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A Note on Kato–Nakayama spaces

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Abstract

In this Note we prove the following result. A fine log scheme over the complex numbers and its saturated have homeomorphic Kato–Nakayama associated spaces. Moreover these spaces are isomorphic as ringed spaces, either with the ring sheaf defined by Kato–Nakayama, or with that defined by Ogus. In the definition of these spaces, non-integral monoids are involved, so that the proof of the result is based on properties of nonnecessarily integral monoids. ***To cite this article: M. Cailotto, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Une Note sur les espaces de Kato–Nakayama. Dans cette Note nous prouvons le resultat suivant. Un log schéma sur le corps des nombres complexes et son saturé ont des espaces de Kato–Nakayama associés qui sont homéomorphes. En plus, ces espaces sont isomorphes en tant qu'espaces annelés, soit avec le faisceau d'anneaux défini par Kato–Nakayama, soit avec le faisceau d'anneaux défini par Ogus. Dans la définition de ces espaces on utilise des monoïdes non intègres, et la démonstation utilise certaines proprietés des monoïdes non nécessairement intègres. ***Pour citer cet article : M. Cailotto, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Introduction

We recall here the basic definition of the Kato–Nakayama spaces associated to a log analytic space (see [1,2,5,3]). Using this ringed spaces for suitable fs (fine and saturated) log spaces they define a Riemann–Hilbert correspondence for a special class of log connections (with nilpotent conditions). These results are then extended by Ogus (see [6]) to the entire class of log connections using a more general ring sheaf on the Kato–Nakayama space, whose construction is similar to the case of indexed algebras of Lorenzon (see [4]).

In this Note we compare the Kato–Nakayama spaces associated to a fine log space and to its saturated log space. It turns out that they are homeomorphic as topological spaces, and they are isomorphic as ringed spaces using either the Kato–Nakayama sheaf $\mathscr{O}_X^{\log}$ or the Ogus sheaf $\widetilde{\mathscr{O}}_X^{\log}$.

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This result is useful, for example, in the computation of Kato–Nakayama spaces for base change in the category of fs log schemes. It suggests also that the logarithmic Riemann–Hilbert correspondences proved by Kato–Nakayama and Ogus for (ideally smooth) fs log schemes cannot be extended directly to the category of fine log schemes.

1. Kato–Nakayama's spaces

1.1. The Kato–Nakayama space [2,3]

Let $\pi : \mathbb{S}^1 \times \mathbb{R}_{\geqslant 0} \to \mathbb{C}$ be the morphism $(\zeta, r) \mapsto r\zeta$ where $\mathbb{S}^1$ and $\mathbb{R}_{\geqslant 0}$ have both the monoidal multiplicative structure (notice that $\mathbb{R}_{\geqslant 0}$ is not integral: $\mathbb{R}_{\geqslant 0}^{\mathrm{gp}} = 0$). Clearly this is a log structure on $\operatorname{Spec}\mathbb{C}$. Call T the log point $(\operatorname{Spec}(\mathbb{C}), \mathbb{S}^1 \times \mathbb{R}_{\geqslant 0})$ and for any coherent log scheme X over $\mathbb{C}$, whose underlying scheme is of finite type, define the set $X^{\log} := \operatorname{Hom}_{\mathbb{C}}(T, X)$. If we have locally a chart of X modeled on a monoid P, we can see that $X^{\log}$ is locally a closed subset of the topological product $X_{\mathrm{an}} \times \operatorname{Hom}(P^{\mathrm{gp}}, \mathbb{S}^1)$. This local definition for the topology globalizes to give a topology on $X^{\log}$. There is, moreover, a canonical surjective morphism $\tau_X : X^{\log} \to X_{\mathrm{an}}$ which is continuous and proper as map of topological spaces.

We refer to [2, § 3] and [6, §3.3] for the definitions of the sheaves of rings $\mathscr{O}_X^{\log}$ and $\widetilde{\mathscr{O}}_X^{\log}$ on $X^{\log}$.

1.2.

For example for $X = \operatorname{Spec}\mathbb{C}[P]$ with P a fine monoid, then $X^{\log} = \operatorname{Hom}_{\mathcal{M}on}(P, \mathbb{R}_{\geqslant 0} \times \mathbb{S}^1) = \operatorname{Hom}_{\mathcal{M}on}(P, \mathbb{R}_{\geqslant 0}) \times \operatorname{Hom}_{\mathcal{M}on}(P^{\mathrm{gp}}, \mathbb{S}^1)$; for $x \in X^{\log}$, $\tau_X(x) = \pi \circ x$ as morphism $P \to \mathbb{C}$. The topology is the natural one, making $X^{\log}$ into a locally compact space.

1.3.

The construction of Kato–Nakayama spaces commutes with projective limits; moreover, if $f : X \to Y$ is a strict morphism of coherent log schemes, then the natural diagram $\tau_Y \circ f^{\log} = f \circ \tau_X$ is a Cartesian square of topological spaces. If X is fs (fine and saturated, see [7]), then the fiber of τ_X at a point $x \in X$ is a torsor under the product of $r(x)$ copies of $\mathbb{S}^1$, where $r(x)$ is the rank of the characteristic sheaf $\overline{\mathscr{M}}_X^{\mathrm{gp}} = \mathscr{M}_X^{\mathrm{gp}} / \mathscr{O}_X^{\times}$ at x.

2. Principal result

Theorem 2.1. *Let X be a fine log scheme over $\mathbb{C}$. Then the canonical morphism $c_X : X^{\mathrm{sat}} \to X$ induces a homeomorphism of topological spaces $c_X^{\log} : (X^{\mathrm{sat}})^{\log} \xrightarrow{\sim} X^{\log}$. Moreover, it induces an isomorphism of ringed spaces if these spaces are endowed with the sheaves of rings $\mathscr{O}_X^{\log}$ defined by Kato–Nakayama or $\widetilde{\mathscr{O}}_X^{\log}$ defined by Ogus.*

For the proof, we need some preliminaries on non-integral monoids.

Definition 2.2. Let M be a monoid (not necessarily integral) and let n be a positive integer; we say that M is quasi-n-saturated if the commutative square given by $c \circ n = n \circ c$ is Cartesian, where $c : M \to M^{\mathrm{gp}}$ is the canonical morphism and $n : M \to M$ (resp. $M^{\mathrm{gp}} \to M^{\mathrm{gp}}$) means "multiplication by n". We say that M is quasi-saturated if it is quasi-p-saturated for every prime p (or equivalently quasi-n-saturated for any $n \in \mathbb{N}$).

Let $M(n) = M \times_{c, M^{\mathrm{gp}}, n} M^{\mathrm{gp}}$ be the fiber product of the diagram $M \xrightarrow{c} M^{\mathrm{gp}} \xleftarrow{n} M^{\mathrm{gp}}$. Then M is quasi-n-saturated if and only if the canonical morphism $(n, c) : M \to M(n)$ is an isomorphism. Notice that $M(n)$ is not quasi-n-saturated in general; however, the canonical map $(n, c)^{\mathrm{gp}} : M^{\mathrm{gp}} \to M(n)^{\mathrm{gp}}$ is a section of the (group extended) second projection.

Remark 1. (i) If M is integral, it is quasi-n-saturated (resp. quasi-saturated) if and only if it is n-saturated (resp. saturated) in the usual sense (see [7]).

(ii) If M is such that $M^{\mathrm{gp}} = 0$, it is quasi-n-saturated (resp. quasi-saturated) if and only if it is uniquely n-divisible (resp. uniquely divisible), i.e. if the multiplication by n is an isomorphism (resp. for any n).

Lemma 2.3. *The inclusion of the category of quasi-n-saturated (resp. quasi-saturated) monoids in that of monoids admits a left adjoint functor, that is for any monoid M there exists functorially a quasi-n-saturated (resp. quasi-saturated) monoid $M^{n\text{-qsat}}$ (resp. M^{qsat}) such that*

$$\mathrm{Hom}(M^{n\text{-qsat}}, P) \cong \mathrm{Hom}(M, P) \quad (\textit{resp. } \mathrm{Hom}(M^{\text{qsat}}, P) \cong \mathrm{Hom}(M, P))$$

for any quasi-n-saturated (resp. quasi-saturated) monoid P.

Proof. Let M be a (not necessarily integral) monoid; we define the quasi-n-saturated of M by $M^{n\text{-qsat}} := \varinjlim_{i\in\mathbb{N}} M(n^i)$ where the inductive system is defined for $i \leqslant j$ by $n^{j-i} \times \mathrm{id} : M(n^i) \to M(n^j)$; and the quasi-saturated of M by $M^{\text{qsat}} := \varinjlim_{n\in(\mathbb{N},|)} M(n) \cong \varinjlim_{n\in(\mathbb{N},|)} M^{n\text{-qsat}}$ where the inductive system in the first expression is defined for $n|m$ by $(m/n) \times \mathrm{id} : M(n) \to M(m)$. We see easily that $M^{n\text{-qsat}}$ is quasi-n-saturated, that M^{qsat} is quasi-saturated, and that the canonical morphisms $M \to M^{n\text{-qsat}}$ and $M \to M^{\text{qsat}}$ (notice that $M = M(1)$) enjoy the universal properties. In fact, e.g. for the second functor, we have $\mathrm{Hom}(M^{\text{qsat}}, P) = \varprojlim_{n\in\mathbb{N}} \mathrm{Hom}(M(n), P)$ and it is enough to prove that for any n the map $\mathrm{Hom}(M(n), P) \to \mathrm{Hom}(M, P)$ is a bijection. Since P is quasi-saturated, for any morphism $\varphi : M \to P$ we have a unique extension $\varphi_n : M(n) \to P$ such that $\varphi_n \circ (n, c) = \varphi$. □

Remark 2. (i) If M is integral, the quasi-n-saturated (resp. quasi-saturated) of M are the usual ones: $M^{n\text{-qsat}} = \{m \in M^{\text{gp}} \mid \exists i \in \mathbb{N},\ m^{n^i} \in M\}$ (resp. $M^{\text{sat}} = \{m \in M^{\text{gp}} \mid \exists n \in \mathbb{N}^*,\ m^n \in M\}$) as submonoids of M^{gp}.

(ii) If M is such that $M^{\text{gp}} = 0$, so that $M(m) \cong M$, then its quasi-n-saturated (resp. quasi-saturated) is given by $M^{n\text{-qsat}} = \varinjlim(M \xrightarrow{n} M \xrightarrow{n} \cdots \xrightarrow{n} M \xrightarrow{n} \cdots)$ (resp. $M^{\text{qsat}} = \varinjlim_{n\in(\mathbb{N},|)} M_n$ where $M_n = M$ for any n and the transition map $M_n \to M_m$ for $n|m$ is the multiplication by m/n).

Proposition 2.4. *Let M be an integral monoid; then the morphism $M \to M^{\text{sat}}$ induces a homeomorphism $(\operatorname{Spec}\mathbb{C}[M^{\text{sat}}])^{\log} \to (\operatorname{Spec}\mathbb{C}[M])^{\log}$ of topological spaces.*

Proof. In fact we can identify the Kato–Nakayama space as

$$\left(\operatorname{Spec}\mathbb{C}[M^{\text{sat}}]\right)^{\log} = \mathrm{Hom}\left(T, \operatorname{Spec}\mathbb{C}[M^{\text{sat}}]\right) = \mathrm{Hom}(M^{\text{sat}}, \mathbb{S}^1 \times \mathbb{R}_{\geqslant 0})$$

the canonical map being induced by the composition with $M \to M^{\text{sat}}$. If we take a morphism $M \to \mathbb{S}^1 \times \mathbb{R}_{\geqslant 0}$ we can extend it to M^{sat} using the fact that $\mathbb{S}^1$ is a group and $\mathbb{R}_{\geqslant 0}$ (as a multiplicative monoid) is a quasi-saturated monoid (or equivalently a uniquely divisible monoid, since $(\mathbb{R}_{\geqslant 0})^{\text{gp}} = 0$). For $X = \operatorname{Spec}\mathbb{C}[M]$ the topology in $X^{\log}$ is that induced as a closed subset of $X(\mathbb{C}) \times \mathrm{Hom}(M^{\text{gp}}, \mathbb{S}^1)$; now the canonical morphism $X^{\text{sat}} \to X$ is finite, so $X^{\text{sat}}(\mathbb{C}) \to X(\mathbb{C})$ is finite and closed, and the topologies induced by $X(\mathbb{C}) \times \mathrm{Hom}(M^{\text{gp}}, \mathbb{S}^1)$ or by $X^{\text{sat}}(\mathbb{C}) \times \mathrm{Hom}(M^{\text{gp}}, \mathbb{S}^1)$ coincide. □

Corollary 2.5. *Let X be a fine log scheme, of finite type as scheme. The (log étale and finite) canonical morphism $X^{\text{sat}} \to X$ induces a homeomorphism $(X^{\text{sat}})^{\log} \to X^{\log}$.*

Proof. In fact X admits locally a chart $X \to \operatorname{Spec}\mathbb{C}[M]$ where M is a fine monoid and X^{sat} is locally defined as $X \times_{\operatorname{Spec}\mathbb{C}[M]} \operatorname{Spec}\mathbb{C}[M^{\text{sat}}]$, so we can apply the proposition: from the Cartesian diagram for X^{sat}, applying $(-)^{\log} = \mathrm{Hom}(T, -)$ we find a Cartesian diagram where on the right we have a homeomorphism. □

Let X be a fine log scheme, of finite type as scheme; we know that the canonical morphism $\tau^{\text{sat}} : (X^{\text{sat}})^{\log} \to X^{\text{sat}}$ is fibered in torsors under a product of $\mathbb{S}^1$; therefore the map $\tau : X^{\log} \to X$ is fibered in torsors under a product of $\mathbb{S}^1$ times a finite group (depending on the point).

2.1. The ringed structures

The sheaf $\mathscr{L}_X$ of logarithmic sections is clearly isomorphic to $\mathscr{L}_{X^{\text{sat}}}$ (see [2, 1.4], [3, 5.6]), since it depends only on $\mathscr{M}_X^{\text{gp}}$ which is isomorphic to $\mathscr{M}_{X^{\text{sat}}}^{\text{gp}}$. We have a natural morphism of structural sheaves $\mathscr{O}_X^{\log} \to \mathscr{O}_{X^{\text{sat}}}^{\log}$. The explicit descriptions (see [2, 3.2]) show immediately that it is an isomorphism, since all sections of $\tau^{-1}\mathscr{O}_X$ (resp. $\tau^{-1}\mathscr{O}_{X^{\mathrm{sat}}}$) are identified with their image in $\mathscr{L}_X$. Since the Ogus ringed structure is constructed using the previous ones (see [6, 3.3.2]), tensorized with the $\mathbb{C}\otimes\overline{\mathscr{M}}_X^{\mathrm{gp}}$-graded algebra $\mathscr{A}_X^{\log}$ (see [6, ante 3.3.1] for the definition), the canonical morphism $\widetilde{\mathscr{O}}_X^{\log}\to\widetilde{\mathscr{O}}_{X^{\mathrm{sat}}}^{\log}$ is also an isomorphism. This complete the proof of the theorem.

3. Applications

3.1.
In many problems, base changes in the category of fs log schemes are involved, and they do not coincide with the base change in the category of log schemes, or even in the category of fine log schemes. In fact, if $X\to Z\leftarrow Y$ are morphisms of fine (resp. fs) log schemes, and $X\times_Z Y$ is the fiber product in the category of log schemes, then the fiber product $X\times_Z^{\mathrm{f}} Y$ (resp. $X\times_Z^{\mathrm{fs}} Y$) in the category of fine (resp. fs) log schemes is $(X\times_Z Y)^{\mathrm{int}}$ (resp. $(X\times_Z Y)^{\mathrm{sat}}$). The following proposition simplifies in some cases the computation of the Kato–Nakayama spaces of a base change.

Proposition 3.1. *Let $X\to Z\leftarrow Y$ be morphisms of fs log schemes, and suppose that the fiber product $X\times_Z Y$ in the category of log schemes is fine* (*e.g. if one of the morphisms is integral*). *Then $(X\times_Z^{\mathrm{fs}} Y)^{\log}\cong X^{\log}\times_{Z^{\log}} Y^{\log}$.*

Proof. The construction of Kato–Nakayama spaces commutes with the fiber product in the category of coherent log schemes, so that we have $(X\times_Z^{\mathrm{fs}} Y)^{\log}\cong((X\times_Z Y)^{\mathrm{sat}})^{\log}\cong(X\times_Z Y)^{\log}\cong X^{\log}\times_{Z^{\log}} Y^{\log}$ (the second isomorphism holds because $X\times_Z Y$ is integral). □

3.2.
The (Kato–Nakayama or Ogus) log Riemann–Hilbert correspondences cannot be extended directly to fine log schemes, as the following example (suggested to me by A. Ogus) shows. Consider the monoid M generated by a and b, subject to the relation $2a=2b$. It is integral but not saturated and M^{sat} is generated by a and $b-a$ inside M^{gp}; in other words, it is generated by two generators α and β subject to the relation $2\beta=0$ (the canonical morphism $M\to M^{\mathrm{sat}}$ sends a to α and b to $\alpha+\beta$). Then the log scheme $\mathrm{Spec}(\mathbb{C}[M])$ is $\mathrm{Spec}(\mathbb{C}[X,Y]/(X^2-Y^2))$ (a pair of intersecting lines), while the log scheme $\mathrm{Spec}(\mathbb{C}[M^{\mathrm{sat}}])$ is $\mathrm{Spec}(\mathbb{C}[X,Z]/(Z^2-1))$ (a disjoint union of two lines). In particular the categories of log connections are not equivalent. By contrast, the associated Kato–Nakayama spaces are isomorphic.

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