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Partial Differential Equations

Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains

Amandine Aftalion [a], Filomena Pacella [b]

[a] *Laboratoire Jacques-Louis Lions, B.C.187, université Paris 6, 175, rue du Chevaleret, 75013 Paris, France*
[b] *Dipartimento di Matematica, Università di Roma "La Sapienza", P.le A. Moro 2, 00185 Roma, Italy*

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Abstract

We study the qualitative properties of sign changing solutions of the Dirichlet problem $\Delta u + f(u) = 0$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a ball or an annulus and f is a C^1 function with $f(0) \geqslant 0$. We prove that any radial sign changing solution has a Morse index bigger or equal to $N + 1$ and give sufficient conditions for the nodal surface of a solution to intersect the boundary. In particular, we prove that any least energy nodal solution is non radial and its nodal surface touches the boundary. ***To cite this article: A. Aftalion, F. Pacella, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Propriétés qualitatives des solutions nodales pour des problèmes elliptiques semilinéaires dans des domaines à symétrie sphérique. Nous étudions les propriétés qualitatives des solutions qui changent de signe du problème de Dirichlet $\Delta u + f(u) = 0$ dans Ω, $u = 0$ sur $\partial\Omega$, où Ω est une boule ou un anneau et f une fonction C^1 avec $f(0) \geqslant 0$. Nous prouvons que toute solution radiale qui change de signe a un indice de Morse supérieur ou égal à $N + 1$ et donnons des conditions suffisantes pour que la surface nodale intersecte le bord. En particulier, nous prouvons que toute solution nodale d'énergie minimale est non radiale et sa surface nodale touche le bord. ***Pour citer cet article : A. Aftalion, F. Pacella, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Dans cette Note, nous étudions les solutions qui changent de signe de problèmes elliptiques semilinéaires :

$$\Delta u + f(u) = 0 \quad \text{dans } \Omega, \quad u = 0 \quad \text{sur } \partial\Omega \tag{1}$$

E-mail addresses: aftalion@ann.jussieu.fr (A. Aftalion), pacella@mat.uniroma1.it (F. Pacella).

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où Ω est un anneau ou une boule de $\mathbb{R}^N$, $N \geqslant 2$, $f:\mathbb{R}\to\mathbb{R}$ vérifie

$$f \in C^1(\mathbb{R}), \quad f(0) \geqslant 0. \tag{2}$$

Nous nous intéressons aux propriétés géométriques des régions nodales, en particulier savoir si la fermeture de l'ensemble où u s'annule

$$\mathcal{N} = \overline{\{x \in \Omega,\ u(x) = 0\}} \tag{3}$$

touche le bord du domaine. Dans la mesure où nous ne nous préoccupons pas de l'existence des solutions de (1), nous supposons uniquement (2) pour f. Cependant, l'existence de solutions est assurée par des conditions supplémentaires sur f. Par exemple, si f se comporte comme $f(s) = |s|^{p-2}s$ avec p souscritique, i.e. f vérifie

(H1) $|f'(s)| \leqslant C(1+|s|^{p-2})$ avec $p \in (2, \frac{2N}{N-2})$ si $N \geqslant 3$ et $p \in (2,\infty)$ si $N = 2$,
(H2) $f'(t) > \frac{f(t)}{t} > 0$, $t \in \mathbb{R}\setminus\{0\}$, $f(0) = 0$,
(H3) $\exists R > 0$, $\theta > 2$, tel que $0 < \theta F(t) \leqslant tf(t)$ pour $|t| \geqslant R$,
(H4) la seconde valeur propre de Dirichlet de $-\Delta - f'(0)$ dans Ω est positive,

alors il est prouvé dans [1,4] qu'il existe une solution de (1) qui change de signe. Cette solution a un indice de Morse égal à 2 et exactement 2 régions nodales. Ce résultat est obtenu en étudiant l'énergie associée

$$J(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2\,\mathrm{d}x - \int_{\Omega} F(u)\,\mathrm{d}x, \tag{4}$$

dans $H_0^1(\Omega)$, où $F(t) = \int_0^t f(s)\,\mathrm{d}s$; les points critiques de J correspondent aux solutions de (5). De plus, les solutions trouvées dans [1] minimisent J sur l'ensemble

$$M = \{u \in H_0^1(\Omega),\ u^+ \neq 0,\ u^- \neq 0,\ J'(u)u^+ = J'(u)u^- = 0\}$$

et sont appelées *solutions nodales d'énergie minimale*. D'autres hypothèses sur f, comme l'imparité, permettent de trouver des solutions qui changent de signe, et qui ont plus de régions nodales [1]. Pour d'autres résultats concernant l'existence de solutions, nous renvoyons aux références citées dans [1].

L'étude des propriétés qualitatives des solutions nodales a été commencée dans [1,4] pour ce qui est de l'indice de Morse et du nombre de régions nodales, comme nous l'avons rappelé plus haut. De plus, dans [2], des résultats partiels de symétrie dans des domaines à symétrie sphérique ont été obtenus pour la solution d'énergie minimale. D'autres questions très intéressantes sont toujours ouvertes. Dans le cas de la seconde fonction propre du Laplacien, une question importante est de comprendre la géometrie de la surface nodale $\mathcal{N}$. Si u est une solution nodale de (1) d'énergie minimale, une question ouverte très intéressante consiste à prouver que $\mathcal{N}$ intersecte le bord du domaine. Pour l'instant, celà n'a été prouvé que dans le cas particulier d'une équation aux perturbations singulières dans la boule [6] et il y a des simulations numériques en dimension 2 qui confirement celà [5]. Une question ouverte moins générale est de prouver que la solution nodale d'énergie minimale dans la boule ou l'anneau n'est pas radiale. A nouveau, des exemples indiquent que ce résultat est attendu [2,3]. Il serait également intéressant d'avoir une borne inférieure sur l'indice de Morse des solutions radiales de (1) puisqu'il est naturel d'attendre que ces solutions radiales aient un indice de Morse plus élevé.

Dans cette Note, nous donnons une réponse positive à ces questions quand Ω est une boule ou un anneau, (ou un domaine plus général convexe et symétrique dans plusieurs directions), et f est une nonlinéarité vérifiant uniquement (2). Plus précisément, nous prouvons

Théorème 0.1. *Sous l'hypothèse* (2)*, si Ω est une boule ou un anneau, toute solution radiale de* (1) *qui change de signe a un indice de Morse supérieur ou égal à $N+1$.*

En particulier, toute solution d'indice 2 dans la boule ou l'anneau n'est pas radiale. Nous nous intéressons maintenant aux propriétés de $\mathcal{N}$:

Théorème 0.2. *Sous l'hypothèse* (2)*, si Ω est une boule, et u une solution qui change de signe, paire en k variables, et d'indice de Morse inférieur ou égal à k, alors l'ensemble $\mathcal{N}$ intersecte le bord de Ω.*

Il est facile de voir que le Théorème 0.2 se généralise à un domaine symétrique et convexe par rapport à k directions orthogonales. Ce théorème permet d'obtenir des propriétés plus précises sur les solutions d'énergie minimale construites dans [1,2] :

Théorème 0.3. *Sous les hypothèses* (H1)–(H4)*, une solution nodale d'énergie minimale de* (1) *n'est pas radiale. De plus, si Ω est une boule, ou $N = 2$ et Ω est un anneau, alors l'ensemble $\mathcal{N}$ intersecte le bord de Ω.*

1. Introduction

In this Note, we study the sign changing solutions of semilinear elliptic problems of the type

$$\Delta u + f(u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{5}$$

where Ω is either an annulus or a ball in $\mathbb{R}^N$, $N \geqslant 2$, $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f \in C^1(\mathbb{R}), \qquad f(0) \geqslant 0. \tag{6}$$

We are interested in geometrical properties of the nodal regions, in particular to know that the closure of the set where u vanishes

$$\mathcal{N} = \overline{\{x \in \Omega,\ u(x) = 0\}} \tag{7}$$

touches the boundary. Since we do not worry about the existence of solutions of (5), we only assume the hypothesis (6) on f. However, it is well-known that existence is insured under additional conditions on f. An interesting case is when f is superlinear. In particular, if f behaves like the model nonlinearity $f(s) = |s|^{p-2}s$ with p subcritical, that is, f satisfies

(H1) $|f'(s)| \leqslant C(1 + |s|^{p-2})$ with $p \in (2, \frac{2N}{N-2})$ if $N \geqslant 3$ and $p \in (2, \infty)$ if $N = 2$,
(H2) $f'(t) > \frac{f(t)}{t} > 0$, $t \in \mathbb{R} \setminus \{0\}$, $f(0) = 0$,
(H3) $\exists R > 0$, $\theta > 2$, such that $0 < \theta F(t) \leqslant t f(t)$ for $|t| \geqslant R$,
(H4) the second Dirichlet eigenvalue of $-\Delta - f'(0)$ in Ω is positive,

it is proved in [1,4] that there exists a sign changing solution of (5) with Morse index 2 and precisely two nodal regions. This is done by studying the associated energy functional

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega F(u) \, dx, \tag{8}$$

in the space $H_0^1(\Omega)$, where $F(t) = \int_0^t f(s)\, ds$, since the critical points of J correspond to the solutions of (5). Moreover, the solutions found in [1] minimize the functional J on the set

$$M = \{u \in H_0^1(\Omega),\ u^+ \neq 0,\ u^- \neq 0,\ J'(u)u^+ = J'(u)u^- = 0\}$$

and are therefore called the least energy nodal solution. Adding other hypotheses, such as f is odd, it is possible to find other sign changing solutions with more nodal regions [1]. Several other results concerning the existence of solutions are available and we refer the reader to the references cited in [1] for more details.

Regarding the study of qualitative properties of the nodal solutions, some results have been achieved in [1,4] on the Morse index and the number of nodal regions as we recalled above. Moreover, in [2], some partial symmetry results in radially symmetric domains have been obtained for the least energy solution. Many other interesting questions are still open. As for the case of the second eigenfunction of the Laplacian, one main question is to understand the geometry of the nodal surface $\mathcal{N}$. In particular, if u is a least energy nodal solution of (5), an interesting open question is to prove that the set $\mathcal{N}$ intersects the boundary. As far as we know, this has been proved only in a particular case concerning a singularly perturbed elliptic equation in a ball [6] and there is a numerical evidence, in dimension 2, that the answer should be positive [5]. A less general open question related to this, is to prove that a least energy nodal solution in a ball, or in an annulus, is not radial. Again there are examples which show that this result should hold [2,3]. More generally it would be important to have a lower bound on the Morse index of radial solutions of (5). Indeed it is natural to expect that radial solutions would have high Morse index. In this Note, with surprisingly simple proofs, we give an answer to this questions in the case when Ω is a ball or an annulus (or more general domains convex and symmetric in several directions) and f is a nonlinearity satisfying only (6). More precisely, we have

Theorem 1.1. *Under the hypothesis* (6)*, if Ω is a ball or an annulus, any radial sign changing solution of* (5) *has Morse index greater than or equal to $N+1$.*

In particular, any index 2 solution in the ball or the annulus is not radial. Now we are interested in the properties of the closure of the nodal set of u, namely

Theorem 1.2. *Under the hypothesis* (6)*, if Ω is a ball, and if u is a sign changing solution which is even in k variables, with Morse index less than or equal to k, then the set $\mathcal{N}$ intersects the boundary of Ω.*

It is easy to see that Theorem 1.2 generalizes to a domain symmetric and convex with respect to k orthogonal directions. This Theorem allows to get more specific properties on the least energy solutions obtained in [1,2].

Theorem 1.3. *If hypotheses* (H1)–(H4) *hold, then a least energy nodal solution of* (5) *is not radial. Moreover if Ω is a ball, or $N=2$ and Ω is an annulus, then the set $\mathcal{N}$ intersects the boundary.*

A natural question, once we know that these least energy solution have their nodal surface $\mathcal{N}$ which touches the boundary, is to prove that they are in fact antisymmetric. This question has been studied for a specific equation with a small diffusion parameter very recently by [6]. We think that the method that we use could provide new results to similar questions in more general domains. Further investigation in these directions is in progress.

2. Proofs

For a fixed sign changing solution u of (5), we denote by L the linearized operator, $L=-\Delta-f'(u)$, and by λ_k the eigenvalues of L in Ω with homogeneous Dirichlet boundary conditions. Moreover, we consider the hyperplanes $T_i=\{x=(x_1,\dots,x_N)\in\Omega,\ x_i=0\}$ and the domains

$$\Omega_i^-=\{x=(x_1,\dots,x_N)\in\Omega,\ x_i<0\},\qquad \Omega_i^+=\{x=(x_1,\dots,x_N)\in\Omega,\ x_i>0\},$$

for $i=1,\dots,N$. Then we denote by μ_i the first eigenvalue of L is Ω_i^-, that is μ_i is such that there exists a function ψ_i solution of

$$-\Delta\psi_i-f'(u)\psi_i=\mu_i\psi_i\quad \text{in } \Omega_i^-,\quad \psi_i=0\quad \text{on } \partial\Omega_i^-,\quad \psi_i>0\quad \text{in } \Omega_i^-. \tag{9}$$

Our proofs are based on the study of the sign of μ_i when u is symmetric with respect to T_i. First let us point out the following properties:

Proposition 2.1. *If u is a solution of* (5) *even in the x_i variable for some i, then the odd extension of ψ_i to Ω, defined by $\tilde{\psi}_i(x) = \psi_i(x)$ if $x \in \Omega_i^-$ and if $x = (x_1, \dots, x_N) \in \Omega_i^+$,*

$$\tilde{\psi}_i(x_1, \dots, x_N) = -\psi_i(x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_N),$$

is an eigenfunction for the linearized operator L in Ω with corresponding eigenvalue μ_i. Hence $\mu_i = \lambda_{\beta(i)}$, with $\beta(i) \geqslant 2$.

If u is even in k variables, $x_1, \dots, x_k$, $1 \leqslant k \leqslant N$, then the eigenfunctions $\tilde{\psi}_1, \dots, \tilde{\psi}_k$ are linearly independent so that they give rise to k eigenvalues $\lambda_{\beta(1)}, \dots, \lambda_{\beta(k)}$ of the operator L in Ω.

Proof. Since ψ_i vanishes on the hyperplane T_i, we have

$$\frac{\partial \psi_i}{\partial x_j} = \frac{\partial^2 \psi_i}{\partial x_j^2} = 0 \quad \text{on } T_i \cap \Omega \quad \text{for } j \neq i, \qquad \frac{\partial^2 \psi_i}{\partial x_i^2} = 0 \quad \text{on } T_i \cap \Omega.$$

The last equality comes from Eq. (9). Hence by the symmetry of u in the x_i variable, it follows easily that the extension $\tilde{\psi}_i$ are Dirichlet eigenfunctions of L in Ω with corresponding eigenvalue $\mu_i = \lambda_{\beta(i)}$. Since $\tilde{\psi}_i$ changes sign in Ω, then $\lambda_{\beta(i)} \geqslant 2$. The second part follows from the oddness of $\tilde{\psi}_i$ in the x_i variable. □

Proof of Theorem 1.1. Since u is radial, u is even in each variable x_i, $i = 1, \dots, N$. Hence by Proposition 2.1, we have $\mu_i = \lambda_{\beta(i)}$, for each i. If we prove that $\mu_i < 0$ for each i, then since $\lambda_1 < \lambda_{\beta(i)} < 0$, this provides $N+1$ negative eigenvalues, hence the assertion.

Let us consider the function $\frac{\partial u}{\partial x_i}$ in Ω_i^-. By the symmetry of u in the x_i variable, we have that $\frac{\partial u}{\partial x_i} = 0$ on $\partial\Omega_i^- \cap T_i$. On the other hand, since u is radial, it does not change sign near $\partial\Omega$. We have to analyze separately the case of the ball and the annulus. In the case of the ball, we can assume that $u > 0$ near the boundary. Then, by (6), we can apply the Hopf Lemma to (5) and deduce that

$$\frac{\partial u}{\partial x_i} > 0 \quad \text{on } (\partial\Omega_i^- \cap \partial\Omega) \setminus T_i. \tag{10}$$

If instead Ω is an annulus, and u has two nodal regions, we can assume that $u > 0$ near the outer boundary while $u < 0$ near the inner boundary. Again by the Hopf Lemma, we get (10). Thus, in any case, $\frac{\partial u}{\partial x_i}$ satisfies

$$-\Delta\left(\frac{\partial u}{\partial x_i}\right) - f'(u)\frac{\partial u}{\partial x_i} = 0 \quad \text{in } \Omega_i^-, \qquad \frac{\partial u}{\partial x_i} \geqslant 0 \quad \text{on } \partial\Omega_i^-. \tag{11}$$

Since u changes sign in Ω_i^-, necessarily $\frac{\partial u}{\partial x_i}$ must change sign and be negative somewhere in Ω_i^-. Hence, there is a region D strictly included in Ω_i^- such that

$$\frac{\partial u}{\partial x_i} < 0 \quad \text{in } D \quad \text{and} \quad \frac{\partial u}{\partial x_i} = 0 \quad \text{on } \partial D. \tag{12}$$

We deduce from (11), (12) that the first eigenvalue of L in D is zero, and as a consequence the first eigenvalue μ_i in Ω_i^- is negative, as we wanted to prove.

If Ω is an annulus and u has more than two nodal regions, we take a smaller annulus, denoted by A, given by two nodal regions of u. We observe that the restriction v of u to the annulus A is a nodal radial solution of (5) in A with only two nodal regions. Applying what we have just proved, we get that the linearized operator, restricted to A, has at least $N+1$ negative eigenvalues. Hence the same holds for L in the whole domain Ω. □

Proof of Theorem 1.2. Assume that u is even in the variables $x_1, \dots, x_k$, for $1 \leqslant k \leqslant N$. Then by Proposition 2.1, each first Dirichlet eigenvalue μ_i of L in Ω_i^-, $i = 1, \dots, k$, gives rise to an eigenvalue $\lambda_{\beta(i)}$ of L in Ω. Assume by contradiction that $\mathcal{N}$ does not intersect $\partial\Omega$. This implies that u has a constant sign near $\partial\Omega$. We can argue as in the proof of Theorem 1.1 to deduce that $\lambda_{\beta(i)} = \mu_i < 0$, for $i = 1, \ldots, k$. Moreover, we also have that $\lambda_1 < 0$, so that the solution has Morse index greater than or equal to $k + 1$. This contradiction to the hypothesis of the Morse index less than or equal to k proves that in fact $\mathcal{N}$ intersects $\partial\Omega$. □

Proof of Theorem 1.3. This uses the previous theorems and properties of the least energy solutions proved in [1,2]. By [1], we know that a least energy solution has Morse index 2 and 2 nodal regions. Hence by Theorem 1.1, it cannot be radial if Ω is either a ball or an annulus.

Recall from [2] that a least energy solution is axially symmetric with respect to an axis containing a point P where u achieves its maximum. This implies that u is even with respect to $N - 1$ variables. Since u has Morse index 2, if Ω is a ball in $\mathbb{R}^N$, by Theorem 1.2, the closure of the nodal domain must intersect the boundary if $N \geqslant 3$.

If $N = 2$, we cannot deduce directly from Theorem 1.2 that the closure of the nodal domains touches the boundary since a priori u is only even in x_1. If Ω is a ball or an annulus, assume that u achieves its maximum at a point P on the x_2 axis where $x_2 \geqslant 0$. First, let us point out that $x_2 > 0$, otherwise, u would be radial, which has been excluded already.

Writing in polar coordinates, we have that $\frac{\partial u}{\partial \theta}$ is a solution of $L(\frac{\partial u}{\partial \theta}) = 0$ in Ω. Moreover we claim that

$$\frac{\partial u}{\partial \theta} < 0 \quad \text{in } \Omega_1^-, \quad \frac{\partial u}{\partial \theta} > 0 \quad \text{in } \Omega_1^+ \quad \text{and} \quad \frac{\partial u}{\partial \theta} = 0 \quad \text{on } T_1.$$

Indeed, let us assume that $\frac{\partial u}{\partial \theta}$ changes sign in Ω_1^-. Then by symmetry, since $\frac{\partial u}{\partial \theta} = 0$ on T_1, $\frac{\partial u}{\partial \theta}$ has at least 4 nodal regions, hence is an eigenfunction for a λ_k with $k \geqslant 4$. This implies that λ_1, λ_2 and λ_3 are negative, which contradicts the hypothesis of index 2. Thus, $\frac{\partial u}{\partial \theta}$ does not change sign in Ω_1^- and is the first eigenfunction of L in Ω_1^-, which implies that $\mu_1 = 0$. If the nodal set $\mathcal{N}$ does not intersect $\partial\Omega$, since u is even in x_1, the same argument as in the proof of Theorem 1.1, yields $\mu_1 < 0$. Thus, we deduce that $\mathcal{N}$ intersects $\partial\Omega$. □

Remark 1. In Theorem 1.3, we have proved that the least energy solution u is not radial. In dimension 2, we know by [2] that u is even in one variable, x_1 for instance. As a consequence of the result that the nodal line touches the boundary, we can exclude that u is even in both variables if Ω is a ball: indeed if this was the case, the nodal line should touch the boundary in 2 points on the same symmetry axis, the x_2 axis, and coincide with the segment T_1, otherwise u would have more than 2 nodal regions. But this is impossible if u is even in the x_1 direction.

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