Available online at www.sciencedirect.com

C. R. Acad. Sci. Paris, Ser. I 341 (2005) 97–102

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Partial Differential Equations

A method for establishing upper bounds for singular perturbation problems

Arkady Poliakovsky

Department of Mathematics, Technion – I.I.T., 32000 Haifa, Israel

Received and accepted 30 May 2005

Available online 5 July 2005

Presented by Haïm Brezis

Abstract

We present two applications of a new method for proving upper bounds for singular perturbation problems involving maps of bounded variation. The two problems are of first and second order, respectively. The first is a minimization problem, related to the question of optimal lifting for BV-maps with values in S^1, for which we prove a Γ-convergence result. The second problem involves the Aviles–Giga functional, $\varepsilon\int_\Omega|\nabla^2 v|^2\,\mathrm{d}x+\frac{1}{\varepsilon}\int_\Omega(1-|\nabla v|^2)^2\,\mathrm{d}x$, for which we construct upper bounds via a sequence of functions whose limit has gradient in BV. ***To cite this article: A. Poliakovsky, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Une méthode de construction des bornes supérieures pour des problèmes de perturbation singulières. On présente deux applications d'une nouvelle méthode pour construire des bornes supérieures pour des problèmes de perturbation singulière où interviennent des applications à variation bornée. On applique cette méthode à deux problèmes, l'un du premier ordre et l'autre du second. Le premier est un problème de minimisation lié à la question de relèvement optimal pour des applications à variation bornée à valeurs dans S^1. Pour ce problème on démontre un théorème de Γ-convergence. Le second problème concerne la fonctionnelle d'Aviles–Giga, $\varepsilon\int_\Omega|\nabla^2 v|^2\,\mathrm{d}x+\frac{1}{\varepsilon}\int_\Omega(1-|\nabla v|^2)^2\,\mathrm{d}x$, pour laquelle on construit une borne supérieure via une suite de fonctions ayant comme limite une fonction dont le gradient est dans BV. ***Pour citer cet article : A. Poliakovsky, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

On donne deux applications d'une nouvelle méthode pour construire des bornes supérieures pour des problèmes de perturbation singulière. Le premier problème est lié à la question de relèvement optimal pour des applications à

E-mail address: maarkady@techunix.technion.ac.il (A. Poliakovsky).

1631-073X/$ – see front matter © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crma.2005.06.009

variation bornée à valeurs dans S^1. Par définition, un relèvement optimal d'une fonction $u=(u_1,u_2)\in \mathrm{BV}(\Omega,S^1)$ est une fonction φ qui réalise le minimum de

$$E(u):=\min\left\{\int_\Omega |D\varphi|:\ \varphi\in \mathrm{BV}(\Omega)\ \text{t.q. } u=\mathrm{e}^{\mathrm{i}\varphi},\ \mathcal{H}^N\text{-p.p. dans } \Omega\right\}.$$

On considère l'approche suivante pour construire une approximation d'un relèvement optimal. Soit $p\geqslant 1$ un paramètre. Pour tout $\varepsilon>0$ on définit la fonctionnelle suivante sur $H^1(\Omega)$:

$$J_\varepsilon(\varphi)=\int_\Omega\left(\varepsilon|\nabla\varphi|^2+\frac{1}{\varepsilon}\left|u(x)-\mathrm{e}^{\mathrm{i}\varphi}\right|^{2p}\right)\mathrm{d}x. \tag{1}$$

On note φ_ε un minimiseur de J_ε, et l'on peut espérer que la limite de $\{\varphi_\varepsilon\}$, quand ε tend vers zéro, soit un relèvement optimal. Le théorème suivant montre que c'est bien le cas si $p=2$. Par contre, un exemple dans [13] montre que pour $p\neq 2$ la limite n'est pas forcément un relèvement optimal. Pour décrire le problème limite on a besoin de définir la fonctionnelle $\mathcal{J}_{0,p}:\mathrm{BV}(\Omega)\to\mathbb{R}$ par

$$\mathcal{J}_{0,p}(\varphi)=\int_{J_\varphi}\left\{\inf_{t\in\mathbb{R}}\left|\int_{\varphi^--t}^{\varphi^+-t}2|\mathrm{e}^{\mathrm{i}s}-1|^p\,\mathrm{d}s\right|\right\}\mathrm{d}\mathcal{H}^{N-1}(x).$$

Théorème 0.1. (i) *Pour tout $\varepsilon>0$ soit φ_ε un minimiseur pour $\mathcal{J}_\varepsilon$, vérifiant $|\int_\Omega\varphi_\varepsilon(x)\,\mathrm{d}x|\leqslant C$, où C est indépendant de ε. Alors, de toute suite $\varepsilon_j\to 0$ on peut extraire une sous-suite vérifiant $\varphi_{\varepsilon_{j_k}}\to\varphi_0$ dans $L^1(\Omega)$.*

(ii) *Si $\varphi_{\varepsilon_j}\to\varphi_0$ dans $L^1(\Omega)$, alors $\varphi_0\in\mathrm{BV}(\Omega)$, $\mathrm{e}^{\mathrm{i}\varphi_0(x)}=u(x)$ p.p. dans Ω et φ_0 est un minimiseur pour le problème*

$$\inf\left\{\mathcal{J}_{0,p}(\varphi):\ \varphi\in\mathrm{BV}(\Omega),\ \mathrm{e}^{\mathrm{i}\varphi}=u \text{ p.p. dans } \Omega\right\}.$$

(iii) *Dans le cas $p=2$, toute fonction φ_0, comme définie dans* (i) *et* (ii)*, est un relèvement optimal.*

Dans le second problème, on s'intéresse à la fonctionnelle d'énergie d'Aviles–Giga :

$$E_\varepsilon(v)=\varepsilon\int_\Omega|\nabla^2 v|^2+\frac{1}{\varepsilon}\int_\Omega\left(1-|\nabla v|^2\right)^2,$$

où Ω est un domaine borné de classe C^2 dans $\mathbb{R}^N$, et v est une fonction scalaire. On s'attend à ce que chaque limite v des minimiseurs vérifie l'équation Eikonale

$$|\nabla v|=1\quad \text{p.p. dans } \Omega. \tag{2}$$

Le problème de Γ-convergence pour cette fonctionnelle est toujours ouvert. On démontre le théorème suivant :

Théorème 0.2. *Soit $\Omega\subset\mathbb{R}^N$ un domaine borné de classe C^2. Soit $v\in W^{1,\infty}(\Omega)$ vérifiant* (2) *t.q. $\nabla v\in$* $\mathrm{BV}(\Omega,\mathbb{R}^N)$*, $v=0$ et $\frac{\partial v}{\partial n}=-1$ sur $\partial\Omega$. Alors, pour tout $p\geqslant 1$ il existe une famille de fonctions $\{v_\varepsilon\}\subset C^2(\mathbb{R}^N)$ vérifiant les mêmes conditions au bord que v t.q.*

$$\lim_{\varepsilon\to 0^+}v_\varepsilon(x)=v(x)\quad \textit{dans } W^{1,p}(\Omega)$$

et

$$\lim_{\varepsilon\to 0}\left\{\varepsilon\int_\Omega\left|\nabla^2 v_\varepsilon(x)\right|^2\mathrm{d}x+\frac{1}{\varepsilon}\int_\Omega\left(1-\left|\nabla v_\varepsilon(x)\right|^2\right)^2\mathrm{d}x\right\}=\frac{1}{3}\int_{J_{\nabla v}}\left|\nabla v^+(x)-\nabla v^-(x)\right|^3\mathrm{d}\mathcal{H}^{N-1}(x).$$

1. Introduction

The aim of this Note is to present two applications of a new method of constructing upper bounds for singular perturbation problems. Our main tool in constructing the upper bounds is convolution with a smoothing kernel. This is of course a standard technique, but the new ingredient in our method is the special selection of the kernel, which is chosen adapted to the particular functional, using an optimization process. The detailed proofs are given in [16,17]. The first problem is related to optimal lifting in BV-space. We refer to the book [3] (see also [11,12,19]) for properties and notations of BV-space that we use throughout this Note. Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. An *optimal lifting* of a function $u=(u_1,u_2)\in \mathrm{BV}(\Omega,S^1)$ (i.e., $u\in \mathrm{BV}(\Omega,\mathbb{C})$, s.t. $|u|=1$ a.e.) is a function φ realizing the minimum for

$$E(u):=\min\left\{\int_\Omega |D\varphi|:\ \varphi\in \mathrm{BV}(\Omega) \text{ such that } u=\mathrm{e}^{\mathrm{i}\varphi},\ \mathcal{H}^N\text{-a.e. in } \Omega\right\}.$$

It was shown by Dávila and Ignat [7] that an optimal lifting satisfying the bound $\int_\Omega |D\varphi|\leqslant 2\int_\Omega |Du|$ always exists, and that the constant 2 is optimal, in general.

A natural approach to approximating optimal liftings is to consider, for a fixed parameter $1\leqslant p<\infty$, the family of energy functionals defined on $H^1(\Omega)$ by

$$\mathcal{J}_\varepsilon(\varphi)=\int_\Omega\left(\varepsilon|\nabla\varphi|^2+\frac{1}{\varepsilon}\left|u(x)-\mathrm{e}^{\mathrm{i}\varphi}\right|^{2p}\right)\mathrm{d}x,\quad \forall\varepsilon>0. \tag{1}$$

Denoting by φ_ε a minimizer for $\mathcal{J}_\varepsilon$, one may hope that due to the presence of the penalizing term in (1), a subsequence of $\{\varphi_\varepsilon\}$ will converge, as ε goes to 0, say in $L^1(\Omega)$, to an optimal lifting of u. Somewhat surprisingly, our result shows that optimal liftings are obtained in the limit, in general, only for $p=2$. However, if we restrict ourselves to $u\in W^{1,1}(\Omega)$, any limit is an optimal lifting. We treat the problem in the framework of Γ-convergence. Similar singular perturbation problems were studied extensively in the past, see Modica [15], Sternberg [18] and Ambrosio [1], to name a few. However, here, due to the high irregularity of the functional, new techniques are needed, and in particular, we make use of our construction method in the proof of the upper bound part.

2. Optimal lifting problem

In order to describe the limiting problem, we define a functional $\mathcal{J}_{0,p}:\mathrm{BV}(\Omega)\to\mathbb{R}$ by

$$\mathcal{J}_{0,p}(\varphi)=\int_{J_\varphi}\left\{\inf_{t\in\mathbb{R}}\left|\int_{\varphi^- -t}^{\varphi^+ -t}2|\mathrm{e}^{\mathrm{i}s}-1|^p\,\mathrm{d}s\right|\right\}\mathrm{d}\mathcal{H}^{N-1}(x).$$

Before stating our main result for this problem, we note that since adding a constant multiple of 2π to φ_ε does not change the value of $\mathcal{J}_\varepsilon(\varphi_\varepsilon)$, we may assume that $0\leqslant\varphi_\varepsilon(x)\leqslant 2\pi|\Omega|$ for every $x\in\Omega$ and every ε.

Theorem 2.1. (i) *For each $\varepsilon>0$ let φ_ε be a minimizer for $\mathcal{J}_\varepsilon$, satisfying $|\int_\Omega\varphi_\varepsilon(x)\,\mathrm{d}x|\leqslant C$, where C does not depend on ε. Then, for any sequence $\varepsilon_j\to 0$ there exists a subsequence satisfying $\varphi_{\varepsilon_{j_k}}\to\varphi_0$ in $L^1(\Omega)$.*

(ii) *If $\varphi_{\varepsilon_j}\to\varphi_0$ in $L^1(\Omega)$, then $\varphi_0\in\mathrm{BV}(\Omega)$, $\mathrm{e}^{\mathrm{i}\varphi_0(x)}=u(x)$ a.e. in Ω and φ_0 is a minimizer for the problem*

$$\inf\left\{\mathcal{J}_{0,p}(\varphi):\ \varphi\in\mathrm{BV}(\Omega),\ \mathrm{e}^{\mathrm{i}\varphi}=u \text{ a.e. in } \Omega\right\}.$$

(iii) *In the case $p=2$, any function φ_0 as in* (i) *and* (ii) *is an optimal lifting.*

The proof of Theorem 2.1 is given in [17]. In [13] it is shown by an example that for $p \neq 2$ and $u \in \mathrm{BV}$ the limit φ_0 in Theorem 2.1 is not necessarily an optimal lifting. On the other hand, for $u \in W^{1,1}$ it is proved in [17] that φ_0 must be an optimal lifting, for any p. Actually, assertion (ii) in Theorem 2.1 is a consequence of the following Γ-convergence result from [17]. Consider an extension of each $\mathcal{J}_\varepsilon$ to a functional $\bar{\mathcal{J}}_\varepsilon$, defined on all of $L^1(\Omega)$, by setting $\bar{\mathcal{J}}_\varepsilon \equiv \infty$ on $L^1(\Omega) \setminus H^1(\Omega)$. Similarly, we define a functional $\bar{\mathcal{J}}_{0,p}$ on $L^1(\Omega)$ by: $\bar{\mathcal{J}}_{0,p}(\varphi) = \mathcal{J}_{0,p}(\varphi)$ if $\varphi \in \mathrm{BV}(\Omega)$ is a lifting of u, and ∞ otherwise. Then, $\{\bar{\mathcal{J}}_\varepsilon\}$ Γ-converges to $\bar{\mathcal{J}}_{0,p}$.

3. Aviles–Giga functional

The second problem that we treat involves the Aviles–Giga energy functional

$$E_\varepsilon(v) = \varepsilon \int_\Omega |\nabla^2 v|^2 + \frac{1}{\varepsilon} \int_\Omega \left(1 - |\nabla v|^2\right)^2. \tag{2}$$

Here we assume that Ω is a C^2 bounded domain in $\mathbb{R}^N$, v is a scalar function and ε is a small parameter. Energies similar to (2) appear in different physical situations: smectic liquid crystals, blisters in thin films, micromagnetics (see [14] and the references therein). Clearly, it is expected that any limit of the minimizers to (2) will satisfy the Eikonal equation:

$$|\nabla v| = 1 \quad \text{a.e. in } \Omega. \tag{3}$$

Aviles and Giga [4] made a conjecture, based on a certain ansatz for the minimizers, that the limiting energy should take the form

$$E(v) = \frac{1}{3} \int_{J_{\nabla v}} |\nabla^+ v - \nabla^- v|^3 \, \mathrm{d}\mathcal{H}^{N-1},$$

where $J_{\nabla v}$ is the jump set of ∇v, and $\nabla^\pm v$ are the traces of ∇v on the two sides of the jump set (cf. [3]). Most of the results for this problem treat the two-dimensional case $N = 2$ (an example due to De Lellis [8] shows that the Aviles–Giga ansatz does not hold for $N \geqslant 3$, so we assume $N = 2$ in the survey of the known results below).

Support for the Aviles–Giga conjecture was given in the work of Jin and Kohn [14] who gave a lower bound for E_ε under the boundary conditions $v = 0$ and $\frac{\partial v}{\partial n} = -1$ on $\partial\Omega$. Aviles and Giga [5] refined the method of [14]. They defined a functional $\mathcal{J}$ on $W^{1,3}(\Omega)$ that coincides with the functional E on functions u with $\nabla u \in \mathrm{BV}$ that satisfy (3). Note that there exist functions u in $W^{1,3}(\Omega)$ satisfying $\mathcal{J}(u) < \infty$, for which ∇u is not in BV (see [2]). Another important contribution is due to Ambrosio, De Lellis and Mantegazza [2] and DeSimone, Kohn, Müller and Otto [9], who independently proved that for any family $\{v_\varepsilon\}$ satisfying $E_\varepsilon(v_\varepsilon) \leqslant C$, $\{\nabla v_\varepsilon\}$ is pre-compact in $L^3(\Omega)$. It was shown by Aviles and Giga [5] that $\mathcal{J}$ is lower semicontinous w.r.t. the strong topology of $W^{1,3}(\Omega)$. However, the Γ-convergence problem for the functionals $\{E_\varepsilon\}$ is still open, since for a given u satisfying $\mathcal{J}(u) < \infty$ and (3), it is not known whether there exists a family $\{u_\varepsilon\}$ satisfying: $u_\varepsilon \to u$ as $\varepsilon \to 0^+$ and $E_\varepsilon(u_\varepsilon) \to \mathcal{J}(u)$.

Our main contribution to this problem is the construction of such a family for u satisfying $\nabla u \in \mathrm{BV}$ and (3). However, we were unable to relax the assumption on u by requiring only $\mathcal{J}(u) < \infty$ instead of $\nabla u \in \mathrm{BV}$. To our knowledge, an upper-bound for the minimization problem (2) was proved only in very special cases, like in the case of u which is the distance to the boundary of an ellipse (see Jin and Kohn [14]), see also Ercolani et al. [10] for a related result. Our main result Theorem 3.1 establishes an upper bound for a more general energy functional than (2), and the latter case is then deduced in Corollary 3.2.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary of class C^2. Let $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}$ be a C^2-function such that $F(a, b, c) \geqslant 0$ for all a, b and c. Let $f \in \mathrm{BV}(\Omega, \mathbb{R}^q) \cap L^\infty(\Omega, \mathbb{R}^q)$ and $v \in W^{1,\infty}(\Omega)$ be such that $\nabla v \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ and $F(\nabla v(x), v(x), f(x)) = 0$ a.e. in Ω. Then,*

(i) *For every* $p \geqslant 1$ *there exists a family of functions* $\{v_\varepsilon\} \subset C^2(\mathbb{R}^N)$ *satisfying*

$$\lim_{\varepsilon\to 0^+} v_\varepsilon(x) = v(x) \quad \text{in } W^{1,p}(\Omega)$$

and

$$\lim_{\varepsilon\to 0}\left\{\varepsilon\int_\Omega \left|\nabla^2 v_\varepsilon(x)\right|^2 \mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega F\big(\nabla v_\varepsilon(x), v_\varepsilon(x), f(x)\big)\,\mathrm{d}x\right\}$$
$$= 2\int_{J_{\nabla v}} \left|\nabla v^+(x) - \nabla v^-(x)\right| \inf_{\tau\in[0,1]}\left\{\int_0^\tau \sqrt{F\big(s\nabla v^-(x) + (1-s)\nabla v^+(x), v(x), f^+(x)\big)}\,\mathrm{d}s\right.$$
$$\left. + \int_\tau^1 \sqrt{F\big(s\nabla v^-(x) + (1-s)\nabla v^+(x), v(x), f^-(x)\big)}\,\mathrm{d}s\right\}\mathrm{d}\mathcal{H}^{N-1}(x),$$

where we assume that the orientation of J_f $\mathcal{H}^{N-1}$ *a.e. coincides with the orientation of* $J_{\nabla v}$ *on* $J_f \cap J_{\nabla v}$.

(ii) *Moreover, if there exists* $h \in C^2(\mathbb{R}^N)$ *which satisfies the boundary conditions* $h = v$ *and* $\nabla h = T\nabla v$ *on* $\partial\Omega$, *then we can choose* v_ε *that satisfies the same boundary conditions,* $v_\varepsilon = v$ *and* $\nabla v_\varepsilon = T\nabla v$ *on* $\partial\Omega$. *If* v *satisfies the additional condition* $v \geqslant 0$ *in* Ω *then we have also* $v_\varepsilon \geqslant 0$ *in* Ω.

Corollary 3.2. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded* C^2 *domain. Let* $v \in W^{1,\infty}(\Omega)$ *satisfy* $\nabla v \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ *and* $|\nabla v| = 1$ *a.e. in* Ω. *Then,*

(i) *For every* $p \geqslant 1$ *there exists a family of functions* $\{v_\varepsilon\} \subset C^2(\mathbb{R}^N)$ *satisfying*

$$\lim_{\varepsilon\to 0^+} v_\varepsilon(x) = v(x) \quad \text{in } W^{1,p}(\Omega)$$

and

$$\lim_{\varepsilon\to 0}\left\{\varepsilon\int_\Omega \left|\nabla^2 v_\varepsilon(x)\right|^2 \mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega \left(1 - \left|\nabla v_\varepsilon(x)\right|^2\right)^2 \mathrm{d}x\right\} = \frac{1}{3}\int_{J_{\nabla v}} \left|\nabla v^+(x) - \nabla v^-(x)\right|^3 \mathrm{d}\mathcal{H}^{N-1}(x).$$

(ii) *Moreover, if* v *satisfies the boundary conditions* $v = 0$ *and* $\frac{\partial v}{\partial n} = -1$ *on* $\partial\Omega$, *then we can choose* v_ε *that satisfies the same boundary conditions,* $v_\varepsilon = 0$ *and* $\frac{\partial v_\varepsilon}{\partial n} = -1$ *on* $\partial\Omega$. *If* v *satisfies the additional condition* $v \geqslant 0$ *in* Ω *then we have also* $v_\varepsilon \geqslant 0$ *in* Ω.

Remark 1. A similar result for the Aviles–Giga functional has been independently obtained by Sergio Conti and Camillo de Lellis [6].

Acknowledgements

Part of this research was done during a visit at the Laboratoire J.-L. Lions of the University Paris VI in the framework of the RTN-Programme Fronts-Singularities. I am grateful to Prof. Haïm Brezis for the invitation, to Prof. Danielle Hilhorst for the tremendous effort she made in arranging this visit, and to Radu Ignat for many interesting discussions.

References

[1] L. Ambrosio, Metric space valued functions of bounded variation, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 17 (1990) 439–478.
[2] L. Ambrosio, C. De Lellis, C. Mantegazza, Line energies for gradient vector fields in the plane, Calc. Var. Partial Differential Equations 9 (1999) 327–355.
[3] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Math. Monographs, Oxford University Press, New York, 2000.
[4] P. Aviles, Y. Giga, A mathematical problem related to the physical theory of liquid crystal configurations, Proc. Centre Math. Anal. Austral. Nat. Univ. 12 (1987) 1–16.
[5] P. Aviles, Y. Giga, On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg–Landau type energy for gradient fields, Proc. Roy. Soc. Edinburgh Sect. A 129 (1999) 1–17.
[6] S. Conti, C. De Lellis, Sharp upper bounds for a variational problem with singular perturbation, Preprint.
[7] J. Dávila, R. Ignat, Lifting of BV functions with values in S^1, C. R. Acad. Sci. Paris, Ser. I 337 (2003) 159–164.
[8] C. De Lellis, An example in the gradient theory of phase transitions, ESAIM Control Optim. Calc. Var. 7 (2002) 285–289 (electronic).
[9] A. DeSimone, S. Müller, R.V. Kohn, F. Otto, A compactness result in the gradient theory of phase transitions, Proc. Roy. Soc. Edinburgh Sect. A 131 (2001) 833–844.
[10] N.M. Ercolani, R. Indik, A.C. Newell, T. Passot, The geometry of the phase diffusion equation, J. Nonlinear Sci. 10 (2000) 223–274.
[11] L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, Stud. Adv. Math., CRC Press, Boca Raton, FL, 1992.
[12] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Monographs Math., vol. 80, Birkhäuser, Basel, 1984.
[13] R. Ignat, A. Poliakovsky, An example related to optimal lifting in BV-space, Preprint.
[14] W. Jin, R.V. Kohn, Singular perturbation and the energy of folds, J. Nonlinear Sci. 10 (2000) 355–390.
[15] L. Modica, The gradient theory of phase transitions and the minimal interface criterion, Arch. Rational Mech. Anal. 98 (1987) 123–142.
[16] A. Poliakovsky, Upper bounds for singular perturbation problems involving gradient fields, Preprint.
[17] A. Poliakovsky, On a singular perturbation problem related to optimal lifting in BV-space, Preprint.
[18] P. Sternberg, The effect of a singular perturbation on nonconvex variational problems, Arch. Rational Mech. Anal. 101 (1988) 209–260.
[19] A.I. Volpert, S.I. Hudjaev, Analysis in Classes of Discontinuous Functions and Equations of Mathematical Physics, Martinus Nijhoff, Dordrecht, 1985.