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Partial Differential Equations

On the Brezis–Nirenberg problem on $\mathbf{S}^3$, and a conjecture of Bandle–Benguria

Wenyi Chen [a], Juncheng Wei [b]

[a] *Department of Mathematics, Wuhan University, Wuhan, Hubei 430072, PR China*
[b] *Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong*

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Abstract

We consider the following Brezis–Nirenberg problem on $\mathbf{S}^3$

$$-\Delta_{\mathbf{S}^3} u = \lambda u + u^5 \quad \text{in } D, \quad u > 0 \quad \text{in } D \quad \text{and} \quad u = 0 \quad \text{on } \partial D,$$

where D is a geodesic ball on $\mathbf{S}^3$ with geodesic radius θ_1, and $\Delta_{\mathbf{S}^3}$ is the Laplace–Beltrami operator on $\mathbf{S}^3$. We prove that for any $\lambda < -\frac{3}{4}$ and for every $\theta_1 < \pi$ with $\pi - \theta_1$ sufficiently small (depending on λ), there exists bubbling solution to the above problem. This solves a conjecture raised by Bandle and Benguria [J. Differential Equations 178 (2002) 264–279] and Brezis and Peletier [C. R. Acad. Sci. Paris, Ser. I 339 (2004) 291–394]. ***To cite this article: W. Chen, J. Wei, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Sur l'équation de Brezis–Nirenberg sur $\mathbf{S}^3$ et une conjecture de Bandle–Benguria. Nous considérons le problème de Brezis–Nirenberg suivant sur $\mathbf{S}^3$

$$-\Delta_{\mathbf{S}^3} u = \lambda u + u^5 \quad \text{dans } D, \quad u > 0 \quad \text{dans } D \quad \text{et} \quad u = 0 \quad \text{sur } \partial D,$$

où D est une boule géodésique sur $\mathbf{S}^3$ de rayon géodésique θ_1, et $-\Delta_{\mathbf{S}^3}$ est l'opérateur de Laplace–Beltrami sur $\mathbf{S}^3$. Nous montrons que pour tout $\lambda < -\frac{3}{4}$ et tout $\theta_1 < \pi$ avec $\pi - \theta_1$ suffisamment petit (dependant de λ), il existe des solutions pour le problème précédent. Ce résultat répond à une conjecture de Bandle et Benguria [J. Differential Equations 178 (2002) 264–279] et de Brezis et Peletier [C. R. Acad. Sci. Paris, Ser. I 339 (2004) 291–394]. ***Pour citer cet article : W. Chen, J. Wei, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

E-mail addresses: wychencn@whu.edu.cn (W. Chen), wei@math.cuhk.edu.hk (J. Wei).

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1. Introduction

We consider the following problem

$$-\Delta_{\mathbf{S}^3} u = \lambda u + u^5, \quad u > 0 \quad \text{in } D \quad \text{and} \quad u = 0 \quad \text{on } \partial D, \tag{1}$$

where $\Delta_{\mathbf{S}^3}$ is the Laplace–Beltrami operator on $\mathbf{S}^3$ and D is the geodesic ball centered at the North Pole with geodesic radius θ_1. Of particular interest is the case of $\theta_1 \in (\frac{\pi}{2}, \pi)$. The analogous problem in $\mathbb{R}^N$

$$-\Delta u = \lambda u + u^5, \quad u > 0 \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, was first studied in a celebrated paper by Brezis and Nirenberg [3]. In particular, they proved that if $\Omega = B_R(0)$, the solutions to (2) exist only if $\lambda \in (0, \lambda_1)$ for $N \geqslant 4$ and $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$ when $N = 3$. Since then, there is a vast literature on many extensions of the problem considered by Brezis and Nirenberg (see, e.g. [7], Chapter 3 and the references therein).

In recent papers by Bandle–Benguria [1] and Bandle–Peletier [2], it was shown that on the sphere $\mathbf{S}^3$ the situation is quite different. In particular, they showed that in the range of $\lambda > -\frac{3}{4}$, there is a solution if and only if $(\pi^2 - 4\theta_1^2)/(4\theta_1^2) < \lambda < (\pi^2 - \theta_1^2)/(\theta_1^2)$. For $\lambda \leqslant -\frac{3}{4}$, it was shown in [1], that there exist no solutions if $\theta_1 \leqslant \frac{\pi}{2}$. Then they conjectured (see a more general conjecture in [4]):

Conjecture. *For every $\lambda < -\frac{3}{4}$ and every $\theta_1 < \pi$ with $\pi - \theta_1$ sufficiently small, there exists a solution to* (1).

In this Note, we solve the conjecture affirmatively. To state our result, we introduce the corresponding equation on $\mathbb{R}^3$. By using stereo-graphic projection at the North Pole, Eq. (1) is equivalent to

$$\Delta u - p(r)u + 3u^5 = 0, \quad u = u(r) > 0, \quad r \geqslant \varepsilon, \quad u(\varepsilon) = 0, \quad u(r) = \mathrm{O}\left(\frac{1}{r}\right) \quad \text{as } r \to +\infty, \tag{3}$$

where $p(r) = \frac{-3/4-\lambda}{(1+r^2)^2}$ and $\varepsilon = \frac{\sin\theta_1}{1-\cos\theta_1}$.

Let $U_\Lambda(r) = (\frac{\Lambda}{\Lambda^2+r^2})^{1/2}$ be the unique radial solution of $\Delta u + 3u^5 = 0, u = u(r) > 0$. Our main result in this Note is the following:

Theorem 1.1. *Let $\lambda < -\frac{3}{4}$ be a fixed number. Then there exists an $\varepsilon_0 = \varepsilon_0(\lambda) > 0$ such that for each $0 < \varepsilon < \varepsilon_0$, problem* (3) *has a solution $u_\varepsilon(r)$ with the following form*

$$u_\varepsilon(r) - U_{\sqrt{\varepsilon}\Lambda_\varepsilon}(r) = \mathrm{O}\left(\frac{\varepsilon^{3/4}}{r}\right), \quad \textit{for } r \geqslant \varepsilon, \textit{ where } \Lambda_\varepsilon \to \Lambda_0 > 0. \tag{4}$$

We remark that Eq. (1) with $\lambda \to -\infty$ is also studied in [4] and [9]. There it is shown that more and more peaked solutions arise when $|\lambda| \to +\infty$.

The proof of Theorem 1.1 mainly relies upon a finite-dimensional reduction procedure. Such a method has been used successfully in many papers, see, e.g. [5,6,8]. In particular, we shall follow that used in [8].

By the scaling $r \to \sqrt{\varepsilon} r$, problem (3) is reduced to the following ODE which we shall work with

$$\Delta u - \varepsilon p(\sqrt{\varepsilon} r)u + 3u^5 = 0, \quad u = u(r) > 0, \quad r \geqslant \sqrt{\varepsilon}, \quad u(\sqrt{\varepsilon}) = 0, \quad u(r) = \mathrm{O}\left(\frac{1}{r}\right) \quad \text{as } r \to +\infty. \tag{5}$$

2. Approximate solutions, some estimates and reduction process

In this section, we first introduce a family of approximate solutions to (5) and derive some useful estimates. Then we perform a finite-dimensional reduction procedure which is similar to that of [8].

Let $\Lambda > 0$ be a fixed positive constant such that $\frac{1}{C} < \Lambda < C$ for some large constant $C > 0$. We define $V_{\varepsilon,\Lambda}$ to be the unique solution satisfying $\Delta v - \varepsilon p(\sqrt{\varepsilon} r)v + 3U_\Lambda^5 = 0,\ r \geqslant \sqrt{\varepsilon},\ v(\sqrt{\varepsilon}) = v(+\infty) = 0$.

To analyze $V_{\varepsilon,\Lambda}$, we introduce two functions: let $\psi_{\varepsilon,\Lambda}$ be the unique solution of $\Delta\psi_{\varepsilon,\Lambda} - p(r)\psi_{\varepsilon,\Lambda} + p(r)U_{\sqrt{\varepsilon}\Lambda} = 0,\ \psi'_{\varepsilon,\Lambda}(0) = \psi_{\varepsilon,\Lambda}(+\infty) = 0$, and $G(r)$ be the Green's function satisfying $\Delta G - p(r)G + 4\pi\delta_0 = 0, G(+\infty) = 0$. (Note that $G(r) = \frac{1}{r} + \mathrm{O}(1)$ for $r \ll 1$ and $\psi_{\varepsilon,\Lambda} = \varepsilon^{1/4}\Lambda^{1/2}\psi_0(r) + \mathrm{o}(\varepsilon^{1/4}(1+r)^{-1})$, where ψ_0 satisfies $\Delta\psi_0 - p(r)\psi_0 + p(r)\frac{1}{r} = 0, \psi'_0(0) = \psi_0(+\infty) = 0$.) It is then easy to see that

$$V_{\varepsilon,\Lambda}(r) = U_\Lambda(r) - \varepsilon^{1/4}\big[\psi_{\varepsilon,\Lambda}(\sqrt{\varepsilon} r) + \beta_{\varepsilon,\Lambda} G(\sqrt{\varepsilon} r)\big], \quad \text{where}$$
$$\beta_{\varepsilon,\Lambda} = \frac{U_{\sqrt{\varepsilon}\Lambda}(\varepsilon) - \psi_{\varepsilon,\Lambda}(\varepsilon)}{G(\varepsilon)} = \varepsilon^{3/4}\Lambda^{-1/2}\big(1 + \mathrm{o}(1)\big). \tag{6}$$

Let $I_\varepsilon = [\sqrt{\varepsilon}, +\infty)$ and $S_\varepsilon[u] = \Delta u - \varepsilon p(\sqrt{\varepsilon} r)u + 3u_+^5$ where $u_+ = \max(u, 0)$. To estimate $S_\varepsilon[V_{\varepsilon,\Lambda}]$, we define two norms $\|\phi\|_* = \sup_{r\in I_\varepsilon}(1+r^2)^{1/2}|\phi(r)|$ and $\|f\|_{**} = \sup_{r\in I_\varepsilon}(r(1+r^2)^{5/4}|f(r)|)$. The reason for defining these two norms lies behind the following lemma whose proof is simple and thus omitted:

Lemma 2.1. *The following holds*: $\|\phi\|_* \leqslant C\|\Delta\phi - \varepsilon p(\sqrt{\varepsilon} r)\phi\|_{**}$ *where* $\phi(\sqrt{\varepsilon}) = \phi(+\infty) = 0$.

Since $S_\varepsilon[V_{\varepsilon,\Lambda}] = 3V_{\varepsilon,\Lambda}^5 - 3U_\Lambda^5$, by (6), it is not difficult to see that

$$\big\|S_\varepsilon[V_{\varepsilon,\Lambda}]\big\|_{**} \leqslant C\varepsilon^{1/2}. \tag{7}$$

Finally we discuss the reduction process. The following lemma can be proved along the same ideas of Proposition 3.2 of [8], using the estimate (7). Interested readers may consult [8]. We omit the details.

Lemma 2.2. *For ε sufficiently small, there exists a unique pair $(\phi_{\varepsilon,\Lambda}, c_\varepsilon(\Lambda))$ satisfying*

$$S_\varepsilon[V_{\varepsilon,\Lambda} + \phi_{\varepsilon,\Lambda}] = c_\varepsilon(\Lambda)Z_\Lambda, \quad \int_{I_\varepsilon} \phi_{\varepsilon,\Lambda} Z_\Lambda r^2\,\mathrm{d}r = 0 \tag{8}$$

where $Z_\Lambda = U_\Lambda^4(\frac{\partial U_\Lambda}{\partial\Lambda})$. Moreover, we also have that $\|\phi_{\varepsilon,\Lambda}\|_* \leqslant C\varepsilon^{1/2}$ and that the map $\Lambda \to c_\varepsilon(\Lambda)$ is continuous.

3. Proof of Theorem 1.1

From (8), we see that, to prove Theorem 1.1, it is enough to find a zero of function $c_\varepsilon(\Lambda)$. To this end, let us expand $c_\varepsilon(\Lambda)$.

Let $L_{\varepsilon,\Lambda} := \Delta - \varepsilon p(\sqrt{\varepsilon} r) + 15V_{\varepsilon,\Lambda}^4$ and $z_{\varepsilon,\Lambda}$ be the unique solution of $\Delta v - \varepsilon p(\sqrt{\varepsilon} r)v + 15U_\Lambda^4(\frac{\partial U_\Lambda}{\partial\Lambda}) = 0$, $r \geqslant \sqrt{\varepsilon},\ v(\sqrt{\varepsilon}) = v(+\infty) = 0$. It is easy to see that $z_{\varepsilon,\Lambda} = \frac{\partial U_\Lambda}{\partial\Lambda} + \mathrm{O}(\varepsilon^{1/4}\frac{1}{r})$.

Multiplying Eq. (8) by $r^2 z_{\varepsilon,\Lambda}(r)$, we obtain, using Lemma 2.2,

$$c_\varepsilon \int_{I_\varepsilon} z_{\varepsilon,\Lambda} Z_\Lambda r^2\,\mathrm{d}r = \int_{I_\varepsilon} S_\varepsilon[V_{\varepsilon,\Lambda}] z_{\varepsilon,\Lambda} r^2\,\mathrm{d}r + \int_{I_\varepsilon} L_{\varepsilon,\Lambda}[\phi_{\varepsilon,\Lambda}] z_{\varepsilon,\Lambda} r^2\,\mathrm{d}r + \mathrm{o}(\varepsilon^{1/2}). \tag{9}$$

By integrating by parts, the second term on the right-hand side of (9) can be estimated as follows:

$$\int_{I_\varepsilon} L_{\varepsilon,\Lambda}[\phi_{\varepsilon,\Lambda}] z_{\varepsilon,\Lambda} r^2\,\mathrm{d}r = \int_{I_\varepsilon} L_{\varepsilon,\Lambda}[z_{\varepsilon,\Lambda}]\phi_{\varepsilon,\Lambda} r^2\,\mathrm{d}r = \int_{I_\varepsilon} 15[V_{\varepsilon,\Lambda}^4 - U_\Lambda^4] z_{\varepsilon,\Lambda}\phi_{\varepsilon,\Lambda} r^2\,\mathrm{d}r + \mathrm{o}(\varepsilon^{1/2}) = \mathrm{o}(\varepsilon^{1/2}).$$

It remains to compute the first term in the right-hand side of (9):

$$\int_{I_\varepsilon} S_\varepsilon[V_{\varepsilon,\Lambda}]z_\varepsilon r^2\,\mathrm{d}r = \int_{I_\varepsilon} 3[V_{\varepsilon,\Lambda}^5 - U_\Lambda^5]z_{\varepsilon,\Lambda}r^2\,\mathrm{d}r$$

$$= -15\varepsilon^{1/2}\Lambda^{1/2}\psi_0(0)\int_0^{+\infty}\left(U_\Lambda^4\frac{\partial U_\Lambda}{\partial\Lambda}\right)r^2\,\mathrm{d}r - 15\varepsilon^{-1/4}\beta_{\varepsilon,\Lambda}\int_0^{+\infty}\left(U_\Lambda^4\frac{\partial U_\Lambda}{\partial\Lambda}\right)r\,\mathrm{d}r + \mathrm{o}(\sqrt{\varepsilon}). \tag{10}$$

By direct computations, we have

$$\int_0^{+\infty}\left(U_\Lambda^4\frac{\partial U_\Lambda}{\partial\Lambda}\right)r^2\,\mathrm{d}r = \frac{1}{10}\left(\int_0^{\infty}U_1^5r^2\,\mathrm{d}r\right)\Lambda^{-1/2}, \quad \int_0^{+\infty}\left(U_\Lambda^4\frac{\partial U_\Lambda}{\partial\Lambda}\right)r\,\mathrm{d}r = -\frac{1}{10}\left(\int_0^{\infty}U_1^5r\,\mathrm{d}r\right)\Lambda^{-3/2}. \tag{11}$$

Substituting (6) and (11) into (10), we arrive at

$$\int_{I_\varepsilon} S_\varepsilon[V_{\varepsilon,\Lambda}]z_{\varepsilon,\Lambda}r^2\,\mathrm{d}r = \varepsilon^{1/2}(-\gamma_0 + \gamma_1\Lambda^{-2}) + \mathrm{o}(\varepsilon^{1/2}) \tag{12}$$

where γ_0, γ_1 are two generic positive constants. We obtain from (9) and (12) that

$$c_\varepsilon(\Lambda) = c_0\varepsilon^{1/2}(\gamma_0 - \gamma_1\Lambda^{-2}) + \mathrm{o}(\varepsilon^{1/2}) \quad \text{for some } c_0 \neq 0. \tag{13}$$

Theorem 1.1 now follows from (13): in fact, (13) implies $c_\varepsilon(\Lambda_0-\delta)c_\varepsilon(\Lambda_0+\delta) < 0$ where $\Lambda_0 = \sqrt{\gamma_1/\gamma_0}$ and δ small. By the continuity of $c_\varepsilon(\Lambda)$, a zero of $c_\varepsilon(\Lambda)$, denoted by $\Lambda_\varepsilon \in (\Lambda_0-\delta, \Lambda_0+\delta)$, is guaranteed. Then $u_\varepsilon = V_{\varepsilon,\Lambda_\varepsilon} + \phi_{\varepsilon,\Lambda_\varepsilon}$ is a solution to (5). This proves Theorem 1.1. □

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