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The zero-one law for a complete orthonormal system

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Abstract

A complete orthonormal system of functions $\Theta = \{\theta_n\}_{n=1}^{\infty}$, $\theta_n \in L^{\infty}_{[0,1]}$ is constructed such that $\sum_{n=1}^{\infty} a_n\theta_n$ converges almost everywhere on [0, 1] if $\{a_n\}_{n=1}^{\infty} \in l^2$ and $\sum_{n=1}^{\infty} a_n\theta_n$ diverges a.e. for any $\{a_n\}_{n=1}^{\infty} \notin l^2$. We also show that for any complete ONS $\{f_n\}_{n=1}^{\infty}$ of functions defined on [0, 1] there exists a fixed non decreasing subsequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that for any $f \in L^0_{[0,1]}$ and some sequence of coefficients $\{b_n\}_{n=1}^{\infty}$,

$$\sum_{n=1}^{n_k} b_n f_n \to f \quad \text{a.e. when } k \to \infty.$$

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Résumé

La loi zéro-un pour un système orthonormal complet. On construit un système orthonormal complet $\Theta = \{\theta_n\}_{n=1}^{\infty}$, $\theta_n \in L^{\infty}_{[0,1]}$ tel que $\sum_{n=1}^{\infty} a_n\theta_n$ converge presque partout pour n'importe quel $\{a_n\}_{n=1}^{\infty} \in l^2$ et diverge presque partout pour n'importe quel $\{a_n\}_{n=1}^{\infty} \notin l^2$. Nous démontrons que pour toute système orthonormal complet $\{f_n\}_{n=1}^{\infty}$ il existe une sous suite croissante $\{n_k\}_{k=1}^{\infty}$ d'entiers naturels tels que pour tout $f \in L^0_{[0,1]}$ il existe une suite de coefficients tels que

$$\sum_{n=1}^{N_k} b_n f_n \to f \quad \text{p.p. si } k \to \infty.$$

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Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal system (ONS) of functions defined on the closed interval $[a,b]$ then we will say that $\{\varphi_n\}_{n=1}^{\infty}$ is a divergence system if the series $\sum_{n=1}^{\infty} a_n\varphi_n$ diverges almost everywhere (a.e.) on $[a,b]$ for any $\{a_n\}_{n=1}^{\infty} \notin l^2$. An ONS $\{\varphi_n\}_{n=1}^{\infty}$ is called a convergence system if $\sum_{n=1}^{\infty} a_n\varphi_n$ converges a.e. for any $\{a_n\}_{n=1}^{\infty} \in l^2$. The given ONS will be called a divergence system in the weak sense if for any $\{a_n\}_{n=1}^{\infty} \notin l^2$ the series $\sum_{n=1}^{\infty} a_n\varphi_n$ diverges on a set of positive measure. In a recent work [6,7] the author has constructed a complete ONS of functions which is a divergence system. A complete ONS which is a divergence system in the weak sense was constructed earlier by Kashin [3,5]. The interest of existence of such systems was indicated by Ulyanov [13], p. 695, who has formulated the following problem: does there exist a complete ONS $\{\varphi_n\}_{n=1}^{\infty}$ of functions defined on the closed interval $[0,1]$ such that $\sum_{n=1}^{\infty} a_n\varphi_n$ converges a.e. for any $\{a_n\}_{n=1}^{\infty} \in l^2$ and $\sum_{n=1}^{\infty} a_n\varphi_n$ diverges a.e. if $\{a_n\}_{n=1}^{\infty} \notin l^2$?

We give a positive answer to Ulyanov's problem. We will say that an ONS $\{\varphi_n\}_{n=1}^{\infty}$ is a *simple* ONS if $\sum_{n=1}^{\infty} a_n\varphi_n$ converges a.e. for any $\{a_n\}_{n=1}^{\infty} \in l^2$ and $\sum_{n=1}^{\infty} a_n\varphi_n$ diverges a.e. for any $\{a_n\}_{n=1}^{\infty} \notin l^2$.

Note that Kashin has proved [3,4] that there exists a complete ONS $\{\psi_n\}_{n=1}^{\infty}$ of functions defined on the closed interval $[0,1]$ such that $\sum_{n=1}^{\infty} a_n\psi_n$ converges a.e. for any $\{a_n\}_{n=1}^{\infty} \in l^2$ and $\sum_{n=1}^{\infty} a_n\psi_n$ diverges on some set of positive measure if $\{a_n\}_{n=1}^{\infty} \notin l^2$. He indicated in [3] that Ulyanov's problem is left open. We prove

Theorem 1. *There exists a complete ONS $\Theta = \{\theta_n\}_{n=1}^{\infty}, \theta_n \in L_{[0,1]}^{\infty}$ such that $\sum_{n=1}^{\infty} a_n\theta_n$ converges almost everywhere on $[0,1]$ if $\{a_n\}_{n=1}^{\infty} \in l^2$ and $\sum_{n=1}^{\infty} a_n\theta_n$ diverges a.e. for any $\{a_n\}_{n=1}^{\infty} \notin l^2$.*

One of the principal problems of the theory of orthogonal series is to describe the class of coefficients for which a given orthonormal system converges in a determinate sense. The constructed system $\{\theta_n\}_{n=1}^{\infty}$ has the following property: a series with respect to our system converges on a set of positive measure if and only if the coefficients belong to l^2, moreover if the series converges on a set of positive measure then it converges a.e. Observe that up to now the mentioned above property was known only for some lacunary orthonormal systems.

In [6,7] we have mentioned the role of the divergence system in the theory of representation of functions by series. Evidently the simple ONS as a system of divergence has the same properties. Moreover it shows the difference that exists between the representation in measure or in some other topology (see [9,8,14]) from one side and the representation in the sense of convergence almost everywhere or on some fixed subset of positive measure from the other side.

Recall that a system of functions $\{f_n\}_1^{\infty}$ defined on $[0,1]$ is called an m-representation system of the space $\mathbf{F}$ if, for every $f \in \mathbf{F}$, there exists a series $\sum_1^{\infty} a_n f_n$, $a_n \in \mathbb{R}$, that converges in measure to the function f. An F-space of functions defined on $[0,1]$ is called an R-space if any m-representation system of the space $\mathbf{F}$ is an m-representation system of the space $L_{[0,1]}^0$. A wide class of F-spaces of functions, including the spaces $L_{[0,1]}^p$, $0 < p < \infty$, are R-spaces (see [9]).

The existence of a simple ONS shows that for the pointwise convergence on a fixed subset of positive measure the described above phenomenon is not true. Combining some theorems of Pogosyan–Arutyunyan [1,12], Bourgain [2] and Marcinkiewicz–Menshov [10,11] we show that for any complete ONS the following result is true.

Theorem 2. *Let $\{f_n\}_{n=1}^{\infty}$ be a complete ONS of functions defined on $[0,1]$. Then there exists a non decreasing subsequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that for any measurable finite a.e. function defined on $[0,1]$ and some sequence of coefficients $\{b_n\}_{n=1}^{\infty}$*

$$\sum_{n=1}^{n_k} b_n f_n(x) \to f(x) \quad \text{a.e. when } k \to \infty.$$

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