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On the ample vector bundles over curves in positive characteristic

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Abstract

Let E be an ample vector bundle over a smooth projective curve defined over an algebraically closed field of positive characteristic. We construct a family of curves in the total space of E, parametrized by an affine space, that surjects onto the total space of E and give a deformation of (nonreduced) zero section of E. ***To cite this article: I. Biswas, A.J. Parameswaran, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

À propos des fibrés vectoriels amples sur les courbes en caractéristique positive. Soit E un fibré vectoriel ample sur une courbe projective et lisse définie sur un corps algébriquement clos de caractéristique positive. Nous construisons une famille de courbes dans l'espace total de E, paramétrisée par un espace affine, qui domine l'espace total de E et qui fournit une déformation de la section nulle (non réduite) du fibré E. ***Pour citer cet article : I. Biswas, A.J. Parameswaran, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

We begin by recalling a theorem proved in [1]. Let E be an ample vector bundle of rank two over a smooth projective curve X defined over the field of complex numbers. Then there is an integer k_0 and an analytic family of curves $\{C_t\}_{t\in T}$ in the total space of E, parametrized by an irreducible variety T, such that the family dominates the total space of E and there is a base point $t_0 \in T$ with $C_{t_0} = k_0 0_X$, where 0_X is the zero section of E. (See [1, Theorem 1.1].)

Recently Langer has proved the following theorem. Let Y be a smooth projective variety over an algebraically closed field of positive characteristic. Let F_Y denote the Frobenius morphism of Y. For any vector bundle V over Y,

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there is an integer k_0 such that the k_0-fold iterated pullback $(F_Y^{k_0})^*V$ has the property that each subsequent quotient of the Harder–Narasimhan filtration of $(F_Y^{k_0})^*V$ is strongly semistable (see [3, Theorem 2.7]).

Our aim here is to show that the positive characteristic version of the earlier mentioned result of [1] can be deduced from the above result of [3]. In fact, the condition $\mathrm{rank}(E) = 2$ in [1, Theorem 1.1] can be removed in the positive characteristic version.

Let X be a smooth projective curve over a field k of positive characteristic and E an ample vector bundle over X. In Theorem 2.2 we prove that there is an integer n_0 such that the vector bundle $(F_X^{n_0})^*E$ is generated by its global sections, where F_X as before is the Frobenius morphism of X.

The family of curves, parametrized by $H^0(X, (F_X^{n_0})^*E)$, in the total space of E surjects onto the total space of E and give a deformation of (nonreduced) zero section of E (Corollary 2.3).

2. Pullback of ample bundle

Let k be an algebraically closed field of characteristic $p > 0$. Let X be an irreducible smooth projective curve over k. Let

$$F_X : X \longrightarrow X$$

be the Frobenius morphism of X. For any $m \geqslant 1$, by F_X^m we will mean the m-fold composition of F_X, and F_X^0 will denote the identity morphism of X.

We recall that a vector bundle E over X is called *strongly semistable* if $(F_X^m)^*E$ is semistable for all $m \in \mathbb{N}$.

The following proposition is proved using Theorem 2.7 of [3].

Proposition 2.1. *Let E be a vector bundle over X. There is $n \in \mathbb{N}$ such that*

$$(F_X^n)^*E \cong \bigoplus_{i=1}^{l} W_i,$$

where each W_i, $i \in [1, l]$, is a strongly semistable vector bundle over X.

Proof. Theorem 2.7 of [3] says that there is $k_0 \in \mathbb{N}$ such that the Harder–Narasimhan filtration

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_l = (F_X^{k_0})^*E$$

of $(F_X^{k_0})^*E$ has the property that each subsequent quotient V_j/V_{j-1}, $j \in [1, l]$, is strongly semistable (see [3, §2.6] for the definition of fdHN in [3, Theorem 2.7]).

For any $j \in [1, l]$ define $\mu_j := \mathrm{degree}(V_j/V_{j-1})/\mathrm{rank}(V_j/V_{j-1})$, and set μ to be the minimum of the $l - 1$ positive numbers $\{\mu_j - \mu_{j+1}\}_{j=1}^{l-1}$.

Take $k_1 \in \mathbb{N}$ such that $\mu \cdot k_1 \cdot p \geqslant 2g_X$, where g_X is the genus of X and p is the characteristic of k. Set $n = k_0 k_1$. We will show that this n satisfies the condition the proposition.

Since each V_j/V_{j-1}, $j \in [1, l]$, is strongly semistable, the filtration

$$0 \subset (F_X^{k_1})^*V_1 \subset (F_X^{k_1})^*V_2 \subset \cdots \subset (F_X^{k_1})^*V_l = (F_X^n)^*E \tag{1}$$

coincides with the Harder–Narasimhan filtration of $(F_X^n)^*E$.

Since $\mu \cdot k_1 \cdot p \geqslant 2g_X$, we have

$$\frac{\mathrm{degree}((F_X^{k_1})^*V_j/(F_X^{k_1})^*V_{j-1})}{\mathrm{rank}((F_X^{k_1})^*V_j/(F_X^{k_1})^*V_{j-1})} - \frac{\mathrm{degree}((F_X^{k_1})^*V_{j+1}/(F_X^{k_1})^*V_j)}{\mathrm{rank}((F_X^{k_1})^*V_{j+1}/(F_X^{k_1})^*V_j)} \geqslant pk_1\mu \geqslant 2g_X$$

for all $j \in [1, l-1]$. On the other hand, if U_1 and U_2 are two strongly semistable vector bundles with $\mathrm{degree}(U_1)/\mathrm{rank}(U_1) - \mathrm{degree}(U_2)/\mathrm{rank}(U_2) > 2(g_X - 1)$, then $\mathrm{Hom}(U_1, U_2)$ is semistable [4, Theorem 3.23], and $\mathrm{degree}(\mathrm{Hom}(U_1, U_2) \otimes K_X) < 0$, where K_X is the canonical line bundle; therefore,

$$H^0\big(X, \mathrm{Hom}(U_1, U_2) \otimes K_X\big) = 0$$

which in turn implies that $H^1(X, \mathrm{Hom}(U_2, U_1)) = 0$ (Serre duality). In other words, there is no nontrivial extension of U_2 by U_1.

These immediately imply that the filtration in (1) splits completely; first $(F_X^{k_1})^* V_2$ splits as $(F_X^{k_1})^* V_1 \oplus ((F_X^{k_1})^* V_2/(F_X^{k_1})^* V_1)$, and then, by induction, up to $(F_X^{k_1})^* V_{j+1}$ splits completely given that up to $(F_X^{k_1})^* V_j$ splits completely. In other words,

$$(F_X^n)^* E \cong \bigoplus_{j=1}^{l} \frac{(F_X^{k_1})^* V_j}{(F_X^{k_1})^* V_{j-1}}.$$

This completes the proof of the proposition. □

We recall that a vector bundle E over X is called *ample* if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$ is ample (see [2, Chapter III, §1] for various equivalent formulations of amplitude).

Let E be an ample vector bundle over X. Take n as in Proposition 2.1 such that

$$(F_X^n)^* E \cong \bigoplus_{i=1}^{l} W_i \tag{2}$$

with each W_i strongly semistable. Since E is ample, the pullback $(F_X^n)^* E$ is ample [2, page 84, Proposition 1.6]. Therefore, each W_i is ample [2, p. 84, Proposition 1.7]. In particular, we have $\mathrm{degree}(W_i) > 0$ for all $i \in [1, l]$.

Set ν to be the minimum of the l positive numbers $\{\mathrm{degree}(W_i)/\mathrm{rank}(W_i)\}_{i=1}^{l}$. Take $k' \in \mathbb{N}$ such that $k'\nu p \geqslant 2g_X$. Set $n_0 = nk'$, where n is as in Proposition 2.1.

Theorem 2.2. *The vector bundle $(F_X^{n_0})^* E$ is globally generated* (*i.e., it is generated by global sections*)*, where n_0 is defined above.*

Proof. From (2) we have

$$(F_X^{n_0})^* E \cong \bigoplus_{i=1}^{l} (F_X^{k'})^* W_i.$$

So it suffices to show that each $(F_X^{k'})^* W_i$ is globally generated.

The vector bundle $(F_X^{k'})^* W_i$ is strongly semistable as W_i is so. Also,

$$\frac{\mathrm{degree}((F_X^{k'})^* W_i)}{\mathrm{rank}((F_X^{k'})^* W_i)} = k'p \frac{\mathrm{degree}(W_i)}{\mathrm{rank}(W_i)} \geqslant k'p\nu > 2g_X - 1.$$

Therefore, we have

$$\frac{\mathrm{degree}(\mathcal{O}_X(-x) \otimes_{\mathcal{O}_X} (F_X^{k'})^* W_i)}{\mathrm{rank}(\mathcal{O}_X(-x) \otimes_{\mathcal{O}_X} (F_X^{k'})^* W_i)} = \frac{\mathrm{degree}((F_X^{k'})^* W_i)}{\mathrm{rank}((F_X^{k'})^* W_i)} - 1 > 2g_X - 2$$

for each closed point $x \in X$. Consequently, we have

$$H^0\big(X, \big(\mathcal{O}_X(-x) \otimes_{\mathcal{O}_X} (F_X^{k'})^* W_i\big)^\vee \otimes K_X\big) = 0$$

(as $(\mathcal{O}_X(-x) \otimes_{\mathcal{O}_X} (F_X^{k'})^* W_i)^\vee \otimes K_X$ is semistable of negative degree). Now Serre duality gives

$$H^1\big(X, \mathcal{O}_X(-x) \otimes_{\mathcal{O}_X} (F_X^{k'})^* W_i\big) = 0.$$

Therefore, using the long exact sequence of cohomologies for the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(-x) \otimes_{\mathcal{O}_X} (F_X^{k'})^* W_i \longrightarrow (F_X^{k'})^* W_i \longrightarrow \big((F_X^{k'})^* W_i\big)_x \longrightarrow 0$$

we conclude that the vector bundle $(F_X^{k'})^* W_i$ is globally generated. This completes the proof of the theorem. □

There is a natural map of total spaces of vector bundles

$$\psi : (F_X^{n_0})^* E \longrightarrow E$$

which projects to the self-map $F_X^{n_0}$ of X. This map ψ is clearly surjective. For a section $s \in H^0(X, (F_X^{n_0})^* E)$, let $Z(s) \subset (F_X^{n_0})^* E$ be the curve in the total space of $(F_X^{n_0})^* E$ defined by the image of s. So $\psi(Z(s))$ is a curve in the total space of E.

Consider $H^0(X, (F_X^{n_0})^* E) \times X$ as a (trivial) family of curves parametrized by the affine space $H^0(X, (F_X^{n_0})^* E)$. Theorem 2.2 says that the natural map

$$H^0\big(X, (F_X^{n_0})^* E\big) \times X \longrightarrow (F_X^{n_0})^* E$$

defined by $(s, x) \mapsto s(x)$ is surjective. Therefore, Theorem 2.2 has the following corollary.

Corollary 2.3. *Let E be an ample vector bundle over the curve X. Then there is a family of curves in the total space of E, parametrized by $H^0(X, (F_X^{n_0})^* E)$, such that the family surjects onto the total space of E. The curve in the total space of E associated to $0 \in H^0(X, (F_X^{n_0})^* E)$ is nonreduced and the corresponding reduced curve coincides with the image of the zero section of E.*

For $k = \mathbb{C}$, Corollary 2.3 was proved in [1] under the assumption that $\operatorname{rank}(E) = 2$ (see [1, Theorem 1.1]).

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