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Probability Theory

Representation theorems for generators of backward stochastic differential equations ☆

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Abstract

It is proved that the generator g of a backward stochastic differential equation (BSDE) can be represented by the solutions of the corresponding BSDEs if and only if g is a Lebesgue generator. ***To cite this article: L. Jiang, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Théorèmes de représentation pour générateurs d'équations différentielles stochastiques rétrogrades. Dans cette Note on montre que le générateur g d'une équation stochastique rétrograde (EDSR) peut être représentré par la solution de l'EDSR correspondante si et seulement si g est un générateur de Lebesgue. ***Pour citer cet article : L. Jiang, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

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Soit $T>0$ un nombre réel donné ; pour chaque $(y,z)\in\mathbf{R}\times\mathbf{R}^d$, $t\in[0,T[$ et $\varepsilon\in]0,T-t]$, notons $(Y_s(g,t+\varepsilon,y+z\cdot(B_{t+\varepsilon}-B_t)),Z_s(g,t+\varepsilon,y+z\cdot(B_{t+\varepsilon}-B_t)))_{s\in[0,t+\varepsilon]}$ la solution unique adapté, carré intégrable de l'equation différentielle stochastique rétrograde suivante :

$$y_s=y+z\cdot(B_{t+\varepsilon}-B_t)+\int_s^{t+\varepsilon}g(u,y_u,z_u)\,\mathrm{d}s-\int_s^{t+\varepsilon}z_u\cdot\mathrm{d}B_u,\quad 0\leqslant s\leqslant t+\varepsilon,$$

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où g est supposé satisfaire les hypothèses usuelles (A1) et (A2) (Section 1). Alors on a :

Théorème 0.1. *Supposons* (A1) *et* (A2) *satisfaites pour* g ; *soit* $1 \leqslant p \leqslant 2$. *Alors, pour chaque triplé* $(t, y, z) \in [0, T[\times \mathbf{R} \times \mathbf{R}^d$, *les deux propriétés suivantes* (i) *et* (ii) *sont equivalentes* :

(i) $g(t, y, z) = L^p\text{-}\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[Y_t(g, t+\varepsilon, y + z \cdot (B_{t+\varepsilon} - B_t)) - y]$;
(ii) $g(t, y, z) = L^p\text{-}\lim_{\varepsilon \to 0^+} \mathbf{E}[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(s, y, z)\,\mathrm{d}s|\mathcal{F}_t]$.

Supposons de plus g *est déterministe et vérifiant* (A3) : $g(\bar{t}, \bar{y}, 0) = 0,\ \forall (\bar{t}, \bar{y}) \in [0, T] \times \mathbf{R}^d$. *Alors pour tout couple* $(y, z) \in \mathbf{R} \times \mathbf{R}^d$, *on a* :

$$g(t, y, z) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\big[Y_0\big(g, t+\varepsilon, y + z \cdot (B_{t+\varepsilon} - B_t)\big) - y\big],\ a.e.,\ \mathrm{d}t.$$

1. Introduction and preliminaries

It is by now well-known (see Pardoux and Peng [6]) that there exists a unique adapted and square integrable solution to a backward stochastic differential equation (BSDE in short) of type,

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad 0 \leqslant t \leqslant T, \tag{1}$$

provided that the function g is Lipschitz in both variables y and z, and that ξ and $(g(t, 0, 0))_{t \in [0,T]}$ are square integrable; g is said to be the generator of the BSDE (1). We denote the unique adapted and square integrable solution of the BSDE (1) by $(Y_t(g, T, \xi), Z_t(g, T, \xi))_{t \in [0,T]}$.

One of the achievements of BSDEs theory is the comparison theorem; some papers, such as Briand et al. [1], Chen [2], Coquet et al. [3], Jiang [5], have been devoted to converse comparison theorem. For studying converse comparison theorem, [1] established the following representation theorem:

$$\forall (t, y, z) \in [0, T[\times \mathbf{R} \times \mathbf{R}^d, \quad g(t, y, z) = L^2\text{-}\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\big[Y_t\big(g, t+\varepsilon, y + z \cdot (B_{t+\varepsilon} - B_t)\big) - y\big], \tag{2}$$

for generators of BSDEs under two additional assumptions that $(g(t, y, z))_{t \in [0,T]}$ is continuous in t for each (y, z) and $\mathbf{E}[\sup_{0 \leqslant t \leqslant T} |g(t, 0, 0)|^2] < \infty$.

This Note proves that, under the usual assumptions, the above equality (2) holds if and only if g is a Lebesgue generator.

Let $T > 0$ be a given real number; let $(\Omega, \mathcal{F}, P)$ be a probability space and $(B_t)_{t \geqslant 0}$ be a d-dimensional standard Brownian motion on this space such that $B_0 = 0$; let $(\mathcal{F}_t)_{t \geqslant 0}$ be the filtration generated by this Brownian motion: $\mathcal{F}_t = \sigma\{B_s,\ s \in [0, t]\} \vee \mathcal{N},\ t \in [0, T]$, where $\mathcal{N}$ is the set of all P-null subsets.

The generator g of a BSDE is a function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \mapsto \mathbf{R}$ such that $(g(t, y, z))_{t \in [0,T]}$ is progressively measurable for each (y, z) and g also satisfies the following assumptions (A1) and (A2):

(A1) (Lipschitz Condition) There exists a constant $K \geqslant 0$, such that P-$a.s.$, we have: $\forall t, \forall y_1, y_2, z_1, z_2$: $|g(t, y_1, z_1) - g(t, y_2, z_2)| \leqslant K(|y_1 - y_2| + |z_1 - z_2|)$.
(A2) $\mathbf{E}[\int_0^T |g(t, 0, 0)|^2\,\mathrm{d}t] < \infty$.

2. Representation theorems for BSDEs

Let $b(\cdot, \cdot) : [0, T] \times \mathbf{R}^n \to \mathbf{R}^n,\ \sigma(\cdot, \cdot) : [0, T] \times \mathbf{R}^n \to \mathbf{R}^{n \times d}$ be two measurable functions satisfying hypotheses (H1), (H2) and (H3):

(H1) $|b(t,x)-b(t,y)|+|\sigma(t,x)-\sigma(t,y)|\leqslant K_1|x-y|,\ \forall x,y\in\mathbf{R}^n, t\in[0,T]$;
(H2) $|b(t,x)|+|\sigma(t,x)|\leqslant K_2(1+|x|),\ \forall x\in\mathbf{R}^n, t\in[0,T]$;
(H3) for each $x\in\mathbf{R}^n$, $t\to b(t,x),\ t\to\sigma(t,x)$ are both right continuous in $t\in[0,T[$,

where K_1 and K_2 are some positive constants.

Given $(t,x)\in[0,T[\times\mathbf{R}^n$, we denote by $X^{t,x}$ the unique solution of the following SDE (3):

$$X_s^{t,x}=x+\int_t^s b(u,X_u^{t,x})\,\mathrm{d}u+\int_t^s\sigma(u,X_u^{t,x})\,\mathrm{d}B_u,\quad s\in[t,T];\quad X_s^{t,x}=x,\quad s\in[0,t]. \tag{3}$$

Then we know that $(X_s^{t,x})_{s\in[0,T]}$ is a continuous and $(\mathcal{F}_s)$-adapted solution with properties that

$$\mathbf{E}\Big[\sup_{0\leqslant s\leqslant T}|X_s^{t,x}|^2\Big]<\infty,\quad\text{and}\quad s\to\mathbf{E}|X_s^{t,x}-x|^2,\quad s\in[0,T],\ \text{is continuous.} \tag{4}$$

Given $(t,x,y,q)\in[0,T[\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n$. Let (A1) and (A2) hold for the generator g. Let $0<\varepsilon\leqslant T-t$; let $(Y_s(g,t+\varepsilon,y+q\cdot(X_{t+\varepsilon}^{t,x}-x)),Z_s(g,t+\varepsilon,y+q\cdot(X_{t+\varepsilon}^{t,x}-x)))_{s\in[0.t+\varepsilon]}$ denote the unique adapted and square integrable solution of the following BSDE:

$$Y_s^\varepsilon=y+q\cdot(X_{t+\varepsilon}^{t,x}-x)+\int_s^{t+\varepsilon}g(u,Y_u^\varepsilon,Z_u^\varepsilon)\,\mathrm{d}u-\int_s^{t+\varepsilon}Z_u^\varepsilon\cdot\mathrm{d}B_u,\quad s\in[0,t+\varepsilon]. \tag{5}$$

Theorem 2.1. *Let* (A1) *and* (A2) *hold for* g; *let* (H1), (H2) *and* (H3) *hold for* b *and* σ; *let* $1\leqslant p\leqslant 2$. *Then for each* $(t,x,y,q)\in[0,T[\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n$, *the following two statements are equivalent*:

(i) $g(t,y,\sigma^*(t,x)q)+q\cdot b(t,x)=L^p\text{-}\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}[Y_t(g,t+\varepsilon,y+q\cdot(X_{t+\varepsilon}^{t,x}-x))-y]$.
(ii) $g(t,y,\sigma^*(t,x)q)=L^p\text{-}\lim_{\varepsilon\to0^+}\mathbf{E}[\frac{1}{\varepsilon}\int_t^{t+\varepsilon}g(u,y,\sigma^*(t,x)q)\,\mathrm{d}u|\mathcal{F}_t]$.

Proof. The main idea of this proof is motivated by [1]. Given $(t,x,y,q)\in[0,T[\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n$. For simplicity we denote by $(Y_s^\varepsilon,Z_s^\varepsilon)_{s\in[0,t+\varepsilon]}$ the solution of the BSDE (5). For $s\in[t,t+\varepsilon]$, we set $\widetilde{Y}_s^\varepsilon:=Y_s^\varepsilon-(y+q\cdot(X_s^{t,x}-x))$, $\widetilde{Z}_s^\varepsilon:=Z_s^\varepsilon-\sigma^*(s,X_s^{t,x})q$. Then applying Itô's formula to $\widetilde{Y}_s^\varepsilon$, we have:

$$\widetilde{Y}_s^\varepsilon=\int_s^{t+\varepsilon}\big[g\big(u,\widetilde{Y}_u^\varepsilon+y+q\cdot(X_u^{t,x}-x),\widetilde{Z}_u^\varepsilon+\sigma^*(u,X_u^{t,x})q\big)+q\cdot b(u,X_u^{t,x})\big]\,\mathrm{d}u-\int_s^{t+\varepsilon}\widetilde{Z}_u^\varepsilon\cdot\mathrm{d}B_u. \tag{6}$$

Now we have the following proposition:

Proposition 2.2. $\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}\mathbf{E}[\sup_{t\leqslant s\leqslant t+\varepsilon}|\widetilde{Y}_s^\varepsilon|^2+\int_t^{t+\varepsilon}|\widetilde{Z}_s^\varepsilon|^2\,\mathrm{d}s]=0$.

Proof. By Proposition 2.2 of [1], (A1), (H1) and Hölder's inequality, we know there exists a universal constant C such that

$$\mathbf{E}\Big[\sup_{t\leqslant s\leqslant t+\varepsilon}|\widetilde{Y}_s^\varepsilon|^2+\int_t^{t+\varepsilon}|\widetilde{Z}_u^\varepsilon|^2\,\mathrm{d}u\Big]$$
$$\leqslant C\,\mathrm{e}^{2(K+K^2)T}\mathbf{E}\Bigg[\bigg(\int_t^{t+\varepsilon}\big|g\big(u,y+q\cdot(X_u^{t,x}-x),\sigma^*(u,X_u^{t,x})q\big)+q\cdot b(u,X_u^{t,x})\big|\,\mathrm{d}u\bigg)^2\Bigg]$$

$$\leqslant C\,\mathrm{e}^{2(K+K^2)T}\mathbf{E}\Bigg[\Bigg(\int\limits_t^{t+\varepsilon}\big(\big|g\big(u,y,\sigma^*(u,x)q\big)\big|+(K+KK_1+K_1)|q||X_u^{t,x}-x|+\big|q\cdot b(u,x)\big|\big)\,\mathrm{d}u\Bigg)^2\Bigg]$$

$$\leqslant 4C_1\varepsilon\mathbf{E}\Bigg[\int\limits_t^{t+\varepsilon}\big(\big|g\big(u,y,\sigma^*(u,x)q\big)\big|^2+|q|^2|X_u^{t,x}-x|^2+\big|q\cdot b(u,x)\big|^2\big)\,\mathrm{d}u\Bigg],$$

where $C_1 := C\mathrm{e}^{2(K+K^2)T}(1+K+KK_1+K_1)^2$ is a positive constant.

By (H2), the Fubini theorem and (4), we have:

$$\mathbf{E}\Bigg[\int\limits_t^{t+\varepsilon}\big|q\cdot b(u,x)\big|^2\,\mathrm{d}u\Bigg]=\int\limits_t^{t+\varepsilon}\big|q\cdot b(u,x)\big|^2\,\mathrm{d}u\leqslant\int\limits_t^{t+\varepsilon}|q|^2K_2^2\big(1+|x|\big)^2\,\mathrm{d}u\to 0\quad(\varepsilon\to 0^+),\tag{7}$$

$$\mathbf{E}\Bigg[\int\limits_t^{t+\varepsilon}|q|^2|X_u^{t,x}-x|^2\,\mathrm{d}u\Bigg]\leqslant\int\limits_t^{t+\varepsilon}|q|^2 2\mathbf{E}\big(|X_u^{t,x}|^2+|x|^2\big)\,\mathrm{d}u\to 0\quad(\varepsilon\to 0^+).\tag{8}$$

By the Lipschitz condition (A1) and (H2), we have:

$$\begin{aligned}\big|g\big(u,y,\sigma^*(u,x)q\big)\big|^2&\leqslant 2\big|g\big(u,y,\sigma^*(t,x)q\big)\big|^2+2K^2\big|\sigma^*(u,x)q-\sigma^*(t,x)q\big|^2\\&\leqslant 2\big|g\big(u,y,\sigma^*(t,x)q\big)\big|^2+8K^2K_2^2\big(1+|x|\big)^2|q|^2.\end{aligned}$$

Thanks to (A2) and the absolute continuity of integral, we have $\lim_{\varepsilon\to 0^+}\mathbf{E}[\int_t^{t+\varepsilon}|g(u,y,\sigma^*(t,x)q)|^2\,\mathrm{d}u]=0$. Thus we also have:

$$\lim_{\varepsilon\to 0^+}\mathbf{E}\Bigg[\int\limits_t^{t+\varepsilon}\big|g\big(u,y,\sigma^*(u,x)q\big)\big|^2\,\mathrm{d}u\Bigg]=0.\tag{9}$$

Thus Proposition 2.2 follows from (7), (8) and (9). □

We set:

$$M_t^\varepsilon := \frac{1}{\varepsilon}\mathbf{E}\Bigg[\int\limits_t^{t+\varepsilon}g\big(u,\widetilde{Y}_u^\varepsilon+y+q\cdot(X_u^{t,x}-x),\widetilde{Z}_u^\varepsilon+\sigma^*(u,X_u^{t,x})q\big)\,\mathrm{d}u\Big|\mathcal{F}_t\Bigg],$$

$$N_t^\varepsilon := \frac{1}{\varepsilon}\mathbf{E}\Bigg[\int\limits_t^{t+\varepsilon}g\big(u,y+q\cdot(X_u^{t,x}-x),\sigma^*(u,X_u^{t,x})q\big)\,\mathrm{d}u\Big|\mathcal{F}_t\Bigg],$$

$$P_t^\varepsilon := \frac{1}{\varepsilon}\mathbf{E}\Bigg[\int\limits_t^{t+\varepsilon}g\big(u,y,\sigma^*(u,x)q\big)\,\mathrm{d}u\Big|\mathcal{F}_t\Bigg],\quad Q_t^\varepsilon := \frac{1}{\varepsilon}\mathbf{E}\Bigg[\int\limits_t^{t+\varepsilon}g\big(u,y,\sigma^*(t,x)q\big)\,\mathrm{d}u\Big|\mathcal{F}_t\Bigg].$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ in the BSDE (6), we get:

$$\frac{1}{\varepsilon}(Y_t^\varepsilon-y)=\frac{1}{\varepsilon}\widetilde{Y}_t^\varepsilon=M_t^\varepsilon+\frac{1}{\varepsilon}\mathbf{E}\Bigg[\int\limits_t^{t+\varepsilon}q\cdot b(u,X_u^{t,x})\,\mathrm{d}u\Big|\mathcal{F}_t\Bigg].\tag{10}$$

Thus we have:

$$\frac{1}{\varepsilon}(Y_t^\varepsilon - y) - \left[g\big(t, y, \sigma^*(t,x)q\big) + q\cdot b(t,x)\right] = (M_t^\varepsilon - N_t^\varepsilon) + (N_t^\varepsilon - P_t^\varepsilon) + (P_t^\varepsilon - Q_t^\varepsilon)$$
$$+ \frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} q\cdot b(u, X_u^{t,x})\,\mathrm{d}u\,\Big|\mathcal{F}_t\right] - q\cdot b(t,x) + \mathbf{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g\big(u,y,\sigma^*(t,x)q\big)\,\mathrm{d}u\,\Big|\mathcal{F}_t\right] - g\big(t,y,\sigma^*(t,x)q\big).$$

By Jensen's inequality, Hölder's inequality and the Lipschitz condition (H1) we conclude:

$$\mathbf{E}\left|\frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} q\cdot b(u, X_u^{t,x})\,\mathrm{d}u\,\Big|\mathcal{F}_t\right] - q\cdot b(t,x)\right|^2$$
$$= \mathbf{E}\left|\frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} q\cdot\big(b(u,X_u^{t,x}) - b(u,x)\big)\,\mathrm{d}u\,\Big|\mathcal{F}_t\right] + \frac{1}{\varepsilon}\int_t^{t+\varepsilon} q\cdot\big(b(u,x)-b(t,x)\big)\,\mathrm{d}u\right|^2$$
$$\leqslant \frac{2}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} |q|^2K_1^2|X_u^{t,x} - x|^2\,\mathrm{d}u\right] + \frac{2}{\varepsilon}\int_t^{t+\varepsilon}|q|^2\big|b(u,x)-b(t,x)\big|^2\,\mathrm{d}u.$$

Noticing that $\mathbf{E}[|X_t^{t,x} - x|^2] = 0$, then by (4) and the right continuity of b, we get:

$$\lim_{\varepsilon\to 0^+}\frac{2}{\varepsilon}\int_t^{t+\varepsilon}|q|^2K_1^2\mathbf{E}\big[|X_u^{t,x}-x|^2\big]\,\mathrm{d}u = 0, \qquad \lim_{\varepsilon\to 0^+}\frac{2}{\varepsilon}\int_t^{t+\varepsilon}|q|^2\big|b(u,x)-b(t,x)\big|^2\,\mathrm{d}u = 0. \tag{11}$$

Therefore,

$$L^2\text{-}\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} q\cdot b(u,X_u^{t,x})\,\mathrm{d}u\,\Big|\mathcal{F}_t\right] = q\cdot b(t,x). \tag{12}$$

By Jensen's inequality, Hölder's inequality, (A1) and Proposition 2.2 we deduce:

$$\mathbf{E}[M_t^\varepsilon - N_t^\varepsilon]^2 \leqslant \frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} 2K^2\big(|\widetilde{Y}_u^\varepsilon|^2 + |\widetilde{Z}_u^\varepsilon|^2\big)\,\mathrm{d}u\right] \to 0 \quad (\varepsilon\to 0^+). \tag{13}$$

By Jensen's inequality, Hölder's inequality, (A1), (H1) and (11) we conclude:

$$\mathbf{E}[N_t^\varepsilon - P_t^\varepsilon]^2 \leqslant \frac{1}{\varepsilon}\int_t^{t+\varepsilon} K^2|q|^2(1+K_1)^2\mathbf{E}|X_u^{t,x}-x|^2\,\mathrm{d}u \to 0 \quad (\varepsilon\to 0^+). \tag{14}$$

By Jensen's inequality, Hölder's inequality and (A1) we conclude:

$$\mathbf{E}[P_t^\varepsilon - Q_t^\varepsilon]^2 \leqslant \frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} K^2|q|^2\big|\sigma^*(u,x) - \sigma^*(t,x)\big|^2\,\mathrm{d}u\right],$$

it follows from the right continuity of $\sigma(\cdot,x)$ that

$$\mathbf{E}[P_t^\varepsilon - Q_t^\varepsilon]^2 \leqslant \frac{1}{\varepsilon}\int_t^{t+\varepsilon} K^2|q|^2\big|\sigma^*(u,x)-\sigma^*(t,x)\big|^2\,\mathrm{d}u \to 0 \quad (\varepsilon\to 0^+). \tag{15}$$

Thus Theorem 2.1 follows from (12)–(14) and (15). □

By Theorem 2.1, we can get the following immediately:

Theorem 2.3. *Let* (A1) *and* (A2) *hold for* g; *let* $1 \leqslant p \leqslant 2$. *Then for each triplet* $(t, y, z) \in [0, T[\times \mathbf{R} \times \mathbf{R}^d$, *the following two statements are equivalent*:

(i) $g(t, y, z) = L^p\text{-}\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}[Y_t(g, t+\varepsilon, y + z\cdot(B_{t+\varepsilon} - B_t)) - y]$;
(ii) $g(t, y, z) = L^p\text{-}\lim_{\varepsilon\to 0^+} \mathbf{E}[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(s, y, z)\,\mathrm{d}s|\mathcal{F}_t]$.

Remark 1. Given $(t, y, z) \in [0, T[\times \mathbf{R} \times \mathbf{R}^d$, if (ii) of Theorem 2.3 holds at (t, y, z), then, we say (t, y, z) is a *conditional Lebesgue point* of g (in the L^p sense). We say g is a *Lebesgue generator* (in the L^p sense) if all $(t, y, z) \in [0, T[\times \mathbf{R} \times \mathbf{R}^d$ are conditional Lebesgue points of g (in the L^p sense).

If g is a deterministic generator, i.e., $g(t, y, z) : [0,T] \times \mathbf{R} \times \mathbf{R}^d \mapsto \mathbf{R}$, then we have the following:

Theorem 2.4. *Let* g *be deterministic and* (A1) *and* (A2) *hold for* g. *Then for each* $(y, z) \in \mathbf{R} \times \mathbf{R}^d$,

(i) $g(t, y, z) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}[Y_t(g, t+\varepsilon, y + z\cdot(B_{t+\varepsilon} - B_t)) - y]$, $a.e.$, $\mathrm{d}t$;
(ii) *Moreover, if* g *also satisfies assumption* (A3): $g(\bar t, \bar y, 0) = 0$, $\forall(\bar t, \bar y) \in [0, T] \times \mathbf{R}^d$. *Then*

$$g(t, y, z) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\big[Y_0\big(g, t+\varepsilon, y + z\cdot(B_{t+\varepsilon} - B_t)\big) - y\big],\ a.e.,\ \mathrm{d}t.$$

Proof. (i): Given $(y, z) \in \mathbf{R} \times \mathbf{R}^d$. Since (A2) holds for g and g is deterministic, then, by Lebesgue Theorem (see Hewitt and Stromberg [4, Lemma 18.4]) we know that $g(t, y, z) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(s, y, z)\,\mathrm{d}s$, $a.e.$, $\mathrm{d}t$. Because g is deterministic and $y + z\cdot(B_{t+\varepsilon} - B_t)$ is independent of $\mathcal{F}_t$, then, by [1, Proposition 3.1], we know that $Y_t(g, t+\varepsilon, y + z\cdot(B_{t+\varepsilon} - B_t))$ is deterministic. Thus (i) follows from Theorem 2.3.

(ii): Since $Y_t(g, t+\varepsilon, y + z\cdot(B_{t+\varepsilon} - B_t))$ is deterministic, (ii) follows from (A3) and (i). □

By Lebesgue's dominated convergence theorem, the following is obvious:

Proposition 2.5. *Let* (A1) *and* (A2) *hold for* g; *let* $1 \leqslant p \leqslant 2$; *let* $(g(t, y, z))_{0\leqslant t\leqslant T}$ *be right continuous in* t *for each* (y, z). *Suppose for each* $t \in [0, T[$, *there exists a positive constant* δ_t *such that* $\delta_t < T - t$ *and* $E[\sup_{s\in[t,t+\delta_t]} |g(s, 0, 0)|^p] < \infty$. *Then* g *is a Lebesgue generator* (*in the* L^p *sense*).

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