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Partial Differential Equations

An extreme variation phenomenon for some nonlinear elliptic problems with boundary blow-up

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Abstract

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ and Γ_∞ be a non-empty open and closed subset of $\partial\Omega$. Denote by $\mathcal{B}$ either the Dirichlet or the mixed boundary operator on $\Gamma_\mathcal{B} := \partial\Omega \setminus \Gamma_\infty$ when $\Gamma_\infty \neq \partial\Omega$. We consider the nonlinear elliptic problem $\Delta u + au = b(x)f(u)$ in Ω, subject to $\mathcal{B}u = 0$ on $\Gamma_\mathcal{B}$ when $\Gamma_\mathcal{B} \neq \emptyset$, where a is a real number, b is a continuous non-negative function on $\overline{\Omega}$, while $f \geqslant 0$ is continuous on $[0,\infty)$ such that $f(u)/u$ is increasing on $(0,\infty)$. Assuming that f varies rapidly at infinity with index ∞ (i.e., $\lim_{u\to\infty} f(\lambda u)/f(u) = \lambda^\infty$ for all $\lambda > 0$), we establish the uniqueness of the positive solution satisfying $u = \infty$ on Γ_∞ and describe its blow-up rate via the extreme value theory. ***To cite this article: F.-C. Cîrstea, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Une phénomène de variation extrême pour quelque problèmes elliptiques non linéaires avec explosion au bord. Soit Ω un domaine borné, régulier de $\mathbb{R}^N$ $(N \geqslant 2)$ et $\Gamma_\infty \neq \emptyset$ un sous-ensemble ouvert et fermé de $\partial\Omega$. On désigne par $\mathcal{B}$ ou bien une condition de Dirichlet ou bien une condition mixte sur $\Gamma_\mathcal{B} := \partial\Omega \setminus \Gamma_\infty$ si $\Gamma_\infty \neq \partial\Omega$. On étudie le problème elliptique non-linéaire $\Delta u + au = b(x)f(u)$ dans Ω, avec la condition $\mathcal{B}u = 0$ sur $\Gamma_\mathcal{B}$ si $\Gamma_\mathcal{B} \neq \emptyset$, où a est un réel, b est une fonction continue non-négative dans $\overline{\Omega}$ et $f \geqslant 0$ est continue sur $[0,\infty)$ telle que $f(u)/u$ est strictement croissante sur $(0,\infty)$. Supposons que f varie rapidement à l'infini d'index ∞ (i.e., $\lim_{u\to\infty} f(\lambda u)/f(u) = \lambda^\infty$ pour tout $\lambda > 0$), on établit alors l'unicité de la solution positive avec $u = \infty$ sur Γ_∞ et on décrit le taux d'explosion au bord en utilisant la théorie des valeurs extrêmes. ***Pour citer cet article : F.-C. Cîrstea, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

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L'étude des solutions explosant au bord a été abordée, pour la première fois, en 1916 par Bieberbach [2] pour l'équation $\Delta u = \mathrm{e}^u$ dans un domaine borné, régulier Ω de $\mathbb{R}^2$. Il a montré qu'il y a une seule solution positive $u \in C^2(\Omega)$ telle que la différence $u(x) - \ln(d(x)^{-2})$ est bornée quand $d(x) := \mathrm{dist}(x, \partial\Omega) \to 0$. Rademacher [7] a montré que ce résultat demeure pour des domaines bornés et réguliers dans $\mathbb{R}^3$.

On établit ici l'unicité et le comportement asymptotique des solutions avec explosion au bord pour quelque problèmes elliptiques avec des non-linéarités $f(u)$ qui *varient rapidement à l'infini* d'index ∞ : $\lim_{u\to\infty} f(\lambda u)/f(u) = \lambda^\infty$ pour tout $\lambda > 0$.

Une fonction croissante f est dite *à variation de type Γ* à l'infini (notée $f \in \Gamma$) si f est définie sur un intervalle (D, ∞), $\lim_{y\to\infty} f(y) = \infty$ et s'il existe une fonction $g : (D, \infty) \to (0, \infty)$ (appelée fonction auxiliaire) telle que $\lim_{y\to\infty} f(y + \lambda g(y))/f(y) = \mathrm{e}^\lambda$, pour tout $\lambda \in \mathbb{R}$ (voir [8]). Supposons que $f \in \Gamma$, alors f varie rapidement à l'infini d'index ∞ (voir [3]).

Soit Ω un domaine borné, régulier de $\mathbb{R}^N$ ($N \geqslant 2$) et $\Gamma_\infty \neq \emptyset$ un sous-ensemble ouvert et fermé de $\partial\Omega$ (éventuellement $\Gamma_\infty = \partial\Omega$). On définit $\Gamma_{\mathcal{B}} = \partial\Omega \setminus \Gamma_\infty$ pour le cas $\Gamma_\infty \neq \partial\Omega$. On désigne par $\mathcal{B}$ l'opérateur de Dirichlet $\mathcal{D}u := u$ ou bien l'opérateur de Neumann/Robin $\mathcal{R}u = \frac{\partial u}{\partial \nu} + \beta(x)u$ sur $\partial\Omega$, où ν est le vecteur unité de la normale extérieure sur $\partial\Omega$ et $0 \leqslant \beta \in C^{1,\mu}(\partial\Omega)$, $0 < \mu < 1$.

Soit $b \in C^{0,\mu}(\overline{\Omega})$ une fonction non négative dans Ω telle que $b > 0$ sur $\Gamma_{\mathcal{B}}$ si $\mathcal{B} = \mathcal{R}$. On définit Ω_0 interieur de $\Omega_{0,b} := \{x \in \Omega\colon\ b(x) = 0\}$. On suppose que $\partial\Omega_0$ est régulier (éventuellement vide) et Ω_0 est un ensemble connexe tel que $\overline{\Omega}_0 \subset \Omega$ et $b > 0$ dans $\Omega \setminus \overline{\Omega}_0$. Soit $\lambda_{\infty,1}$ la première valeur propre de $(-\Delta)$ dans $H_0^1(\Omega_0)$ (avec $\lambda_{\infty,1} = +\infty$ si $\Omega_0 = \emptyset$). On définit $\mathcal{K}$ l'ensemble des fonctions $k : (0, \nu) \to (0, \infty)$ de classe C^1, croissantes, telles que $\lim_{t\searrow 0}(\int_0^t k(s)\,\mathrm{d}s/k(t))^{(i)} := \ell_i$, pour $i = 0, 1$.

Etant donnée une fonction H croissante sur $\mathbb{R}$, on définit $H^\leftarrow$ comme l'inverse (continue à gauche) $H^\leftarrow(y) := \inf\{s\colon\ H(s) \geqslant y\}$. On considère le problème elliptique singulier (P_a), plus précisément :

$$\begin{cases} -\Delta u = au - b(x)f(u) \quad \text{dans } \Omega, \quad u = +\infty \quad \text{sur } \Gamma_\infty, & \text{si } \Gamma_\infty = \partial\Omega, \\ -\Delta u = au - b(x)f(u) \quad \text{dans } \Omega, \quad u = +\infty \quad \text{sur } \Gamma_\infty, \quad \mathcal{B}u = 0 \quad \text{sur } \Gamma_{\mathcal{B}}, & \text{si } \Gamma_\infty \neq \partial\Omega, \end{cases}$$

où $a \in \mathbb{R}$, $f \geqslant 0$ est une fonction localement Lipschitz sur $[0, \infty)$ telle que l'application $f(u)/u$ soit strictement croissante sur $(0, \infty)$. On démontre le résultat suivant d'unicité :

Théorème 0.1. *Soit f une fonction à variation de type Γ à l'infini avec la fonction auxiliaire g. Supposons que pour tout ensemble connexe ouvert et fermé Γ_∞^c de Γ_∞ il existe $k \in \mathcal{K}$ avec $\ell_1 \neq 0$ tel que*

$$0 < \liminf_{d(x)\to 0} b(x)/k^2\big(d(x)\big) \quad \textit{et} \quad \limsup_{d(x)\to 0} b(x)/k^2\big(d(x)\big) < \infty, \quad \textit{où } d(x) = \mathrm{dist}(x, \Gamma_\infty^c).$$

Alors, pour chaque $a < \lambda_{\infty,1}$, le problème (P_a) admet une seule solution positive u_a et, de plus,

$$u_a(x)/\phi(d(x)) \to 1 \quad \textit{quand } d(x) \to 0, \ \textit{où } \phi(t) = \psi^\leftarrow\big(1/\big[tk(t)\big]^2\big) \ (t > 0 \textit{ assez petit})$$

et $\psi(u) = \sup\{f(y)/g(y)\colon\ \alpha \leqslant y \leqslant u\}$ est défini pour $u \geqslant \alpha$ ($\alpha > 0$ assez grand).

1. Introduction and main result

The topic of blow-up solutions has been initiated in 1916 by Bieberbach [2] for the equation $\Delta u = \mathrm{e}^u$ in a smooth bounded domain $\Omega \subset \mathbb{R}^2$. He showed that there is a unique positive solution $u \in C^2(\Omega)$ such that $u(x) - \ln(d(x)^{-2})$ is bounded as $d(x) = \mathrm{dist}(x, \partial\Omega) \to 0$. Problems of this type arise in Riemannian geometry; if a Riemannian metric of the form $|\mathrm{d}s|^2 = \mathrm{e}^{2u(x)}|\mathrm{d}x|^2$ has constant Gaussian curvature $-c^2$ then $\Delta u = c^2\mathrm{e}^{2u}$. Rademacher [7] extended the result of Bieberbach on smooth bounded domains in $\mathbb{R}^3$.

Our aim is to give the uniqueness and asymptotic behaviour of blow-up solutions to a general class of semilinear elliptic problems involving non-linearities $f(u)$ *rapidly varying* (*at infinity*) with index ∞, i.e.,

$$\lim_{u\to\infty}\frac{f(\lambda u)}{f(u)}=\begin{cases}\infty, & \text{if } \lambda>1,\\ 1, & \text{if } \lambda=1,\\ 0, & \text{if } 0<\lambda<1.\end{cases}$$

This study answers a research question formulated to the author by Prof. N. Dancer in 2003.

In this Note we establish a subtle connection between the blow-up rate of the solution and the rapid variation of f at infinity using the *extreme value theory* (in [8]).

Definition 1.1 (see [8]). A non-decreasing function f is Γ-varying at ∞ (written $f\in\Gamma$) if f is defined on (D,∞), $f(\infty)=\infty$ and there is $g:(D,\infty)\to(0,\infty)$ such that $\lim_{y\to\infty} f(y+\lambda g(y))/f(y)=e^{\lambda}$, $\forall\lambda\in\mathbb{R}$.

The function g is called an *auxiliary function* and is unique up to asymptotic equivalence (see [8]).

Remark 1. If $f\in\Gamma$, then f is *rapidly varying* (*at infinity*) *of index* ∞, cf. Proposition 3.10.3 in [3].

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N\geqslant 2)$ and Γ_∞ be a non-empty open and closed subset of $\partial\Omega$ (possibly, $\Gamma_\infty=\partial\Omega$). Set $\Gamma_{\mathcal{B}}=\partial\Omega\setminus\Gamma_\infty$ when $\Gamma_\infty\neq\partial\Omega$. Denote by $\mathcal{B}$ either the Dirichlet boundary operator $\mathcal{D}u:=u$ or the Neumann/Robin boundary operator $\mathcal{R}u=\frac{\partial u}{\partial\nu}+\beta(x)u$, where ν is the outward unit normal to $\partial\Omega$ and $\beta\geqslant 0$ is in $C^{1,\mu}(\partial\Omega)$, $\mu\in(0,1)$. We consider the elliptic problem (M_a), namely:

$$-\Delta u=au-b(x)f(u)\quad\text{in }\Omega,\tag{1}$$

if $\Gamma_\infty=\partial\Omega$, and the boundary value problem,

$$-\Delta u=au-b(x)f(u)\quad\text{in }\Omega,\qquad \mathcal{B}u=0\quad\text{on }\Gamma_{\mathcal{B}},\tag{2}$$

if $\Gamma_\infty\neq\partial\Omega$, where $f\in C[0,\infty)$ is locally Lipschitz, $a\in\mathbb{R}$ is a parameter and $b\geqslant 0$ is in $C^{0,\mu}(\overline{\Omega})$.

A $C^2(\Omega)$-solution of (1) and $C^2(\Omega\cup\Gamma_{\mathcal{B}})$-solution of (2), respectively satisfying $u(x)\geqslant 0$ in Ω and $u(x)\to\infty$ as $\operatorname{dist}(x,\Gamma_\infty)\to 0$ is called a *blow-up solution* of (1) and (2), respectively.

Let Ω_0 be the interior of the set $\Omega_{0,b}:=\{x\in\Omega:\ b(x)=0\}$. We assume, throughout, that $\partial\Omega_0$ satisfies the exterior cone condition (possibly, $\Omega_0=\emptyset$), Ω_0 is connected, $\overline{\Omega}_0\subset\Omega$ and $b>0$ on $\Omega\setminus\overline{\Omega}_0$. If $\Gamma_\infty\neq\partial\Omega$, then we require $b>0$ on $\Gamma_{\mathcal{B}}$ if $\mathcal{B}=\mathcal{R}$. Note that we allow $b\geqslant 0$ on Γ_∞ and on $\Gamma_{\mathcal{B}}$ when $\mathcal{B}=\mathcal{D}$.

Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of $(-\Delta)$ in $H_0^1(\Omega_0)$ (with $\lambda_{\infty,1}=\infty$ if $\Omega_0=\emptyset$).

As in [5], $\mathcal{K}$ denotes the set of all positive, non-decreasing functions $k\in C^1(0,\nu)$, for some $\nu>0$, that satisfy $\lim_{t\to 0^+}(\int_0^t k(s)\,ds/k(t))^{(i)}:=\ell_i$, with $i=0,1$. Recall that $\ell_0=0$ and $\ell_1\in[0,1]$, for every $k\in\mathcal{K}$.

A positive measurable function Z defined on $[D,\infty)$, for some $D>0$, is called *regularly varying* (*at infinity*) *with index* $q\in\mathbb{R}$, written $Z\in RV_q$, if $\lim_{u\to\infty}Z(\xi u)/Z(u)=\xi^q$, for all $\xi>0$. When $q=0$ we say that Z is slowly varying (see [8]). By $f_1(u)\sim f_2(u)$ as $u\to\infty$ we mean $\lim_{u\to\infty}f_1(u)/f_2(u)=1$.

When f varies regularly at ∞ with (real) index greater than 1, the uniqueness and asymptotic behaviour of the blow-up solution to problems like (M_a) has been treated in [4–6].

In [4], the authors prove the uniqueness of the blow-up solution u_a to (M_a), for any $a<\lambda_{\infty,1}$, provided that for each connected open and closed subset Γ_∞^c of Γ_∞ there exists $k\in\mathcal{K}$ such that

$$0<\liminf_{d(x)\to 0}b(x)/k^2\big(d(x)\big)\quad\text{and}\quad\limsup_{d(x)\to 0}b(x)/k^2\big(d(x)\big)<\infty,\quad\text{where } d(x):=\operatorname{dist}(x,\Gamma_\infty^c),\tag{3}$$

while $f\in RV_{\rho+1}$ $(\rho>0)$ satisfies (A): $f\geqslant 0$ is locally Lipschitz continuous on $[0,\infty)$ and $f(u)/u$ is increasing for $u>0$. The blow-up rate of u_a is also given when (3) is slightly more restrictive.

If H is a non-decreasing function on $\mathbb{R}$, then we define the (left continuous) inverse of H by: $H^{\leftarrow}(y)=\inf\{s:\ H(s)\geqslant y\}$.

In this Note we treat the *extreme* case when $f\in\Gamma$ (instead of $f\in RV_{\rho+1}$) and obtain the following:

Theorem 1.2. *Let* (A) *hold and* f *be* $\varGamma$*-varying at* ∞ *with auxiliary function* g*. Assume that for each connected open and closed subset* $\varGamma_\infty^c$ *of* $\varGamma_\infty$ *there exists* $k \in \mathcal{K}$ *with* $\ell_1 \neq 0$ *such that* (3) *is fulfilled.*

Then, for any $a < \lambda_{\infty,1}$*,* (M_a) *has a unique blow-up solution* u_a*, which satisfies*

$$u_a(x)/\phi\big(d(x)\big) \to 1 \quad \text{as } d(x) := \operatorname{dist}(x, \varGamma_\infty^c) \to 0, \tag{4}$$

where ϕ *is given by,*

$$\phi(t) = \psi^{\leftarrow}\big(1/\big[tk(t)\big]^2\big) \quad \text{for } t > 0 \text{ small}, \tag{5}$$

and ψ *is defined on some interval* $[\alpha, \infty) \subset (0, \infty)$ *by,*

$$\psi(u) = \sup\big\{f(y)/g(y)\colon\ \alpha \leqslant y \leqslant u\big\}, \quad \forall u \geqslant \alpha. \tag{6}$$

Corollary 1.1. *If* $f(u) = \mathrm{e}^{cu} - 1$ $(c > 0)$ *in Theorem* 1.2*, then the unique blow-up solution* u_a *satisfies:* $\frac{u_a(x)}{\ln d(x)} \to -\frac{2}{c\,\ell_1}$ *as* $d(x) \to 0$.

We point out that Theorem 1.2 does not concern the quotient of $u_a(x)$ and $\varUpsilon(d(x))$, as established in Bandle–Marcus [1] (for $a = 0$ and $b = 1$), where $\varUpsilon$ is a chosen solution of the singular problem: $u''(r) = f(u(r))$ on $(0, \tau)$ for some $\tau > 0$, $u(r) \to \infty$ as $r \to 0^+$. In contrast, the function ϕ in (4) does not have enough regularity to use it directly in constructing upper and lower solutions near $\varGamma_\infty^c$. The idea is to build smoother versions of ϕ which are asymptotically equivalent to ϕ at the origin. This will be achieved in Lemmas 2.2 and 2.3 via the extreme value theory.

We note an extreme variation phenomenon given that the solution u_a blows-up at $\varGamma_\infty$ in a slow fashion (cf. Remark 3) while f varies rapidly at infinity.

2. Approach

We recall some concepts which appear in the extreme value theory (see [8] or [3]).

Definition 2.1. A non-negative, non-decreasing function V defined on (z, ∞) is $\varPi$-varying (written $V \in \varPi$) if there exists a function $\theta(u) > 0$ such that $\lim_{u\to\infty}(V(\lambda u) - V(u))/\theta(u) = \log\lambda$, for $\lambda > 0$.

The function θ is called an *auxiliary function* and is unique up to asymptotic equivalence.

If $V_1 \in \varPi$, with auxiliary function $\theta(u)$, we say V_1 and V_2 are $\varPi$-equivalent (written $V_1 \overset{\varPi}{\sim} V_2$) if $(V_1(u) - V_2(u))/\theta(u) \to c \in \mathbb{R}$ as $u \to \infty$. In this case $V_2 \in \varPi$ with auxiliary function $\theta(u)$.

Lemma 2.2. *If* $f \in \varGamma$*, with auxiliary function* g*, then there exists a twice differentiable* $V_2 \overset{\varPi}{\sim} f^{\leftarrow}$ *with* $V_2(u) > f^{\leftarrow}(u)$*,* $V_2' \in RV_{-1}$*,* $\lim_{u\to\infty} -uV_2''(u)/V_2'(u) = 1$*, and* $\lim_{u\to\infty} V_2(u)/f^{\leftarrow}(u) = 1$*. Furthermore, if* f *is continuous and increasing on* (D, ∞)*, then* $\lim_{u\to\infty} f(V_2(u))/u = C(\text{Const.}) > 0$ *and*

$$\big(V_2 \circ (1/V_2')^{\leftarrow}\big)(u) \sim \psi^{\leftarrow}(u) \quad \text{as } u \to \infty, \ \text{where } \psi \text{ is defined by (6)}. \tag{7}$$

Proof. By [8, Propositions 0.9 and 0.12], $f^{\leftarrow} \in \varPi$ with auxiliary function $g \circ f^{\leftarrow} \in RV_0$. Thus, by Proposition 0.16 in [8], there exists a twice differentiable $V_2 \overset{\varPi}{\sim} f^{\leftarrow}$ with $V_2(u) > f^{\leftarrow}(u)$, $V_2' \in RV_{-1}$, $\lim_{u\to\infty} -u\frac{V_2''(u)}{V_2'(u)} = 1$. Since $V_2 \in \varPi$ is increasing, we have $\lim_{u\to\infty} V_2(u)/(g \circ f^{\leftarrow})(u) = \infty$ and $V_2 \in RV_0$ (see p. 35 in [8]). Using $V_2 \overset{\varPi}{\sim} f^{\leftarrow}$, we deduce $\lim_{u\to\infty} V_2(u)/f^{\leftarrow}(u) = 1$.

Assuming that f is continuous and increasing on (D, ∞), then $f^{\leftarrow}(u)$ coincides with $f^{-1}(u)$ (the inverse of f at u) for $u > 0$ large. By $V_2 \overset{\varPi}{\sim} f^{\leftarrow}$, we have $\lim_{u\to\infty}(V_2(u) - f^{\leftarrow}(u))/(g \circ f^{\leftarrow})(u) = c \in \mathbb{R}$. By Definition 1.1, we get $\lim_{u\to\infty} f(V_2(u))/u = \mathrm{e}^c > 0$. By (6), we infer that $(\psi \circ f^{\leftarrow})(u) = \sup\{z/(g \circ f^{\leftarrow})(z)\colon\ f(\alpha) \leqslant z \leqslant u\}$ ($\alpha > 0$ is large), so that $\psi \circ f^{\leftarrow} \in RV_1$ and $(\psi \circ f^{\leftarrow})(u) \sim u/(g \circ f^{\leftarrow})(u)$ as $u \to \infty$ (use Theorem 1.5.3 in [3]).

By the construction of V_2 in [8, p. 34] and Proposition 0.15 in [8], we get $\lim_{u\to\infty} uV_2'(u)/(g \circ f^{\leftarrow})(u) = 1$. Consequently, $(\psi \circ f^{\leftarrow})(u) \sim 1/V_2'(u)$ as $u \to \infty$. It follows that $(\psi \circ f^{\leftarrow})^{\leftarrow}(u) = (f \circ \psi^{\leftarrow})(u) \sim (1/V_2')^{\leftarrow}(u)$ as $u \to \infty$. By the Uniform Convergence Theorem (see [3] or [8]) and $V_2(u) \sim f^{-1}(u)$ as $u \to \infty$, we achieve (7). □

We say $\widehat{Z}(u)$, defined for $u > D$, is a *normalised* regularly varying function of index q (in short, $\widehat{Z} \in NRV_q$) if $\widehat{Z}$ is a positive C^1-function such that $\lim_{u\to\infty} u\widehat{Z}'(u)/\widehat{Z}(u) = q$. By the Karamata Representation Theorem (see [8, p. 17]), we have:

Remark 2. For each $Z \in RV_q$, there exists $\widehat{Z} \in NRV_q$ such that $\widehat{Z}(u) \sim Z(u)$ as $u \to \infty$.

If $f \in \Gamma$ and $k \in \mathcal{K}$, set $\chi(t) = (1/V_2')^{\leftarrow}(1/[tk(t)]^2)$, for $t > 0$ small (with V_2 from Lemma 2.2).

Lemma 2.3. *Suppose $f \in \Gamma$ is continuous and increasing on some interval (D, ∞). If $k \in \mathcal{K}$ with $\ell_1 \neq 0$, then there exists $\hat{\chi} \in C^2(0, \tau)$ satisfying $\lim_{t\to 0^+} \hat{\chi}(t)/\chi(t) = 1$ and the following*: (i) $\lim_{t\to 0^+} \frac{\hat{\chi}(t)}{\hat{\chi}'(t)} = \lim_{t\to 0^+} \frac{\hat{\chi}'(t)}{\hat{\chi}''(t)} = 0$ *and* $\lim_{t\to 0^+} \frac{\hat{\chi}(t)\hat{\chi}''(t)}{[\hat{\chi}'(t)]^2} = \frac{2+\ell_1}{2}$; (ii) $\lim_{t\to 0^+} P_1(t) := \lim_{t\to 0^+} \frac{V_2(\hat{\chi}(t))}{V_2'(\hat{\chi}(t))} \frac{\hat{\chi}(t)}{[\hat{\chi}'(t)]^2} = 0$ *and* $\lim_{t\to 0^+} P_2(t) := \lim_{t\to 0^+} \frac{k^2(t)(f\circ V_2)(\hat{\chi}(t))}{\hat{\chi}''(t)V_2'(\hat{\chi}(t))} = \frac{C\ell_1^2}{2(2+\ell_1)}$.

Proof. By Lemma 2.2, $1/V_2'(u) \in NRV_1$ so that $(1/V_2')^{\leftarrow}(u) \in NRV_1$. Since $k \in \mathcal{K}$ with $\ell_1 \neq 0$, we have $k(1/u) \in NRV_{1-1/\ell_1}$ (see [4]). Therefore, $\chi(1/u) \in NRV_{2/\ell_1}$. By Karamata's Theorem [8, p. 17], we get $\frac{\mathrm{d}}{\mathrm{d}u}[\chi(1/u)] \in RV_{-1+2/\ell_1}$. Hence, $-\chi'(1/u) \in RV_{1+2/\ell_1}$. By Remark 2, there exists $\hat{\chi} \in C^2(0, \tau)$ such that $-\hat{\chi}'(1/u) \in NRV_{1+2/\ell_1}$ and $\hat{\chi}'(1/u) \sim \chi'(1/u)$ as $u \to \infty$.

It follows that $\lim_{t\to 0^+} \hat{\chi}'(t)/\chi'(t) = 1 = \lim_{t\to 0^+} \hat{\chi}(t)/\chi(t)$ and $\lim_{t\to 0^+} t\hat{\chi}''(t)/\hat{\chi}'(t) = -(1 + 2/\ell_1)$. Consequently, $\hat{\chi}(1/u) \in NRV_{2/\ell_1}$ (that is, $\lim_{t\to 0^+} t\widehat{\chi}'(t)/\chi(t) = -2/\ell_1$). Thus, (i) follows. Moreover, we have $\lim_{t\to 0^+} \log \hat{\chi}(t)/\log t = -2/\ell_1$ and $\lim_{t\to 0^+} \log(-\hat{\chi}'(t))/\log t = -(1 + 2/\ell_1)$.

Since $\lim_{u\to\infty} \log V_2'(u)/\log u = -1$ and $\lim_{u\to\infty} \log V_2(u)/\log u = 0$, we find $\lim_{t\to 0^+} \log P_1(t) = -\infty$.

Using $V_2' \in NRV_{-1}$ and $\hat{\chi}(t) \sim \chi(t)$ as $t \to 0^+$, by the Uniform Convergence Theorem, we obtain $t^2k^2(t)/V_2'(\hat{\chi}(t)) \sim t^2k^2(t)/V_2'(\chi(t)) = 1$ as $t \to 0^+$. From this and Lemma 2.2, we infer that $\lim_{t\to 0^+} P_2(t) = \lim_{t\to 0^+} \frac{\hat{\chi}(t)}{t^2\hat{\chi}''(t)} \frac{(f\circ V_2)(\hat{\chi}(t))}{\hat{\chi}(t)} = \frac{C\ell_1^2}{2(2+\ell_1)}$. □

Remark 3. If $f \in \Gamma$ is continuous and increasing on (D, ∞) and $k \in \mathcal{K}$ with $\ell_1 \neq 0$, then by Lemmas 2.2 and 2.3, we have $\lim_{t\to 0^+} (V_2 \circ \hat{\chi})(t)/\phi(t) = 1$, where ϕ is given by (5) and $(V_2 \circ \hat{\chi})(1/u)$ belongs to RV_0.

Proof of Theorem 1.2. By Lemma 2.2, $f(V_2(u)) \sim Cu$ as $u \to \infty$ and $(V_2(u))^q \in RV_0$, for any $q \in \mathbb{R}$. Thus, $\lim_{u\to\infty} f(u)/u^2 = \infty$ so that the Keller–Osserman condition holds (i.e., $\int_1^\infty [F(s)]^{-1/2}\,\mathrm{d}s < \infty$, where $F(t) = \int_0^t f(s)\,\mathrm{d}s$). Hence, (M_a) possesses blow-up solutions if and only if $a < \lambda_{\infty,1}$ (see [4] or [6]).

Fix $a < \lambda_{\infty,1}$. Let Γ_∞^c be an arbitrary connected open and closed subset of Γ_∞. Set $d(x) = \mathrm{dist}(x, \Gamma_\infty^c)$.

By (3), there exist some positive constants γ_-, γ_+ and δ such that $\gamma_- \leqslant b(x)/k^2(d(x)) \leqslant \gamma_+$, for all $x \in \Omega$ with $d(x) \leqslant 2\delta$. Choose $\beta_- \in (0, \gamma_-/2)$ and $\beta_+ \in (2\gamma_+, \infty)$. We diminish $\delta > 0$ such that: (i) $d(x)$ is a C^2-function on $\{x \in \Omega\colon\ d(x) < 2\delta\}$; (ii) k is non-decreasing on $(0, 2\delta)$; (iii) $\hat{\chi}''(t) > 0$ on $(0, 2\delta)$, where $\hat{\chi}$ is provided by Lemma 2.3. Let $\sigma \in (0, \delta)$ be arbitrary. With V_2 given by Lemma 2.2, we define

$$u_\sigma^\pm(x) := V_2\big(m(\beta_\mp)^{-1}\hat{\chi}\big(d(x) \mp \sigma\big)\big) > 0, \quad \forall x \in \Omega \text{ with } \sigma/2 < d(x) \mp \sigma/2 < 2\delta - \sigma/2, \tag{8}$$

where $m := (C\ell_1/2)^{-1}$ ($C > 0$ from Lemma 2.2). For simplicity, we put $J^\pm(x) := m(\beta_\mp)^{-1}\hat{\chi}(d(x) \mp \sigma)$.

We prove that, by diminishing $\delta > 0$, u_σ^+ and u_σ^- become upper and lower solutions near the boundary:

$$\pm\big[-\Delta u_\sigma^\pm - au_\sigma^\pm + b(x)f(u_\sigma^\pm)\big] \geqslant 0, \quad \forall x \in \Omega \text{ with } \sigma/2 < d(x) \mp \sigma/2 < 2\delta - \sigma/2. \tag{9}$$

One can see that

$$\Delta u_\sigma^\pm = m(\beta_\mp)^{-1}\hat{\chi}''\big(d(x)\mp\sigma\big)V_2'(J^\pm)\left[1+\frac{J^\pm V_2''(J^\pm)}{V_2'(J^\pm)}\frac{[\hat{\chi}']^2}{\hat{\chi}\hat{\chi}''}\big(d(x)\mp\sigma\big)+\Delta d(x)\frac{\hat{\chi}'}{\hat{\chi}''}\big(d(x)\mp\sigma\big)\right]. \quad (10)$$

We denote by $S^\pm(d\mp\sigma)$ the last factor in the right-hand side of (10). It follows that $\pm[-\Delta u_\sigma^\pm - au_\sigma^\pm + b(x)f(u_\sigma^\pm)] \geqslant \pm m(\beta_\mp)^{-1}\hat{\chi}''(d\mp\sigma)V_2'(J^\pm)K^\pm(d\mp\sigma)$, where

$$\begin{aligned}K^\pm(d\mp\sigma) &= \frac{\gamma_\mp\beta_\mp}{m}\frac{k^2(d\mp\sigma)}{\hat{\chi}''(d\mp\sigma)}\frac{f(u_\sigma^\pm)}{V_2'(J^\pm(x))} - \frac{a}{m}\frac{\beta_\mp}{\hat{\chi}''(d\mp\sigma)}\frac{V_2(J^\pm(x))}{V_2'(J^\pm(x))} - S^\pm(d\mp\sigma)\\ &=: T_1(d\mp\sigma)+T_2(d\mp\sigma)-S^\pm(d\mp\sigma).\end{aligned}$$

By Lemmas 2.2 and 2.3, $\lim_{d\mp\sigma\to 0} T_1(d\mp\sigma) = (\gamma_\mp/\beta_\mp)\ell_1/(2+\ell_1)$, $\lim_{d\mp\sigma\to 0} S^\pm(d\mp\sigma) = \ell_1/(2+\ell_1)$ and $\lim_{d\mp\sigma\to 0} T_2(d\mp\sigma)=0$. Hence $\lim_{d\mp\sigma\to 0} K^\pm(d\mp\sigma) = (\gamma_\mp/\beta_\mp - 1)\ell_1/(2+\ell_1)$. This proves (9).

Proof of (4). Let $\zeta>0$ be small such that a is less than the first Dirichlet eigenvalue of $(-\Delta)$ in the domain $E_\zeta := \{x\in\mathbb{R}^N\setminus\overline{\Omega}\colon d(x)<\zeta\}$. Set $I_\delta=\{x\in\Omega\colon d(x)<\delta\}$ and $\Omega_1 := E_{2\zeta}\cup\{x\in\overline{\Omega}\colon d(x)<\delta\}$, (where $\delta>0$ is as in (9)). Let $p\in C^{0,\mu}(\overline{\Omega}_1)$ be such that $0<p(x)\leqslant b(x)$ for $x\in\Omega$ with $d(x)\leqslant\delta$, $p=0$ in $\overline{E}_\zeta$ and $p>0$ in $\overline{E}_{2\zeta}\setminus\overline{E}_\zeta$. Denote by w a blow-up solution of $-\Delta u = au - p(x)f(u)$ in Ω_1. Note that w is uniformly bounded on Γ_∞^c and $w=\infty$ on $\partial I_\delta\cap\Omega$.

Let u_a be an arbitrary blow-up solution of (M_a). By (9) and (A), we find:

$$\begin{cases} -\Delta(u_a+w)-a(u_a+w)+b(x)f(u_a+w)\geqslant 0\geqslant -\Delta u_\sigma^- - au_\sigma^- + b(x)f(u_\sigma^-) & \text{in } I_\delta,\\ -\Delta(u_\sigma^+ + w)-a(u_\sigma^+ + w)+b(x)f(u_\sigma^+ + w)\geqslant 0\geqslant -\Delta u_a - au_a + b(x)f(u_a) & \text{in } I_\delta\setminus\overline{I}_\sigma,\\ (u_a+w)|_{\partial I_\delta} = \infty > u_\sigma^-|_{\partial I_\delta} \quad\text{and}\quad (u_\sigma^+ + w)|_{\partial(I_\delta\setminus\bar{I}_\sigma)} = \infty > u_a|_{\partial(I_\delta\setminus\bar{I}_\sigma)}. \end{cases}$$

By Lemma 2.1 in [6], we get $u_a + w \geqslant u_\sigma^-$ in I_δ and $u_\sigma^+ + w\geqslant u_a$ in $I_\delta\setminus\bar{I}_\sigma$. Letting $\sigma\to 0$, we arrive at $V_2(m(\beta_+)^{-1}\hat{\chi}(d(x))) - w(x)\leqslant u_a\leqslant V_2(m(\beta_-)^{-1}\hat{\chi}(d(x)))+w(x)$, for each $x\in\Omega$ with $0<d(x)<\delta$. Since $V_2\in RV_0$, by the Uniform Convergence Theorem and Remark 3, we conclude (4). The uniqueness of the blow-up solution follows in a standard way (see e.g., [4] or [6]). □

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