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Numerical Analysis

Numerical solution of the two-dimensional elliptic Monge–Ampère equation with Dirichlet boundary conditions: a least-squares approach

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Abstract

We addressed, in a previous note [C. R. Acad. Sci. Paris, Ser. I 336 (2003) 779–784], the numerical solution of the Dirichlet problem for the two-dimensional elliptic Monge–Ampère equation, namely: $\det D^2\psi = f$ in Ω, $\psi = g$ on $\partial\Omega$ ($\Omega \subset \mathbb{R}^2$ and $f > 0$, here). The method discussed previously relies on an augmented Lagrangian algorithm operating in the space $H^2(\Omega)$ and related functional spaces of symmetric tensor-valued functions. In the particular case where the above problem has no solution in $H^2(\Omega)$, while the data f and g verify $\{f, g\} \in L^1(\Omega) \times H^{3/2}(\partial\Omega)$, there is strong evidence that the augmented Lagrangian algorithm discussed in previously converges-in some sense-to a least squares solution belonging to $V_g = \{\varphi \mid \varphi \in H^2(\Omega), \varphi = g \text{ on } \partial\Omega\}$. Our goal in this note is to discuss a least-squares based alternative solution method for the Monge–Ampère Dirichlet problem. This method relies on the minimization on the set $V_g \times \mathbf{Q}_f$ (with $\mathbf{Q}_f = \{\mathbf{q} \mid \mathbf{q} = (q_{ij})_{1\leqslant i,j\leqslant 2}, q_{ij} \in L^2(\Omega), \forall i, j, 1 \leqslant i, j \leqslant 2, \mathbf{q} = \mathbf{q}^t, \det\mathbf{q} = f\}$) of a well-chosen least-squares functional. From a practical point of view we solve the above minimization problem via a relaxation type algorithm, operating alternatively in V_g and $\mathbf{Q}_f$ and very easy to combine to the mixed finite element approximations employed in the earlier work. Numerical experiments show that the above method has good convergence properties when the Monge–Ampère Dirichlet problem has solutions in V_g; they show also that, for cases where the above problem has no solution in V_g, while neither V_g nor $\mathbf{Q}_f$ are empty, the new method reproduces the solutions obtained via the augmented Lagrangian approach, but faster. ***To cite this article: E.J. Dean, R. Glowinski, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Résolution numérique du problème de Dirichlet pour l'équation de Monge–Ampère elliptique en dimension deux par une méthode de moindres carrés. La résolution numérique du problème de Dirichlet pour l'équation de Monge–Ampère elliptique bi-dimensionelle, soit : $\det D^2\psi = f$ in Ω, $\psi = g$ on $\partial\Omega$ (ici, $\Omega \subset \mathbb{R}^2$ et $f > 0$), a été étudiée dans une note précédente [C. R. Acad. Sci. Paris, Ser. I 336 (2003) 779–784]. La méthode décrite là, repose sur un algorithme de Lagrangien

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augmenté opérant dans l'espace $H^2(\Omega)$ et des espaces associés de fonctions à valeurs tensorielles symétriques. Dans les cas où le problème ci-dessus n'a pas de solution dans $H^2(\Omega)$, alors que les données f and g verifient $\{f,g\} \in L^1(\Omega) \times H^{3/2}(\partial\Omega)$, diverses observations et analogies suggèrent fortement que l'algorithme de Lagrangien augmenté décrit dans notre note précédente converge-en un certain sens-vers une solution appartenant à $V_g = \{\varphi \mid \varphi \in H^2(\Omega), \varphi = g \text{ on } \partial\Omega\}$ et du type moindres carrés. L'objet de cette note est la résolution du problème de Monge–Ampère Dirichlet, directement par une méthode de moindres carrés. Cette méthode repose sur la minimisation sur l'ensemble $V_g \times \mathbf{Q}_f$ (avec $\mathbf{Q}_f = \{\mathbf{q} \mid \mathbf{q} = (q_{ij})_{1 \leqslant i,j \leqslant 2}, q_{ij} \in L^2(\Omega), \forall i,j, 1 \leqslant i,j \leqslant 2, \mathbf{q} = \mathbf{q}^t, \det \mathbf{q} = f\}$), d'une fonction coût bien choisie, de type moindres carrés. D'un point de vue pratique, on résout le problème de minimisation ci-dessus par un algorithme de type relaxation qui opère alternativement dans V_g et $\mathbf{Q}_f$; cet algorithme est facile à combiner aux approximations par élements finis mixtes utilisées dans la note précédente. Des essais numériques montrent que la méthode de moindres carrés ci-dessus a de bonnes propriétés de convergence quand le problème de Monge–Ampère Dirichlet a des solutions dans V_g ; ces essais montrent également que lorsque problème ci-dessus n'a pas de solution dans V_g, bien que V_g et $\mathbf{Q}_f$ soient non vides, la nouvelle méthode reproduit les solutions obtenues par Lagrangien augmenté, mais ce plus rapidement. ***Pour citer cet article : E.J. Dean, R. Glowinski, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction: summary of previous results

In a preceding publication [1], we discussed the solution in $H^2(\Omega)$ of the *Dirichlet problem* for the *two-dimensional elliptic Monge–Ampère equation*, namely

$$\det D^2\psi = f \quad \text{in } \Omega, \qquad \psi = g \quad \text{on } \partial\Omega, \tag{E-MAD}$$

where, in (E-MAD), $D^2\psi$ is the *Hessian* of the unknown function ψ, Ω is a bounded domain of $\mathbb{R}^2$, $\partial\Omega$ is the boundary of Ω, $\{f,g\} \in L^1(\Omega) \times H^{3/2}(\partial\Omega)$ with $f > 0$. Let us define the (affine) space V_g, the space $\mathbf{Q}$ and the nonlinear manifold $\mathbf{Q}_f$ by, respectively,

$$V_g = \{\varphi \mid \varphi \in H^2(\Omega),\ \varphi = g \text{ on } \partial\Omega\}, \tag{1}$$

$$\mathbf{Q} = \{\mathbf{q} \mid \mathbf{q} = (q_{ij})_{1 \leqslant i,j \leqslant 2},\ q_{ij} \in L^2(\Omega),\ \forall i,j,\ 1 \leqslant i,j \leqslant 2,\ \mathbf{q} = \mathbf{q}^t\}, \tag{2}$$

$$\mathbf{Q}_f = \{\mathbf{q} \mid \mathbf{q} \in \mathbf{Q}, \det \mathbf{q} = f\}. \tag{3}$$

In [1], the key idea was to reduce the solution of (E-MAD) to the solution of the following saddle-point problem:

$$\begin{cases} \text{Find } \{\{\psi,\mathbf{p}\},\boldsymbol{\lambda}\} \in (V_g \times \mathbf{Q}_f) \times \mathbf{Q} \text{ such that} \\ \mathcal{L}_r(\{\psi,\mathbf{p}\},\boldsymbol{\mu}) \leqslant \mathcal{L}_r(\{\psi,\mathbf{p}\},\boldsymbol{\lambda}) \leqslant \mathcal{L}_r(\{\varphi,\mathbf{q}\},\lambda),\ \forall \{\{\varphi,\mathbf{q}\},\boldsymbol{\mu}\} \in (V_g \times \mathbf{Q}_f) \times \mathbf{Q}, \end{cases} \tag{SDP}$$

where, in (SDP), the *augmented Lagrangian functional* $\mathcal{L}_r$ is defined (with $r > 0$, $\mathbf{S}:\mathbf{T} = \Sigma s_{ij}t_{ij}$, if $\mathbf{S} = (s_{ij})$ and $\mathbf{T} = (t_{ij})$, and $|\mathbf{S}| = \sqrt{\mathbf{S}:\mathbf{S}}$) by

$$\mathcal{L}_r(\{\varphi,\mathbf{q}\},\boldsymbol{\mu}) = \frac{1}{2}\int_\Omega |\Delta\varphi|^2\,\mathrm{d}x + \frac{r}{2}\int_\Omega |D^2\varphi - \mathbf{q}|^2\,\mathrm{d}x + \int_\Omega \boldsymbol{\mu}:(D^2\varphi - \mathbf{q})\,\mathrm{d}x. \tag{4}$$

Indeed, if (SDP) has a solution, we have $\mathbf{p} = D^2\psi$ with ψ a solution of (E-MAD). To solve (SDP), we advocated in [1], among other possible algorithms, the following one (of the Douglas–Rachford–Uzawa type; cf., e.g., [2,3]):

$$\{\psi^{-1},\boldsymbol{\lambda}^0\} \text{ is given in } V_g \times \mathbf{Q}; \tag{5}$$

for $n \geqslant 0$, $\{\psi^{n-1},\boldsymbol{\lambda}^n\}$ being known, solve

$$\mathbf{p}^n \in \mathbf{Q}_f;\ \mathcal{L}_r(\{\psi^{n-1},\mathbf{p}^n\},\boldsymbol{\lambda}^n) \leqslant \mathcal{L}_r(\{\psi^{n-1},\mathbf{q}\},\boldsymbol{\lambda}^n),\ \forall \mathbf{q} \in \mathbf{Q}_f, \tag{6}$$

$$\psi^n \in V_g;\ \mathcal{L}_r(\{\psi^n,\mathbf{p}^n\},\boldsymbol{\lambda}^n) \leqslant \mathcal{L}_r(\{\varphi,\mathbf{p}^n\},\boldsymbol{\lambda}^n),\ \forall \varphi \in V_g, \tag{7}$$

and update $\boldsymbol{\lambda}^n$ via $\boldsymbol{\lambda}^{n+1} = \boldsymbol{\lambda}^n + r(D^2\psi^n - \mathbf{p}^n)$. (8)

Remark 1. In [1], all calculations were done with algorithm (5)–(8) initialized by $\boldsymbol{\lambda}^0 = \mathbf{0}$ and ψ^{-1} the solution in V_g of the Dirichlet problem $-\Delta\psi^{-1} = \sqrt{f}$ in Ω, $\psi^{-1} = g$ on $\partial\Omega$ (see [4] for the rational of this choice).

Numerical experiments realized with a mixed finite implementation of algorithm (5)–(8) lead to the following conclusions (see [1,4] and [5] for details): (i) If (E-MAD) has a solution in V_g, the corresponding discrete analogue of (5)–(8) is convergent and produces, at the limit, ψ_h such that $\|\psi_h - \psi\|_{L^2(\Omega)} = 0(h^2)$, with ψ solution to (E-MAD). (ii) If (E-MAD) has no solution in V_g, with V_g and $\mathbf{Q}_f$ both non-empty, then (with obvious notation) the sequence $\{\boldsymbol{\lambda}^n\}_{n\geqslant 0}$ is *divergent*, while $\{\{\psi^n, \mathbf{p}^n\}\}_{n\geqslant 0}$ converges to a pair $\{\psi, \mathbf{p}\}$ which minimizes (locally or globally) the functional $\{\varphi, \mathbf{q}\} \to \|D^2\varphi - \mathbf{q}\|_{\mathbf{Q}}$ over the set $V_g \times \mathbf{Q}_f$.

2. On two least squares formulations of (E-MAD)

The above mentioned behavior of algorithm (5)–(8) strongly suggests to look at least-squares methods for the solution of (E-MAD). Such a method has been investigated in [4]; it relies on the following brute force least-squares formulation of (E-MAD):

$$\min_{\varphi\in V_g} j_1(\varphi), \tag{LSQ1}$$

with

$$j_1(\varphi) = \begin{cases} \frac{1}{2}\int_\Omega |\det D^2\varphi - f|^2\,\mathrm{d}x, & \text{if } (\det D^2\varphi - f) \in L^2(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

The solution of (E-MAD) via (LSQ1) is discussed in [4]; it relies on iterative methods whose convergence, however, is not as clear cut as the convergence of the discrete variants of algorithm (5)–(8) (see [4] for details). Actually, Section 1 suggests an alternative (and more natural) least squares formulation, namely

$$\min_{\{\varphi,\mathbf{q}\}\in V_g\times\mathbf{Q}_f} j_2(\varphi, \mathbf{q}), \tag{LSQ2}$$

with

$$j_2(\varphi, \mathbf{q}) = \frac{1}{2}\int_\Omega \left|D^2\varphi - \mathbf{q}\right|^2 \mathrm{d}x. \tag{9}$$

3. On the iterative solution of problem (LSQ2) and related issues

Let us define the (non-convex) functional $I_{\mathbf{Q}_f} : \mathbf{Q} \to \mathbb{R} \cup \{+\infty\}$ by

$$I_{\mathbf{Q}_f}(\mathbf{q}) = \begin{cases} 0, & \text{if } \mathbf{q} \in \mathbf{Q}_f; \\ +\infty, & \text{otherwise,} \end{cases}$$

namely, $I_{\mathbf{Q}_f}(\cdot)$ is the *indicator functional* of the set $\mathbf{Q}_f$ in $\mathbf{Q}$. Problem (LSQ2) is thus clearly *equivalent* to the following minimization problem in $V_g \times \mathbf{Q}$:

$$\min_{\{\varphi,\mathbf{q}\}\in V_g\times\mathbf{Q}} \left[j_2(\varphi, \mathbf{q}) + I_{\mathbf{Q}_f}(\mathbf{q})\right], \tag{10}$$

whose (formal) *Euler–Lagrange* equation reads as follows at a solution $\{\psi, \mathbf{p}\}$ of problem (LSQ2):

$$\begin{cases} \{\psi, \mathbf{p}\} \in V_g \times \mathbf{Q}, \\ \int_\Omega (D^2\psi - \mathbf{p}) : (D^2\varphi - \mathbf{q})\,\mathrm{d}x + \langle \partial I_{\mathbf{Q}_f}(\mathbf{p}), \mathbf{q}\rangle = 0, \ \forall\{\varphi, \mathbf{q}\} \in V_0 \times \mathbf{Q}, \end{cases} \tag{11}$$

with $V_0 = H^2(\Omega) \cap H_0^1(\Omega)$ and $\partial I_{\mathbf{Q}_f}(\mathbf{p})$ a (kind of) generalized differential of $I_{\mathbf{Q}_f}(\cdot)$ at $\mathbf{p}$. Classically, we associate to (11) the following initial value problem (flow in the terminology of Dynamical Systems) since its steady state solutions solve problem (11):

$$\big\{\psi(0), \mathbf{p}(0)\big\} = \{\psi_0, \mathbf{p}_0\}(\in V_g \times \mathbf{Q}), \tag{12}$$

$$\begin{cases} \{\psi(t), \mathbf{p}(t)\} \in V_g \times \mathbf{Q},\ \forall t \in (0, +\infty), \\ \int_\Omega \Delta(\partial\psi/\partial t)\Delta\varphi \,\mathrm{d}x + \int_\Omega (D^2\psi - \mathbf{p}) : D^2\varphi \,\mathrm{d}x = 0,\ \forall \varphi \in V_0, \\ \int_\Omega (\partial\mathbf{p}/\partial t) : \mathbf{q} \,\mathrm{d}x + \int_\Omega (\mathbf{p} - D^2\psi) : \mathbf{q} \,\mathrm{d}x + \langle \partial I_{\mathbf{Q}_f}(\mathbf{p}), \mathbf{q}\rangle = 0,\ \forall \mathbf{q} \in \mathbf{Q}, \end{cases} \tag{13}$$

the idea being to capture the steady state solutions of (13) via the integration of (12), (13) from $t = 0$ to $t = +\infty$. Concerning the initialization of (12), (13), following Remark 1 we advocate for ψ_0 the unique solution in V_g of the Dirichlet problem $-\Delta\psi_0 = \sqrt{f}$ in Ω, $\psi_0 = g$ on $\partial\Omega$ and $\mathbf{p}_0 = D^2\psi_0$. Let τ (> 0) be a *time-discretization step*. Applying to (12), (13) an operator-splitting scheme à la Marchuk–Yanenko (see, e.g., [6, Chapter VI] and the references therein) we obtain the following iterative method:

$$\big\{\psi^0, \mathbf{p}^0\big\} = \{\psi_0, \mathbf{p}_0\}; \tag{14}$$

for $n \geqslant 0$, $\{\psi^n, \mathbf{p}^n\}$ being known, compute $\{\psi^{n+1}, \mathbf{p}^{n+1}\}$ as follows

$$\big(\mathbf{p}^{n+1} - \mathbf{p}^n\big)/\tau + \mathbf{p}^{n+1} + \partial I_{\mathbf{Q}_f}\big(\mathbf{p}^{n+1}\big) = D^2\psi^n, \tag{15}$$

$$\begin{cases} \psi^{n+1} \in V_g, \\ \int_\Omega \Delta[(\psi^{n+1} - \psi^n)/\tau]\Delta\varphi \,\mathrm{d}x + \int_\Omega D^2\psi^{n+1} : D^2\varphi \,\mathrm{d}x = \int_\Omega \mathbf{p}^{n+1} : D^2\varphi \,\mathrm{d}x,\ \forall \varphi \in V_0. \end{cases} \tag{16}$$

Relation (15) is a *necessary optimality condition* for the following minimization problem:

$$\min_{\mathbf{q}\in\mathbf{Q}_f} \Big[\tfrac{1}{2}(1+\tau)\int_\Omega |\mathbf{q}|^2 \,\mathrm{d}x - \int_\Omega (\mathbf{p}^n + \tau D^2\psi^n) : \mathbf{q} \,\mathrm{d}x\Big], \tag{NLP}$$

while (16) characterizes ψ^{n+1} as the solution of

$$\min_{\varphi\in V_g} \Big[\tfrac{1}{2}\int_\Omega |\Delta\varphi|^2 + \tau|D^2\varphi|^2 \,\mathrm{d}x - \int_\Omega (\Delta\psi^n\Delta\varphi + \tau\mathbf{p}^{n+1} : D^2\varphi) \,\mathrm{d}x\Big]. \tag{LQP}$$

Each problem (NLP) can be solved pointwise (in practice at the vertices of a finite element or finite difference mesh); to obtain $\mathbf{p}^{n+1}$ from $\mathbf{p}^n$ and ψ^n we have to minimize, pointwise on Ω, a three-variable polynomial of the following type $\mathbf{z}(= \{z_i\}_{i=1}^3) \to \frac{1}{2}(1+\tau)(z_1^2 + z_2^2 + 2z_3^2) - \mathbf{b}_n(x)\cdot\mathbf{z}$ over the set defined by $z_1z_2 - z_3^2 = f(x)$. The above problem is a *generalized eigenvalue problem* which can be solved by a variant of the *Newton's method*. Each problem (LQP) is equivalent to (16), a *well-posed linear variational problem*. Problem (16) can be solved by a *conjugate gradient algorithm* operating in V_g and V_0 equipped with the scalar product $\{v, w\} \to \int_\Omega \Delta v\Delta w \,\mathrm{d}x$. As in [1,4], we have used, for the space approximation of (LSQ2), a mixed finite element discretization closely related to the one employed in [2,3,7] for the numerical simulation of two-dimensional *Bingham visco-plastic flow* using the *stream function formulation*. With this approach φ, $\mathbf{q}$, ψ, $\mathbf{p}$ are approximated by continuous piecewise linear approximations associated to a finite element triangulation of Ω. The condition $\det\mathbf{q} = f$ is imposed at the vertices of this triangulation.

Remark 2. Algorithm (14)–(16) is clearly of the *relaxation* type. Actually, when $\tau \to +\infty$, we recover at the limit an algorithm very close to the *block Gauss–Seidel* one discussed in, e.g., [7,8].

4. Numerical experiments

The least-squares method discussed in Sections 2 and 3 has been applied to the solution of three E-MAD test problems with $\Omega = (0,1)^2$. The *first test problem* can be expressed as follows (with $|x| = (x_1^2 + x_2^2)^{1/2}$ and $R \geqslant \sqrt{2}$):

$$\det D^2\psi = R^2/\big(R^2 - |x|^2\big)^{1/2} \quad \text{in } \Omega, \qquad \psi = \big(R^2 - |x|^2\big)^{1/2} \quad \text{on } \partial\Omega. \tag{17}$$

Table 1
First test problem

h	τ	n_{it}	$\lVert D_h^2\psi_h^c - \mathbf{p}_h^c\rVert_{\mathbf{Q}}$	$\lVert \psi_h^c - \psi\rVert_{L^2(\Omega)}$
1/32	0.1	517	0.9813×10^{-6}	0.450×10^{-5}
1/32	1	73	0.9618×10^{-6}	0.449×10^{-5}
1/32	10	28	0.7045×10^{-6}	0.450×10^{-5}
1/32	100	21	0.6773×10^{-6}	0.449×10^{-5}
1/32	1000	22	0.8508×10^{-6}	0.449×10^{-5}
1/32	10 000	22	0.8301×10^{-6}	0.449×10^{-5}
1/64	1	76	0.9624×10^{-6}	0.113×10^{-5}
1/64	10	29	0.8547×10^{-6}	0.113×10^{-5}
1/64	100	24	0.8094×10^{-6}	0.113×10^{-5}

Table 2
Second test problem

h	τ	n_{it}	$\lVert D_h^2\psi_h^c - \mathbf{p}_h^c\rVert_{\mathbf{Q}}$	$\lVert \psi_h^c - \psi\rVert_{L^2(\Omega)}$
1/32	1	145	0.9381×10^{-6}	0.556×10^{-4}
1/32	10	56	0.9290×10^{-6}	0.556×10^{-4}
1/32	100	46	0.9285×10^{-6}	0.556×10^{-4}
1/32	1000	45	0.9405×10^{-6}	0.556×10^{-4}
1/64	1	151	0.9500×10^{-6}	0.145×10^{-4}
1/64	10	58	0.9974×10^{-6}	0.145×10^{-4}
1/64	100	49	0.9531×10^{-6}	0.145×10^{-4}
1/64	1000	48	0.9884×10^{-6}	0.145×10^{-4}

The function ψ defined by $\psi(x) = (R^2 - |x|^2)^{1/2}$ is a solution of problem (17) (the graph of ψ is thus a piece of the *sphere* of center $\mathbf{0}$ and radius R). The above function $\psi \in C^\infty(\bar{\Omega})$ if $R > \sqrt{2}$ (if $R = \sqrt{2}$, we have no better than $\psi \in W^{1,p}(\Omega),\ \forall p < 4$). We have discretized problem (17) relying, as in [1], on a mixed variational formulation associated to uniform triangulations of Ω, allowing us to solve the various elliptic problems encountered at each iteration of (14)–(16) by fast Poisson and Helmholtz solvers taking advantage of the decomposition properties of biharmonic problems such as (16). The finite element analogue of algorithm (14)–(16) diverges if $R = \sqrt{2}$ (which is not surprising since the corresponding $\psi \notin H^2(\Omega)$); on the other hand, for $R = 2$ we have a quite fast convergence as soon as τ is large enough, the corresponding results being reported on Table 1, below (we stopped iterating as soon as $\lVert D_h^2\psi_h^n - \mathbf{p}_h^n\rVert_{\mathbf{Q}} \leqslant 10^{-6}$, ψ_h^n and $\mathbf{p}_h^n$ being the computed approximations of ψ^n and $\mathbf{p}^n$, respectively).

Above, $\{\psi_h^c, \mathbf{p}_h^c\}$ is the computed approximate solution, h the space discretization step and n_{it} the number of iterations necessary to achieve convergence. Table 1 clearly suggests that: (i) For τ large enough the speed of convergence is essentially independent of τ. (ii) The speed of convergence is essentially independent of h. (iii) The $L^2(\Omega)$-approximation error is $0(h^2)$. By comparing the above results to those reported in [4], concerning the solution of problem (17) by the augmented Lagrangian algorithm (5)–(8), we can add to (i)–(iii), above, that the new approach is easier to implement, is more robust, and provides the same approximate solutions, but faster (for τ large enough); it avoids also the adjustment of parameter r, a delicate issue, particularly if one looks for an optimal value. Similarly, the new methodology is easier to implement and leads to faster algorithms than those derived from (LSQ1), another least-squares approach. The *second test problem* is defined by

$$\det D^2\psi = 1/|x| \quad \text{in } \Omega, \qquad \psi = 2\sqrt{2}|x|^{3/2}/3 \quad \text{on } \partial\Omega. \tag{18}$$

With these data, ψ defined by $\psi(x) = 2\sqrt{2}|x|^{3/2}/3$ is solution of (18). We can easily show that $\psi \in W^{2,p}(\Omega),\ \forall p < 4$, but does not have the $C^2(\bar{\Omega})$-regularity. Using the same algorithm and approximation than for the first test problem, we obtain then the results reported in Table 2.

The various comments we have done concerning the solution of the first test problem still apply here. The *third test problem*, namely

$$\det D^2\psi = 1 \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega, \tag{19}$$

has no solution in $H^2(\Omega)$, despite the smoothness of the data (see [1] for details). On the other hand, since the corresponding V_g ($= H^2(\Omega) \cap H_0^1(\Omega)$, here) and $\mathbf{Q}_f$ are both non-empty, it makes sense to solve (19) in a least squares sense via formulation (LSQ2) and algorithm (14)–(16). We obtain then the results reported in Table 3.

For this test problem we have used $\lVert \psi_h^{n+1} - \psi_h^n\rVert_{L^2(\Omega)} \leqslant 10^{-7}$ as the stopping criterion. The convergence is clearly slower than for the two first test problems, however some important features remain such as: the number of iterations necessary to achieve convergence is essentially independent of τ as soon as this last parameter is large enough and increases slowly with h (actually like $\sqrt{h}$). Most importantly (from a conceptual point of view), the solutions computed via formulation (LSQ2) and algorithm (14)–(16) coincide, essentially, with those obtained via

Table 3
Third test problem

h	τ	n_{it}	$\lVert D_h^2\psi_h^c - \mathbf{p}_h^c\rVert_{\mathbf{Q}}$
$1/32$	1	4977	0.1054×10^{-1}
$1/32$	100	3297	0.4980×10^{-2}
$1/32$	1000	3275	0.4904×10^{-2}
$1/32$	10 000	3273	0.4896×10^{-2}
$1/64$	1	6575	0.1993×10^{-1}
$1/64$	100	4555	0.1321×10^{-1}
$1/64$	1000	4527	0.1312×10^{-1}
$1/128$	100	5402	0.1841×10^{-1}
$1/128$	1000	5372	0.1830×10^{-1}

the augmented Lagrangian algorithm (5)–(8); this is a result we were looking for, in order to clarify the convergence properties of algorithm (5)–(8) when (E-MAD) has no solution in $H^2(\Omega)$ while V_g and $\mathbf{Q}_f$ are both non-empty.

Remark 3. An evidence that both approaches produce, essentially, the same results for the third test problem is the fact that $\lVert\psi_h^{\mathrm{LS}} - \psi_h^{\mathrm{AL}}\rVert_{L^2(\Omega)}$ is of the order of 10^{-5} (the superscript LS (respectively, AL) being associated to the least-squares (respectively, augmented Lagrangian) solution).

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