Functional Analysis

A new approximation result for BV-functions

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Abstract

This Note is devoted to obtaining an approximation result for BV-functions by means of a *quasi-polyhedral* sequence of BV-functions. This approximation could have interesting applications in some problems of the Calculus of Variations. ***To cite this article: M. Amar, V. De Cicco, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***
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Résumé

Un nouveau résultat d'approximation pour fonctions BV. On démontre un théorème de approximation pour une fonction qui appartient à l'espace BV avec une suite quasi-polyédriques de fonctions BV. Cette approximation peut être très utile pour quelques problèmes du Calcul des Variations. ***Pour citer cet article : M. Amar, V. De Cicco, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***
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1. Introduction

In this Note, we prove an approximation result for BV-functions. In this direction, the first classical result is due to Anzellotti–Giaquinta, who proved that BV-functions can be approximated by means of smooth functions u_ε which are essentially obtained by mollification, so that their main interest lies on the $\mathcal{C}^\infty$-regularity.

In the framework of the Calculus of Variations, this type of approximation has been usefully applied in various problems concerning relaxation and Γ-convergence, where $\{u_\varepsilon\}$ plays the role of the 'recovery sequence'. However, in some recent problems it seems more useful to have an approximation of BV-functions which takes into account not so much the $\mathcal{C}^\infty$-regularity of the approximating sequence, as the geometric properties of its discontinuity set. This idea has been firstly developed by Dibos and Séré, in the context of the approximation of minimizers for the Mumford–Shah functional (see [5]). They proved an approximation result for *SBV*-functions by means of functions, still belonging to *SBV*, with their jump set contained in a finite union of smooth hypersurfaces, included in hyperplanes.

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Following the outlines of Dibos and Séré, Cortesani and Toader in [4] (see also [3]) proved that those functions, which have a polyhedral jump set and are of class $\mathcal{C}^\infty$ outside, are dense in $SBV^p \cap L^\infty$, in an appropriate sense connected with the Mumford–Shah functional.

In view of similar possible applications, we propose a new approximation result for general BV-functions, which therefore could also be of Cantor type. This implies that the approximating functions are slightly more general than those proposed by Cortesani and Toader. More precisely, given $u \in BV \cap L^\infty$ we construct a sequence of BV-functions, strictly converging to u, such that their set of approximate discontinuity points is "almost" a polyhedron, in the sense that the $\mathcal{H}^{N-1}$-measure of the non-polyhedral part is small. This result is obtained by refining a classical theorem, due to Federer, of approximation of a countable $\mathcal{H}^{N-1}$-rectifiable set by means of smooth compact manifolds which are arbitrarily close to a polyhedron, apart from a set of small $\mathcal{H}^{N-1}$-measure.

We think that this type of approximation could be useful to study problems of relaxation and Γ-convergence for functionals defined on the whole BV. On the other hand, this approach, being very intrinsic, should also permit to simplify the study of similar problems on SBV.

2. Preliminaries

In the following, we assume that $N \geqslant 2$ be a fixed integer and Ω be an open bounded set in $\mathbb{R}^N$.

Let $u \in L^1_{\mathrm{loc}}(\Omega)$; we say that u has an *approximate limit* at $x \in \Omega$, if there exists a unique value $\tilde{u}(x) \in \mathbb{R}$ such that

$$\lim_{r \to 0^+} \fint_{B_r(x)} \big|u(y) - \tilde{u}(x)\big| \, \mathrm{d}x = 0, \tag{1}$$

where $\fint_{B_r(x)}$ stands for $\frac{1}{\mathcal{L}^N(B_r(x))} \int_{B_r(x)}$ and $B_r(x)$ denotes the ball in $\mathbb{R}^N$ centered in x with radius r. Let S_u be the set of points where the previous property does not hold, the so-called *approximate discontinuity set*. Note that S_u is a Borel set and $\tilde{u} : \Omega \setminus S_u \to \mathbb{R}$ is a Borel function. Clearly, if x is a Lebesgue point of u, then (1) holds with $\tilde{u}(x)$ replaced by $u(x)$. Moreover, we recall that $\mathcal{L}^N$-almost every $x \in \Omega$ is a Lebesgue point of u.

Let us recall a useful property for the composition of BV-functions with Lipschitz functions.

Theorem 2.1 (see Theorem 3.16 in [2]). *Let Ω, Ω' be open subsets of $\mathbb{R}^N$ and $\phi : \Omega' \to \Omega$ be a bijective Lipschitz function, whose inverse is Lipschitz, too. Let $u \in \mathrm{BV}(\Omega)$ and $v = u \circ \phi$. Then $v \in \mathrm{BV}(\Omega')$ and*

$$\frac{1}{[\mathrm{Lip}(\phi)]^{N-1}} |Du|\big(\phi(B)\big) \leqslant |Dv|(B) \leqslant \big[\mathrm{Lip}(\phi^{-1})\big]^{N-1} |Du|\big(\phi(B)\big)$$

for every Borel subset B of Ω'.

3. Main results

In the first theorem, we improve a fine property of countable $\mathcal{H}^{N-1}$-rectifiable sets stated in Theorem 4.2.19 of [6] (see also Theorem 3.2 in [1]). This result is the crucial tool in order to obtain the quasi-polyhedral approximation of BV-functions, which will be stated in Theorem 3.3.

Theorem 3.1. *Let B_R be a ball in $\mathbb{R}^N$ and $S \subset B_R$ be a countable $\mathcal{H}^{N-1}$-rectifiable set. Then for every $\varepsilon > 0$ there exists a diffeomorphism $\phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ satisfying the following properties*:

(i) $\phi_\varepsilon : B_R \to B_R$ *and, outside B_R, $\phi_\varepsilon = \mathrm{Id}$ (where Id denotes the identity function $\mathrm{Id}(x) = x$);*
(ii) $\mathrm{Lip}(\phi_\varepsilon)$, $\mathrm{Lip}(\phi_\varepsilon^{-1}) \leqslant 1 + \varepsilon$;
(iii) *there exists a compact set $K_\varepsilon \subset B_R$, such that K_ε is a polyhedron composed by a finite number of cubes in $(N-1)$ dimensions and $\mathcal{H}^{N-1}(S \bigtriangleup \phi_\varepsilon(K_\varepsilon)) < \varepsilon$.*

Moreover for any family $\{\phi_\varepsilon\}$ *satisfying* (i) *and* (ii) *we have*:

(iv) *as* $\varepsilon \to 0^+$, $\phi_\varepsilon \to \mathrm{Id}$, *uniformly on* $\mathbb{R}^N$;
(v) *as* $\varepsilon \to 0^+$, $|J(\phi_\varepsilon^{-1})|$ *and* $|J(\phi_\varepsilon)|$ *tend to* 1 *uniformly on* Ω, *where* $J(\phi_\varepsilon^{-1})$ *and* $J(\phi_\varepsilon)$ *denote the determinants of the Jacobian matrices of* ϕ_ε^{-1} *and* ϕ_ε, *respectively.*

Proof. The existence of the diffeomorphism ϕ_ε with the properties (i)–(iii) is proved in Theorem 4.2.19 of [6] (see also Theorem 3.2 in [1]). Assume now that $\{\phi_\varepsilon\}$ is a family satisfying (i) and (ii); we will show that it satisfies also (iv) and (v). By (ii), it follows $(1+\varepsilon)^{-1}|x-y| \leqslant |\phi_\varepsilon(x) - \phi_\varepsilon(y)| \leqslant (1+\varepsilon)|x-y|$ and $|\phi_\varepsilon(x)| \leqslant 2R + 3|y_0|$, where $x \in B_R$ and $y_0 \in \mathbb{R}^N \setminus B_R$; the same holds for ϕ_ε^{-1}. Hence, the sequences $\{\phi_\varepsilon\}$, $\{\phi_\varepsilon^{-1}\}$ are equibounded and equicontinuous, then there exist $\phi_0, \psi_0 : \mathbb{R}^N \to \mathbb{R}^N$ such that, up to a subsequence, $\phi_\varepsilon \to \phi_0$ and $\phi_\varepsilon^{-1} \to \psi_0$ uniformly on B_R (and so on the whole of $\mathbb{R}^N$, since $\phi_\varepsilon = \phi_\varepsilon^{-1} = \mathrm{Id}$ outside B_R). It is easy to see that $\phi_0(\psi_0(x)) = \psi_0(\phi_0(x)) = x$, so that $\psi_0 = \phi_0^{-1}$. Moreover, $|\phi_0(x) - \phi_0(y)| = |x-y|$, which implies that ϕ_0 (and hence ϕ_0^{-1}, too) is a linear isometric map. Indeed, assume firstly that $\phi_0(0) = 0$; then we have $|\phi_0(x)| = |x|$, which implies also that $\langle x, y \rangle = \langle \phi_0(x), \phi_0(y) \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^N$; hence, if $\{e_i\}_{i=1,\dots,N}$ is an orthonormal basis in $\mathbb{R}^N$, then $\{\phi_0(e_i)\}_{i=1,\dots,N}$ is an orthonormal basis in $\mathbb{R}^N$, too. Given $x = \sum \alpha_i e_i$, we can write $\phi_0(x) = \sum \beta_i \phi_0(e_i)$, where $\alpha_i = \langle x, e_i \rangle = \langle \phi_0(x), \phi_0(e_i) \rangle = \beta_i$. This proves that ϕ_0 is a linear isometric map, which coincides with the identity outside B_R, hence $\phi_0 \equiv \mathrm{Id}$. If $\phi_0(0) = x_0 \neq 0$, it is enough to replace ϕ_0 with $\tilde{\phi}_0(\cdot) = \phi_0(\cdot) - x_0$. Hence, the whole sequence $\{\phi_\varepsilon\}$, and not only a subsequence, tends to Id. In order to prove (v), we note that, by (ii) and recalling that for every invertible matrix A, the $\det(A^{-1}) = 1/\det(A)$, it follows $\frac{1}{(1+\varepsilon)^N} \leqslant |J(\phi_\varepsilon)| \leqslant (1+\varepsilon)^N$ and $\frac{1}{(1+\varepsilon)^N} \leqslant |J(\phi_\varepsilon^{-1})| \leqslant (1+\varepsilon)^N$, which concludes the proof. $\quad\square$

In order to state our main result (Theorem 3.3), we need the following definition.

Definition 3.2. Given $u \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$, we say that a sequence $\{u_\varepsilon\} \subseteq \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ is a *quasi-polyhedral* approximation of u, if the following properties hold:

- the sequence $\{u_\varepsilon\}$ is equibounded in $L^\infty(\Omega)$;
- $u_\varepsilon \to u$ strongly in $L^1(\Omega)$, as $\varepsilon \to 0$;
- $|Du_\varepsilon|(\Omega) \to |Du|(\Omega)$, as $\varepsilon \to 0$;
- there exists a sequence of polyhedra $\{K_\varepsilon\} \subseteq \Omega$ such that $\mathcal{H}^{N-1}(S_{u_\varepsilon} \triangle K_\varepsilon) \to 0$, as $\varepsilon \to 0$;
- $\mathcal{H}^{N-1}(S_{u_\varepsilon}) \to \mathcal{H}^{N-1}(S_u)$, as $\varepsilon \to 0$.

Theorem 3.3. *Let* $\Omega = B_R$. *For every* $u \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$, *there exists a quasi-polyhedral approximation* $\{u_\varepsilon\}$ *of* u. *More precisely, for every* $\varepsilon > 0$, *setting* $u_\varepsilon := u \circ \phi_\varepsilon$ *(where* ϕ_ε *is given in Theorem* 3.1*), it follows that* $u_\varepsilon \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ *and*

(i) $\|u_\varepsilon\|_\infty = \|u\|_\infty$ *and* $\|u_\varepsilon\|_{\mathrm{BV}} \leqslant (1+\varepsilon)^N \|u\|_{\mathrm{BV}}$;
(ii) $u_\varepsilon \to u$ *strongly in* $L^1(\Omega)$, *as* $\varepsilon \to 0$;
(iii) $|Du_\varepsilon|(\Omega) \to |Du|(\Omega)$, *as* $\varepsilon \to 0$;
(iv) *there exists a polyhedron* $K_\varepsilon \subset \Omega$ *and a constant* $c > 0$ *such that* $\mathcal{H}^{N-1}(S_{u_\varepsilon} \triangle K_\varepsilon) \leqslant c\varepsilon$;
(v) $(1+\varepsilon)^{1-N}\mathcal{H}^{N-1}(S_u) \leqslant \mathcal{H}^{N-1}(S_{u_\varepsilon}) \leqslant (1+\varepsilon)^{N-1}\mathcal{H}^{N-1}(S_u)$.

Proof. For every $\varepsilon > 0$, define $u_\varepsilon := u \circ \phi_\varepsilon$, where $\phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ is the diffeomorphism given in Theorem 3.1, with $S := S_u$, which is a countable $\mathcal{H}^{N-1}$-rectifiable subset of Ω. Clearly, by definition, $\|u_\varepsilon\|_\infty = \|u\|_\infty$. By Theorem 2.1 the function u_ε belongs to $\mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ and, for every $0 < \varepsilon \leqslant 1$, an easy calculations yields $\|u_\varepsilon\|_{\mathrm{BV}} \leqslant c$, where c is a positive constant independent of ε. Hence, there exists $u_0 \in \mathrm{BV}(\Omega)$ such that, up to a subsequence, $u_\varepsilon \to u_0$ strongly in $L^1(\Omega)$ and $Du_\varepsilon \rightharpoonup Du_0$ weakly* in the sense of measures. Firstly, we will show

that $u_0 = u$, which implies also that the whole sequence, not only a subsequence, converges to u. Let $N \subset \Omega$, with $\mathcal{L}^N(N) = 0$, be such that for every $x \in \Omega \setminus N$, up to another subsequence, $u_\varepsilon(x) \to u_0(x)$ and

$$u_0(x) = \lim_{r \to 0} \frac{1}{\omega_N r^N} \int_{B_r(x)} u_0(y)\,\mathrm{d}y \quad \text{and} \quad u(x) = \lim_{r \to 0} \frac{1}{\omega_N r^N} \int_{B_r(x)} u(y)\,\mathrm{d}y, \tag{2}$$

where ω_N denotes the Lebesgue measure of the unit ball in $\mathbb{R}^N$. Recalling Theorem 3.1, we have that $|J\phi_\varepsilon^{-1}|$ and $|J\phi_\varepsilon|$ tend to 1, as $\varepsilon \to 0^+$, uniformly on Ω. Moreover, for every $r > 0$ and every $x \in \Omega \setminus N$, we have

$$\left| u_\varepsilon(x) - u(x) \right| \leqslant \left| u_\varepsilon(x) - \frac{1}{\omega_N r^N} \int_{B_r(x)} u_\varepsilon(y)\,\mathrm{d}y \right| + \left| \frac{1}{\omega_N r^N} \int_{\phi_\varepsilon(B_r(x))} u(y)\big[\big|J\phi_\varepsilon^{-1}(y)\big| - 1\big]\,\mathrm{d}y \right|$$

$$+ \left| \frac{1}{\omega_N r^N} \left[\int_{\phi_\varepsilon(B_r(x))} u(y)\,\mathrm{d}y - \int_{B_r(x)} u(y)\,\mathrm{d}y \right] \right| + \left| \frac{1}{\omega_N r^N} \int_{B_r(x)} u(y)\,\mathrm{d}y - u(x) \right|.$$

Passing to the $\limsup$ as $\varepsilon \to 0^+$ and taking into account that $\{u_\varepsilon\}$ tends to u_0 strongly in $L^1(\Omega)$ and pointwise in $\Omega \setminus N$, we obtain

$$\limsup_{\varepsilon \to 0^+} \left| u_\varepsilon(x) - u(x) \right| \leqslant \left| u_0(x) - \fint_{B_r(x)} u_0(y)\,\mathrm{d}y \right| + \frac{\|u\|_\infty}{\omega_N r^N} \limsup_{\varepsilon \to 0^+} \mathcal{L}^N\big(\phi_\varepsilon\big(B_r(x)\big)\big) \Big[\sup_\Omega \big|\big|J\phi_\varepsilon^{-1}(y)\big| - 1\big| \Big]$$

$$+ \frac{\|u\|_\infty}{\omega_N r^N} \limsup_{\varepsilon \to 0^+} \mathcal{L}^N\big(\phi_\varepsilon\big(B_r(x)\big) \triangle B_r(x)\big) + \left| \frac{1}{\omega_N r^N} \int_{B_r(x)} u(y)\,\mathrm{d}y - u(x) \right|$$

$$= \left| u_0(x) - \frac{1}{\omega_N r^N} \int_{B_r(x)} u_0(y)\,\mathrm{d}y \right| + \left| \frac{1}{\omega_N r^N} \int_{B_r(x)} u(y)\,\mathrm{d}y - u(x) \right|,$$

for every $r > 0$. Now, letting $r \to 0^+$ and taking into account (2), it follows $|u_\varepsilon(x) - u(x)| \to 0$, when $\varepsilon \to 0$. This implies that $u_\varepsilon(x) \to u(x)$ for a.e. $x \in \Omega$ and since we have also that $u_\varepsilon(x) \to u_0(x)$ for every $x \in \Omega \setminus N$, it follows that $u = u_0$ almost everywhere and that the whole sequence $\{u_\varepsilon\}$ tends to u in $L^1(\Omega)$. Now, taking into account the lower semicontinuity of the total variation, (i) and (ii) of Theorem 3.1, and Theorem 2.1, we have also that $|Du_\varepsilon|(\Omega) \to |Du|(\Omega)$, when $\varepsilon \to 0$. Hence, (iii) is proven.

Finally, by (iii) of Theorem 3.1, there exists a polyhedron $K_\varepsilon \subset \Omega$ such that $\mathcal{H}^{N-1}(S_u \triangle \phi_\varepsilon(K_\varepsilon)) < \varepsilon$. Hence, taking into account that $S_{u_\varepsilon} = \phi_\varepsilon^{-1}(S_u)$, the properties of Hausdorff measures (see Proposition 2.49 in [2]), and (ii) of Theorem 3.1, it follows $\mathcal{H}^{N-1}(S_{u_\varepsilon} \triangle K_\varepsilon) = \mathcal{H}^{N-1}(\phi_\varepsilon^{-1}(S_u \triangle \phi_\varepsilon(K_\varepsilon))) \leqslant (1+\varepsilon)^{N-1}\mathcal{H}^{N-1}(S_u \triangle \phi_\varepsilon(K_\varepsilon)) \leqslant (1+\varepsilon)^{N-1}\varepsilon$. In a similar way we obtain (v), which concludes the proof. $\quad\square$

Acknowledgements

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