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Mathematical Problems in Mechanics/Dynamical Systems

Periodic orbits of a one-dimensional inelastic particle system

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Abstract

The dynamical behavior of a one-dimensional inelastic particle system with two particles of different masses traveling between two walls is investigated. Energy is added at only one of the walls, which is oscillating, while the other wall is stationary. We show that if the particle nearer to the stationary wall is slightly lighter than the other particle and collisions between particles tend to the elastic limit, there are an infinite number of stable orbits. We also show that the widely studied situation of equal masses is an extremely special case, in which all the orbits are degenerate and collapse to a single trivial orbit in which one of the particles is trapped against the stationary wall. ***To cite this article: J.J. Wylie, Q. Zhang, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Orbites périodiques pour un système inélastique uni-dimensionnel de particules. On étudie le comportement dynamique d'un système inélastique uni-dimensionnel formé de deux particules de masses différentes se déplaçant entre deux murs. De l'énergie est ajoutée à l'un des murs, qui oscille, alors que l'autre est stationnaire. On montre que, si la particule proche du mur stationnaire est un peu plus légère que l'autre, et si les collisions entre les particules tendent vers la limite élastique, alors il y a un nombre infini d'orbites stables. On montre également que la situation couramment étudiée où les masses sont égales est un cas très particulier, dans lequel toutes les orbites sont dégénérées et tendent vers une orbit e triviale unique où l'une des particules est piégée par le mur stationnaire. ***Pour citer cet article : J.J. Wylie, Q. Zhang, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

In recent years there has been significant interest in one-dimensional (1D) models of dissipative particle systems. The understanding obtained in these relatively simple systems has led to a number of important insights into the problem in higher dimensions. Particular interest has been aroused by the problems that occur when developing

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equations to describe the macroscopic scale of systems with particles of equal mass [1–3,7,9,10]. An understanding of the particle-scale dynamics is fundamental to ensuring the validity of assumptions frequently used when deriving macroscopic equations. The particle-scale dynamics has only been studied for systems with only one particle [4], or systems with particles of equal mass [6,8].

In this Note we investigate how the dynamics of these systems are affected when the particles have different masses. It is well known that when the masses are equal, the particle nearest the oscillating wall moves rapidly, whereas the remaining particles are typically trapped close to the stationary wall [1]. We show that this widely studied phenomena in equal mass systems is an extremely special case. When the masses are not equal, multiple stable periodic orbits can be realized. In fact, as the coefficient of restitution tends to the elastic limit, the number of periodic orbits becomes unbounded. As the mass ratio tends to unity, all of these orbits collapse to a single degenerate orbit and the trivial dynamics observed in the equal mass case are recovered.

2. Formulation

We consider the motion of two particles constrained on a line between two walls that are separated by a distance l. For definiteness we assume that wall-particle collisions are elastic. Energy is added at the left wall only and the right wall is fixed. The left and right walls are referred to as the 'oscillating wall' and the 'stationary wall', respectively. In applications involving granular materials, it is often the case that energy is added through vibrations at a boundary [5]. Thus, we adopt a 'saw-tooth' motion [5] for the oscillating wall, in which the wall moves with a constant speed v over a distance a before executing an instantaneous jump back to its starting position. We further assume that $a \ll l$, and so, to the leading order, all collisions with the oscillating wall occur at the same location. When two particles collide, momentum is conserved and their new relative velocity is just $-e$ times their old relative velocity, where e is the coefficient of restitution. Since the physical size of the particles does not play a role in 1D we consider point particles. We choose our length and time scales such that $l = 1$ and $v = 1$. We assume the left particle has unit mass and the right particle has mass m.

There are three types of collisions: C – collisions between particles; R – collisions between the right particle and the stationary wall; and L – collisions between the left particle and the oscillating wall. When a collision occurs, the velocities of the left and right particles v_1 and v_2, are updated according to the following rules,

$$C:\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longmapsto \frac{1}{1+m}\begin{pmatrix} 1-em & m(1+e) \\ 1+e & m-e \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$
$$R:\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longmapsto \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}, \quad \text{and} \quad L:\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longmapsto \begin{pmatrix} 2-v_1 \\ v_2 \end{pmatrix}.$$

3. Construction of periodic orbits

For a given sequence of collisions one simply needs to solve a set of linear equations to determine the velocities and locations of the collisions. The nonlinearity in the system arises exactly because the sequence of collisions is unknown. This is because finding the appropriate collision requires choosing the minimum time amongst the three possible collisions, and this is inherently nonlinear.

However, not all collision sequences will be feasible, since collisions can only occur when the two involved objects are moving towards each other. Even if the sequence gives velocities that are consistent, one still needs to check that the order of collisions is consistent. In the case of two particles this reduces to checking the particles are approaching each other and that collisions between the two particles occur between the two walls.

It is easy to see that any sequence must include at least one collision between particles and at least one collision with each wall. Between any two inter-particle collisions, there are three possibilities: a collision with the oscillating wall, a collision with the stationary wall or collisions with both walls. These represent the three basic subsequences that are the building blocks from which any sequence must be constructed. We denote these subsequences as $\{LC, RC, LRC(=RLC)\}$, where the sequence is read from right to left. In addition, since this problem is linear, any sequence that is made up of a number of repetitions of a subsequence must be exactly the same as its subsequence, e.g. *LRCLRC* must be made up of two identical sets of the solution of *LRC*.

Given a sequence, we need to check the consistency and stability of the sequence via the following procedure. Firstly, we need to solve a linear system to ensure that the velocities are periodic. For inelastic collisions ($e < 1$) it is easy to show that this linear system always has a unique solution. Secondly, we compute the transit times from one inter-particle collision to the next for each of the two particles. By equating the transit times and requiring that the sequence is periodic, we can compute all of the collision locations. Thirdly, we need to check the particles are approaching each other and that the locations of the collisions are between the two walls. Finally, we need to ensure that instabilities do not arise in the locations of the collisions. If any of these requirements fails, the given periodic orbit cannot be realized.

Following this procedure, one can show that the *LRC* orbit is always unstable. The *LCRC* is consistent and stable only if $1 \leqslant m < \frac{1+e^2}{2e}$, and the *LRCRC* is consistent and stable only if $m \leqslant 1$, $m > \frac{1+e^2}{1+4e+e^2}$ and $m \geqslant \frac{1-e}{1+e}$.

4. Multiple orbits near $m = 1$

The *LCRC* orbit can only be realized for $m \geqslant 1$, whereas the *LRCRC* orbit can only be realized for $m \leqslant 1$. If m is decreased through the transition $m = 1$ an extra collision with the right wall is added to the sequence. When $m = 1$, the right particle is trapped tightly against the wall and has zero velocity for most of the period. In this case, the two orbits are identical since the collision with the right wall becomes degenerate. It is natural to consider whether combinations of these two sequences can give rise to consistent and stable orbits. Therefore, we examine collision sequences of the form $(LRCRC)(LCRC)^{N-1}$. Following the procedure outlined in the previous section we can check the consistency and stability of such orbits.

If we consider the operators R and C to be matrices, the velocities v_1 and v_2 satisfy

$$[R - (-RCRC)^N][v_1, v_2]^{\mathrm{T}} = \sum_{i=0}^{N-1} (-RCRC)^i [2, 0]^{\mathrm{T}}.$$

Here T denotes transpose. After solving for v_1 and v_2 we can calculate the locations of the collisions. We are interested in the case when $m - 1$ is small and we therefore expand the results in powers of $m - 1$. We find that the location of the final collision, that is, the last collision in the only (*LRCRC*) block, will have the largest value and the location of the first collision will have the smallest value. This is because the net effect of each of the (*LCRC*) blocks is to move the locations of subsequent collisions toward the right.

In the limit as $m \to 1$, the stability condition is automatically satisfied and so this type of orbit is always stable. We also find that the location of the first collision in the sequence occurs at a location given by $p^{(1)} = 1 - (m-1)S^{(1)} + \mathrm{O}(m-1)^2$, where $S^{(1)} > 0$. This implies that the first inter-particle collision will occur between the two walls for $m < 1$. However, for $m > 1$ the location will occur behind the stationary wall and therefore orbits of this type can never occur for $m > 1$.

We find that the location of the last collision, i.e. the $2N$-th inter-particle collision in the sequence, $p^{(2N)}$, is given by $p^{(2N)} = 1 + (m-1)S^{(2N)} + \cdots$, where $S^{(2N)}$ is the gradient of the location of the final collision in the sequence with respect to m,

$$S^{(2N)} = \begin{cases} \dfrac{(1+e)^3[1 - e^{2N} - 2Ne^{N-1}(1-e)]}{4(1-e^2)[1-(-1)^{N/2}e^N]^2}, & \text{if } N \text{ is even,} \\[2ex] \dfrac{(1+e)^3[1 - e^{2N} - 2Ne^{N-1}(1-e)]}{4(1-e^2)[1-(-1)^{N}e^N]^2}, & \text{if } N \text{ is odd.} \end{cases}$$

The requirement that all collisions occur between the two walls leads to the condition $S^{(2N)} > 0$ when $m < 1$. For e sufficiently close to unity, this will always be true and so the orbit will be consistent. The gradient, $S^{(2N)}$ can only change sign at the zeros of the polynomial, $1 - e^{2N} - 2Ne^{N-1}(1-e)$. As $N \to \infty$ this polynomial has a root e_N near unity. A straightforward asymptotic expansion gives $e_N = 1 - \frac{3}{N^2} + \mathrm{O}(\frac{1}{N^4})$. Hence, for $e > e_N$, there will be a solution with $2N$ inter-particle collisions. Asymptotically, e_N is an increasing function of N, and so if the N-th orbit exists then the lower orbits must also exist. Hence, asymptotically, the number of periodic orbits will be at least of order $\sqrt{3/(1-e)}$ as $e \to 1$.

When $m = 1$, all of the collisions in any of these periodic orbits will occur exactly at the stationary wall. The final right collision in the *LRCRC* block thus becomes degenerate and the $(LRCRC)(LCRC)^{N-1}$ orbit becomes the same as the $(LCRC)^N$ orbit. That is, an orbit is made up of N repeated *LCRC* sequences. Since the problem is linear, the velocities and locations for the orbit must be made up of N repeated solutions of the *LCRC* problem. This orbit is just the trivial case in which the right particle is trapped against the stationary wall. So, all of the orbits of the form $(LRCRC)(LCRC)^{N-1}$ that exist for $m < 1$ become degenerate when $m = 1$ and collapse to the trivial *LCRC* orbit.

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