



ELSEVIER

Available online at www.sciencedirect.com



C. R. Acad. Sci. Paris, Ser. I 341 (2005) 53–58



<http://france.elsevier.com/direct/CRASSI/>

Statistics/Probability Theory

Asymptotic distribution of a Pickands-type estimator of the extreme-value index

Laurent Gardes^a, Stéphane Girard^b

^a *Université Grenoble 2, LABSAD, B.P. 47, 38040 Grenoble cedex 9, France*

^b *Université Grenoble 1, LMC-IMAG, B.P. 53, 38041 Grenoble cedex 9, France*

Received 18 January 2005; accepted after revision 24 May 2005

Presented by Paul Deheuvels

Abstract

One of the main goals of extreme-value analysis is to estimate the probability of rare events given a sample from an unknown distribution. The upper tail behavior of this distribution is described by the extreme-value index ξ . The aim of this Note is to establish the asymptotic distribution of the estimator of $\xi \in \mathbb{R}$ introduced in Gardes and Girard [A Pickands-type estimator of the extreme-value index, Technical Report LMC-IMAG, RR-1063, 2004]. We also give its rate of convergence in some typical situations. **To cite this article:** *L. Gardes, S. Girard, C. R. Acad. Sci. Paris, Ser. I 341 (2005)*.

© 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

Loi asymptotique d'un estimateur de type Pickands de l'indice des valeurs extrêmes. Dans de nombreuses applications, on s'intéresse à l'estimation de probabilités d'événements rares étant donné un échantillon d'une loi inconnue. Pour ce faire, il est généralement indispensable d'estimer au préalable l'indice des valeurs extrêmes ξ qui caractérise le type de décroissance de la queue de distribution. L'objectif de cette Note est de déterminer la loi asymptotique de l'estimateur de $\xi \in \mathbb{R}$ défini dans Gardes et Girard [A Pickands-type estimator of the extreme-value index, Technical Report LMC-IMAG, RR-1063, 2004] et d'en déduire sa vitesse de convergence pour certains modèles. **Pour citer cet article :** *L. Gardes, S. Girard, C. R. Acad. Sci. Paris, Ser. I 341 (2005)*.

© 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.

E-mail addresses: laurent.gardes@upmf-grenoble.fr (L. Gardes), stephane.girard@imag.fr (S. Girard).

1631-073X/\$ – see front matter © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crma.2005.05.023

Version française abrégée

Considérons n variables aléatoires indépendantes X_1, \dots, X_n de fonction de répartition commune F et supposons qu’il existe deux suites $a_n > 0$ et b_n et un réel ξ tels que :

$$\lim_{n \rightarrow \infty} P \left[\frac{\max(X_1, \dots, X_n) - b_n}{a_n} \leq x \right] = G_\xi(x), \tag{1}$$

avec $G_\xi(x) = \exp[-(1 + \xi x)_+^{-1/\xi}]$ si $\xi \neq 0$ et $G_0(x) = \exp[-e^{-x}]$, où $y_+ = \max(0, y)$. Le résultat principal de cette Note donne la loi asymptotique de l’estimateur de l’indice des valeurs extrêmes $\xi \in \mathbb{R}$ proposé par Gardes et Girard [3]. Cet estimateur, noté $\hat{\xi}_{k,n}$, est défini comme étant l’unique solution (cf. [2], Annexe B) de l’équation en θ :

$$\left\{ \begin{array}{l} \varphi_\theta(1/k') \\ \varphi_\theta(1/k) \end{array} \right\} \frac{X_{n-k+1,n} - X_{n,n}}{X_{n-k'+1,n} - X_{n,n}} = 1, \quad \text{pour } 1 < k' < k < n,$$

où $\varphi_t(x) = \int_1^x u^{t-1} du$, $x > 0$, $t \in \mathbb{R}$. Cet estimateur est similaire à (4) introduit par Pickands [6], mais il exploite l’information sur la queue de distribution contenue entre $X_{n-k+1,n}$ et $X_{n,n}$. Dans la suite, on pose $U(x) = (1/[1 - F(x)])^{\leftarrow}$. Pour démontrer le résultat de convergence en loi, on introduit les deux hypothèses suivantes :

(H1) : U a une dérivée positive et il existe une fonction à variations lentes ℓ monotone à l’infini, $\rho \leq 0$ et $b(x) \rightarrow 0$ tels que $U'(x) = x^\xi \ell(x)$. De plus, si $b(x)\varphi_\rho(t) \rightarrow 0$ lorsque $x, t \rightarrow \infty$, on a alors uniformément localement $\ell(tx)/\ell(x) - 1 \sim b(x)\varphi_\rho(t)$.

(H2) : $\varphi_\delta(k')\varphi_\rho(k')b(n/k') \rightarrow 0$ quand $n \rightarrow \infty$, où $\delta = \min(-\xi, 1/2)$.

Le résultat principal de cette Note est le suivant :

Théorème 0.1. Posons $V_k(\xi) = \varphi_\delta(k)[(\ln(k) - 1)\mathbf{1}\{\xi \geq 0\} + 1]$. Supposons que F vérifie la relation (1). Si $k/k' = c > 1$, $k \rightarrow \infty$, $k/n \rightarrow 0$ lorsque $n \rightarrow \infty$, et sous (H1) et (H2), on a pour tout $t \in \mathbb{R}$:

$$\lim_{n \rightarrow \infty} P[V_k(\hat{\xi}_{k,n}) - \xi \leq t] = \begin{cases} \exp(-e^{-t}) & \text{si } 0 < \xi, \\ \exp(-e^{-t/2}) & \text{si } \xi = 0, \\ \exp[-[1 + t \ln(c)/\varphi_\xi(1/c)]^{-1/\xi}] & \text{si } -1/2 < \xi < 0, \\ \Phi[-tc^{-\xi} \ln(c)/(2\xi\sigma)] & \text{si } \xi < -1/2, \end{cases}$$

où $\sigma = c^{-\xi}(c - 1)^{1/2}$ et Φ est la fonction de répartition d’une loi normale centrée et réduite.

Ainsi, la loi asymptotique de $\hat{\xi}_{k,n}$ est normale si $\xi < -1/2$ ou une loi des valeurs extrêmes si $\xi > -1/2$. Nous déduisons de ce théorème la vitesse de convergence de l’estimateur $\hat{\xi}_{k,n}$. Soit $\alpha, \beta > 0$, $\theta \in \mathbb{R} \setminus \{0\}$ et posons $\ln_2(x) = \ln(\ln(x))$, $x > 1$. On considère les deux modèles suivants :

$$\ell(x) = \alpha + \theta x^{-\beta} + o(x^{-\beta}), \quad (\text{Modèle A}) \quad \text{et} \quad \ell(x) = \theta [\ln(x)]^{-\beta} \left\{ 1 + O\left(\frac{\ln_2(x)}{\ln(x)}\right) \right\}, \quad (\text{Modèle B}).$$

Dans la suite, on utilise la notation suivante. Soit (u_n) et (v_n) deux suites positives. L’écriture $u_n \asymp v_n$ signifie que $0 < \liminf u_n/v_n \leq \limsup u_n/v_n < \infty$.

Corollaire 0.2. Supposons que $k = ck'$, $k \rightarrow \infty$, $k/n \rightarrow 0$ et que F vérifie l’hypothèse (H1).

(i) Si ℓ appartient au modèle A et si $\varphi_\delta(k)(k/n)^\beta \rightarrow 0$ alors les hypothèses du Théorème 0.1 sont satisfaites et la vitesse de convergence de $\hat{\xi}_{k,n}$ est donnée par :

$$V_k(\hat{\xi}) \asymp \ln(n)\mathbf{1}\{0 < \xi\} + \ln^2(n)\mathbf{1}\{\xi = 0\} + n^{\delta/[1+\delta/\beta]-\varepsilon}\mathbf{1}\{\xi < 0\}$$

où $\varepsilon \in]0, \delta/(1 + \delta/\beta)[$ est arbitrairement petit.

(ii) Si ℓ appartient au modèle B et si $\varphi_\delta(k) \ln(k)/\ln(n) \rightarrow 0$ et $\varphi_\delta(k) \ln_2(n)/\ln(n) \rightarrow 0$ alors les hypothèses du Théorème 0.1 sont satisfaites et la vitesse de convergence de $\hat{\xi}_{k,n}$ est donnée par :

$$V_k(\xi) = \ln_2(n)\mathbf{1}\{0 < \xi\} + \ln_2^2(n)\mathbf{1}\{\xi = 0\} + \ln^{1-\varepsilon}(n)\mathbf{1}\{\xi < 0\}$$

où $\varepsilon \in]0, 1[$ est arbitrairement petit.

Nous renvoyons le lecteur à [3] pour une illustration du comportement de l'estimateur sur simulations. Son bon comportement dans le domaine $\xi < 0$ y est souligné.

1. Definitions

Suppose one is given a sequence X_1, \dots, X_n of independent and identically distributed (i.i.d.) observations from some distribution function F with associated survival distribution function \bar{F} . Suppose there exist sequences $a_n > 0$ and b_n and some $\xi \in \mathbb{R}$ such that:

$$\lim_{n \rightarrow \infty} P \left[\frac{\max(X_1, \dots, X_n) - b_n}{a_n} \leq x \right] = G_\xi(x), \tag{2}$$

with $G_\xi(x) = \exp[-(1 + \xi x)_+^{-1/\xi}]$ if $\xi \neq 0$ and $G_0(x) = \exp[-e^{-x}]$, where $y_+ = \max(0, y)$. Necessary and sufficient conditions on F for the convergence (2) can be found in [4]. The aim of this Note is to establish the asymptotic distribution of the extreme-value index estimator proposed in [3] and to give its rate of convergence. Let U be the tail quantile function of the distribution function F defined by $U(x) = (1/[1 - F(x)])^\leftarrow$ and let $\varphi_t(x) = \int_1^x u^{t-1} du$, $x > 0$, $t \in \mathbb{R}$. Gardes and Girard [3] propose to estimate the extreme-value index $\xi \in \mathbb{R}$ by $\hat{\xi}_{k,n}$ defined as the unique root of the equation in θ (see [2], Appendix B for a proof):

$$\left\{ \frac{\varphi_\theta(1/k')}{\varphi_\theta(1/k)} \right\} \frac{X_{n-k+1,n} - X_{n,n}}{X_{n-k'+1,n} - X_{n,n}} = 1, \quad \text{for } 1 < k' < k < n. \tag{3}$$

This estimator is similar to the one of Pickands [6]:

$$\hat{\xi}_{k,n}^P = \frac{1}{\ln(2)} \ln \left(\frac{X_{n-k+1,n} - X_{n-2k+1,n}}{X_{n-2k+1,n} - X_{n-4k+1,n}} \right), \quad \text{for } k = 1, \dots, \left\lfloor \frac{n}{4} \right\rfloor, \tag{4}$$

but it takes into account the information on the distribution tail given by the spacing between $X_{n-k+1,n}$ and $X_{n,n}$.

2. Main result

To establish the asymptotic distribution of the estimator $\hat{\xi}_{k,n}$, two conditions are introduced:

(H1): U has a positive derivative and there exist a slowly varying function ℓ ultimately monotone, $\rho \leq 0$ and $b(x) \rightarrow 0$ such that $U'(x) = x^{\xi-1} \ell(x)$. Moreover, if $b(x)\varphi_\rho(t) \rightarrow 0$ as $x, t \rightarrow \infty$, then uniformly locally $\ell(tx)/\ell(x) - 1 \sim b(x)\varphi_\rho(t)$.

(H2): $\varphi_\delta(k')\varphi_\rho(k')b(n/k') \rightarrow 0$ as $n \rightarrow \infty$, where $\delta = \min(-\xi, 1/2)$.

The second order parameter $\rho \leq 0$ tunes the rate of convergence of $\ell(tx)/\ell(x)$ to 1. We refer to [1] for more details on slow variation theory. Our main result is the following:

Theorem 2.1. Let $V_k(\xi) = \varphi_\delta(k)[(\ln(k) - 1)\mathbf{1}\{\xi \geq 0\} + 1]$. Suppose that relation (2) holds. If $k/k' = c > 1$, $k \rightarrow \infty$, $k/n \rightarrow 0$ as $n \rightarrow \infty$, and (H1), (H2) are satisfied, we have for all $t \in \mathbb{R}$:

$$\lim_{n \rightarrow \infty} P[V_k(\xi)(\hat{\xi}_{k,n} - \xi) \leq t] = \begin{cases} \exp(-e^{-t}) & \text{if } 0 < \xi, \\ \exp(-e^{-t/2}) & \text{if } \xi = 0, \\ \exp[-[1 + t \ln(c)/\varphi_\xi(1/c)]^{-1/\xi}] & \text{if } -1/2 < \xi < 0, \\ \Phi[-tc^{-\xi} \ln(c)/(2\xi\sigma)] & \text{if } \xi < -1/2, \end{cases} \tag{5}$$

where $\sigma = c^{-\xi}(c - 1)^{1/2}$ and Φ is the cumulative distribution function of the standard normal distribution.

Theorem 2.1 states that the asymptotic distribution of $\hat{\xi}_{k,n}$ is Gaussian if $\xi < -1/2$ and an extreme-value distribution if $\xi > -1/2$. If $\xi = -1/2$, we prove that $V_k(\xi)(\hat{\xi}_{k,n} - \xi)$ converges to a non-degenerate distribution with non-explicit cumulative distribution function.

Now, we deduce from Theorem 2.1 the rate of convergence of the estimator $\hat{\xi}_{k,n}$. Let $\alpha, \beta > 0$, $\theta \in \mathbb{R} \setminus \{0\}$ and define $\ln_2(x) = \ln(\ln(x))$, $x > 1$. The following models of slowly varying functions ℓ are considered:

$$\ell(x) = \alpha + \theta x^{-\beta} + o(x^{-\beta}), \quad (\text{Model A}) \quad \text{and} \quad \ell(x) = \theta [\ln(x)]^{-\beta} \left\{ 1 + O\left(\frac{\ln_2(x)}{\ln(x)}\right) \right\}, \quad (\text{Model B}).$$

Model A has been first introduced by Hall [5]. It includes for instance Burr and Fréchet distributions. Model B includes for instance Gaussian and Weibull distributions. In both models, the decay of the slowly varying function ℓ is described by the parameter β .

In the sequel, the following notation is adopted. Let (u_n) and (v_n) be two non-negative deterministic sequences. The notation $u_n \asymp v_n$ means that $0 < \liminf u_n/v_n \leq \limsup u_n/v_n < \infty$. The following corollary gives the rate of convergence of $\hat{\xi}_{k,n}$ for Model A and Model B.

Corollary 2.2. Suppose that $k = ck'$, $k \rightarrow \infty$ and that F satisfies assumption (H1).

(i) If ℓ belongs to Model A and if $\varphi_\delta(k)(k/n)^\beta \rightarrow 0$ then convergence (5) holds. In this case, the rate of convergence of $\hat{\xi}_{k,n}$ is given by:

$$V_k(\xi) \asymp \ln(n)\mathbf{1}\{0 < \xi\} + \ln^2(n)\mathbf{1}\{\xi = 0\} + n^{\delta/[1+\delta/\beta]-\varepsilon}\mathbf{1}\{\xi < 0\}$$

where $\varepsilon \in]0, \delta/(1 + \delta/\beta)[$ is arbitrarily small.

(ii) If ℓ belongs to Model B and if $\varphi_\delta(k) \ln(k)/\ln(n) \rightarrow 0$ and $\varphi_\delta(k) \ln_2(n)/\ln(n) \rightarrow 0$ then, convergence (5) holds. Furthermore, the rate of convergence of $\hat{\xi}_{k,n}$ is given by:

$$V_k(\xi) \asymp \ln_2(n)\mathbf{1}\{0 < \xi\} + \ln_2^2(n)\mathbf{1}\{\xi = 0\} + \ln^{1-\varepsilon}(n)\mathbf{1}\{\xi < 0\}$$

where $\varepsilon \in]0, 1[$ is arbitrarily small.

We refer to [3] for an illustration of these results on finite sample situations. The estimator $\hat{\xi}_{k,n}$ reveals good performances for $\xi < 0$.

3. Proofs

Let us define $\varphi_t^*(x) = (1 + tx)\mathbf{1}\{t \neq 0\} + e^x\mathbf{1}\{t = 0\}$. The following function will play an important role:

$$H_n(x) = \left\{ \frac{\varphi_x(1/k')}{\varphi_x(1/k)} \right\} \frac{X_{n-k+1,n} - X_{n,n}}{X_{n-k'+1,n} - X_{n,n}} = \left\{ \frac{\varphi_x(1/k')}{\varphi_x(1/k)} \right\} (1 + Z_n),$$

with $Z_n = (X_{n-k+1,n} - X_{n-k'+1,n})/(X_{n-k'+1,n} - X_{n,n})$ and $x \in \mathbb{R}$.

To prove Theorem 2.1, two auxiliary results are necessary. Lemma 3.1 is dedicated to the study of φ_t^* .

Lemma 3.1. Let (u_n) and (v_n) be two sequences such that $u_n \sim v_n$ (i.e. $u_n/v_n \rightarrow 1$).

(I) For $x \in (0, \infty)$, $\varphi_t^*[\varphi_t(x)] = x^{t+1\{t=0\}}$.

(II) Let $t = 0$. If $u_n \rightarrow \infty$ and $u_n - v_n \rightarrow \alpha$ then $\varphi_t^*(v_n) \sim \varphi_t^*(u_n) e^{-\alpha}$.

(III) Let $t \neq 0$. If $u_n \rightarrow \infty$ then $\varphi_t^*(v_n) \sim \varphi_t^*(u_n)$.

(IV) Let $t \neq 0$. If $u_n \rightarrow -1/t$ with $v_n = u_n(1 + \varepsilon_n)$, then:

(i) If moreover $\varepsilon_n/\varphi_t^*(u_n) \sim \alpha_n$ where α_n does not converge to ∞ or to 1, then $\varphi_t^*(v_n) \sim \varphi_t^*(u_n)(1 - \alpha_n)$.

(ii) If moreover $\varepsilon_n/\varphi_t^*(u_n) \rightarrow \infty$ then $\varphi_t^*(v_n) \sim -\varepsilon_n$.

The proof of this basic result is not detailed here. Clearly, the distribution of $H_n(x)$ is determined by Z_n . The following lemma provides the asymptotic distribution of Z_n (see [3], Lemma 6 for a proof).

Lemma 3.2. Under the conditions of Theorem 2.1,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[k'^{\delta-1\{\xi=0\}} \varphi_\xi^* \left(-\frac{\varphi_\xi(1/c)}{Z_n} \right) \leq t \right] = \begin{cases} \exp(-t^{-1/\xi}) & \text{if } 0 < \xi, \\ \exp(-t^{-1}) & \text{if } \xi = 0, \\ 1 - \exp(-t^{-1/\xi}) & \text{if } -1/2 < \xi < 0, \\ \mathbb{P}[T < t\sqrt{c}] & \text{if } \xi = -1/2, \\ \Phi[-t\varphi_\xi(1/c)\sqrt{c}/\sigma] & \text{if } \xi < -1/2, \end{cases}$$

where the random variable T is the (non-degenerate) limit in distribution of

$$T_n = \sqrt{\frac{k\bar{F}(X_{n,n})}{\bar{F}(X_{n-k'+1,n})}} + \frac{\sqrt{k}}{\varphi_\xi(1/c)} \left(\varphi_\xi \left(\frac{\bar{F}(X_{n-k'+1,n})}{\bar{F}(X_{n-k+1,n})} \right) - \varphi_\xi \left(\frac{1}{c} \right) \right).$$

Proof of Theorem 2.1. Let $F_n(t) = \mathbb{P}[V_k(\xi)(\hat{\xi}_{k,n} - \xi) \leq t]$. We have,

$$F_n(t) = \mathbb{P} \left[\hat{\xi}_{k,n} \leq \xi + \frac{t}{V_k(\xi)} \right] = \mathbb{P} \left[H_n \left(\xi + \frac{t}{V_k(\xi)} \right) \geq 1 \right] = \mathbb{P} \left[(1 + Z_n) \frac{\varphi_{-\xi-t/V_k(\xi)}(k/c)}{\varphi_{-\xi-t/V_k(\xi)}(k)} \geq 1 \right],$$

since H_n is a non-decreasing function and since $H_n(\hat{\xi}_{k,n}) = 1$. Routine calculations yield:

$$F_n(t) = \mathbb{P} \left[-\frac{\varphi_\xi(1/c)}{Z_n} \leq t_n \right], \quad \text{with } t_n = -\varphi_\xi(1/c) \frac{\varphi_{-\xi-t/V_k(\xi)}(k/c)}{\varphi_{-\xi-t/V_k(\xi)}(k) - \varphi_{-\xi-t/V_k(\xi)}(k/c)}.$$

Remarking that φ_ξ^* is an increasing function for $\xi \geq 0$ and decreasing for $\xi < 0$, we have,

$$F_n(t) = \begin{cases} \mathbb{P} \left[(k')^{\delta-1\{\xi=0\}} \varphi_\xi^* \left(-\frac{\varphi_\xi(1/c)}{Z_n} \right) \leq (k')^{\delta-1\{\xi=0\}} \varphi_\xi^*(t_n) \right] & \text{if } \xi \geq 0, \\ \mathbb{P} \left[(k')^\delta \varphi_\xi^* \left(-\frac{\varphi_\xi(1/c)}{Z_n} \right) \geq (k')^\delta \varphi_\xi^*(t_n) \right] & \text{if } \xi < 0. \end{cases} \tag{6}$$

The asymptotic behavior of the left-hand side random term $(k')^{\delta-1\{\xi=0\}} \varphi_\xi^* (-\varphi_\xi(1/c)/Z_n)$ is given by Lemma 3.2. Let us now focus on the right-hand side deterministic term $(k')^{\delta-1\{\xi=0\}} \varphi_\xi^*(t_n)$. Different cases have to be considered:

If $\xi > 0$, the following sequence of asymptotic equivalences holds

$$t_n \sim k^{\xi+t/V_k(\xi)} \frac{\varphi_\xi(1/c)}{1 - c^{\xi+t/V_k(\xi)}} \sim k^{\xi+t/V_k(\xi)} \frac{c^{-\xi}}{\xi}.$$

Since $V_k(\xi) \sim \ln(k)/\xi$, we have that $k^{t/V_k(\xi)} \rightarrow e^{t\xi}$ as $n \rightarrow \infty$. Thus, $t_n \sim (k')^\xi/\xi e^{t\xi} \rightarrow \infty$ as $n \rightarrow \infty$. Lemma 3.1(III) implies that $\varphi_\xi^*(t_n) \sim 1 + (k')^\xi e^{t\xi} \sim (k')^\xi e^{t\xi}$. Thus,

$$\lim_{n \rightarrow \infty} (k')^{\delta-1\{\xi=0\}} \varphi_\xi^*(t_n) = e^{t\xi}. \tag{7}$$

If $\xi = 0$, we have $t_n = \ln(c) \frac{c^{t/\ln^2(k)} - k^{t/\ln^2(k)}}{1 - c^{t/\ln^2(k)}}$. Using the expansions,

$$c^{t/\ln^2(k)} = 1 + t \frac{\ln(c)}{\ln^2(k)} + O\left(\frac{1}{\ln^4(k)}\right) \quad \text{and} \quad k^{t/\ln^2(k)} = 1 + t \frac{1}{\ln(k)} + t^2 \frac{1}{2\ln^2(k)} + o\left(\frac{1}{\ln^2(k)}\right),$$

we find that $t_n = \ln(k)[1 + \frac{t/2 - \ln(c)}{\ln(k)} + o(\frac{1}{\ln(k)})]$. Since $t_n \rightarrow \infty$ and $\ln(k) - t_n \rightarrow -t/2 + \ln(c)$ as $n \rightarrow \infty$, Lemma 3.1 (II) implies that $\varphi_\xi^*(t_n) \sim \varphi_\xi^*(\ln(k)) \exp[t/2 - \ln(c)] = k' \exp(t/2)$. Thus,

$$\lim_{n \rightarrow \infty} (k')^{\delta - 1\{\xi=0\}} \varphi_\xi^*(t_n) = e^{t/2}. \tag{8}$$

When $\xi < 0$, we have: $t_n = -\frac{1}{\xi} \frac{c^{-\xi} - 1}{c^{-\xi} - c^{t/V_k(\xi)}} c^{t/V_k(\xi)} [1 - (k')^{\xi+t/V_k(\xi)}]$. Remarking that $c^{t/V_k(\xi)} = 1 + t \frac{\ln(c)}{V_k(\xi)} + O(\frac{1}{V_k(\xi)})$, and that $(k')^{t/V_k(\xi)} = 1 + o(1)$ lead to the following expansion:

$$t_n = -\frac{1}{\xi} \left[1 + t \frac{\ln(c)}{V_k(\xi)} \right] \left[1 + o\left(\frac{1}{V_k(\xi)}\right) \right] \left[1 + t \frac{\ln(c)}{(c^{-\xi} - 1)V_k(\xi)} + o\left(\frac{1}{V_k(\xi)}\right) \right] [1 - (k')^\xi + o(k^\xi)].$$

Then, $t_n = u_n(1 + \varepsilon_n)$, with $u_n = -\frac{1}{\xi} [1 + \frac{t \ln(c)}{V_k(\xi)}]$ and $\varepsilon_n = t \frac{\delta \ln(c)}{c^{-\xi} - 1} k^{-\delta} - (k')^\xi + o(k^{-\delta}) + o(k^\xi)$. Two situations have to be considered:

If $-1/2 \leq \xi < 0$, it follows that $\lim_{n \rightarrow \infty} \frac{\varepsilon_n}{\varphi_\xi^*(u_n)} = -\frac{1}{t\xi \ln(c)} [\frac{t\xi \ln(c)}{c^{-\xi} - 1} + c^{-\xi}]$, and Lemma 3.1 (IV) (i) implies $\varphi_\xi^*(t_n) \sim \varphi_\xi^*(u_n)[1 + \frac{1}{t\xi \ln(c)} (\frac{t\xi \ln(c)}{c^{-\xi} - 1} + c^{-\xi})]$ and thus

$$\lim_{n \rightarrow \infty} (k')^{\delta - 1\{\xi=0\}} \varphi_\xi^*(t_n) = 1 + t \frac{\xi \ln(c)}{c^{-\xi} - 1}. \tag{9}$$

If $\xi < -1/2$, $\varepsilon_n/\varphi_\xi^*(u_n) \rightarrow (1 - c^{-\xi})^{-1}$ as $n \rightarrow \infty$. Thus, Lemma 3.1(IV)(i) yields

$$\lim_{n \rightarrow \infty} (k')^{\delta - 1\{\xi=0\}} \varphi_\xi^*(t_n) = -\frac{t}{2} \ln(c) \frac{c^{-\xi} - 1/2}{1 - c^{-\xi}}. \tag{10}$$

Collecting (6)–(10) with Lemma 3.2 concludes the proof. \square

References

[1] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, Cambridge University Press, 1987.
 [2] L. Gardes, Estimation d’une fonction quantile extrême, PhD Thesis, Université Montpellier 2, 2003.
 [3] L. Gardes, S. Girard, A Pickands-type estimator of the extreme-value index, Technical Report LMC-IMAG, RR-1063, 2004.
 [4] B. Gnedenko, Sur la distribution limite du terme maximum d’une série aléatoire, Ann. Math. 44 (1943) 423–453.
 [5] P. Hall, On some simple estimates of an exponent of regular variation, J. Roy. Statist. Soc. Ser. B 44 (1982) 37–42.
 [6] J. Pickands III, Statistical inference using extreme-order statistics, Ann. Statist. 3 (1975) 119–131.