslant 2$. The secondary class associated to $H^1(\mathrm{Vect}(0,1))$ is the class described in Theorem 3.1.

Example 5. It is not difficult to see (see [6]) that $H^3(\mathrm{Vect}(1,1)_0) \simeq \mathbb{R}^3$ and $H^i(\mathrm{Vect}(1,1)) = 0$. The secondary class associated to $H^1(\mathrm{Vect}(1,1))$ are the classes described in Theorem 3.2 for $i \geqslant 3$.

It is natural to ask whether it is possible to construct using similar techniques a natural homomorphism $\psi_{\mathcal{M},\mathcal{F}}$: $H^*(\mathrm{Vect}(n,m)) \to H^*(\mathcal{M})$ where $H^*(\mathrm{Vect}(n,m))$ is the Gelfand–Fuks cohomology of $\mathrm{Vect}(n,m)$. The answer is that it possible and, moreover, that this construction has been done in [5] in the case of a super-foliation of codimension $0+\epsilon m$ on a super-manifold of dimension $n+\epsilon m$. The following proposition relates both construction and explains why all the classes that can be constructed through the homomorphism $\psi_{\mathcal{M},\mathcal{F}}$ (using therefore the cohomology of $\mathrm{Vect}(n,m)$) are indeed among the classes that can be constructed through the homomorphism $\phi_{\mathcal{M},\mathcal{F}}$ (using therefore the cohomology of $\mathrm{Vect}(n,m)_0$).

Proposition 4.2. (i) *For any super-foliation $\mathcal{F}$ of codimension $n+\epsilon m$ with a trivialized normal bundle, the following diagram is commutative*

$$\begin{array}{ccc} H^*\big(\mathrm{Vect}(n,m)\big) & \xrightarrow{\psi_{\mathcal{M},\mathcal{F}}} & H^*(\mathcal{M}) \\ \downarrow J & & \downarrow \hat{p} \\ H^*\big(\mathrm{Vect}(n,m)_0\big) & \xrightarrow{\phi_{\mathcal{M},\mathcal{F}}} & H^*(M) \end{array} \tag{1}$$

where $J : H^*(\mathrm{Vect}(n,m)) \to H^*(\mathrm{Vect}(n,m)_0)$ is induced by the inclusion of $\mathrm{Vect}(n,m)_0$ *into* $\mathrm{Vect}(n,m)$ *and $\hat{p} : H^*(\mathcal{M}) \to H^*(M)$ is the isomorphism of Batchelor.*

(ii) *The Godbillon–Vey classes constructed in* [5] *are among the secondary classes of super-foliation constructed above. In particular, the secondary class for super-foliations of codimension* $0+\epsilon 1$ *of Theorem* 3.1 *is equal to the class constructed in* [5]*, Example* 1*, for super-foliations of codimension* $0+\epsilon 1$ *on super-manifolds of dimension $n+\epsilon m$.*

5. Super-foliation on a super-manifold and flat foliated connection on its basis manifold

5.1. Characteristic classes of trivial flat foliated connections

Let F be a foliation of codimension n on a smooth manifold M and E a vector bundle over M. We recall that a *foliated connection* is a bilinear map $\nabla : \mathrm{Vect}(F) \otimes \Gamma(E) \to \Gamma(E)$ that satisfies the usual properties of a connection. We say that this foliated connection is *flat* if $\forall X, Y \in \mathrm{Vect}(F)$, $\forall s \in \Gamma(E)$ we have $\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = 0$. By a *trivial* foliated connection (F, E, ∇), we mean a foliation F with a trivialized normal bundle on M, a trivial vector bundle E over M and a foliated connection ∇.

There is a natural application from the set of flat trivial foliated connections (F, E, ∇) to the set of super-foliations $\mathcal{F}$ with a trivialized normal bundle: given a trivial foliated connection (F, E, ∇) we define first the super-manifold $\mathcal{M}$ whose super-functions are the sections of $\wedge^* E$ and the super-foliation generated by the even derivations ∇_X of $\mathcal{O}(\mathcal{M})$ for all $X \in \mathrm{Vect}(F)$. By Theorem 4.1, we can therefore associate to any flat trivial foliated connection a homomorphism from $H^*(\mathrm{Vect}(n,m)_0)$ to $H^*(M, \mathbb{R})$. This defines *characteristic classes of a trivial flat foliated connection*, which can be shown to be non-trivial in general.

5.2. A functor from super-foliations to flat foliated connections

Given a super-foliations with a trivialized normal bundle on a super-manifold $\mathcal{M}$ one can construct a flat trivial foliated connection as follows. Let $\mathcal{F}$ be a super-foliation $\mathcal{F}$ with a trivialized normal bundle.

- First a regular foliation F on M is defined by the Lie algebra $p(\mathrm{Vect}(\mathcal{F})_0)$, p being here the natural projection from the even part of $\mathrm{Vect}(\mathcal{M})$ onto $\mathrm{Vect}(M)$ and $\mathrm{Vect}(\mathcal{F})_0$ being the even part of $\mathrm{Vect}(\mathcal{F})$.
- Second a vector-bundle E_F is defined to be the vector bundle whose sections are of the form $\frac{\mathcal{J}_F\mathcal{O}(\mathcal{M})}{\mathcal{I}^2}$, where $\mathcal{I}$ is the ideal of nilpotent elements of $\mathcal{O}(\mathcal{M})$ and $\mathcal{J}_F$ the algebra of odd super-functions constant on the leaves of $\mathcal{F}$ (i.e. the algebra of super-functions f such that $\mathrm{d}f(\widetilde{X}) = 0$ for any $\widetilde{X} \in \mathrm{Vect}(\mathcal{F})_0$).
- The foliated connection is then defined $\forall X \in \mathrm{Vect}(F), \forall s \in \Gamma(E)$ by $\nabla_X s = \mathrm{d}f(\widetilde{X}) \bmod \mathcal{I}^2$ where $f \in \mathcal{J}_F\mathcal{O}(\mathcal{M})$ is a super-function with $s = f \bmod \mathcal{I}^2$ and $\widetilde{X} \in \mathrm{Vect}(\mathcal{M})$ is an even super-vector field that projects naturally on X. We let the reader check that this defines a foliated connection.
- If the super-foliation $\mathcal{F}$ has a trivialized normal bundle then the foliation F has a trivialized normal bundle also and the vector bundle E_F is trivial.

We have therefore constructed a trivial foliated connection. It is easy now to check that the foliated connection ∇ on E_F is flat (see [6]).

This last proposition finishes our construction. Note that this construction is not the inverse of the construction of Section 5.1, see [6].

Theorem 5.1 [6]. *Two super-foliations $\mathcal{F}$ and $\mathcal{F}'$ with a trivialized normal bundle on a super-manifold $\mathcal{M}$ defining isomorphic flat trivial foliated connections have the same secondary classes, i.e., the homomorphisms $\phi_{\mathcal{M},\mathcal{F}}$ and $\phi_{\mathcal{M},\mathcal{F}'}$ from $H^*(\mathrm{Vect}(n,m)_0)$ to $H^*(M,\mathbb{R})$ associated to these super-foliations are equal.*

The foliation F previously defined has a trivialized normal bundle: therefore the theory of secondary classes of [2,4] applies and provides a homomorphism $\phi_{M,F} : H^*(\mathrm{Vect}(n)^*) \to H^*(M,\mathbb{R})$. These secondary classes are among those constructed in Theorem 4.1. More precisely, there is a natural inclusion $i : H^*(\mathrm{Vect}(n)^*) \to H^*(\mathrm{Vect}(n,m)_0^*)$, see [6], and it can be shown that

Proposition 5.2 [6]. *The following diagram is commutative*

$$\begin{array}{ccc} H^*\big(\mathrm{Vect}(n)^*\big) & \stackrel{i}{\to} & H^*\big(\mathrm{Vect}(n,m)_0^*\big) \\ & \searrow \phi_{M,F} & \downarrow \phi_{\mathcal{M},\mathcal{F}} \\ & & H^*(M,\mathbb{R}) \end{array}.$$

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