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Algebra

Characterizing type I C^*-algebras via entropy

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Abstract

Let A be a separable unital C^*-algebra. It is shown that A is type I if and only if the CNT-entropy of every inner automorphism of A is zero. ***To cite this article: N.P. Brown, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Charactérisation des algèbres de type I par l'entropie. Soit A une C^*-algèbre avec unité, séparable, nous montrons que A est de type I si et seulement si la CNT-entropie de tout automorphisme intérieur de A est nulle. ***Pour citer cet article : N.P. Brown, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

The theory of noncommutative entropy began in the context of noncommutative dynamical systems. However, in this short note we observe that entropy is also closely connected to the structure theory of C^*-algebras. Taking this point of view it will be useful to regard entropy as an invariant of an algebra, as opposed to an invariant of a dynamical system. Consider the following analogue of the topological entropy invariants discussed in [1, 6.4.1].

Definition 1.1. Let A be a unital C^*-algebra. Let $CNT_{\mathrm{Inn}}(A)$ denote the set of real numbers t such that there exists a unitary $u \in A$ and a state $\varphi \in S(A)$ such that $\varphi \circ \mathrm{Ad}\, u = \varphi$ and $h_\varphi(\mathrm{Ad}\, u) = t$, where $h_\varphi(\mathrm{Ad}\, u)$ denotes the CNT-entropy [2].

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In [4] it is shown that the CNT-entropy in a type I W^*-algebra can always be computed by looking at the restriction to the center. Since inner automorphisms act trivially on the center this implies that *if A is a type I C^*-algebra then $CNT_{\mathrm{Inn}}(A) = \{0\}$* (see [4, Cor. 7]). Hence this entropy invariant provides a natural dynamical obstruction to an algebra being type I. Our main result shows that this is the only obstruction, i.e. we observe the converse of Neshveyev and Størmer's result.

Theorem 1.2. *Let A be a separable unital C^*-algebra. Then A is type I if and only if $CNT_{\mathrm{Inn}}(A) = \{0\}$.*

Proof. We refer to [2] or [7] for all relevant definitions and the notation which follows.

In light of [4, Cor. 7], it suffices to show that if A is unital, separable and *not* type I then there exists a unitary $u \in A$ and a state $\varphi \in S(A)$ such that $\varphi \circ \mathrm{Ad}\, u = \varphi$ and $h_\varphi(\mathrm{Ad}\, u) > 0$. This will follow from Glimm's theorem and a few other well known results.

So assume A is unital, separable and *not* type I. Let $\mathcal{U} = \bigotimes_{n\in\mathbb{N}} M_n(\mathbb{C})$, $M_{2^\infty} = \bigotimes_{\mathbb{Z}} M_2(\mathbb{C})$ be the CAR algebra and $\gamma \in Aut(M_{2^\infty})$ be the noncommutative Bernoulli shift on M_{2^∞} arising from the map $\mathbb{Z} \to \mathbb{Z}$, $i \mapsto i+1$. By [8] we may regard the crossed product $M_{2^\infty} \rtimes_\gamma \mathbb{Z}$ as a unital subalgebra of $\mathcal{U}$. By Glimm's theorem [6, Thm. 6.7.3] we can find a unital subalgebra $\widetilde{B} \subset A$ and a surjective $*$-homomorphism $\pi : \widetilde{B} \to \mathcal{U}$. Let $B = \pi^{-1}(M_{2^\infty} \rtimes_\gamma \mathbb{Z}) \subset \widetilde{B} \subset A$. Since the unitary group of $\mathcal{U}$ is connected, any unitary in $\mathcal{U}$ lifts to a unitary in $\widetilde{B}$. So, letting $v \in M_{2^\infty} \rtimes_\gamma \mathbb{Z}$ be the implementing unitary, we can find a unitary $u \in B$ such that $\pi(u) = v$.

It should now be clear what to do: use the trace on $M_{2^\infty} \rtimes_\gamma \mathbb{Z}$ to construct an $\mathrm{Ad}\, u$-invariant state on A with positive entropy. This requires a bit of care, but is essentially straightforward.

Let τ be the unique tracial state on $M_{2^\infty} \rtimes_\gamma \mathbb{Z}$ and $(\pi_\tau, L^2(M_{2^\infty} \rtimes_\gamma \mathbb{Z}, \tau), \eta_\tau)$ be the associated GNS representation (here, η_τ denotes the canonical cyclic vector in $L^2(M_{2^\infty} \rtimes_\gamma \mathbb{Z}, \tau)$). Since $M_{2^\infty} \rtimes_\gamma \mathbb{Z}$ is nuclear, we can apply Arveson's extension theorem followed by a conditional expectation to construct a unital completely positive map $\Phi : A \to \pi_\tau(M_{2^\infty} \rtimes_\gamma \mathbb{Z})'' \subset B(L^2(M_{2^\infty} \rtimes_\gamma \mathbb{Z}, \tau))$ which is an extension of the map $\pi_\tau \circ \pi : B \to \pi_\tau(M_{2^\infty} \rtimes_\gamma \mathbb{Z})$. Define a state φ on A by

$$\varphi(a) = \langle \Phi(a)\eta_\tau, \eta_\tau \rangle.$$

Since Φ maps u to a unitary operator, u lies in the multiplicative domain of Φ (see [5, Exercise 4.2]). Because we have arranged that Φ takes values in $\pi_\tau(M_{2^\infty} \rtimes_\gamma \mathbb{Z})''$, this will imply that $\varphi \circ \mathrm{Ad}\, u = \varphi$. To see this we first observe that $\pi_\tau(v)T\pi_\tau(v)^* = U_\gamma T U_\gamma^*$, for all $T \in \pi_\tau(M_{2^\infty} \rtimes_\gamma \mathbb{Z})''$ where $U_\gamma : L^2(M_{2^\infty} \rtimes_\gamma \mathbb{Z}, \tau) \to L^2(M_{2^\infty} \rtimes_\gamma \mathbb{Z}, \tau)$ is the unitary operator defined by $U_\gamma(x) = vxv^*$. Hence we have

$$\begin{aligned} \varphi(uau^*) &= \langle \Phi(uau*)\eta_\tau, \eta_\tau \rangle \\ &= \langle \pi_\tau(v)\Phi(a)\pi_\tau(v)^*\eta_\tau, \eta_\tau \rangle \\ &= \langle U_\gamma \Phi(a) U_\gamma^* \eta_\tau, \eta_\tau \rangle \\ &= \varphi(a), \end{aligned}$$

for all $a \in A$.

To finish the proof we must observe that $h_\varphi(\mathrm{Ad}\, u) \geqslant h_\tau(\mathrm{Ad}\, v) = \log 2$. This almost follows immediately from the definitions. The only observation which needs to be made is that the CNT-entropy of the systems $(\pi_\tau(M_{2^\infty} \rtimes_\gamma \mathbb{Z}), \mathrm{Ad}\, \pi_\tau(v), \langle \cdot\eta_\tau, \eta_\tau \rangle)$ and $(\pi_\tau(M_{2^\infty} \rtimes_\gamma \mathbb{Z})'', \mathrm{Ad}\, \pi_\tau(v), \langle \cdot\eta_\tau, \eta_\tau \rangle)$ naturally agree. Hence when computing the CNT-entropy of the dynamical system $(\pi_\tau(M_{2^\infty} \rtimes_\gamma \mathbb{Z})'', \mathrm{Ad}\, \pi_\tau(v), \langle \cdot\eta_\tau, \eta_\tau \rangle)$ it suffices to consider unital completely positive maps from matrices into $\pi_\tau(M_{2^\infty} \rtimes_\gamma \mathbb{Z})$. But all of these maps lift to unital completely positive maps into $B \subset A$. Finally, it is clear that any Abelian model defined on $\pi_\tau(M_{2^\infty} \rtimes_\gamma \mathbb{Z})''$ can be lifted to an Abelian model on A via the map $\Phi : A \to \pi_\tau(M_{2^\infty} \rtimes_\gamma \mathbb{Z})''$. The desired inequality now follows easily from the definition of CNT-entropy. □

Corollary 1.3. *If A is a unital non-type I C^*-algebra, then $\infty \in TE_{\mathrm{Inn}}(A)$, where $TE_{\mathrm{Inn}}(\cdot)$ denotes the inner topological entropy invariant* [1, 6.4.1].

Proof. Considering crossed products by automorphisms with infinite CNT-entropy (for example $(\bigotimes_{\mathbb{N}} M_{2^\infty}, \bigotimes_{n\in\mathbb{N}} \gamma^n)$) it is clear that $\infty \in CNT_{\mathrm{Inn}}(A)$. The corollary then follows from [3, Prop. 9] in the case that A is exact or the remark that even the identity automorphism has infinite topological entropy when A is not exact. □

We conjecture that type I C^*-algebras are also completely determined by the topological entropy invariant $TE_{\mathrm{Inn}}(\cdot)$ (i.e. we conjecture that the topological analogue of [4, Cor. 7] is also true). There is a natural strategy for proving this by using a composition series to reduce to the case of continuous trace C^*-algebras. However, this would require understanding the behavior of topological entropy in extensions and this appears to be a highly non-trivial problem. Another approach would be to prove that in type I C^*-algebras one always has a CNT-variational principle and hence reduce the problem to [4, Cor. 7]. However, proving a CNT-variational principle for all type I C^*-algebras does not appear very easy either.

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