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C. R. Acad. Sci. Paris, Ser. I 340 (2005) 479–482

COMPTES RENDUS MATHEMATIQUE

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Complex Analysis

Green functions with analytic singularities

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Received 14 June 2004; accepted after revision 3 February 2005

Available online 16 March 2005

Presented by Jean-Pierre Demailly

Abstract

We study properties of a Green function with singularities determined by a closed complex subspace A of a complex manifold X. It is defined as the largest negative plurisubharmonic function u satisfying locally $u \leqslant \log|\psi| + \mathrm{O}(1)$, where $\psi = (\psi_1, \ldots, \psi_m)$ with $\psi_1, \ldots, \psi_m$ local generators for the ideal sheaf $\mathcal{I}_A$ of A. ***To cite this article: A. Rashkovskii, R. Sigurdsson, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Fonctions de Green avec singularités analytiques. Nous étudions des propriétés d'une fonction de Green avec des singularités determinées par un sous-espace fermé complexe A d'une variété complexe lisse X. Elle est définie comme la plus grande fonction plurisouharmonique u négative vérifiant $u \leqslant \log|\psi| + \mathrm{O}(1)$, où $\psi = (\psi_1, \ldots, \psi_m)$ avec $\psi_1, \ldots, \psi_m$ générateurs locaux du faisceau d'idéaux $\mathcal{I}_A$ de A. ***Pour citer cet article : A. Rashkovskii, R. Sigurdsson, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

Pluricomplex Green functions are, generally speaking, the largest negative plurisubharmonic functions with prescribed singularities; they can be thought of as 'pluripotentials' of the singularities. The most-well known is the pluricomplex Green function $G_{\Omega,a}$ of a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with logarithmic pole at $a \in \Omega$; it can be defined as the upper envelope of the class of negative plurisubharmonic functions u in Ω with the Lelong number $\nu_u(a) \geqslant 1$ or, equivalently, such that $u(z) \leqslant \log|z-a| + \mathrm{O}(1)$ near a. Then $G_{\Omega,a}(z) = \log|z-a| + \mathrm{O}(1)$ near a and $(dd^c G_{\Omega,a})^n = \delta_a$; here $d = \partial + \bar\partial$, $d^c = (\partial - \bar\partial)/2\pi \mathrm{i}$.

Different kinds of singularities determine different types of Green functions (e.g., Green functions with several weighted logarithmic poles [7] or with nonlogarithmic multicircled singularities [8]) which, due to the non-linear

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.02.019

character of pluripotential theory, cannot be reproduced from one another. The case of *analytic* singularities is certainly of major importance because of, for example, applications to estimates of holomorphic functions with zeros of given multiplicities in a prescribed set.

Given a complex manifold X, let $\mathrm{PSH}(X)$ denote the class of all plurisubharmonic functions on X and $\mathrm{PSH}^-(X)$ its subclass of all non-positive functions. For a closed complex subspace A of X, let $\mathcal{I}_A = (\mathcal{I}_{A,x})_{x\in X}$ be the associated coherent sheaf of ideals in the sheaf $\mathcal{O}_X$ of germs of holomorphic functions on X, and let $|A|$ be the variety in X locally defined as the common set of zeros of holomorphic functions with germs in $\mathcal{I}_A$. For every point $x \in X$, we denote by $\tilde\nu_A(x)$ the Lelong number $\nu_{\log|\psi|}(x)$ of $\log|\psi|$ at x, where $\psi = (\psi_1,\dots,\psi_m)$ and $\psi_1,\dots,\psi_m$ are generators of $\mathcal{I}_{A,x}$.

A Green function $\widetilde{G}_A = \widetilde{G}_{\tilde\nu_A}$ of a complex subspace A of X was defined in [5] as the upper envelope of the class $\widetilde{\mathcal{F}}_A = \{u \in \mathrm{PSH}^-(X)\colon\ \nu_u \geqslant \tilde\nu_A\}$. It was proved in [4–6] that $\widetilde{G}_A \in \widetilde{\mathcal{F}}_A$ and,

$$\widetilde{G}_A(x) = \inf\left\{\widetilde{G}_{f^*\tilde\nu_A}(0) = \sum_{w\in\mathbb{D}} \tilde\nu_A\big(f(w)\big)\log|w|;\ f \in \mathcal{O}(\overline{\mathbb{D}}, X),\ f(0) = x\right\}, \tag{1}$$

where $\mathcal{O}(\overline{\mathbb{D}}, X)$ is the family of all closed analytic discs in X, i.e., maps from the unit disc $\mathbb{D} \subset \mathbb{C}$ to X that can be extended to holomorphic maps in some neighbourhood of the closed disc. If X is a relatively compact domain in a Stein manifold Y and A is the intersection with X of a complex subspace B of Y, then $\nu_{G_A} = \tilde\nu_A$, the function $\widetilde{G}_A$ is locally bounded and maximal on $X \setminus |A|$, and $\widetilde{G}_A(x) \to 0$ as $x \to p$ at each $p \in \partial X \setminus |B|$ with strong plurisubharmonic barrier. Finally, if A is an effective divisor generated on an open set U by a holomorphic function f, then $\widetilde{G}_A \leqslant \log|f| + \mathrm{O}(1)$ locally on U and thus $dd^c\widetilde{G}_A \geqslant [A]$.

The statements concerning effective divisors generally have no higher codimensional analogues (an example: $\psi(z) = (z_1^2, z_2)$). To overcome this, we introduce here a new Green function.

2. Definition and main properties of the Green function

Throughout the Note, A is a closed complex subspace of a complex manifold X of dimension n.

The Green function G_A with singularities along A is the upper envelope of the class $\mathcal{F}_A$ of all functions $u \in \mathrm{PSH}^-(X)$ such that $u \leqslant \log|\psi| + C$ locally for any choice of local generators $\psi_1,\dots,\psi_m$ for $\mathcal{I}_A$, with a constant C depending on u and the generators.

Evidently, $G_A \leqslant \widetilde{G}_A$ for any A, and it follows from [5], Proposition 3.2, that $G_A = \widetilde{G}_A$ when A is a divisor.

The definition of the Green function gives immediately the following variant of the Schwarz lemma.

Proposition 2.1. *Let A' be the restriction of A to a submanifold X' of X. Then any $f \in \mathcal{I}_A$ has the bound,*

$$\big|f(x)\big| \leqslant \mathrm{e}^{G_{A'}(x)} \sup_{X'}|f|, \quad \forall x \in X'.$$

We pass to the properties of the Green function.

2.1. Upper bounds

The first main property is given by:

Theorem 2.2. $G_A \in \mathcal{F}_A$.

One way to prove it is by reducing to the divisor case. If $\Phi : Y \to X$ is a holomorphic map between complex manifolds X and Y, then the pullback Φ^*A of A is a complex subspace of Y with the ideal sheaf $\mathcal{I}_{\Phi^*A}$, locally generated at a point y_0 by $\Phi^*\psi_1,\dots,\Phi^*\psi_m$, if $\psi_1,\dots,\psi_m$ are local generators for $\mathcal{I}_A$ at $\Phi(y_0)$. It is evident that $\Phi^*u \in \mathcal{F}_{\Phi^*A}$ for all $u \in \mathcal{F}_A$, so $\Phi^*G_A \leqslant G_{\Phi^*A}$. Moreover, the Green function is in fact invariant with respect to finite branched coverings:

Proposition 2.3. *Let X and Y be complex manifolds of the same dimension, A be a closed complex subspace of X, and $\Phi : Y \to X$ be a proper surjective holomorphic map. Then $\Phi^* G_A = G_{\Phi^* A}$.*

By a variant of the Hironaka desingularization theorem (see, e.g., [1], Th. 1.10 and 13.4), there exists a proper surjective holomorphic map Φ from a complex manifold $\widehat{X}$ to X, which is an isomorphism outside $\Phi^{-1}(|A|)$ and such that $\hat{A} = \Phi^* A$ is a normal-crossing divisor. Then $\Phi^* G_A = G_{\hat{A}} = \widetilde{G}_{\hat{A}} \in \widetilde{\mathcal{F}}_{\hat{A}} = \mathcal{F}_{\hat{A}}$. Since Φ is an isomorphism outside $\Phi^{-1}(|A|)$, this implies $G_A \in \mathcal{F}_A$.

A more elementary way (using neither the desingularization nor referring to the divisor case) is as follows. In our definition of the class $\mathcal{F}_A$, the constant C in the local estimates $u \leqslant \log|\psi| + C$ is allowed to depend on the function u. The crucial point is that they are indeed *locally uniform*. This is quite easy to prove in the case of complete intersection. The general situation can be reduced to this case by means of the following result which is of independent interest; its proof uses a method from [9].

Lemma 2.4. *The variety $|A|$ can be decomposed into a disjoint union of local analytic varieties (not necessarily closed) J^k, $1 \leqslant k \leqslant n$, such that* $\operatorname{codim} J^k \geqslant k$ *and each $a \in J^k$ has a neighbourhood U in X and k holomorphic functions $\xi_1, \ldots, \xi_k \in \mathcal{I}_{A,U}$ such that* $\log|\xi| = \log|\psi| + O(1)$ *on U for any generators ψ of A on U (in other words, the local ideal $\mathcal{I}_{A,a}$ has analytic spread at most k).*

2.2. Analytic discs

A fundamental property of the Green function is that it can be obtained as the lower envelope for the disc functional $f \mapsto G_{f^*A}(0)$. If $f : \mathbb{D} \to X$ is an analytic disc, $f(0) = x$, then: $G_A(x) \leqslant G_{f^*A}(0) = \sum_{a\in\mathbb{D}} \tilde{\nu}_{f^*A}(a) \log|a|$. Let $\Phi : \widehat{X} \to X$ be the desingularization map as above with $\hat{A} = \Phi^* A$ an effective divisor. By using the disc formula (1) and the fact that $G_{\hat{A}} = \widetilde{G}_{\hat{A}}$, we get:

Theorem 2.5. $G_A(x) = \inf\{G_{f^*A}(0);\ f \in \mathcal{O}(\overline{\mathbb{D}}, X),\ x = f(0)\}, \forall x \in X$.

2.3. Product property

Let X_1 and X_2 be complex manifolds, A_1 and A_2 be closed complex subspaces of X_1 and X_2 respectively, $X = X_1 \times X_2$, and $A = A_1 \times A_2$. Then $x = (x_1, x_2) \mapsto \max\{u_1(x_1), u_2(x_2)\} \in \mathcal{F}_A$ for all $u_1 \in \mathcal{F}_{A_1}$ and $u_2 \in \mathcal{F}_{A_2}$, so we obviously have $G_A(x) \geqslant \max\{G_{A_1}(x_1), G_{A_2}(x_2)\}$.

The following is called the *product property* for Green functions; its proof is based on Theorem 2.5 and is a modification of the corresponding proofs from [5] and [3].

Theorem 2.6. *Let A_1 and A_2 be closed complex subspaces of complex manifolds X_1 and X_2, respectively, and let A be the product of A_1 and A_2 in $X = X_1 \times X_2$. Then $G_A = \max\{G_{A_1}, G_{A_2}\}$.*

2.4. Maximality

One of the main properties of the 'standard' pluricomplex Green function $G_{\Omega,a}$ is that it satisfies the homogeneous Monge–Ampère equation $(dd^c G_{\Omega,a})^n = 0$, outside its pole a; this means that $G_{\Omega,a}$ is a maximal plurisubharmonic function on $\Omega \setminus \{a\}$. We recall that a function $u \in \mathrm{PSH}(\Omega)$ is called *maximal* in Ω if for any $v \in \mathrm{PSH}(\Omega)$ the relation $\{v > u\} \Subset \Omega$ implies $v \leqslant u$ in Ω. A function $u \in \mathrm{PSH}(\Omega)$ is called *locally maximal* if each point of Ω has a neighbourhood where u is maximal; we do not know if every locally maximal function is maximal (however, this is true for the locally bounded functions).

Lemma 2.4 and the maximality of $\log|\xi|$ for a map ξ with less than n components imply:

Theorem 2.7. *The function G_A is maximal on $X \setminus |A|$ and locally maximal outside a discrete subset of $|A|$ (actually, outside the set J^n from Lemma* 2.4*). If A has $m < n$ global generators on X, then G_A is maximal on the whole X.*

2.5. Spaces with bounded global generators

If A has bounded global generators $\psi_1, \dots, \psi_m$ on X (for example, if X is a relatively compact domain in a Stein manifold Y and A is the intersection with X of a complex subspace B of Y), then the function $\log|\psi| - \sup_X \log|\psi| \in \mathcal{F}_A$. This implies:

Theorem 2.8. *If A has bounded global generators $\psi_1, \dots, \psi_m$ in X, then*: (1) $G_A = \log|\psi| + \mathrm{O}(1)$ *locally in X*; (2) *if X has a strong plurisubharmonic barrier at $p \in \partial X \setminus |A|$, then $G_A(x) \to 0$ as $x \to p$.*

If ψ_i generate $\mathcal{I}_{A,a}$ and $\operatorname{codim}_a|A| = p$, then by the King-Demailly formula ([2], Theorem 6.20), $(dd^c \log|\psi|)^p = Z_A^p + R$ on a neighbourhood U of a, where $Z_A^p = \sum_i m_{i,p}[A_i^p]$, A_i^p are p-codimensional components of $|A| \cap U$, $m_{i,p} \in \mathbb{Z}^+$ is the generic multiplicity of ψ along A_i^p, and R is a positive closed current such that $\chi_{|A|}R = 0$ and $\operatorname{codim} E_c(R) > p$ for every $c > 0$. Here $\chi_{|A|}$ is the characteristic function of the set $|A|$, $E_c(R) = \{x\colon\ \nu(R,x) \geqslant c\}$ and $\nu(R,x)$ is the Lelong number of the current R at x.

In view of Siu's structural formula for closed positive currents and Demailly's Comparison theorem for Lelong numbers ([2], Theorem 5.9), we have:

Theorem 2.9. *Let A have bounded global generators in X. If* $\operatorname{codim}|A| = p$ *on an open set U, then $(dd^c G_A)^p = Z_A^p + Q$ on U, where Q is a positive closed current such that $\chi_{|A|}Q = 0$ and* $\operatorname{codim} E_c(Q) > p$ *for every $c > 0$; if the analytic spread of $\mathcal{I}_{A,U}$ equals p, then Q has zero Lelong numbers.*

2.6. Reduced spaces

Let A be a reduced complex subspace of X, so it can be identified with the analytic variety $|A|$. In this case, the Green function can be described without referring to the generators. In addition, its maximality property has a global character (compare with Theorem 2.7).

Theorem 2.10. *If A is a reduced complex subspace of X with bounded global generators, then*: (1) $G_A(x) = \widetilde{G}_A(x) = \sup\{u(x)\colon\ u \in \mathrm{PSH}^-(X),\ \nu_u(a) \geqslant 1\ \forall a \in \operatorname{Reg} A\}$, $\forall x \in X$; (2) G_A *is a maximal plurisubharmonic function on $X \setminus A_0$, where A_0 is the collection of* 0*-dimensional components of A.*

The first statement can be proved by using either the desingularization procedure as above or Lemma 2.4, and the second follows from Siu's theorem on analyticity of the upper-level sets for Lelong numbers.

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