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Mathematical Analysis

A new estimate for the topological degree

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Abstract

We establish a new estimate for the topological degree of continuous maps from the sphere $\mathbb{S}^N$ into itself, which answers a question raised in Bourgain, Brezis, and Mironescu [Commun. Pure Appl. Math. 58 (2005) 529–551] and extends some of the results proved there, as well as in recent work by these authors (Lifting, degree, and distributional Jacobian revisited, http://ann.jussieu.fr/publications). ***To cite this article: J. Bourgain et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Une nouvelle estimée du degré topologique. Nous présentons une nouvelle estimée du degré topologique pour des applications continues de la sphère $\mathbb{S}^N$ dans elle-même. Celle-ci répond à une question posée dans Bourgain, Brezis, et Mironescu [Commun. Pure Appl. Math. 58 (2005) 529–551] et généralise certains résultats de cet article ainsi que du travail récent de ces auteurs (Lifting, degree, and distributional Jacobian revisited, http://ann.jussieu.fr/publications). ***Pour citer cet article : J. Bourgain et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Notre résultat principal est l'estimée (2), valable pour toute fonction continue $g:\mathbb{S}^N \to \mathbb{S}^N$, et tout $0 < \delta < \sqrt{2}$, où $C = C(\delta, N)$ est une constante indépendante de g. Elle répond à une question posée dans [1] et généralise l'inégalité suivante prouvée dans [1], pour tout $p > N$,

$$|\deg g| \leqslant C(p, N) \int_{\mathbb{S}^N}\int_{\mathbb{S}^N} \frac{|g(x) - g(y)|^p}{|x - y|^{2N}} \, \mathrm{d}x \, \mathrm{d}y. \tag{1}$$

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L'estimée (2) était déja connue en dimension $N = 1$ (voir [2]) mais la preuve donnée dans [2] était assez complexe et ne pouvait pas s'étendre aux dimensions $N > 1$.

1. Introduction

In this Note we prove the following

Theorem 1.1. *Let $g : \mathbb{S}^N \to \mathbb{S}^N$ be a continuous function. Then, for every $0 < \delta < \sqrt{2}$, there exists a constant $C = C(\delta, N)$, independent of g, such that*

$$|\deg g| \leqslant C \iint\limits_{\substack{\mathbb{S}^N \mathbb{S}^N \\ |g(x)-g(y)|>\delta}} \frac{1}{|x-y|^{2N}} \,\mathrm{d}x\,\mathrm{d}y. \tag{2}$$

Estimate (2) trivially implies (1) which was proved in [1]. Estimate (2) was already known for $N = 1$ (see [2]), but the proof given in [2] was quite involved and could not be extended to higher dimensions.

2. Proof of Theorem 1.1

2.1. Step 1: Proof of (2) when $g \in \mathrm{Lip}(\mathbb{S}^N, \mathbb{S}^N)$

Consider the function $u : B \to B$, where $B = \{X \in \mathbb{R}^{N+1};\ |X| \leqslant 1\}$, defined by

$$u(X) = \fint_{B(x,r)} g(s)\,\mathrm{d}s, \quad \text{when } X \neq 0, \tag{3}$$

where $x = \frac{X}{|X|}$, $r = 2(1 - |X|)$ and $B(x, r)$ denotes the spherical cap, $B(x, r) = \{y \in \mathbb{S}^N;\ |y - x| \leqslant r\}$, and

$$u(0) = \fint_{\mathbb{S}^N} g(s)\,\mathrm{d}s.$$

Note that $u_{|\mathbb{S}^N} = g$ and $u \in \mathrm{Lip}(B, B)$. We now apply the same strategy as in [1], except that we use the function u in place of the harmonic extension of g. We have

$$\left|\nabla u(X)\right| \leqslant \frac{C}{1 - |X|} \quad \text{for all } X,\ |X| < 1, \tag{4}$$

where C depends only on N. For every $x \in \mathbb{S}^N$, let $\rho(x)$ be the length of the largest radial interval coming from $x \in \mathbb{S}^N$ on which $|u| > \alpha$ (possibly $\rho(x) = 1$), where $0 < \alpha < 1$ will be chosen later. Set

$$\tilde{u}(X) = \begin{cases} \dfrac{u(X)}{|u(X)|} & \text{if } \left|u(X)\right| \geqslant \alpha, \\[2mm] \dfrac{1}{\alpha} u(X) & \text{otherwise.} \end{cases}$$

By Kronecker's formula we have

$$\deg g = \fint_B \det\bigl(\nabla \tilde{u}(X)\bigr)\,\mathrm{d}X.$$

Set

$$G = \{X \in B;\ |u(X)| < \alpha\},$$

so that

$$G \subset \bigcup_{x\in\mathbb{S}^N} \big[0, (1-\rho(x))x\big]. \tag{5}$$

Since $|\tilde{u}| = 1$ in $B \setminus G$ we have $\det(\nabla\tilde{u}) = 0$ a.e. in $B \setminus G$ and thus

$$\deg g = \frac{1}{\alpha^{N+1}|B|}\int_G \det(\nabla u(X))\,\mathrm{d}X. \tag{6}$$

From (4), (5) and (6) we have

$$|\deg g| \leqslant \frac{C}{\alpha^{N+1}|B|}\int_{\mathbb{S}^N}\int_0^{1-\rho(x)} \frac{r^N\,\mathrm{d}x\,\mathrm{d}r}{(1-r)^{N+1}}$$

and therefore

$$|\deg g| \leqslant \frac{C}{\alpha^{N+1}|B|}\int_{\substack{\mathbb{S}^N\\ \rho(x)<1}} \frac{\mathrm{d}x}{\rho(x)^N}. \tag{7}$$

Choosing $\alpha = \frac{1}{4}(2-\delta^2)$, we have

Lemma 2.1. *Assume $x \in \mathbb{S}^N$ is such that $\rho(x) < 1$. Then*

$$\frac{1}{\rho(x)^N} \leqslant \frac{C}{2-\delta^2}\int_{\substack{y\in\mathbb{S}^N\\ |g(y)-g(x)|>\delta}} \frac{1}{|y-x|^{2N}}\,\mathrm{d}y, \tag{8}$$

where C depends only on N.

Proof. Let $X = (1-\rho(x))x$. Since $\rho(x) < 1$ we have

$$|u(X)| = \alpha. \tag{9}$$

Therefore, by (3) and (9),

$$\begin{aligned}
2(1-\alpha) &\leqslant 2\big(1 - g(x)\cdot u(X)\big) = \fint_{B(x,r)} |g(y)-g(x)|^2\,\mathrm{d}y\\
&\leqslant \delta^2 + \frac{4}{|B(x,r)|}\int_{\substack{y\in B(x,r)\\ |g(y)-g(x)|>\delta}} \mathrm{d}y\\
&= \delta^2 + \frac{4}{|B(x,r)|}\,\mathrm{meas}\{y \in B(x,r);\ |g(y)-g(x)| > \delta\},
\end{aligned}$$

where $r = 2(1-|X|) = 2\rho(x)$.

From the choice $\alpha = \frac{1}{4}(2-\delta^2)$, we see that

$$\frac{1}{2}(2-\delta^2) \leqslant \frac{4}{|B(x,r)|}\,\mathrm{meas}\{y \in B(x,r);\ |g(y)-g(x)| > \delta\},$$

and thus

$$\int\limits_{\substack{y\in\mathbb{S}^N\\|g(y)-g(x)|>\delta}} \frac{1}{|y-x|^{2N}}\,\mathrm{d}y \geqslant \frac{1}{r^{2N}}\,\mathrm{meas}\bigl\{y\in B(x,r);\ \bigl|g(y)-g(x)\bigr|>\delta\bigr\} \geqslant \frac{C(2-\delta^2)}{\rho(x)^N},$$

which is precisely (8). □

Combining (7) and (8) yields

$$|\deg g| \leqslant C(\delta,N) \int\limits_{\mathbb{S}^N}\int\limits_{\substack{\mathbb{S}^N\\|g(x)-g(y)|>\delta}} \frac{1}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y, \quad \forall g\in \mathrm{Lip}\bigl(\mathbb{S}^N,\mathbb{S}^N\bigr). \tag{10}$$

2.2. *Step 2: Proof of (2) when g is only continuous from $\mathbb{S}^N$ to $\mathbb{S}^N$*

Choose any sequence $(g_j)\subset \mathrm{Lip}(\mathbb{S}^N,\mathbb{S}^N)$ such that $g_j\to g$ uniformly. Given $\delta\in(0,\sqrt{2})$, set $\delta'=\frac{1}{2}(\sqrt{2}+\delta)$. Choose j so large that

$$\deg g_j = \deg g$$

and

$$\|g_j-g\|_{L^\infty(\mathbb{S}^N)} \leqslant \frac{\sqrt{2}-\delta}{4}.$$

Then, by Step 1,

$$|\deg g_j| \leqslant C(\delta',N) \int\limits_{\mathbb{S}^N}\int\limits_{\substack{\mathbb{S}^N\\|g_j(x)-g_j(y)|>\delta'}} \frac{1}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y.$$

Note that if $|g_j(x)-g_j(y)|>\delta'$, then

$$\begin{aligned}\bigl|g(x)-g(y)\bigr| &\geqslant \bigl|g_j(x)-g_j(y)\bigr| - 2\|g_j-g\|_{L^\infty(\mathbb{S}^N)}\\ &> \delta' - \frac{1}{2}(\sqrt{2}-\delta) = \delta\end{aligned}$$

and the desired estimate follows. □

Remark 1. The optimality of the condition $\delta<\sqrt{2}$ in Theorem 1.1 will be studied in [3].

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