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Complex Analysis

Gevrey properties of real planar singularly perturbed systems

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Abstract

By applying geometric techniques to real analytic singularly perturbed vector fields on the plane, we develop a way to give a bound on the Gevrey type of the Taylor development of center manifolds at normally hyperbolic turning points, and show that the same technique is useful in the study of degenerate planar turning points and their corresponding canard manifolds. At the end of the Note, we motivate the interest in Gevrey asymptotics by briefly discussing its relation with bifurcation delay. ***To cite this article: P. De Maesschalck, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Propriétés Gevrey de systèmes singulièrement perturbées. Suivant l'approche géométrique dans l'étude de problèmes de perturbations singulières dans le plan, nous développons une méthode pour majorer le type Gevrey des variétés centrales aux points normalement hyperboliques, et des variétés canards aux points tournants. A la fin de la note, nous motivons l'intérêt de l'asymptotique Gevrey en décrivant le rapport avec le retard à la bifurcation. ***Pour citer cet article : P. De Maesschalck, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

Consider a real analytic family of vector fields X_ϵ on the plane. We suppose that γ is an isolated curve of singular points of X_0, that does not persist for $\epsilon > 0$ (e.g. the family is singularly perturbed). In general, points of γ are normally hyperbolic w.r.t. X_0. This situation is locally modelled by the nonlinear analytic

$$X_\epsilon:\quad \begin{cases} \dot{x} = \epsilon^\sigma, \\ \dot{y} = F(x)y + \epsilon G(x, y, \epsilon), \end{cases} \tag{1}$$

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with $\sigma \in \mathbf{N}_1$, F and G real analytic near $(x, y, \epsilon) = (0, 0, 0)$ and with $F(0) \neq 0$. It is well known that there exists a unique formal expansion $y = \sum_{n=1}^{\infty} y_n(x)\epsilon^n$ that is defined for all $x \in D$, where D is some open region containing the origin, for which this series is a formal solution to $\epsilon^\sigma \frac{\mathrm{d}y}{\mathrm{d}x} = yF + \epsilon G$. Also the asymptotics of such series is known (see for example [5]): it is Gevrey-$1/\sigma$ of type T for some $T > 0$, meaning that $\sup_x |y_n(x)| \leqslant CT^{n/\sigma}\Gamma(\alpha + n/\sigma)$ for some $C, \alpha > 0$. Our aim is to develop a method to give a bound for the type T at $x = 0$, in a way that serves for the study of turning point situations. Such situations are modelled by systems where an isolated point of γ is not normally hyperbolic. To fix the ideas, we concentrate on the model (2), although our method is applicable to a wider range of systems:

$$X_{\epsilon,a}\colon \quad \begin{cases} \dot{x} = \epsilon^\sigma, \\ \dot{y} = a + f(x)y + \epsilon^\sigma g(x, y, \epsilon), \end{cases} \tag{2}$$

where $f(x) = \lambda x^p + o(x^p)$ as $x \to 0$ (with $\lambda > 0$ and p odd). For $p = 1$, this turning point system has a formal canard solution $y = \sum_{n=1}^{\infty} y_n(x)\epsilon^n$ and control curve $a = \sum_{n=1}^{\infty} a_n\epsilon^n$. It is known that there exist actual overstable solutions (these are canard solutions defined in a full complex neighbourhood of the turning point) and control curves, asymptotic to the formal series. The presence of overstable solutions is established in a small neighbourhood (exponentially small w.r.t. ϵ) of a control curve in the (ϵ, a)-plane. For $p > 1$, a similar result is true [1], provided that one looks for overstable solutions in more-dimensional parameter spaces $(\epsilon, a_1, \ldots, a_p)$: overstable solutions are found in an exponentially small neighbourhood of a (codimension-p) curve in the $(\epsilon, a_1, \ldots, a_p)$-plane. On the other hand, using C^∞ geometric singular perturbation theory (see [3,2]), one can prove (for $p = 1$ as well as for $p > 1$) the existence of real canard solutions $y = \phi(x, \epsilon)$, in the neighbourhood of a codimension-1 manifold $a = \mathcal{A}(\epsilon)$ in the parameter space (ϵ, a). By searching for canard solutions in $\mathbf{R}$ instead of overstable solutions, a broader class of solutions is found. However, manifolds of such canard solutions are not necessarily differentiable at the turning point and hence do not necessarily have a formal expansion. Indeed, using the geometric singular perturbation theory, canard solutions are obtained in a blown up phase space. The smoothness of the canards is expressed w.r.t. blow up coordinates, and is lost after blowing down. Nevertheless, the control curve $a = \mathcal{A}(\epsilon)$ in parameter space has a well-defined asymptotic expansion w.r.t. ϵ. A combination of analytic techniques with the notion 'family blow up' leads to a proof that also this control curve satisfies Gevrey-$1/\sigma$ estimates. In Section 3, we determine a bound on its Gevrey type.

2. Treating normally hyperbolic points

We consider (1) and define $X = X_\epsilon + 0\frac{\partial}{\partial\epsilon}$. In the next section, we will show how a similar treatment applies to (2). We blow up γ with the quasi-homogeneous cylindrical blow up formulas

$$(x, \epsilon) = (u^\sigma \bar{x}, u\bar{\epsilon}), \quad u \geqslant 0,\ (\bar{x}, \bar{\epsilon}) \in S^1_+ := \{(\bar{x}, \bar{\epsilon}) \in S^1 \subset \mathbf{R}^2\colon\ \bar{\epsilon} \geqslant 0\}.$$

The study of the blown up vector field is done in charts. We are in particular interested in the phase-directional rescaling chart $\{\bar{x} = -1\}$ and the family rescaling chart $\{\bar{\epsilon} = 1\}$. In the first chart, we use the transformation formulas $\{x = -u^\sigma, \epsilon = u\bar{\epsilon}\}$ so that there, X is given by (after multiplication with σ):

$$\sigma X\colon \quad \begin{cases} \dot{u} = -u\bar{\epsilon}^\sigma, \\ \dot{\bar{\epsilon}} = \bar{\epsilon}^{\sigma+1}, \\ \dot{y} = \sigma F(-u^\sigma)y + \sigma u\bar{\epsilon}G(-u^\sigma, y, u\bar{\epsilon}). \end{cases} \tag{3}$$

The results that we present are formulated in terms of (3). We complexify σX, i.e. consider it for $u, \bar{\epsilon}, y \in \mathbf{C}$, but to keep $\bar{\epsilon}$ close to the positive real axis, and let u vary in the complex plane. For all $u \in \mathbf{C}$ where it makes sense, i.e. for which $[0, u] \times \{0\} \times \{0\}$ lies in the interior of the domain of (3), we define

$$R(u) := \Re\left(\frac{1}{u^\sigma}\int_0^u \sigma\tilde{u}^{\sigma-1}F(-\tilde{u}^\sigma)\,\mathrm{d}\tilde{u}\right).$$

Theorem 2.1 [2]. *Consider the boundary curve Σ: $\{u = u_0, y = s(\bar\epsilon)\}$, with s analytic at $\bar\epsilon = 0$ and with $u_0 \in \mathbf{C}$ such that $R(u_0) < 0$. There exists a manifold W: $y = \Phi(u, \bar\epsilon)$ with $\Sigma \subset W$, and such that W is invariant under* (3). *The manifold is defined for $\bar\epsilon$ in some local sectorial neighbourhood of $\bar\epsilon = 0$ containing the positive real axis, and for u in a local sectorial neighbourhood of $u = 0$ containing the complex segment* $[0, u_0]$.

Let $T > 0$ be such that $-1/T = |u_0|^\sigma R(u_0)$. Suppose that $\mathcal{C}(T) := \{u \in \mathbf{C}\colon\ |u|^\sigma R(u) = -1/T\}$ is angle-parametrizable[1] *around the origin. Then, for all $T' > T$ the manifold W is Gevrey-$1/\sigma$ at $u = 0$ with type $T'|\bar\epsilon|^\sigma$ (but as $T' \to T$, the sectorial neighbourhood for $(u, \bar\epsilon)$ may shrink to* $[0, u_0] \times \{0\}$).

The existence of W follows from the saddle behaviour of (3) at the origin, and is based on direct majorations of solutions to (3). The Gevrey estimates are based on a theorem of Ramis–Sibuya, relating sectorial coverings of analytic bounded functions to the Gevrey type. To guarantee the existence of such sectorial coverings, it suffices that $\mathcal{C}(T)$ is angle-parametrizable around the origin $u = 0$. We remark that one can prove that for $T > 0$ large enough, $\mathcal{C}(T)$ is angle-parametrizable around the origin.

Let us now explain how such manifolds are important in the study of the original vector field (1). To that end, we intersect the manifold W with $\{\bar\epsilon = \bar\epsilon_0\}$. This yields a Gevrey-curve $y = \Phi(u, \bar\epsilon_0)$ that is also visible in the family rescaling chart $\{\bar\epsilon = 1\}$. In that chart, we use the transformation formulas $\{x = u^\sigma \bar x,\ \epsilon = u\}$, and it can easily be verified that the above section of W is seen as a curve $\{\bar x = -\bar\epsilon_0^{-\sigma},\ y = \Phi(u/\bar\epsilon_0, \bar\epsilon_0)\}$. This shows that if $y = \Phi(u, \bar\epsilon_0)$ is Gevrey-$1/\sigma$ with type $T'|\bar\epsilon_0|^\sigma$, then in the family rescaling chart this yields a curve that is Gevrey-$1/\sigma$ of type T'. In the family rescaling chart, X is determined by the family

$$X_u\colon\quad \begin{cases} \dot{\bar x} = 1, \\ \dot y = F(u^\sigma \bar x)y + uG(u^\sigma \bar x, y, u) \end{cases}$$

making it clear that in this chart, there are no singular points. Hence, the transition from the plane $\{\bar x = -\bar\epsilon_0^{-\sigma}\}$ to the plane $\{\bar x = 0\}$ is given by an analytic diffeomorphism. Hence, continuing the orbits of the invariant manifold W in this chart, we find a curve $\{y = \theta(u),\ \bar x = 0\}$ with θ Gevrey-$1/\sigma$ of type T', for all $T' > T$. Blowing down this curve gives the curve $\{y = \theta(\epsilon), x = 0\}$. This yields:

Theorem 2.2. *Let T be a value for which $\mathcal{C}(T)$ is angle-parametrizable around $u = 0$. Then, the unique formal expansion solving the o.d.e. associated to* (1) *is, at $x = 0$, Gevrey-$1/\sigma$ of type T', for any $T' > T$.*

3. Treating turning points

A similar technique permits to treat systems like (2). The blow up is not cylindrical, but we blow up the turning point, using

$$(x, y, \epsilon) = (u^\sigma \bar x, u^\sigma \bar y, u^m \bar\epsilon), \quad u \geqslant 0,\ (\bar x, \bar y, \bar\epsilon) \in S^2,\ \bar\epsilon \geqslant 0,$$

with $m = p + 1$. In the search for canards, we restrict the parameter space (ϵ, a) to a region $a\epsilon^{-\sigma} \in [-A_0, A_0]$. To that end, we replace a by $A\epsilon^\sigma$. In the phase-directional rescaling chart $\{\bar x = -1\}$, the vector field (2) yields (after division by $u^{p\sigma}/\sigma$):

$$\sigma u^{-p\sigma} X\colon\quad \begin{cases} \dot u = -u\bar\epsilon^\sigma, \\ \dot{\bar\epsilon} = m\bar\epsilon^{\sigma+1}, \\ \dot{\bar y} = \sigma\bar\epsilon^\sigma A + \sigma u^{-p\sigma} f(-u^\sigma)\bar y + \sigma\bar\epsilon^\sigma g(-u^\sigma, u^\sigma \bar y, u^m \bar\epsilon) + \sigma\bar\epsilon^\sigma \bar y. \end{cases} \tag{4}$$

[1] A set $\Omega \subset \mathbf{C}$ is *angle-parametrizable* around $0 \in \mathbf{C}$ if Ω contains the image of $\theta \to r(\theta)\,\mathrm{e}^{\mathrm{i}\theta}$ for some positive 2π-periodic C^0-function r.

Write $F(u) = u^{-p\sigma} f(-u^\sigma)$. (Then, F is analytic and $F(0) = -\lambda < 0$.) The new equation is of the same form as (3), except that the invariant foliation $\mathrm{d}(u\bar\epsilon) = 0$ in (3) is replaced by $\mathrm{d}(u^m\bar\epsilon) = 0$. We define $R_m(u) := \Re(\frac{1}{u^{m\sigma}} \int_0^u \sigma \tilde u^{m\sigma-1} F(\tilde u)\,\mathrm{d}\tilde u)$, and draw conclusions similar to those in Theorem 2.1:

Theorem 3.1 [2]. *Consider the boundary curve Σ: $\{u = u_0, y = s(\bar\epsilon, A)\}$, with s analytic at $\bar\epsilon = 0$ and with $u_0 \in \mathbf{C}$ such that $R_m(u_0) < 0$. There exists a manifold W: $y = \Phi(u, \bar\epsilon, A)$ invariant under* (4) *and with $\Sigma \subset W$. It is defined for $\bar\epsilon$ in a local sectorial neighbourhood of $\bar\epsilon = 0$ containing the positive real axis, and for u in some local sectorial neighbourhood of $u = 0$ containing the segment* $[0, u_0]$.

Let $T > 0$ be such that $-1/T = |u_0|^{m\sigma} R_m(u_0)$. Suppose that $\mathcal{C}_m(T) := \{u \in \mathbf{C}\colon |u|^{m\sigma} R_m(u) = -1/T\}$ is angle-parametrizable. Then the manifold W is Gevrey-$1/m\sigma$ at $u = 0$ with type $T'|\bar\epsilon|^{m\sigma}$, for all $T' > T$.

As before, the invariant manifolds W are continued in the chart $\bar\epsilon = 1$, where there are no further singularities. This means that the invariant manifold can be continued until its intersection with the plane $\{x = 0\}$ (or in blown up coordinates, with the plane $\{\bar x = 0\}$ in the chart $\{\bar\epsilon = 1\}$). Such intersection is a Gevrey-$1/m\sigma$ curve $\bar y = \theta(u, A)$, uniform in A. As our interest goes to real dynamics, one chooses in particular $u_0 \in \mathbf{R}^+$, so that the manifold W is a real manifold over the attracting real branch of γ. The same construction can be repeated if one looks in the chart $\{\bar x = 1\}$, where the repelling real branch of γ is seen for $u \geqslant 0$. One can now take two boundary curves Σ_- (in the chart $\{\bar x = -1\}$, at $u_0 \in \mathbf{R}^+$) and Σ_+ (in the chart $\{\bar x = +1\}$, at $u_0' \in \mathbf{R}^+$), and 'match' both invariant manifolds in the plane $\{\bar x = 0\}$, using an implicit function argument w.r.t. A. A Gevrey-version of the implicit function theorem allows to find a curve in parameter space $A = \mathcal{A}(u)$, Gevrey-$1/m\sigma$ in u of type T' for all $T' > T$:

Theorem 3.2 [2]. *Let T be a value for which $\mathcal{C}_m(T)$ is angle-parametrizable around $u = 0$. Then, the unique formal control curve (w.r.t. $\epsilon^{1/m}$, with $m = p + 1$) along which canard solutions are found in* (2) *is Gevrey-$1/m\sigma$ w.r.t. $\epsilon^{1/m}$, of type T', for all $T' > T$. All canard manifolds of* (2) *intersect $\{x = 0\}$ in a curve that is smooth w.r.t. $\epsilon^{1/m}$, which is formally Gevrey-$1/m\sigma$ of type T', for all $T' > T$.*

Systems with canard solutions are systems exhibiting bifurcation delay: orbits tend to follow the repelling branch of γ for a while, instead of immediately diverting from it after passing the turning point. For vector fields X_ϵ, the following question is interesting: does X_ϵ exhibit bifurcation delay, and if so, what is the maximum delay? An equivalent formulation of the question is the following: does X_ϵ have canard solutions, and if so, what is the maximum size of such a canard solution? In general, there is a direct relation between the Gevrey type of $\mathcal{A}$ and the maximum size of complex 'overstable' solutions: one can think of $1/T$ as the distance to the turning point of the first obstruction in resummating formal series to analytic functions in sectors. This first obstruction will limit the size of overstable solutions, but the obstruction may be complex. In that case, it is possible to find better resummation methods in $\mathbf{R}$ (through analytic continuation of the Borel transforms). For a study of real canards, the determination of the type hence no longer suffices to find the maximum canard: a more detailed analysis of the location of obstructions in $\mathbf{R}$ is needed. Therefore, Theorem 3.2 is only a starting point for such an analysis. Nevertheless, it already implies good results for important classes of systems (for example, when $f(x) = \lambda x^p$, g polynomial in (2)). Interesting related results can also be found in [4], where 'S^∞-canards' are studied.

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