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Killing tensors as irreducible representations of the general linear group[☆]

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Abstract

We show that the vector space of fixed valence Killing tensors on a space of constant curvature is naturally isomorphic to a certain highest weight, irreducible representation of the general linear group. The isomorphism is equivariant in the sense that the natural action of the isometry group corresponds to the restriction of the linear action to the appropriate subgroup. As an application, we deduce the Delong–Takeuchi–Thompson formula on the dimension of the vector space of Killing tensors from the classical Weyl dimension formula. *To cite this article: R.G. McLenaghan et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).*
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Résumé

Tenseurs de Killing comme des représentations irréductibles du groupe linéaire. Nous démontrons que l'espace des tenseurs de Killing d'un ordre donné est naturellement isomorphe à une représentation irréductible de plus haut poids du groupe linéaire. L'isomorphisme est équivariant ; les transformations par isométries correspondent à l'inclusion du groupe des isométries comme un sous-groupe particulier du groupe linéaire. Comme application de cet isomorphisme nous obtenons la formule de Delong–Takeuchi–Thompson sur la dimension de l'espace des tenseurs de Killing à partir de la formule classique de dimension de Weyl. *Pour citer cet article : R.G. McLenaghan et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).*
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1. Introduction and preliminaries

Let $(M^n, g_{\alpha\beta})$ be an n -dimensional, pseudo-Riemannian manifold of constant curvature. Various geometric structures on M^n can be described in terms of \mathfrak{G} , the corresponding Lie group of isometries [4,8]. Recently, it has

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been shown that a group and invariant-theoretic approach can also be applied to the study of Killing tensors [5]. Continuing in the same direction, we announce a characterization of Killing tensors on pseudo-Riemannian spaces of constant curvature as certain irreducible representations of the general linear group. Our result is valid for all dimensions, all signatures of the metric, and for all values (positive, negative, and zero) of the curvature.

A Killing tensor of valence p is a symmetric tensor field $h_{\alpha_1 \dots \alpha_p}$ satisfying

$$\nabla_{(\alpha_0} h_{\alpha_1 \dots \alpha_p)} = 0. \quad (1)$$

Let \mathcal{K}^p denote the vector space of such tensor fields on M^n , with \mathcal{K}^1 being the Lie algebra of Killing vector fields. On spaces of constant curvature every Killing tensor can be represented as a sum of symmetric products of Killing vectors [2]. However, certain such products vanish identically; these form the so-called syzygy module [6]. Thus, \mathcal{K}^p is a quotient of $\text{Sym}^p \mathcal{K}^1$ by the syzygy module. It is desirable to describe this quotient explicitly.

Takeuchi took a step in this direction by showing [6] that \mathcal{K}^p is isomorphic to a certain representation of the linear group by invoking the Bott–Borel–Weil theorem [1]. We go further by exhibiting an elementary isomorphism between \mathcal{K}^p and this irreducible representation. Our construction uses a Young symmetrizer to define a complementary subspace to the syzygy module.

The main result resolves an outstanding conjecture about the equivariance of the action of the isometry group [5]. An additional application is a simple proof of the Delong–Takeuchi–Thompson dimension formula [2,6,7].

2. Representation theory of the general linear group

In this section we recall some basic facts from the representation theory of the general linear group [3]. Let V be a finite-dimensional, real vector space, \mathfrak{S}_m the symmetric group on m elements, and $\mathbb{Q}\mathfrak{S}_m$ the group algebra with rational coefficients. Let $m = \lambda_1 + \dots + \lambda_\ell$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$, be a partition of an integer m . Let $\mathbf{a} = (a_1, \dots, a_n)$, where $a_j = \lambda_j - \lambda_{j+1}$, be the corresponding weight sequence. In the sequel τ_{ij} denotes the standard tableau, defined by $\tau_{ij} = \lambda_1 + \dots + \lambda_{i-1} + j$, where $i = 1, \dots, m$ is the row index and $j = 1, \dots, \lambda_i$ is the column index.

Let $\mathbb{S}^{\mathbf{a}}V$ denote the irreducible representation of $\text{GL}(V)$ generated by the highest-weight element of $\text{Sym}^{a_1} \Lambda^1 V \otimes \dots \otimes \text{Sym}^{a_n} \Lambda^n V$. Let $c_{\mathbf{a}} \in \mathbb{Q}\mathfrak{S}_m$ be the Young symmetrizer (i.e., $c_{\mathbf{a}}^2 = c_{\mathbf{a}}$) defined by

$$c_{\mathbf{a}} = k_{\mathbf{a}} \sum_{q,r} (-1)^q q r. \quad (2)$$

In the above summation q and r range over all permutations preserving, respectively, the columns, and the rows of the tableau. The normalization constant is given by

$$k_{\mathbf{a}} = \frac{1}{b_1! \dots b_\ell!} \prod_{1 \leq i < j \leq \ell} (b_i - b_j), \quad \text{with } b_j = \lambda_j + \ell - j. \quad (3)$$

The representation $\mathbb{S}^{\mathbf{a}}V$ consists of all tensors satisfying $c_{\mathbf{a}}A = A$. The dimension of the representation is given by the Weyl dimension formula:

$$\dim \mathbb{S}^{\mathbf{a}}V = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \quad (4)$$

We will be particularly interested in weights of the form $\mathbf{a} = (0, p, 0, \dots, 0)$, and denote the corresponding representation, concisely, by $\mathbb{S}^{\{p\}}V$. The corresponding symmetry condition admits the following description. Fix p and let $\mathcal{E} = \Lambda$ if p is odd, and $\mathcal{E} = \text{Sym}$ if p is even. Let $\sigma : \text{Sym}^p \Lambda^2 V \rightarrow \mathcal{E}^2 \text{Sym}^p V$, be defined by

$$\sigma(K)_{a_1 a_2 \dots a_p b_1 b_2 \dots b_p} = \sum_{r \in \mathfrak{S}_p} K_{a_1 b_{r(1)} a_2 b_{r(2)} \dots a_p b_{r(p)}}, \quad K \in \text{Sym}^p \Lambda^2 V, \quad (5)$$

and let $\kappa : \mathcal{E}^2 \text{Sym}^p V \rightarrow \text{Sym}^p \Lambda^2 V$, be defined by

$$\kappa(S)_{a_1 b_1 a_2 b_2 \dots a_p b_p} = 2^p \widehat{S}_{[a_1 b_1][a_2 b_2] \dots [a_p b_p]}, \tag{6}$$

where we define

$$\widehat{S}_{a_1 b_1 a_2 b_2 \dots a_p b_p} = S_{a_1 a_2 \dots a_p b_1 b_2 \dots b_p}, \quad S \in \mathcal{E}^2 \text{Sym}^p V.$$

Proposition 2.1. *The representation $\mathbb{S}^{(p)}V$ consists of $K \in \text{Sym}^p \Lambda^2 V$ satisfying*

$$\kappa(\sigma(K)) = (p + 1)!K. \tag{7}$$

Thus, the tensors in questions have p skew-symmetric pairs of indices, but are symmetric with respect to interchange of the undisturbed pairs. Eq. (7) means that these tensors also satisfy a number of other symmetry conditions, e.g., the cyclic condition

$$K_{abc\dots} + K_{bca\dots} + K_{cab\dots} = 0. \tag{8}$$

Indeed, for $p = 2$, conditions (7) and (8) are equivalent. The corresponding irreducible representation consists of type (4,0) tensors having the symmetry type of the Riemann curvature tensor. However, for $p > 2$, Eq. (7) implies additional symmetry conditions, which we will not analyze here.

Let us also note that a direct application of (4) with $n = \dim V - 1$ gives

$$\dim \mathbb{S}^{(p)}V = \frac{1}{n} \binom{n+p}{p+1} \binom{n+p-1}{p}. \tag{9}$$

3. Killing tensors

Henceforth, we assume $\dim V = n + 1$. Let

$$(\mathbf{u}, \mathbf{v}) = u^a v_a, \quad \mathbf{u}, \mathbf{v} \in V$$

be a signature $(n + 1 - q, q)$ inner product, which we use to endow V with a flat metric. We consider the standard models [8] of constant-curvature submanifolds $\iota: M^n \rightarrow V$. These are either a generalized unit sphere,

$$M^n = \{x \in V: x^a x_a = 1\}; \tag{10}$$

or a unittally displaced hyperplane,

$$M^n = \{x \in V: u^a x_a = 1\}, \quad u^a u_a = 1. \tag{11}$$

Let $\mathfrak{O} < \text{GL}(V)$ be the corresponding group of orientation preserving isometries. In the first case, $\mathfrak{O} \cong \text{SO}(n + 1 - q, q)$. In the second case, \mathfrak{O} is isomorphic to the semi-direct product $\text{SO}(n - q, q) \ltimes V$.

Use ϕ_A , $A \in \text{Sym}^p V^*$ to denote the type $(0, p)$ symmetric tensor field on V with constant components. Let π_A be the corresponding degree p , homogeneous polynomial

$$\pi_A(\mathbf{x}) = A(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x}), \quad \mathbf{x} \in V.$$

Use $\phi_{A \otimes B}$, $A, B \in \text{Sym}^p V^*$ to denote the type $(0, p)$ symmetric tensor field defined by

$$\phi_{A \otimes B}(\mathbf{x}) = \pi_A(\mathbf{x})\phi_B, \quad \mathbf{x} \in V.$$

Since $\mathcal{E}^2 \text{Sym}^p V^*$ is a subspace of $\text{Sym}^p V^* \otimes \text{Sym}^p V^*$, the above definition extends to give a type $(0, p)$ tensor field ϕ_S for every $S \in \mathcal{E}^2 \text{Sym}^p V^*$. For $K \in \text{Sym}^p \Lambda^2 V^*$, use \widehat{K} to denote the symmetric type $(0, p)$ tensor field on M^n defined by

$$\widehat{K} = \iota^*(\phi_{\sigma(K)}),$$

where ι^* denotes the inclusion pullback map, and where σ is the symmetrization operator defined in (5). Treating the vector space components x^a , $a = 1, \dots, n + 1$ as functions of local coordinates on M^n , and using $\partial_\alpha x^a$, $\alpha = 1, \dots, n$ to denote the corresponding partial derivatives, we have

$$\widehat{K}_{\alpha_1 \dots \alpha_p}(\mathbf{x}) = K_{a_1 b_1 \dots a_p b_p}(\partial_{\alpha_1} x^{a_1}) \dots (\partial_{\alpha_p} x^{a_p}) x^{b_1} \dots x^{b_p}, \quad \mathbf{x} = (x^1, \dots, x^{n+1}) \in M^n \subset V.$$

Proposition 3.1. *Every such \widehat{K} is a Killing tensor on M^n .*

The above follows from (1), (10), (11) by local coordinate calculations.

Proposition 3.2. *The mapping $\Upsilon : \text{Sym}^p \Lambda^2 V^* \rightarrow \mathcal{K}^p$ defined by $\Upsilon(K) = \widehat{K}$ is a linear surjection.*

The case $p = 1$ corresponds to the isomorphism between $\Lambda^2 V^*$ and \mathcal{K}^1 , and can be established directly. For $p > 1$, we prove the proposition by noting that, on a spaces of constant curvature, all Killing tensors can be generated using symmetric products of Killing vectors [2,6].

Proposition 3.3. *The kernel of Υ is isomorphic to the kernel of σ , as defined in Eq. (5).*

Let Ad_g , $g \in \text{GL}(V)$, denote the usual adjoint action on V^* , namely

$$(\text{Ad}_g \mathbf{u})(\mathbf{v}) = \mathbf{u}(g^{-1} \mathbf{v}), \quad \mathbf{v} \in V, \mathbf{u} \in V^*.$$

We also let Ad_g denote the corresponding action of g on the tensor algebra of V^* . For an isometry $g \in \mathfrak{G}$, the pull-back map $(g^{-1})^*$ is an invertible linear transformation of \mathcal{K}^p , which defines a representation of \mathfrak{G} on \mathcal{K}^p .

Proposition 3.4. *For $g \in \mathfrak{G}$, $K \in \text{Sym}^p \Lambda^2 V^*$, we have*

$$(g^{-1})^*(\widehat{K}) = \widehat{\text{Ad}_g K}.$$

Here is our main theorem. It follows directly from Propositions 3.3 and 3.4. A form of this result, expressed in terms of the infinitesimal action on the parameter space induced by the isometry group, was conjectured in [5], and verified for a number of particular cases.

Let Π^p denote the restriction Υ to $\mathbb{S}^{\{p\}} V^*$.

Theorem 3.5. *The linear map $\Pi^p : \mathbb{S}^{\{p\}} V^* \rightarrow \mathcal{K}^p$ is a \mathfrak{G} -representation isomorphism.*

As per the comment at the end of the preceding section, we also obtain a direct proof of the Delong–Takeuchi–Thompson formula [2,6,7].

Corollary 3.6. *The dimension of \mathcal{K}^p is given by the right-hand side of Eq. (9).*

References

- [1] R. Bott, Homogeneous vector bundles, *Ann. Math.* 66 (1957) 203–247.
- [2] R. Delong, Killing tensors and the Hamilton–Jacobi equation, Ph.D. thesis, University of Minnesota, 1982.
- [3] W. Fulton, J. Harris, *Representation Theory*, Springer, 1991.
- [4] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, American Mathematical Society, Providence, RI, 2001.
- [5] R.G. McLenaghan, R.G. Smirnov, D. The, An extension of the classical theory of algebraic invariants to pseudo-Riemannian geometry and Hamiltonian mechanics, *J. Math. Phys.* 45 (2004) 1079–1120.
- [6] M. Takeuchi, Killing tensor fields on spaces of constant curvature, *Tsukuba J. Math.* 7 (1983) 233–255.
- [7] G. Thompson, Killing tensors in spaces of constant curvature, *J. Math. Phys.* 27 (1986) 2693–2699.
- [8] J. Wolf, *Spaces of Constant Curvature*, Publish or Perish, Houston, TX, 1984.