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Partial Differential Equations/Optimal Control

Remarks on the null controllability of the Burgers equation

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Abstract

In the context of the Burgers equation with distributed controls, we present optimal estimates for the minimal time of controllability $T(r)$ of the initial data of norm $\leqslant r$ in L^2. ***To cite this article: E. Fernández-Cara, S. Guerrero, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Remarques sur la contrôlabilité exacte à zéro de l'équation de Burgers. Dans le contexte de l'équation de Burgers avec contrôles distribués, on présente une estimation optimale du temps minimal de contrôlabilité $T(r)$ des données initiales de norme $\leqslant r$ dans L^2. ***Pour citer cet article : E. Fernández-Cara, S. Guerrero, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction and main results

Let $T > 0$ be an arbitrary positive time and let us assume that $\omega \subset (0,1)$ is a nonempty open set, with $0 \notin \overline{\omega}$. In this Note, we will be concerned with the null controllability of the following system for the Burgers equation:

$$\begin{cases} y_t - y_{xx} + yy_x = v1_\omega, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,1). \end{cases} \tag{1}$$

Here, $v = v(x,t)$ denotes the control and $y = y(x,t)$ denotes the state. It will be said that (1) is *null controllable at time T* if, for every $y^0 \in L^2(0,1)$, there exists $v \in L^2((0,1) \times (0,T))$ such that

$$y(x,T) = 0 \quad \text{in } (0,1). \tag{2}$$

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Some controllability properties of (1) have been studied in [2] (see Chapter 1, Theorems 6.3 and 6.4). There, it is shown that one cannot reach (even approximately) stationary solutions of (1) with large L^2-norm at any time T. In other words, with the help of one control, the solutions of the Burgers equation cannot go anywhere at any time.

For each $y^0 \in L^2(0,1)$, let us introduce $T(y^0) = \inf\{T > 0\colon$ (1) is null controllable at time $T\}$. Then, for each $r > 0$, we define the quantity $T^*(r) = \sup\{T(y^0)\colon \|y^0\|_{L^2(0,1)} \leqslant r\}$. Our main purpose in this Note is to prove that $T^*(r) > 0$ with an explicit sharp estimate in terms of r, which in particular implies that (global) null controllability at any positive time does not hold for (1).

More precisely, let us set $\phi(r) = (\log \frac{1}{r})^{-1}$. We have the following:

Theorem 1.1. *There exist positive constants C_0 and C_1 independent of r such that*

$$C_0\phi(r) \leqslant T^*(r) \leqslant C_1\phi(r) \quad \text{as } r \to 0. \tag{3}$$

Remark 1. The same estimates hold when the control v acts on system (1) through the boundary *only* at $x = 1$ (or only at $x = 0$). When (1) is controlled at both points $x = 0$ and $x = 1$, it is unknown whether we still have an estimate from below for $T(r)$.

The main ideas of the proof of Theorem 1.1 will be presented in the following section. More details will be given in a forthcoming paper.

2. Sketch of the proof of Theorem 1.1

The proof of the estimate from above in (3) can be obtained by solving (1), (2) with a (more or less) standard fixed point argument, using global Carleman inequalities to estimate the control and energy inequalities to estimate the state and being very careful with the role of T in these inequalities.

We will concentrate in the proof of the other estimate, that has been inspired by the arguments in [1].

We will prove that there exist positive constants C_0 and C_0' such that, for any sufficiently small $r > 0$, we can find initial data y^0 satisfying $\|y^0\|_{L^2(0,1)} \leqslant r$ with the following property: for any state y associated to y^0, one has

$$\left|y(x,t)\right| \geqslant C_0' r \quad \text{for some } x \in (0,1) \text{ and any } t\colon\ 0 < t < C_0\phi(r).$$

Let us set $T = \phi(r)$ and let $\rho_0 \in (0,1)$ be such that $(0,\rho_0) \cap \omega = \emptyset$. We can suppose that $0 < r < \rho_0$. Let us choose $y^0 \in L^2(0,1)$ such that $y^0(x) = -r$ for all $x \in (0,\rho_0)$ and let us denote by y an associated solution of (1).

Let us introduce the function $Z = Z(x,t)$, with

$$Z(x,t) = \exp\left\{-\frac{2}{t}\left(1 - \mathrm{e}^{-\rho_0^2(\rho_0 - x)^3/(\rho_0/2 - x)^2}\right) + \frac{1}{\rho_0 - x}\right\}. \tag{4}$$

Then one has $Z_t - Z_{xx} + ZZ_x \geqslant 0$.

Let us now set $w(x,t) = Z(x,t) - y(x,t)$. It is immediate that

$$\begin{cases} w_t - w_{xx} + ZZ_x - yy_x \geqslant 0, & (x,t) \in (0,\rho_0) \times (0,T), \\ w(0,t) > 0, \quad w(\rho_0,t) = +\infty, & t \in (0,T), \\ w(x,0) = r, & x \in (0,\rho_0) \end{cases} \tag{5}$$

and, consequently, $w^-(x,t) \equiv 0$. Indeed, let us multiply the differential equation in (5) by $-w^-$ and let us integrate in $(0,\rho_0)$. Since w^- vanishes at $x = 0$ and $x = \rho_0$, after some manipulation we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\rho_0} |w^-|^2\,\mathrm{d}x + \int_0^{\rho_0} |w_x^-|^2\,\mathrm{d}x = \int_0^{\rho_0} w^-(ZZ_x - yy_x)\,\mathrm{d}x \leqslant C\int_0^{\rho_0} |w^-|^2\,\mathrm{d}x. \tag{6}$$

Hence,

$$y \leqslant Z \quad \text{in } (0,\rho_0)\times(0,T). \tag{7}$$

Let us set $\rho_1 = \rho_0/2$ and let us introduce the solution u of the auxiliary system

$$\begin{cases} u_t - u_{xx} + uu_x = 0, & (x,t)\in(0,\rho_1)\times(0,T),\\ u(0,t) = Z(\rho_1,t), \quad u(\rho_1,t) = Z(\rho_1,t), & t\in(0,T),\\ u(x,0) = -\tilde r(x), & x\in(0,\rho_1), \end{cases} \tag{8}$$

where $\tilde r$ is any regular function satisfying the following: $\tilde r(0) = \tilde r(\rho_1) = 0$; $\tilde r(x) = r$ for all $x\in(\delta\rho_1,(1-\delta)\rho_1)$ and some $\delta\in(0,1/4)$; $-r \leqslant -\tilde r(x) \leqslant 0$;

$$|\tilde r_x| \leqslant Cr \quad \text{and} \quad |\tilde r_{xx}| \leqslant C \quad \text{in } (0,\rho_1), \tag{9}$$

where $C = C(\rho_1)$ is independent of r. Taking into account (7) and that $u_x, y \in L^\infty((0,\rho_1)\times(0,T))$ (see Lemma 2.1 below), a standard application of Gronwall's lemma shows that

$$y \leqslant u \quad \text{in } (0,\rho_1)\times(0,T). \tag{10}$$

We will prove that, for some appropriate choices of C_0 and C_0', $u(\rho_1/2,t)$ remains below $-C_0'r$ for any time $t < C_0\phi(r)$. This, together with (10), will prove Theorem 1.1.

We will need the following lemma:

Lemma 2.1. *One has*

$$|u| \leqslant Cr \quad \textit{and} \quad |u_x| \leqslant Cr^{1/2} \quad \textit{in } (0,\rho_1)\times\big(0,\phi(r)\big), \tag{11}$$

where C is independent of r.

A consequence of (11) is that $u_t - u_{xx} \leqslant C^* r^{3/2}$ in $(0,\rho_1)\times(0,\phi(r))$ for some $C^* > 0$. Let us consider the functions p and q, given by $p(t) = C^* r^{3/2} t - r$ and $q(x,t) = c(\mathrm{e}^{-(x-(\rho_1/4))^2/4t} + \mathrm{e}^{-(x-3(\rho_1/4))^2/4t})$. It is then clear that $b = u - p - q$ satisfies

$$\begin{cases} b_t - b_{xx} \leqslant 0, & (x,t)\in(\rho_1/4, 3\rho_1/4)\times\big(0,\phi(r)\big),\\ b(\rho_1/4,t) \leqslant Z(\rho_1,t) - C^* r^{3/2} t + r - c\big(1+\mathrm{e}^{-\rho_1^2/(16t)}\big), & t\in\big(0,\phi(r)\big),\\ b(3\rho_1/4,t) \leqslant Z(\rho_1,t) - C^* r^{3/2} t + r - c\big(1+\mathrm{e}^{-\rho_1^2/(16t)}\big), & t\in\big(0,\phi(r)\big),\\ b(x,0) = 0, & x\in(\rho_1/4, 3\rho_1/4). \end{cases} \tag{12}$$

Obviously, in the definition of q, the constant c can be chosen large enough to have $Z(\rho_1,t) - C^* r^{3/2} t + r - c(1+\mathrm{e}^{-\rho_1^2/(16t)}) < 0$ for any $t\in(0,\phi(r))$. If this is the case, we get $u \leqslant p + q$ and, in particular,

$$u(\rho_1/2,t) \leqslant (p+q)(\rho_1/2,t) = 2c\,\mathrm{e}^{-\rho_1^2/(64t)} + C^* r^{3/2} t - r.$$

Therefore, we see that there exist C_0 and C_0' such that $u(\rho_1/2,t) < -C_0'r$ for any $t\in(0,C_0\phi(r))$.

This proves (3) and, consequently, ends the proof of Theorem 1.1.

Proof of Lemma 2.1. The first estimate in (11) can be obtained in a classical way, using arguments based on the maximum principle for the heat equation and the facts that $Z(\rho_1,t) \leqslant Cr^2$ and $Z_t(\rho_1,t) \leqslant Cr^2\phi(r)^{-2}$ for $t\in(0,\phi(r))$. Let us explain how the second estimate in (11) can be deduced. Thus, let us set $\tilde u(x,t) = u(x,t) - Z(\rho_1,t)$. This function satisfies

$$\begin{cases} \tilde u_t - \tilde u_{xx} + \big(\tilde u + Z(\rho_1,t)\big)\tilde u_x = -Z_t(\rho_1,t), & (x,t)\in(0,\rho_1)\times\big(0,\phi(r)\big),\\ \tilde u(0,t) = 0, \quad \tilde u(\rho_1,t) = 0, & t\in\big(0,\phi(r)\big),\\ \tilde u(x,0) = -\tilde r(x), & x\in(0,\rho_1). \end{cases} \tag{13}$$

- In a classical way, we can deduce energy estimates for $\tilde{u}$:

$$\|\tilde{u}\|^2_{L^\infty(0,T;L^2(0,\rho_1))} + \|\tilde{u}_x\|^2_{L^2((0,\rho_1)\times(0,T))} \leqslant C\|\tilde{r}\|^2_{L^2(0,\rho_1)} + Cr\int_0^{\rho_1}\int_0^{\phi(r)} \left|Z_t(\rho_1,t)\right| \mathrm{d}t\,\mathrm{d}x \leqslant Cr^2. \tag{14}$$

From the definition of $\tilde{u}$, a similar estimate holds for u. Multiplying the equation satisfied by $\tilde{u}$ by $\tilde{u}_t$, we also get $\tilde{u}_t \in L^2((0,\rho_1)\times(0,T))$, $\tilde{u}_x \in C([0,T];L^2(0,\rho_1))$ and

$$\begin{aligned}&\|\tilde{u}_t\|^2_{L^2((0,\rho_1)\times(0,T))} + \|\tilde{u}_x\|^2_{L^\infty(0,T;L^2(0,\rho_1))}\\ &\quad\leqslant C\left(\left\|\left(\tilde{u}+Z(\rho_1,t)\right)\tilde{u}_x\right\|^2_{L^2((0,\rho_1)\times(0,T))} + \left\|Z_t(\rho_1,\cdot)\right\|^2_{L^2(0,\phi(r))} + \|\tilde{r}_x\|^2_{L^2(0,\rho_1)}\right) \leqslant Cr^2.\end{aligned} \tag{15}$$

Here, we have used (9), the first estimate in (11) and (14). Obviously, this also holds for the norm of $\tilde{u}_{xx}$ in $L^2((0,\rho_1)\times(0,T))$. Again, these estimates are satisfied by u.

- Next, multiplying the equation satisfied by $\tilde{u}$ by $-\tilde{u}_{txx}$ and integrating in $(0,\rho_1)$, we have $\int_0^{\rho_1}|\tilde{u}_{tx}|^2\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\rho_1}|\tilde{u}_{xx}|^2\,\mathrm{d}x = \int_0^{\rho_1}\tilde{u}_{txx}(\tilde{u}+Z(\rho_1,t))\tilde{u}_x\,\mathrm{d}x + \int_0^{\rho_1}\tilde{u}_{txx}Z_t(\rho_1,t)\,\mathrm{d}x$. Integrating in $(0,t)$, we obtain the following after several integration by parts:

$$\begin{aligned}&\int_0^t\int_0^{\rho_1}|\tilde{u}_{tx}|^2\,\mathrm{d}x\,\mathrm{d}s + \left(\int_0^{\rho_1}|\tilde{u}_{xx}|^2\,\mathrm{d}x\right)(t) \leqslant C\left(\left(\int_0^{\rho_1}\left(|\tilde{u}+Z(\rho_1,t)|^2|\tilde{u}_x|^2\right)\mathrm{d}x\right)(t)\right.\\ &\quad + \int_0^{\rho_1}\tilde{r}\tilde{r}_x\tilde{r}_{xx}\,\mathrm{d}x + \int_0^{\rho_1}|\tilde{r}_{xx}|^2\,\mathrm{d}x + \int_0^t\int_0^{\rho_1}|\tilde{u}_{xx}|^2\left|\tilde{u}+Z(\rho_1,s)\right|^2\mathrm{d}x\,\mathrm{d}s + r^2\\ &\quad\left. + \int_0^t\int_0^{\rho_1}\left(|\tilde{u}_t|^2 + \left|Z_t(\rho_1,s)\right|^2\right)|\tilde{u}_x|^2\,\mathrm{d}x\,\mathrm{d}s + \left|Z_t(\rho_1,t)\right|^2 + \int_0^t\left|Z_{tt}(\rho_1,s)\right|^2\mathrm{d}s\right).\end{aligned}$$

Using again the first estimate in (11) and (15), we deduce that

$$\|\tilde{u}_{tx}\|^2_{L^2((0,\rho_1)\times(0,T))} + \|\tilde{u}_{xx}\|^2_{L^\infty(0,T;L^2(0,\rho_1))} \leqslant C\left(r^4 + r^2 + 1 + r^4\phi(r)^{-4} + r^4\phi(r)^{-8}\right). \tag{16}$$

As a consequence, (16) implies that

$$\|\tilde{u}_{tx}\|^2_{L^2((0,\rho_1)\times(0,T))} + \|\tilde{u}_{xx}\|^2_{L^\infty(0,T;L^2(0,\rho_1))} \leqslant C. \tag{17}$$

- Finally, in order to estimate $\tilde{u}_x$ in $L^\infty((0,\rho_1)\times(0,T))$, we observe that for each $t\in(0,T)$ there exists $a(t)\in(0,\rho_1)$ such that $\tilde{u}_x(a(t),t)=0$. Using this fact, we obtain: $\left|\tilde{u}_x(x,t)\right|^2 = \frac{1}{2}\int_{a(t)}^x \tilde{u}_x(\xi,t)\tilde{u}_{xx}(\xi,t)\,\mathrm{d}\xi$.

Applying the estimates (15) and (17) to $\tilde{u}_x \in L^\infty(0,T;L^2(0,\rho_1))$ and $\tilde{u}_{xx}\in L^\infty(0,T;L^2(0,\rho_1))$ respectively, we readily deduce that $\|\tilde{u}_x\|^2_{L^\infty((0,\rho_1)\times(0,T))} \leqslant Cr$ which, in particular, implies the second estimate in (11).

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